Optimal error regions for quantum state estimation: They are as hard to compute as this title is long

Daniel Suess, Łukasz Rudnicki, David Gross

Outline

- error regions crash course
- error regions for quantum state estimation (QSE)
- complexity result for Bayesian credible regions
- comments on Frequentist’s confidence regions

Error Regions

Frequentist
→ confidence regions

Bayesian
→ credible regions

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★

Error Regions

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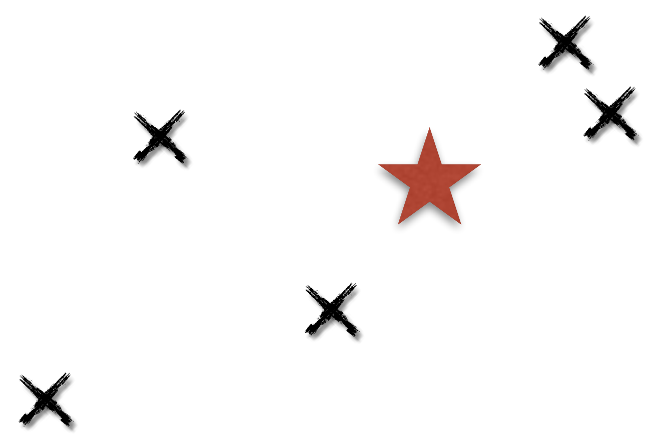

Error Regions

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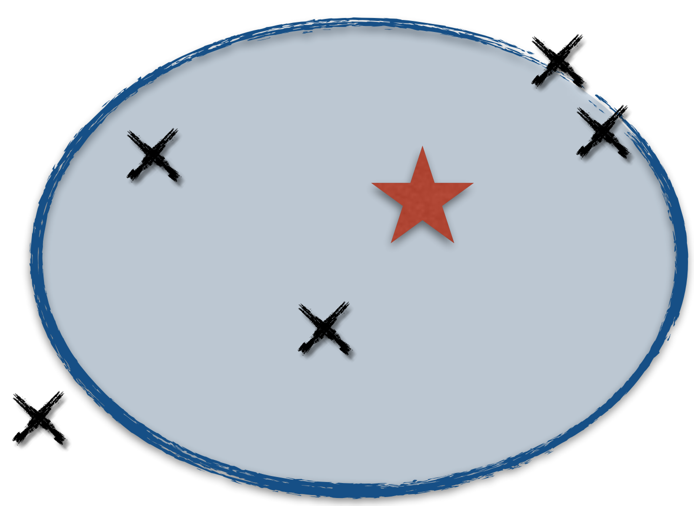

c.r. should contain true value with probability ≥ *1 - α*

Error Regions

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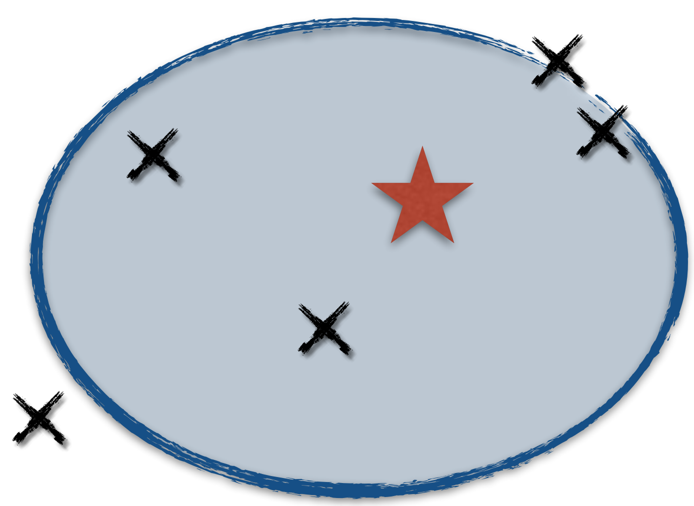

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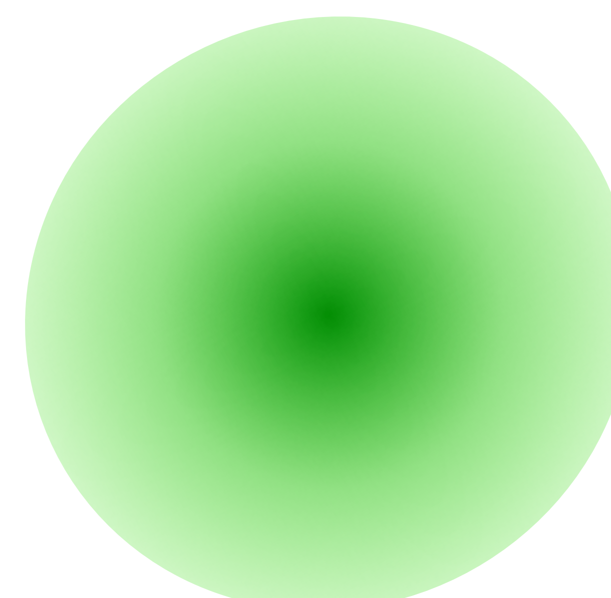

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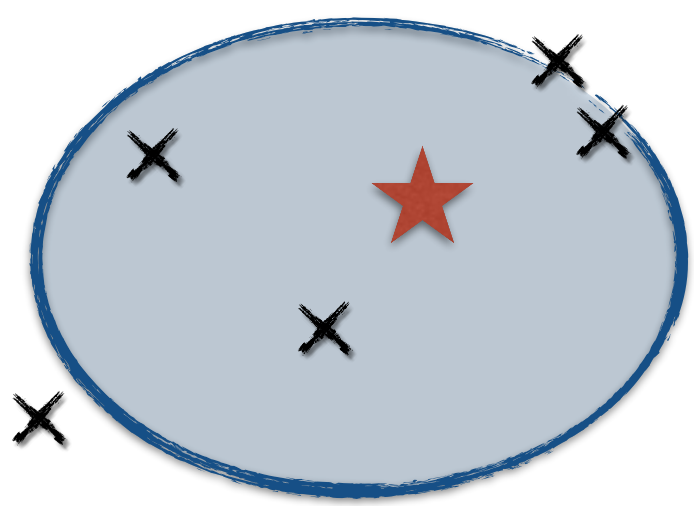

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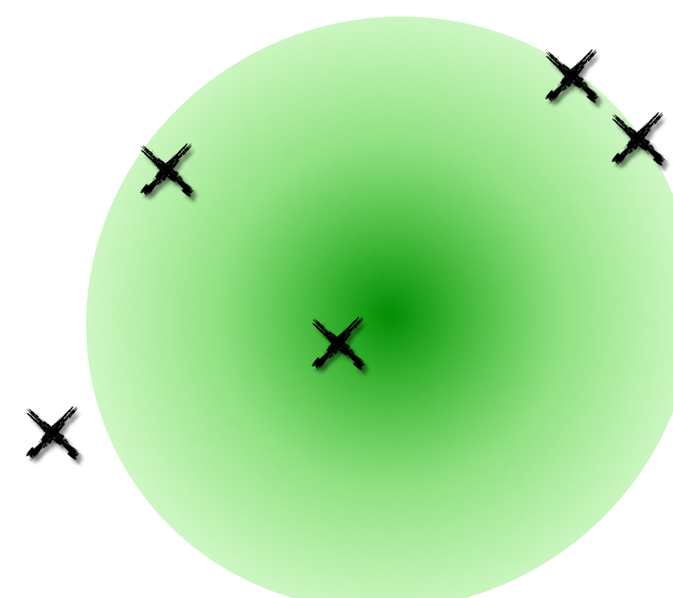

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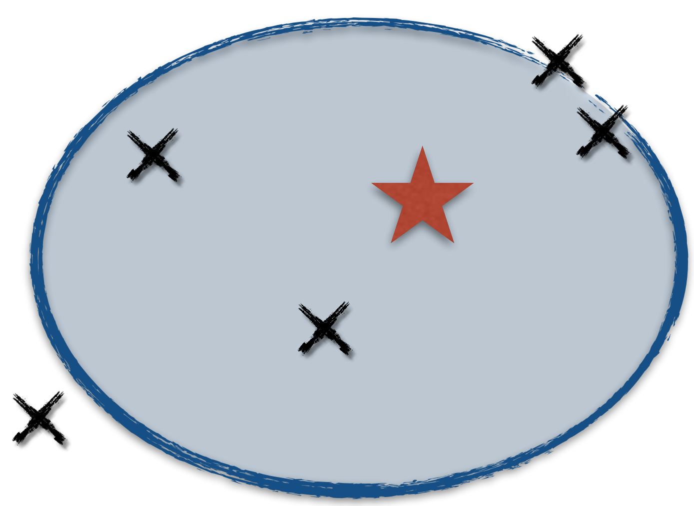

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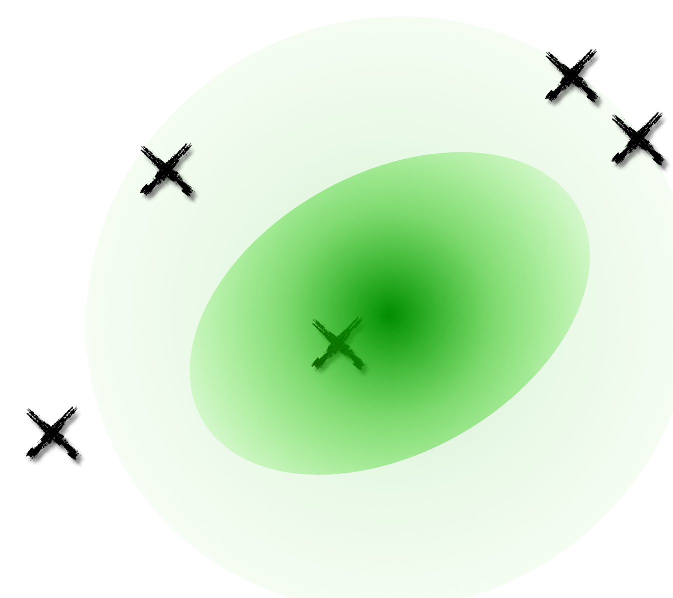

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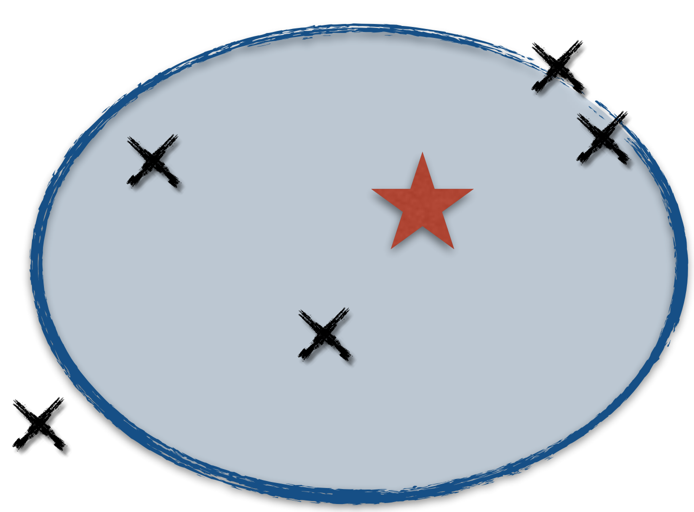

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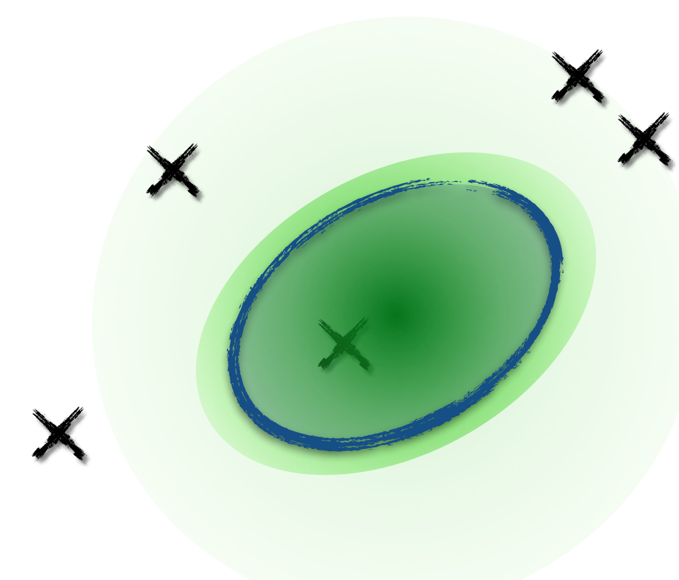

mass of posterior contained in c.r. should be ≥ *1 - α*

Error Regions

Frequentist
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c.r. should contain true value with probability ≥ *1 - α*

- minimal volume (uniform, average, or minimax)
- unbiasedness
- …

Bayesian
→ credible regions

mass of posterior contained in c.r. should be ≥ *1 - α*

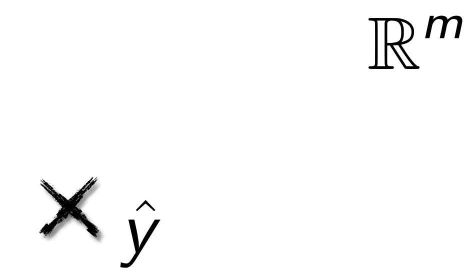
- minimal volume

Geometric construction of Error Regions in QSE

$\mathbb{R}^m$

$\hat{y}$

Geometric construction of Error Regions in QSE



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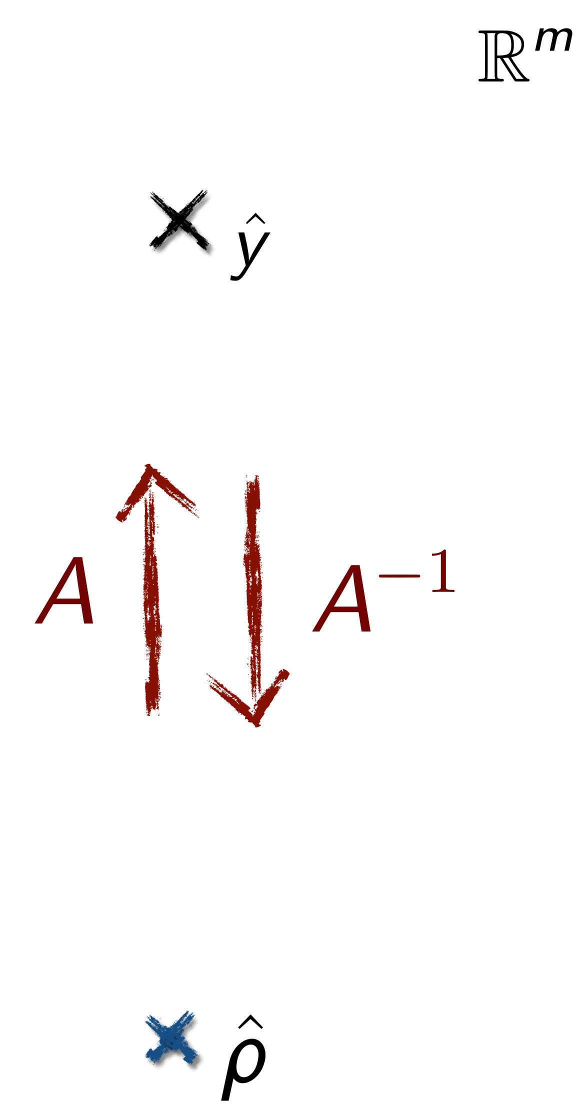
Geometric construction of Error Regions in QSE

- task: estimate state ρ from measurement outcomes y
- simplest case: linear inversion of Gaussian measurements



$$\mathcal{S}^d = \{X \in \mathbb{H}^{d\times d} : \operatorname{tr} X = 1\}$$

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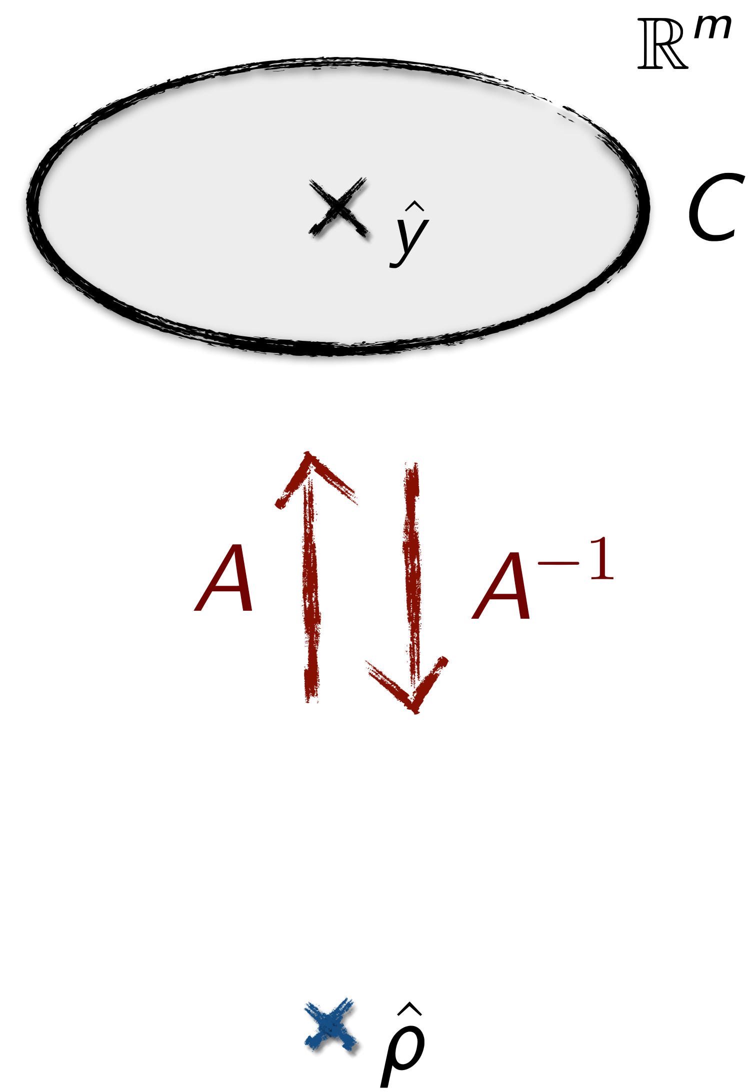

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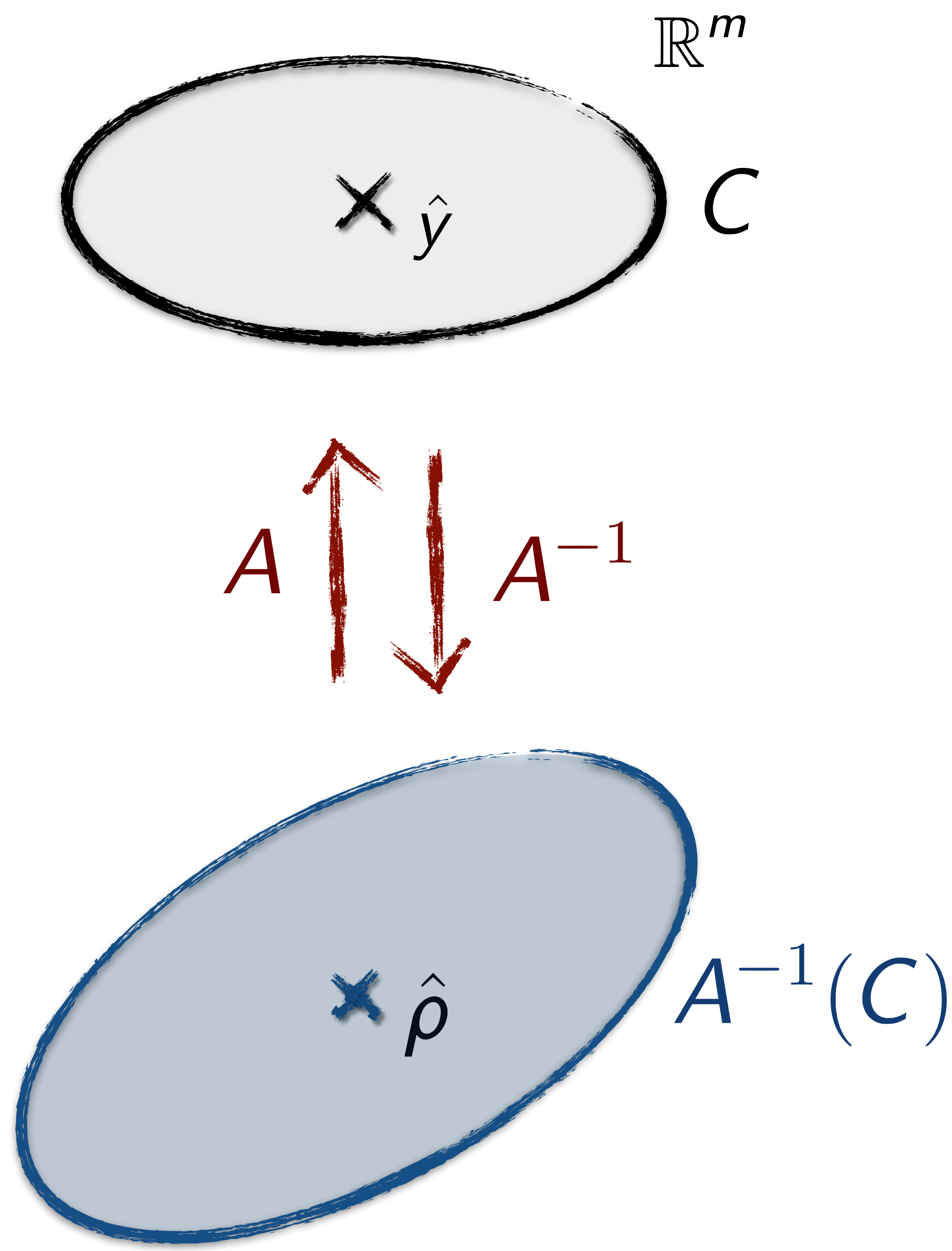

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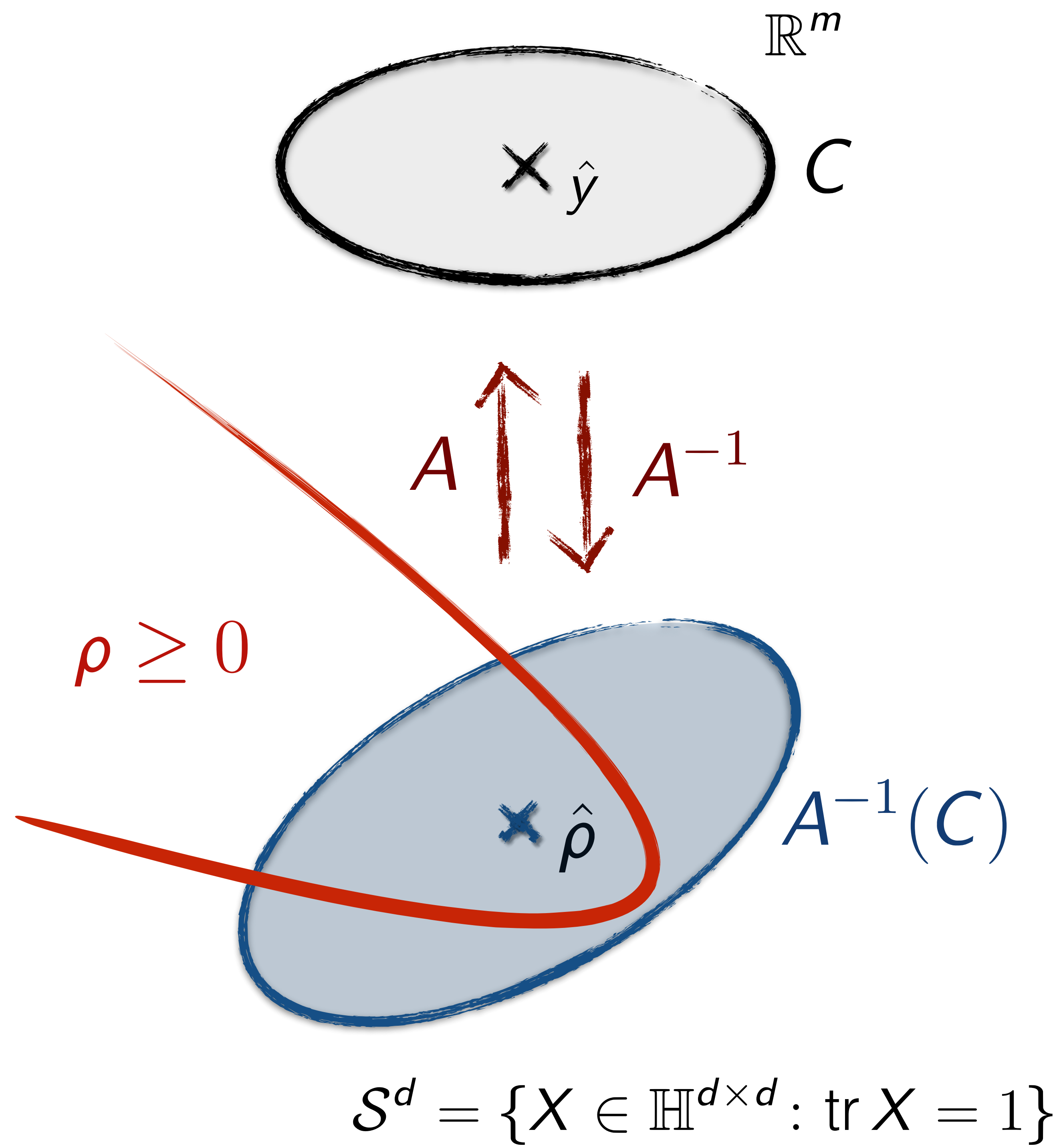

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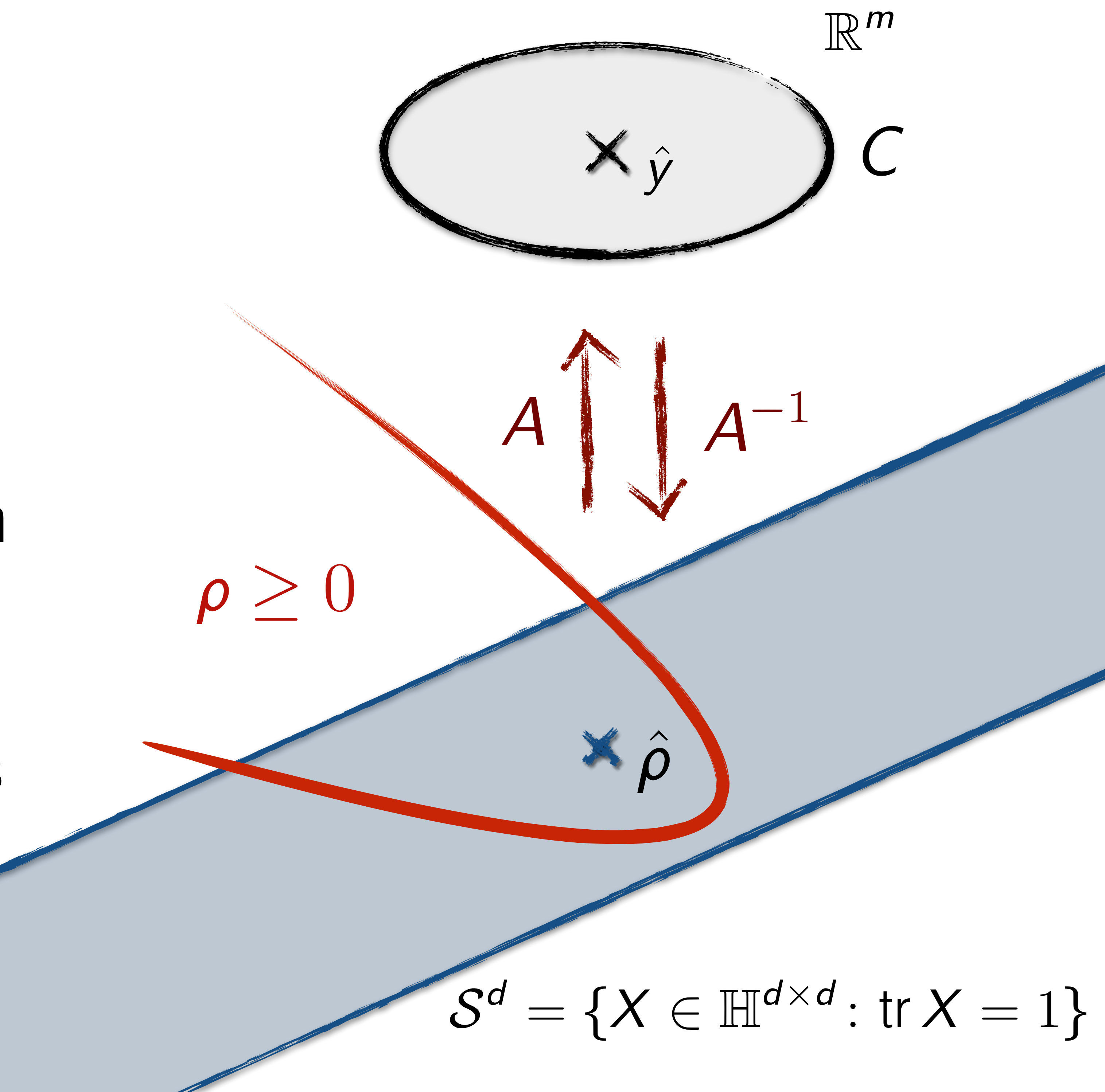

Result I

Problem 1 For given ellipsoid $\mathcal{E} \subset \mathcal{S}^d$, decide whether $\mathcal{E} \subset \text{PSD}$ or $\mathcal{E} \not\subset \text{PSD}$.

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- coNP complete $\Rightarrow$ no efficient algorithm
- **Corollary:** No efficient algorithm for computing volume of truncated ellipsoid
- statistical significance?

Bayesian POV

- Gaussian prior with mean θ and covariance matrix Σ

$$\pi_{\theta,\Sigma}(\rho) = (2\pi)^{-\frac{N}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\|\rho - \theta\|_{\Sigma}^{2}\right)$$

- Bayesian update with Gaussian observation
  $\Longrightarrow$ Gaussian posterior $\pi_{\theta',\Sigma'}$

- minimal volume credible regions from greedy construction

$$\mathcal{E}(R_{1-\alpha}) = \left\{\rho \in \mathcal{S}^d \colon \|\rho - \theta'\|_{\Sigma'} \leq R_{1-\alpha}\right\}$$

- $R_{1-\alpha}$ determined by $\int_{\mathcal{E}} \pi_{\theta',\Sigma'}(\rho)\mathrm{d}\rho = 1 - \alpha$

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efficiently computable

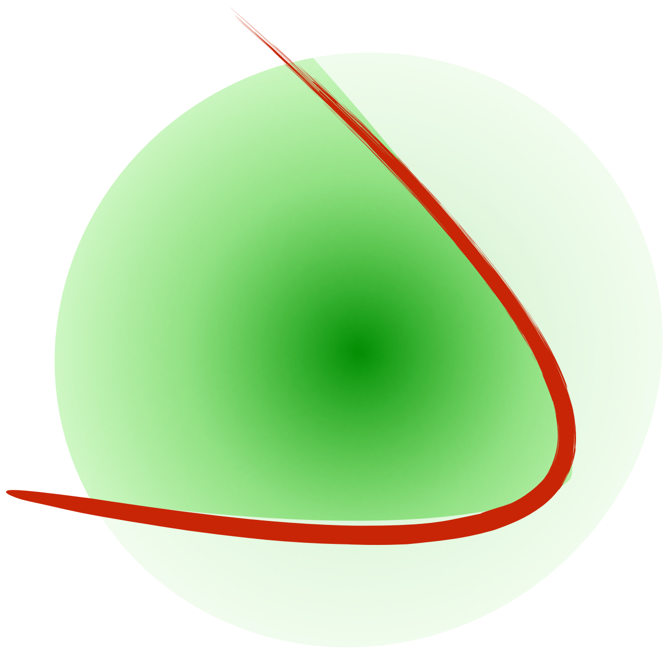
Bayesian POV: Physical Constraints

- truncated Gaussian prior

$$\pi^+_{\theta,\Sigma}(\rho) = C \times \pi_{\theta,\Sigma}(\rho) \times \chi^+(\rho)$$

$\Longrightarrow$ truncated Gaussian posterior

$$\pi^+_{\theta',\Sigma'}(\rho) = C' \times \pi_{\theta',\Sigma'}(\rho) \times \chi^+(\rho)$$



- optimal error regions: truncated ellipsoids with radius $R^+_{1-\alpha}$

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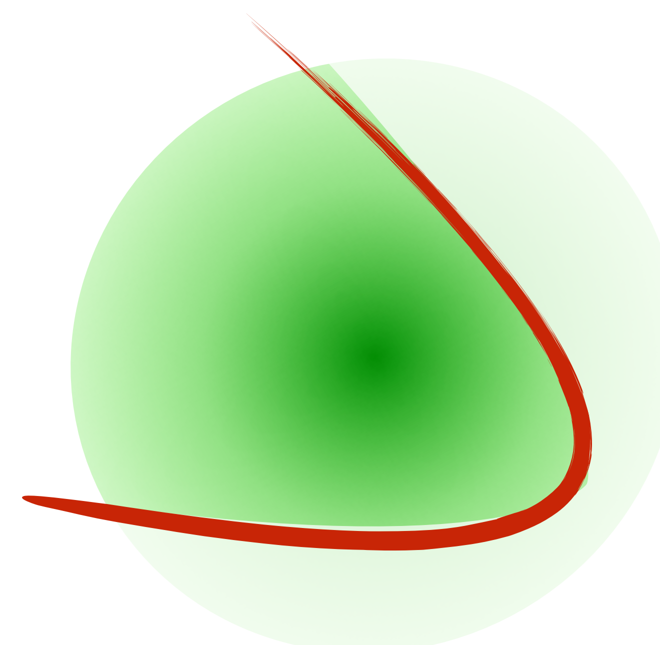

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Bayesian POV: Physical Constraints

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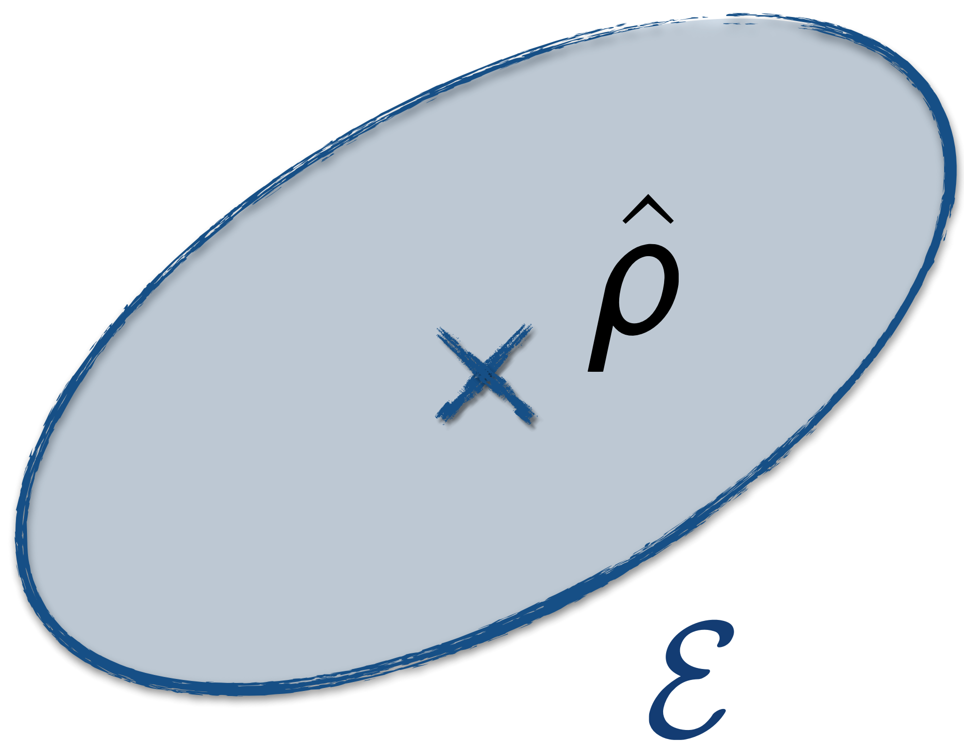
Problem 2 For given mean $\theta \in \mathcal{S}^d$, covariance matrix Σ, credibility $1-\alpha \in [0,1]$, and accuracy δ with $\delta^{-1} \in \mathbb{N}$, determine the radius of the minimal-volume credible ellipsoid $R^{+}_{1-\alpha}$ with given accuracy.

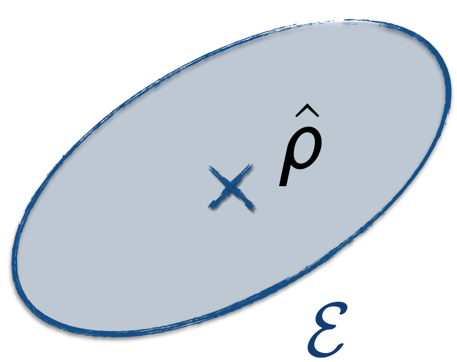
not efficiently computable

Frequentist Optimal Confidence Region?



$$\hat{\rho} = \frac{1}{d}\,\mathrm{I} + \sum_i \langle \hat{\rho} \rangle_i \, \sigma_i \qquad \text{with } \langle \hat{\rho} \rangle_i := \frac{1}{N_i} \sum_{k=1}^{N_i} y_{i,k}$$

Frequentist Optimal Confidence Region?

	MinMax Expected Volume	Admissible	Expected Volume + Unbiased
			

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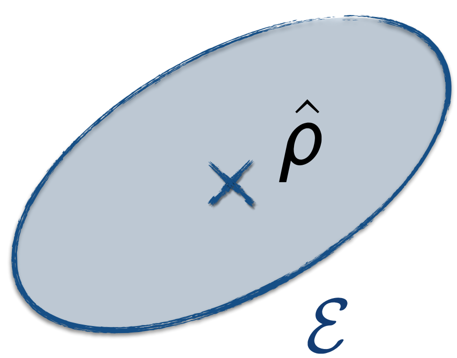
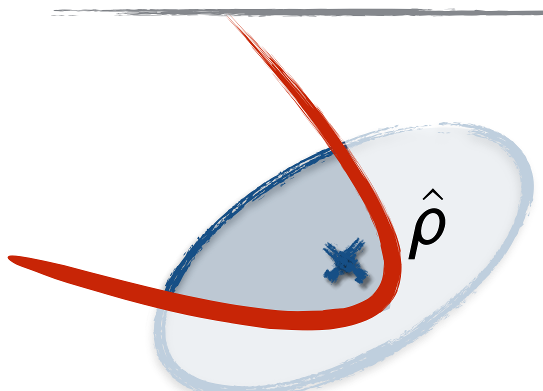
 $\hat{\rho}$ $\mathcal{E}$	✗	✗	✓
 $\hat{\rho}$ $\mathcal{E}^+ = \mathcal{E} \cap \mathrm{PSD}$			

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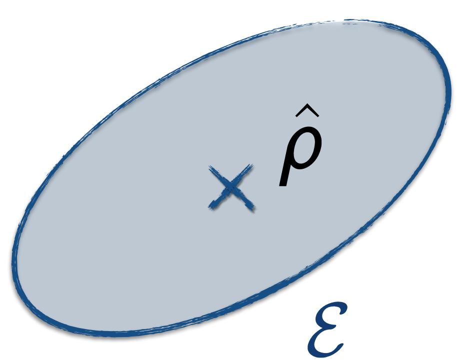
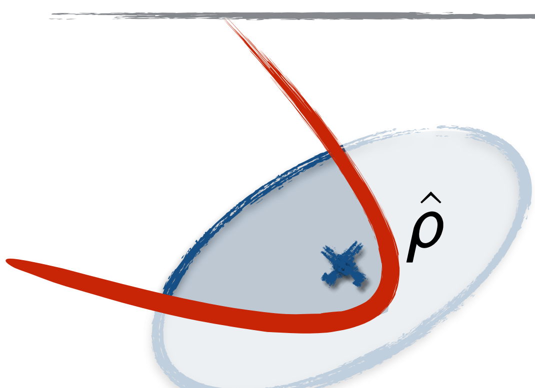
	✗	✗	✓
 $\mathcal{E}^+ = \mathcal{E} \cap \mathrm{PSD}$	✗	?	¯_(ツ)_/¯

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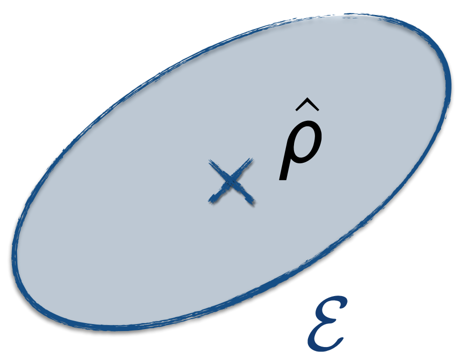
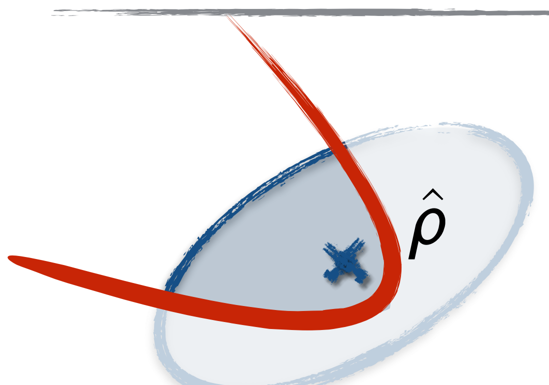
	✗	✗	✓
 $\mathcal{E}^+ = \mathcal{E} \cap \mathrm{PSD}$	✗	?	¯_(ツ)_/¯

Further reading: https://en.wikipedia.org/wiki/Stein%27s_example

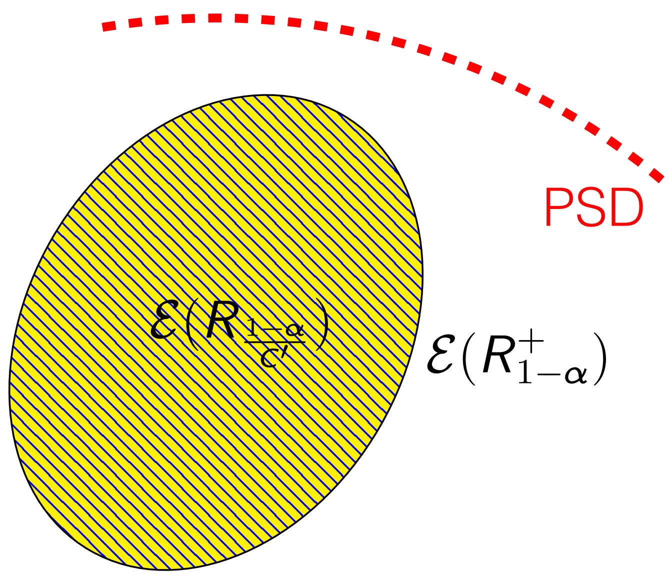
Conclusion

- truncated ellipsoids as error regions for QSE
- constitute optimal Bayesian credible regions for Gaussian posterior distribution
- computing radius computational hard problem
- application to Frequentists’ confidence regions: ¯_(ツ)_/¯

Let's look at credibility level $\frac{1-\alpha}{C'}$ & recall truncated posterior

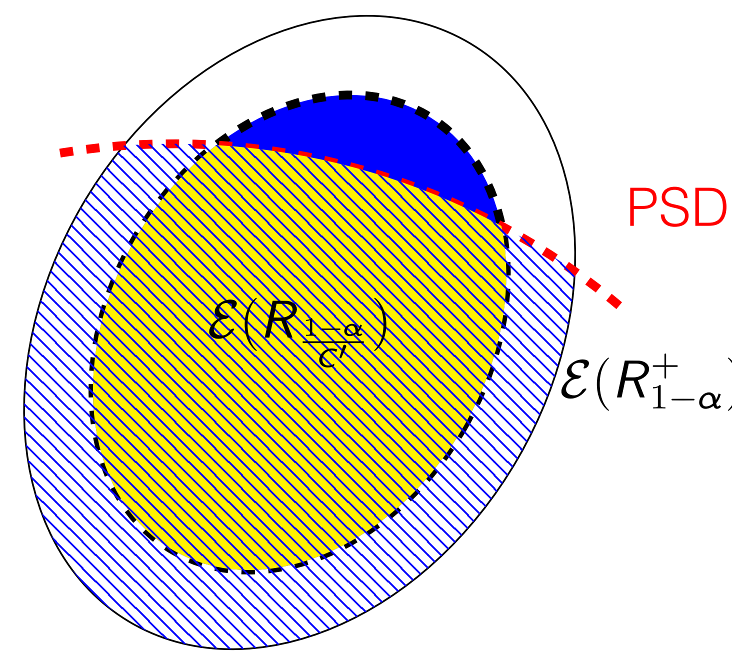
$$\pi^{+}_{\theta',\Sigma'}(\rho) = C' \times \pi_{\theta',\Sigma'}(\rho) \times \chi^{+}(\rho)$$

$\mathcal{E}(R_{\frac{1-\alpha}{C'}}) \subset \text{PSD}$



$\implies R_{\frac{1-\alpha}{C'}} = R^{+}_{1-\alpha}$

$\mathcal{E}(R_{\frac{1-\alpha}{C'}}) \not\subset \text{PSD}$



$\implies R_{\frac{1-\alpha}{C'}} < R^{+}_{1-\alpha}$

Frequentist POV

- special case: Pauli measurements $\implies$ $\rho = \frac{1}{d}\,\mathrm{I} + \sum_i \langle \rho \rangle_i \, \sigma_i$
- estimate $\langle \rho \rangle_i$ by empirical mean

$$\langle \hat{\rho} \rangle_i := \frac{1}{N_i} \sum_{k=1}^{N_i} y_{i,k}$$

where $y_{i,k} \in \{\pm 1\}$…outcome of the k-th measurement of σ_i

- Gaussian assumption $\langle \hat{\rho} \rangle_i \sim \mathcal{N}(\langle \hat{\rho} \rangle_i, \Sigma_i)$ $\implies$ "natural" confidence region

$$\mathcal{E} = \left\{ \hat{\rho} + \sum_i \Sigma_i \, u_i \, \sigma_i : \ \sum u_i^2 \leq R^2 \right\}$$