Optimal error regions for quantum state estimation: They are as hard to compute as this title is long

Daniel Suess, Łukasz Rudnicki, David Gross

arXiv:1608.00374

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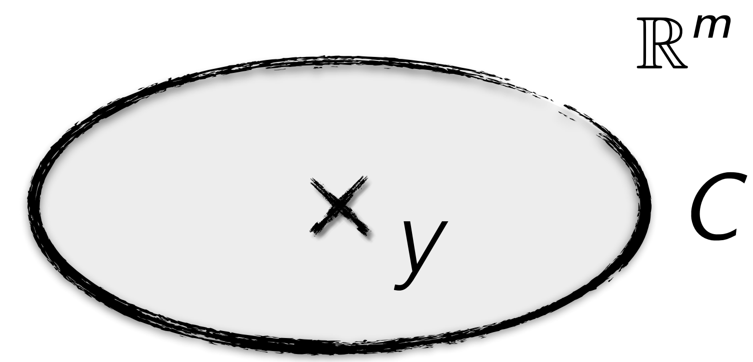

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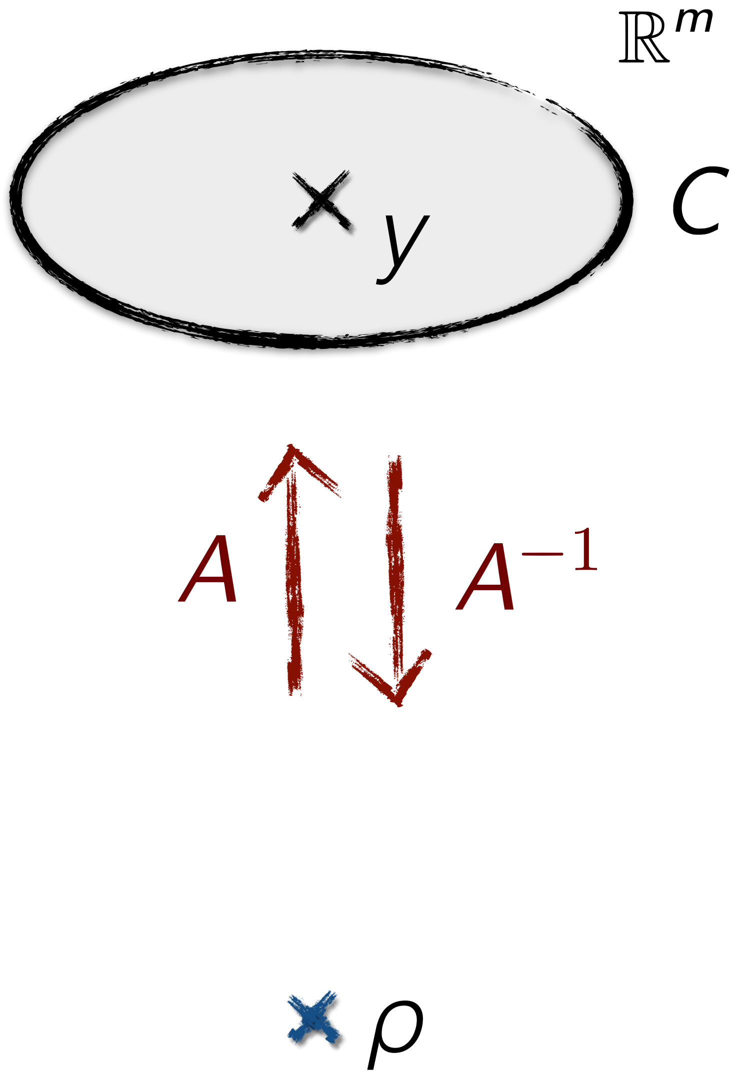

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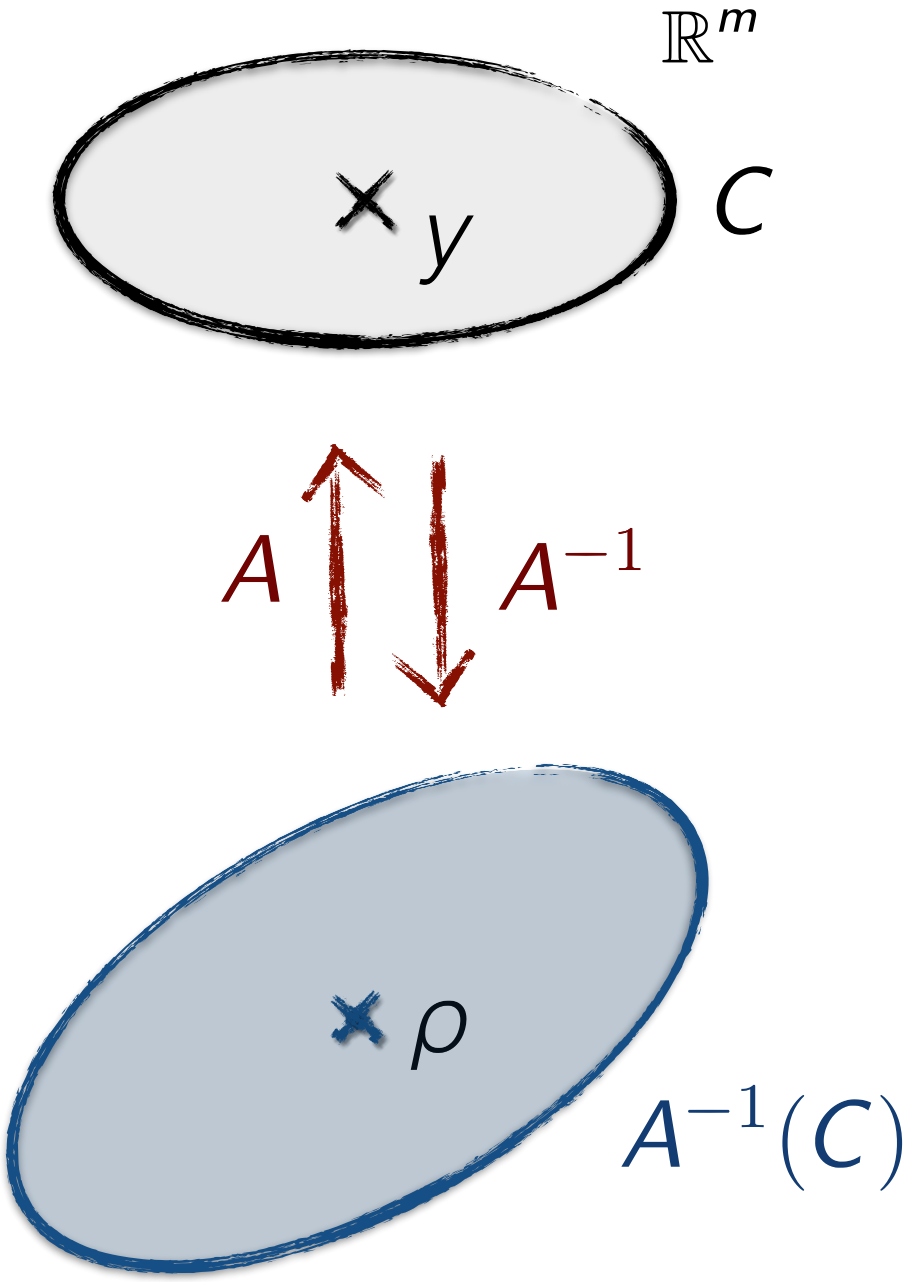

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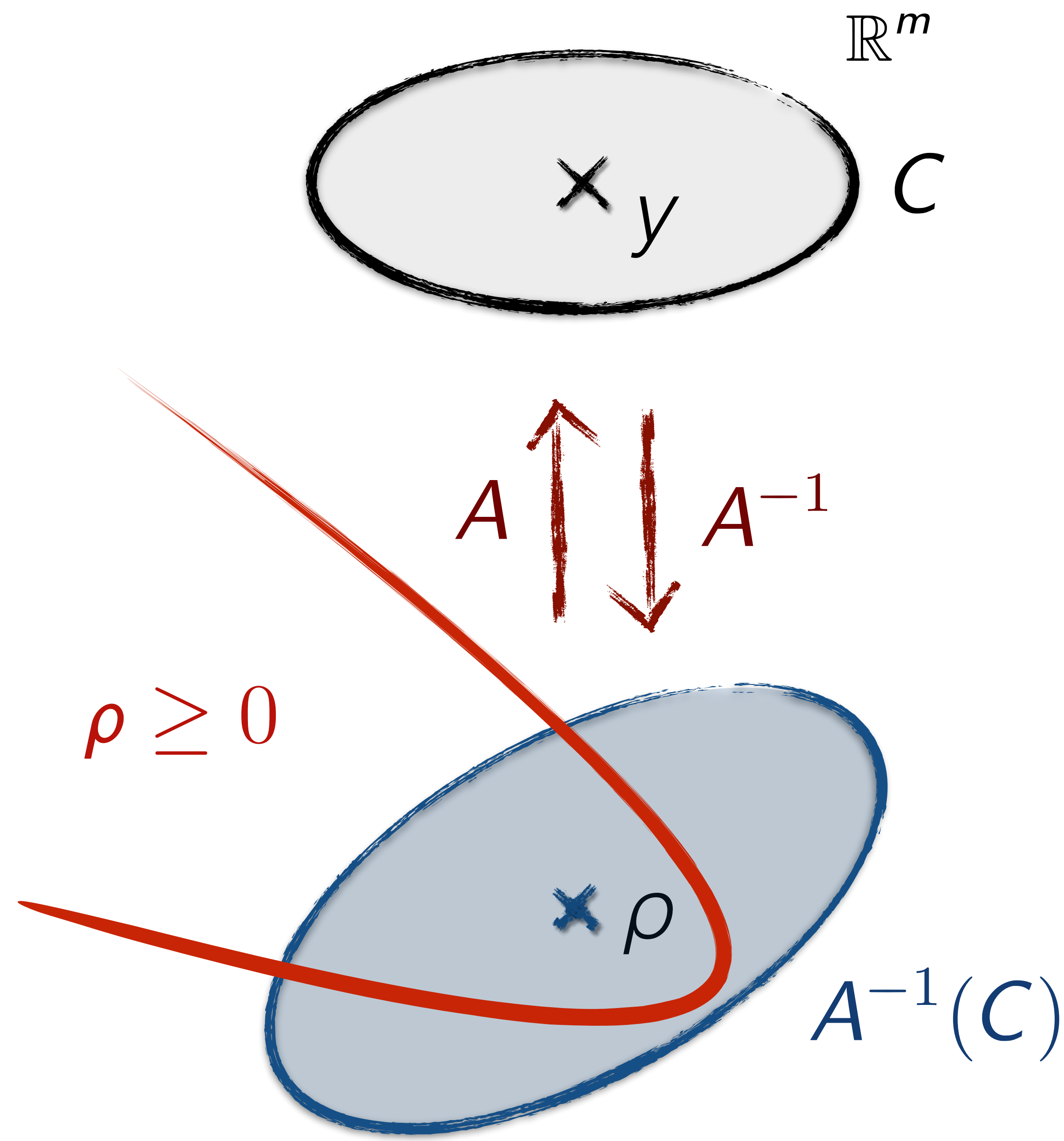

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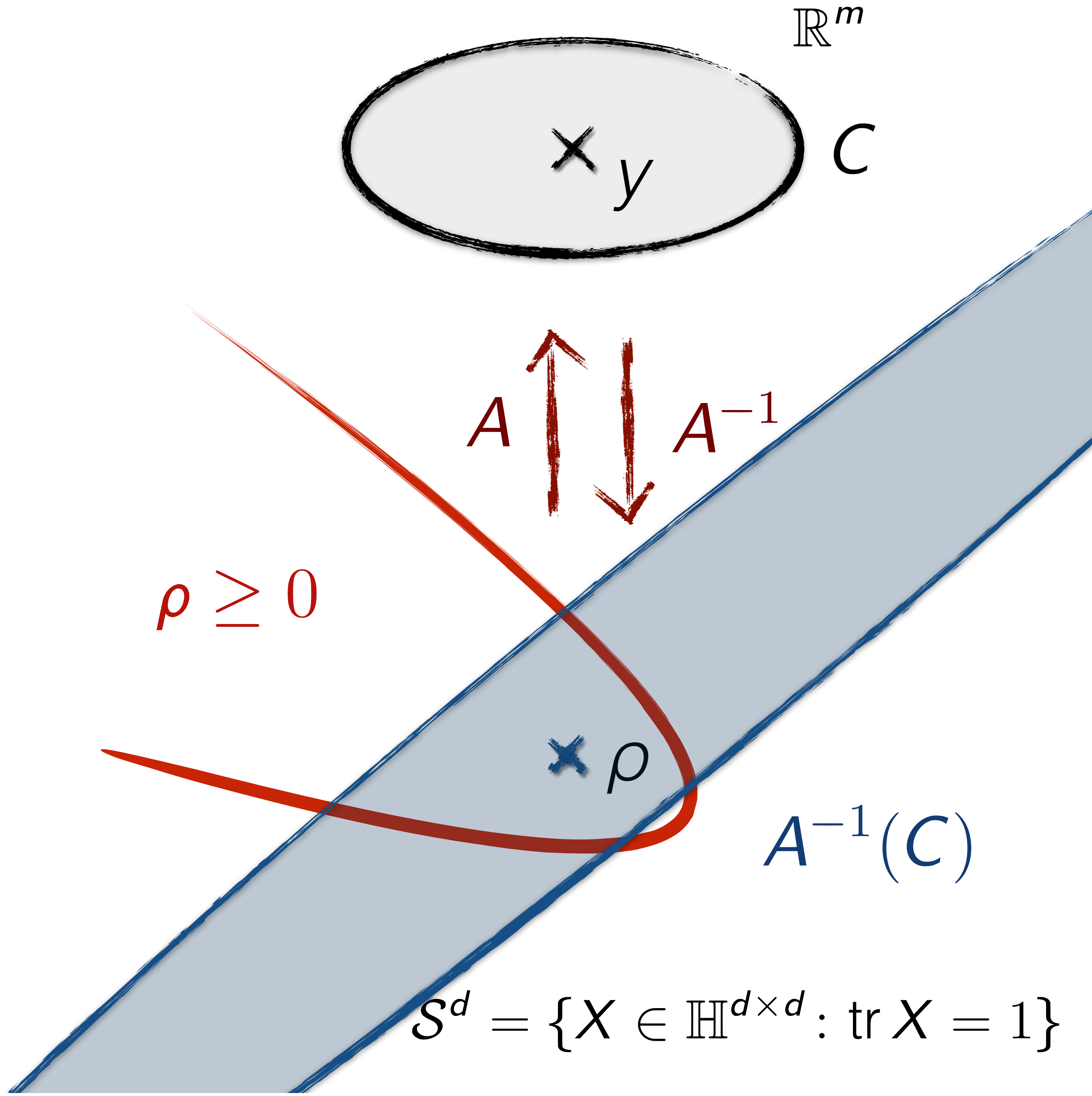

A. Carpentier, J. Eisert, D. Gross, and R. Nickl, arXiv:1504.03234

Bayesian Credible Regions

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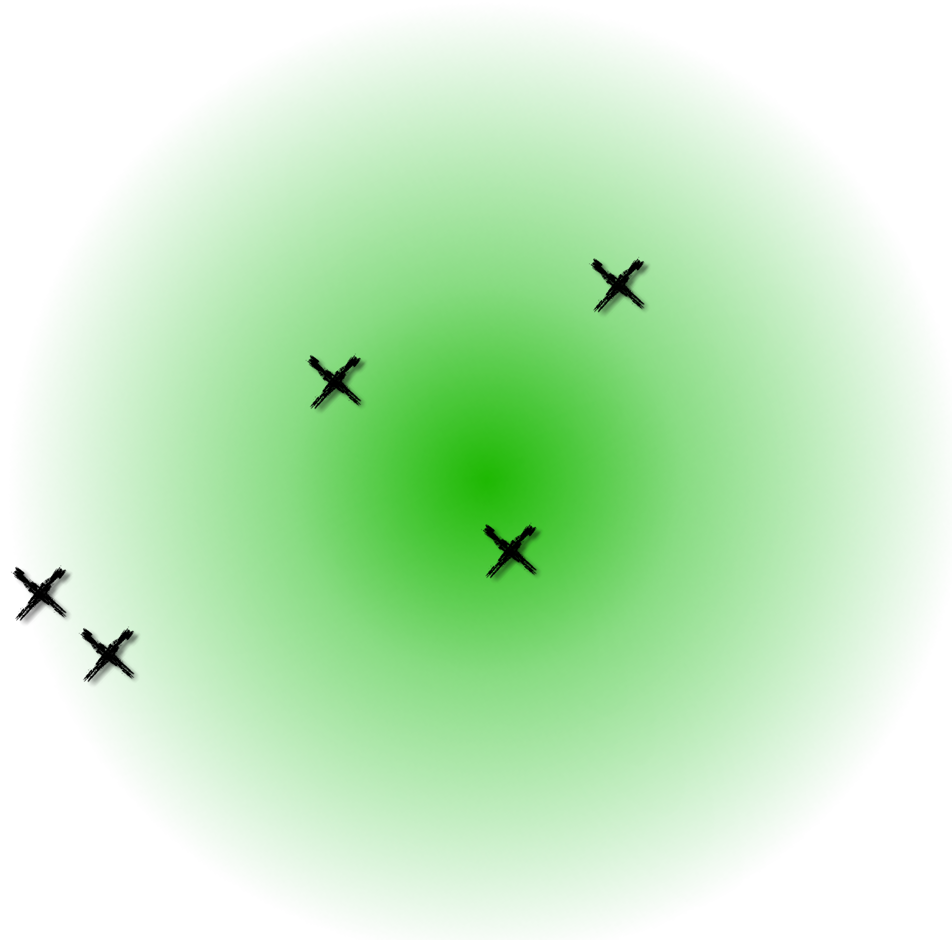

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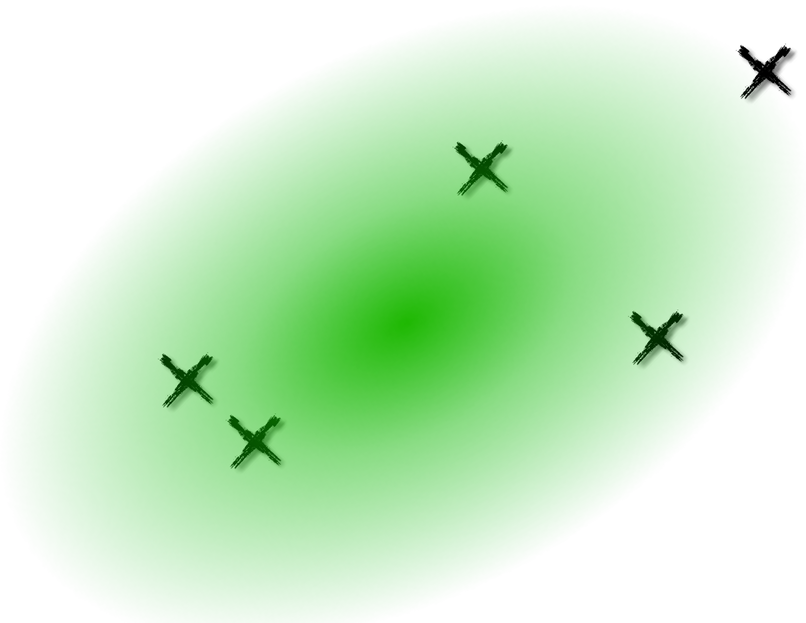

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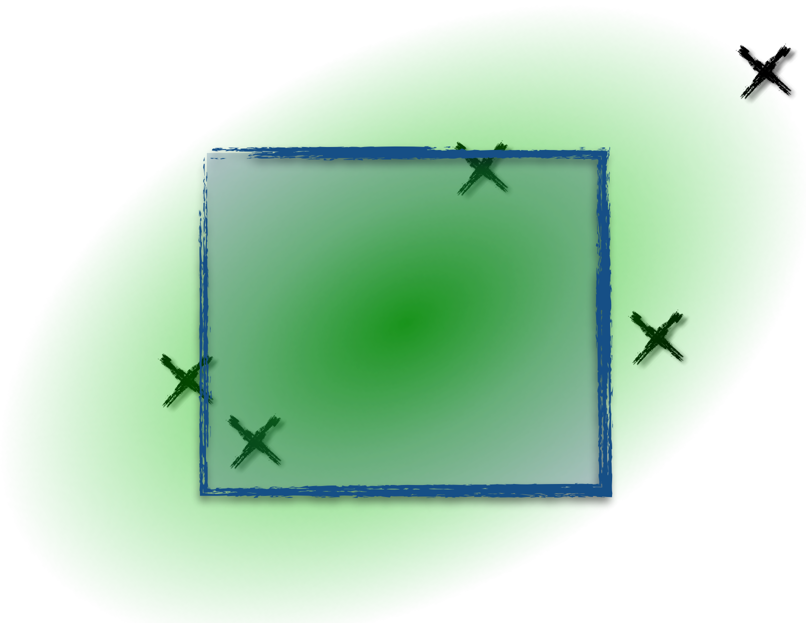

- mass of posterior contained in c.r. should be ≥ *1 - α*

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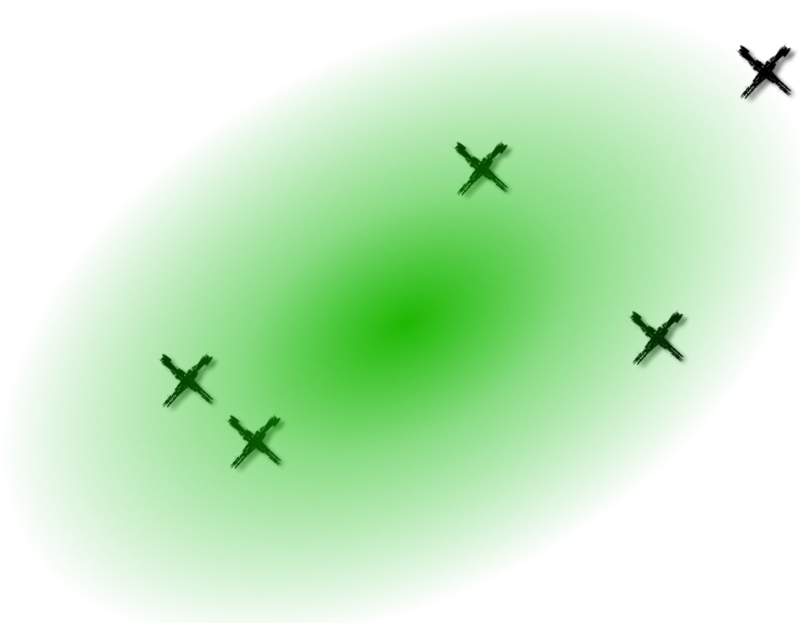

- mass of posterior contained in c.r. should be $\geq$ *1 - α*
- optimality: e.g. minimal volume

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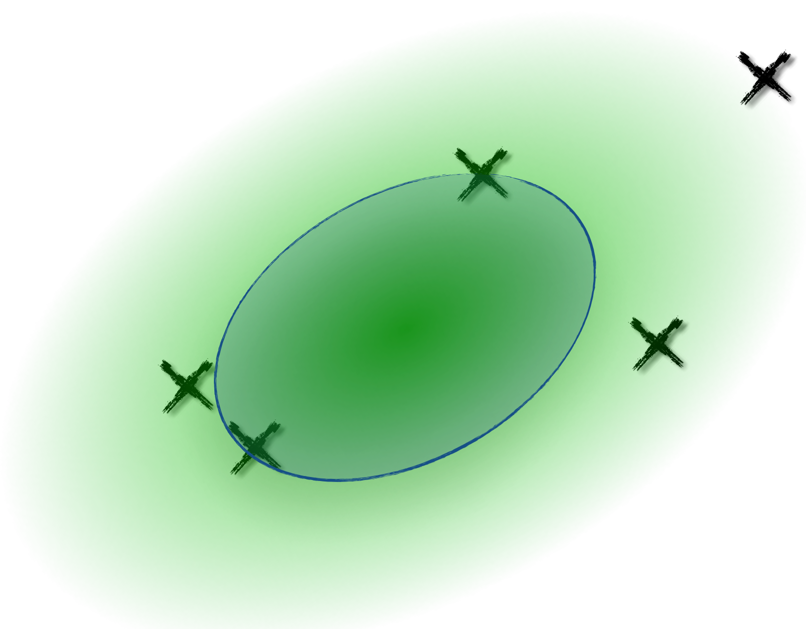

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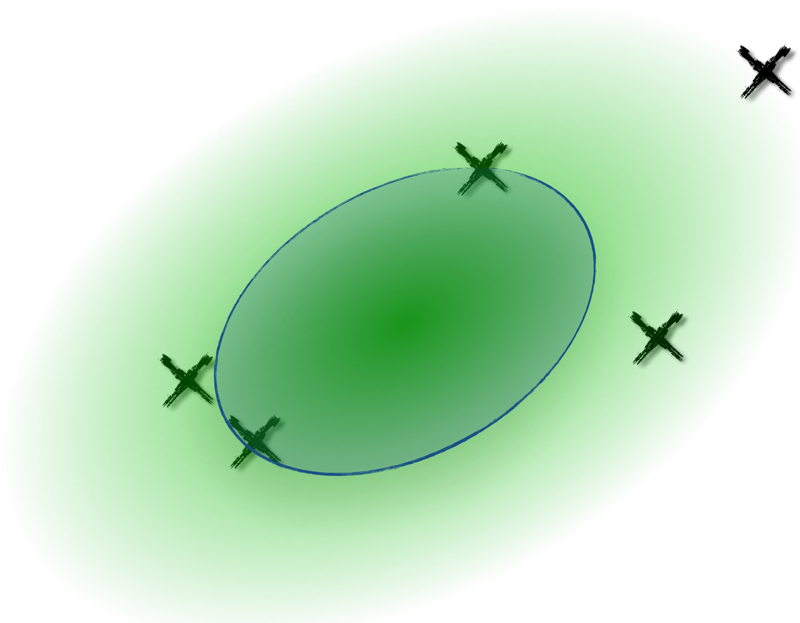

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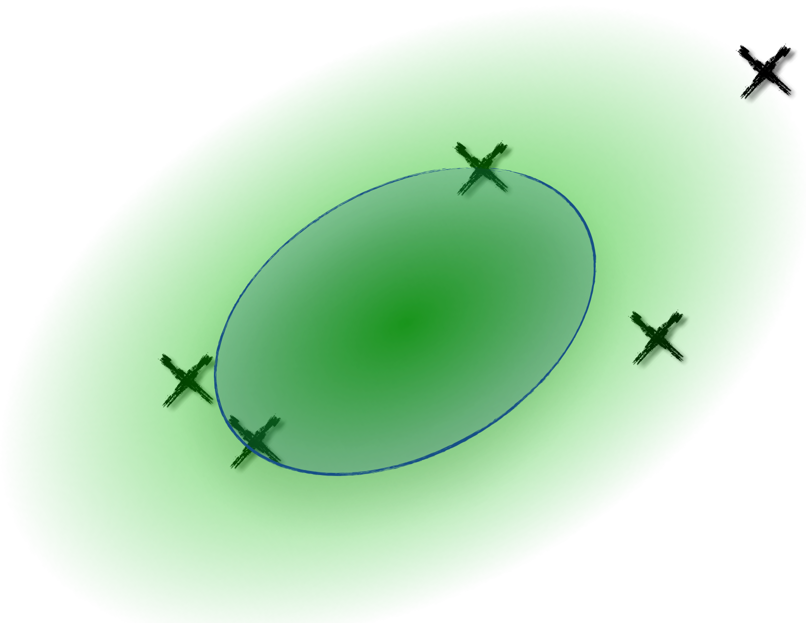

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$$\mathcal{E}(R_{1-\alpha}) = \left\{\rho \in \mathcal{S}^d : \|\rho-\theta\|_{\Sigma} \leq R_{1-\alpha}\right\}$$

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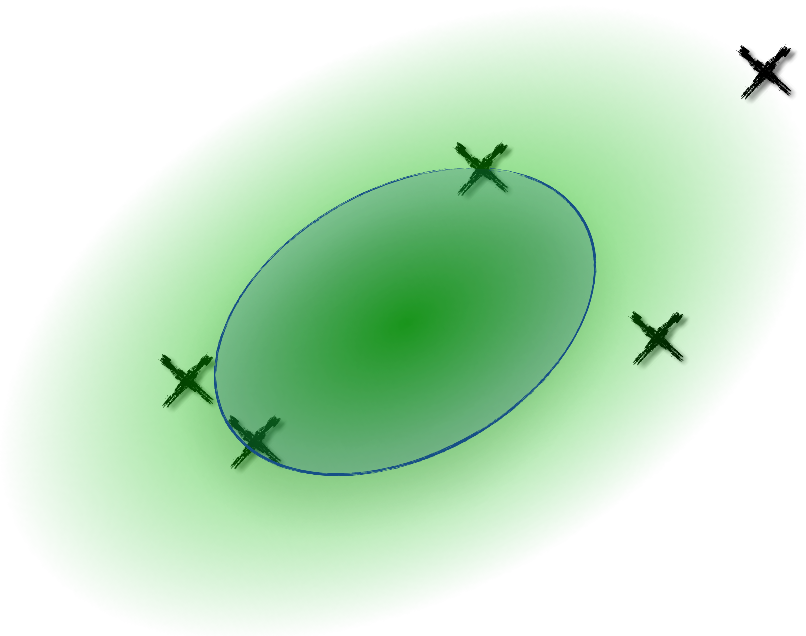

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Credible Regions for QSE

- truncated Gaussian prior

$$\pi^+_{\theta_0,\Sigma_0}(\rho) = C_{\theta_0,\Sigma_0} \times \pi_{\theta_0,\Sigma_0}(\rho) \times \mathbb{1}_{\mathrm{PSD}}(\rho)$$

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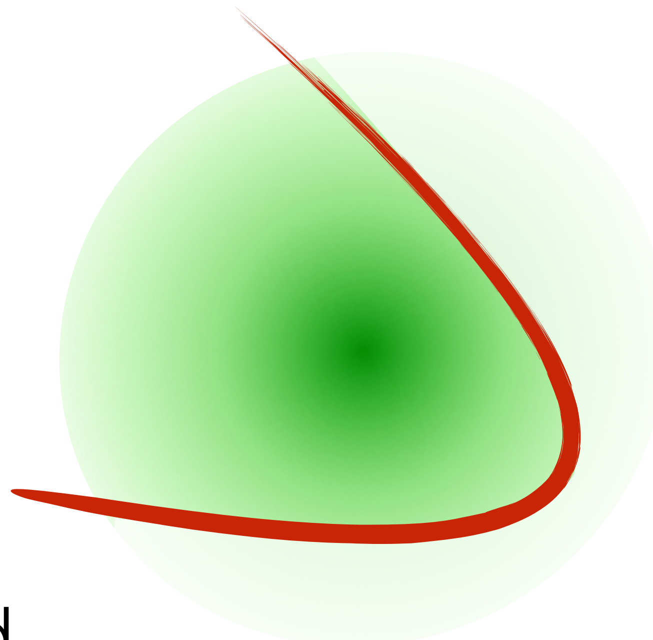
- minimal volume error region: truncated ellipsoid

$$\mathcal{E}(R^+_{1-\alpha}) \cap \mathrm{PSD}$$



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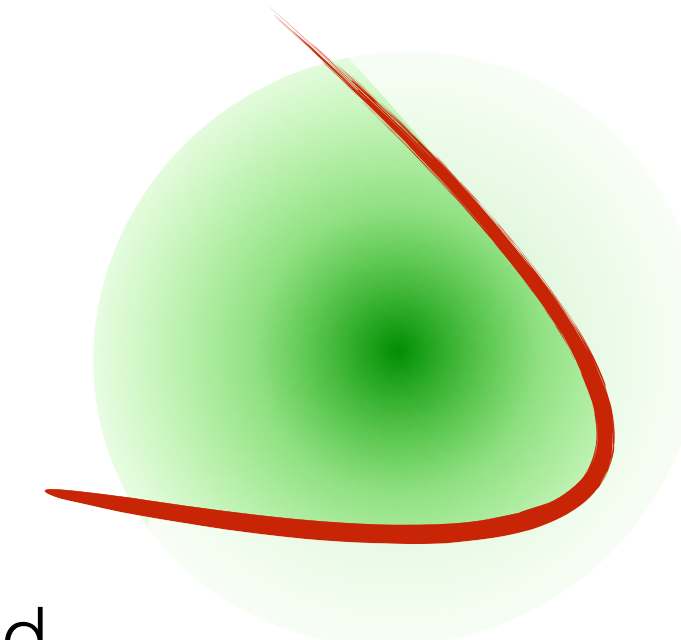

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$$\mathcal{E}(R^{+}_{1-\alpha}) \cap \mathrm{PSD}$$

not efficiently computable

Credible Regions for QSE

- truncated Gaussian prior
- in
- min

Problem 1 For given mean $\theta \in \mathcal{S}^d$, covariance matrix Σ, credibility $1-\alpha \in [0,1]$, and accuracy δ with $\delta^{-1} \in \mathbb{N}$, determine the radius of the minimal-volume credible ellipsoid $R^+_{1-\alpha}$ with given accuracy.

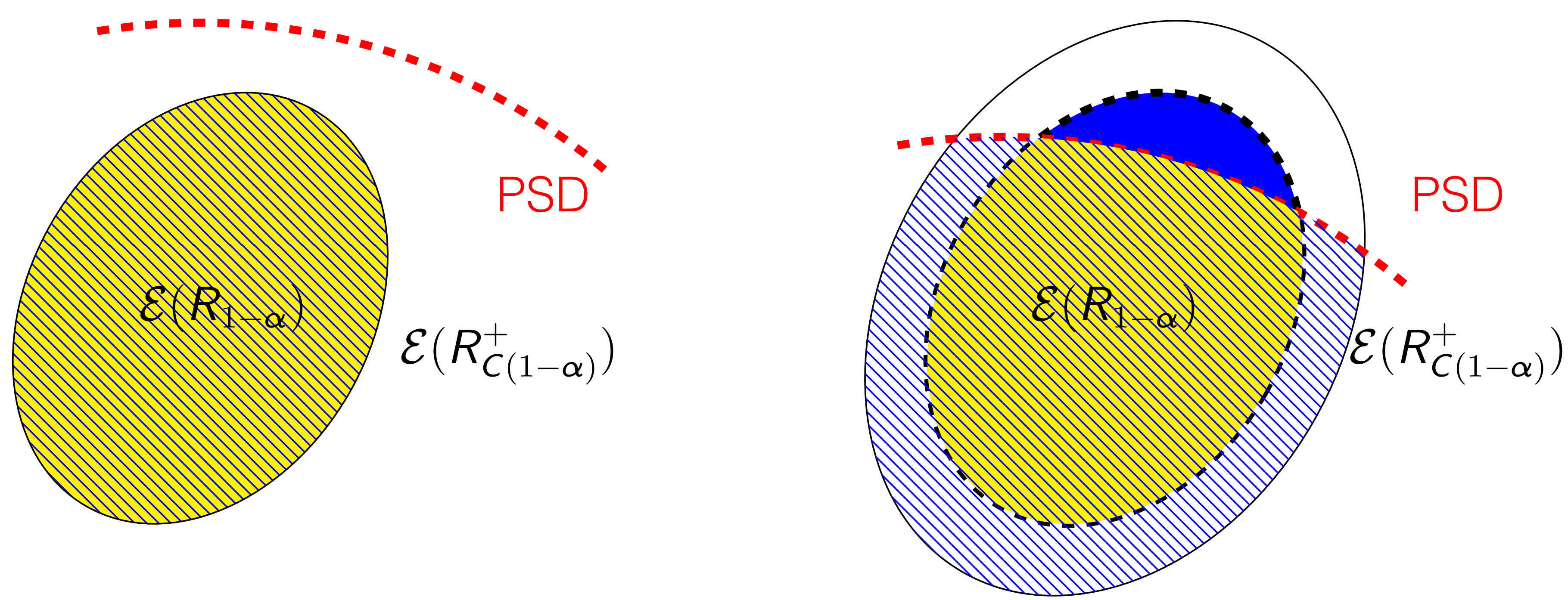
not efficiently computable

Proof Outline I

Problem 2 For given ellipsoid $\mathcal{E} \subset \mathcal{S}^d$, decide whether $\mathcal{E} \subset \text{PSD}$ or $\mathcal{E} \not\subset \text{PSD}$.

- computationally hard (coNP complete)
- proof by reduction to balanced sum problem

Proof Outline II



- given any ellipsoid $\mathcal{E}$, encode it as a MVCR of a Gaussian distribution $\pi_{\theta,\Sigma}$ with credibility $1-\alpha$
- algorithm to problem 1 allows you to compute normalisation constant C of truncated distribution and, hence, $R^+_{C(1-\alpha)}$
- check whether $R_{1-\alpha} = R^+_{C(1-\alpha)}$

Frequentist Confidence Regions

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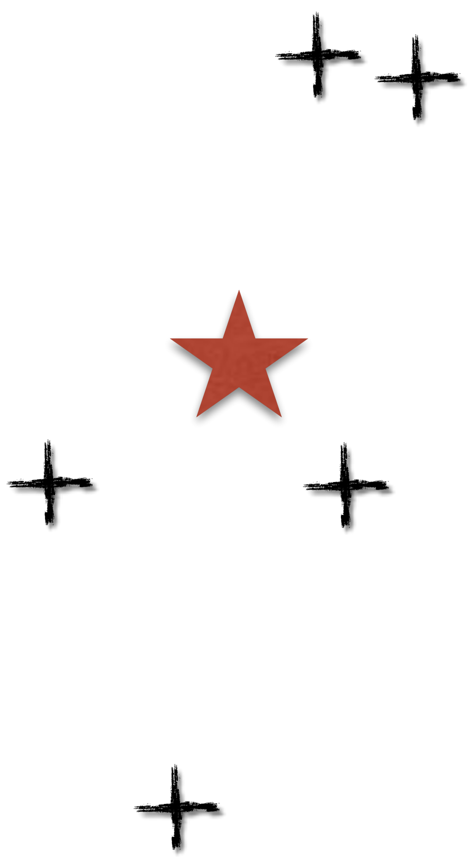

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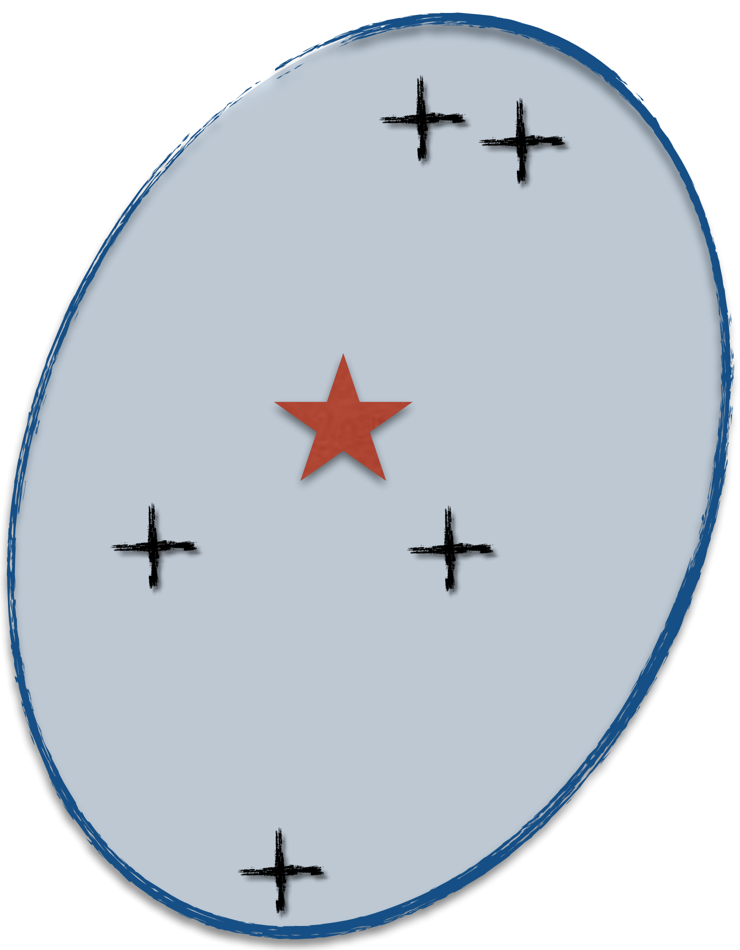

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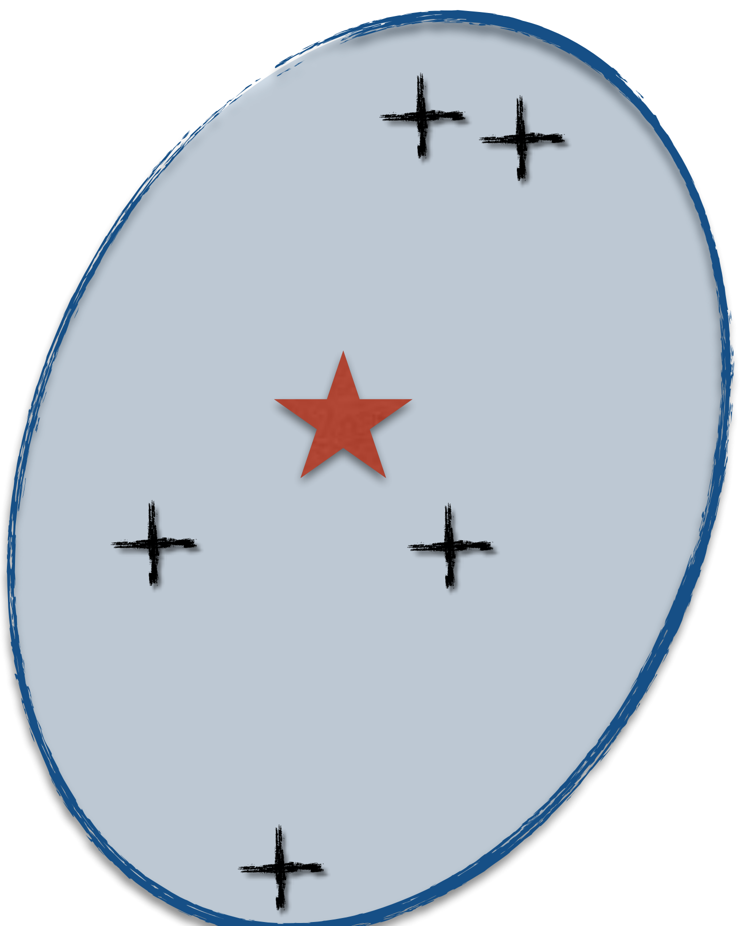

- c.r. should contain true value with probability $\geq 1 - \alpha$
- various optimality criteria:
  - minimal expected volume (uniform, average, minimax)
  - unbiased, admissible
- assuming Gaussian data for simplicity

Truncated Confidence Ellipsoids

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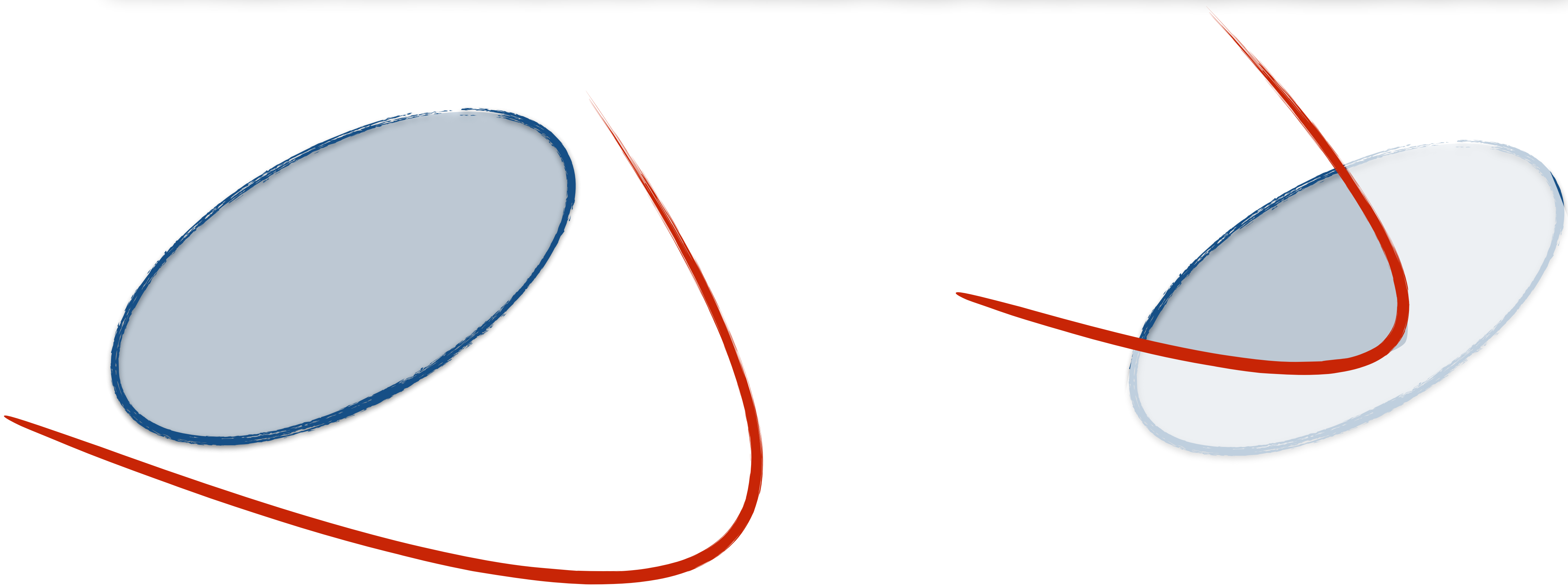
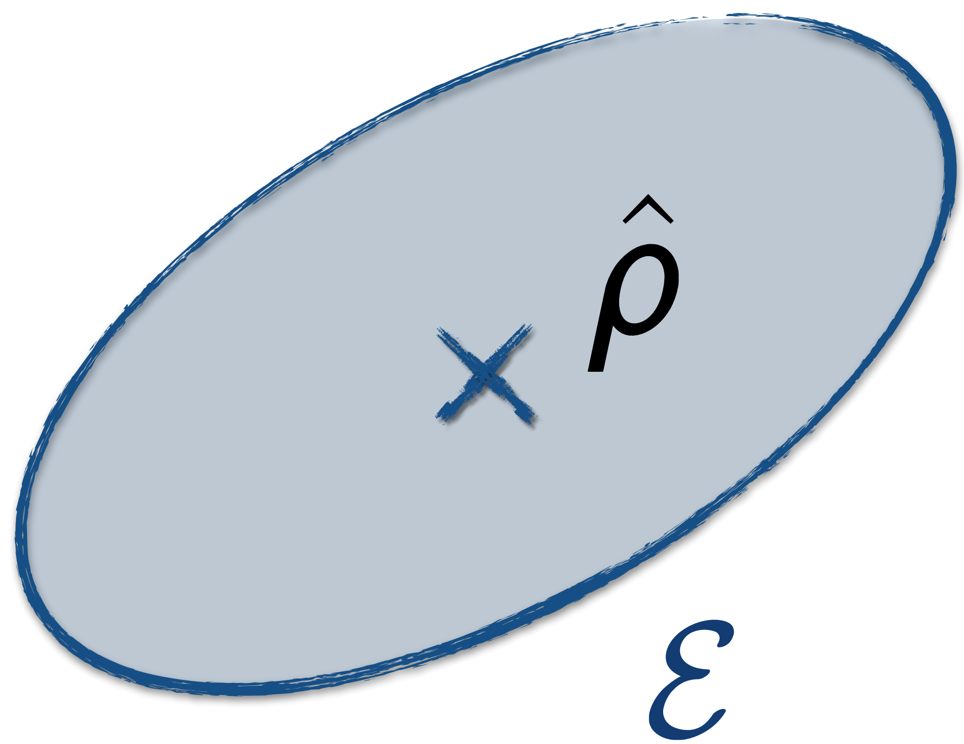
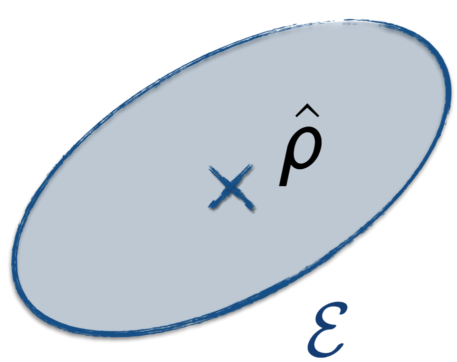


Optimal Confidence Region?



$$\hat{\rho} = \frac{1}{d}\,\mathrm{I} + \sum_i \langle \hat{\rho} \rangle_i \, \sigma_i \qquad \text{with } \langle \hat{\rho} \rangle_i := \frac{1}{N_i} \sum_{k=1}^{N_i} y_{i,k}$$

Optimal Confidence Region?

	MinMax Expected Volume	Admissible	Expected Volume + Unbiased
	✗	✗	✓

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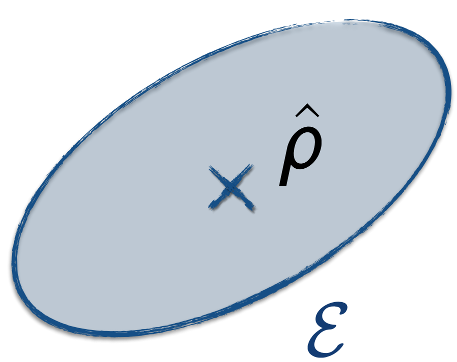
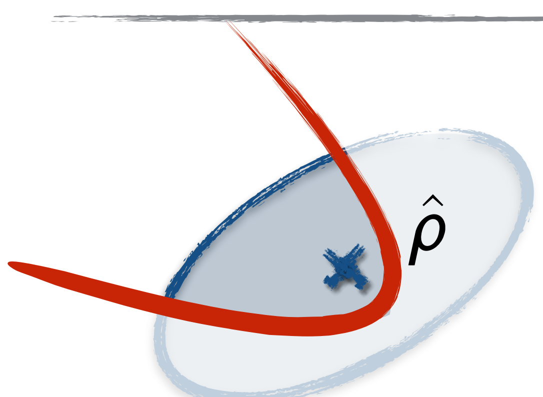
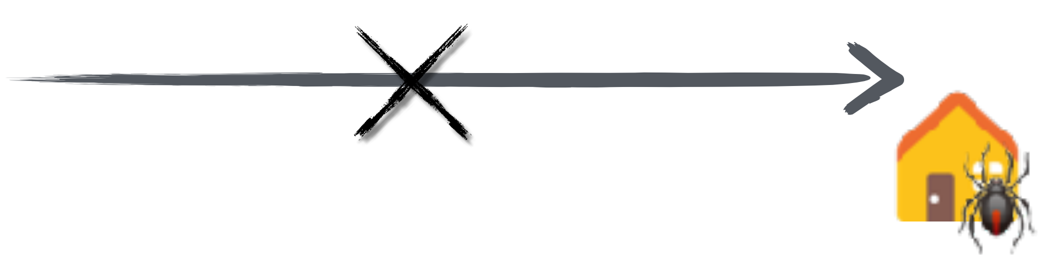
	✗	✗	✓
 $\mathcal{E}^+ = \mathcal{E} \cap \mathrm{PSD}$	✗	?	¯_(ツ)_/¯

Stein's paradox

What is the estimator $\hat{\theta}$ for the mean of a Gaussian random vector $X \sim \mathcal{N}(\theta, \mathbb{1})$ with independent components that has the smallest least square error $\mathbb{E}||\theta - \hat{\theta}||_2^2$?



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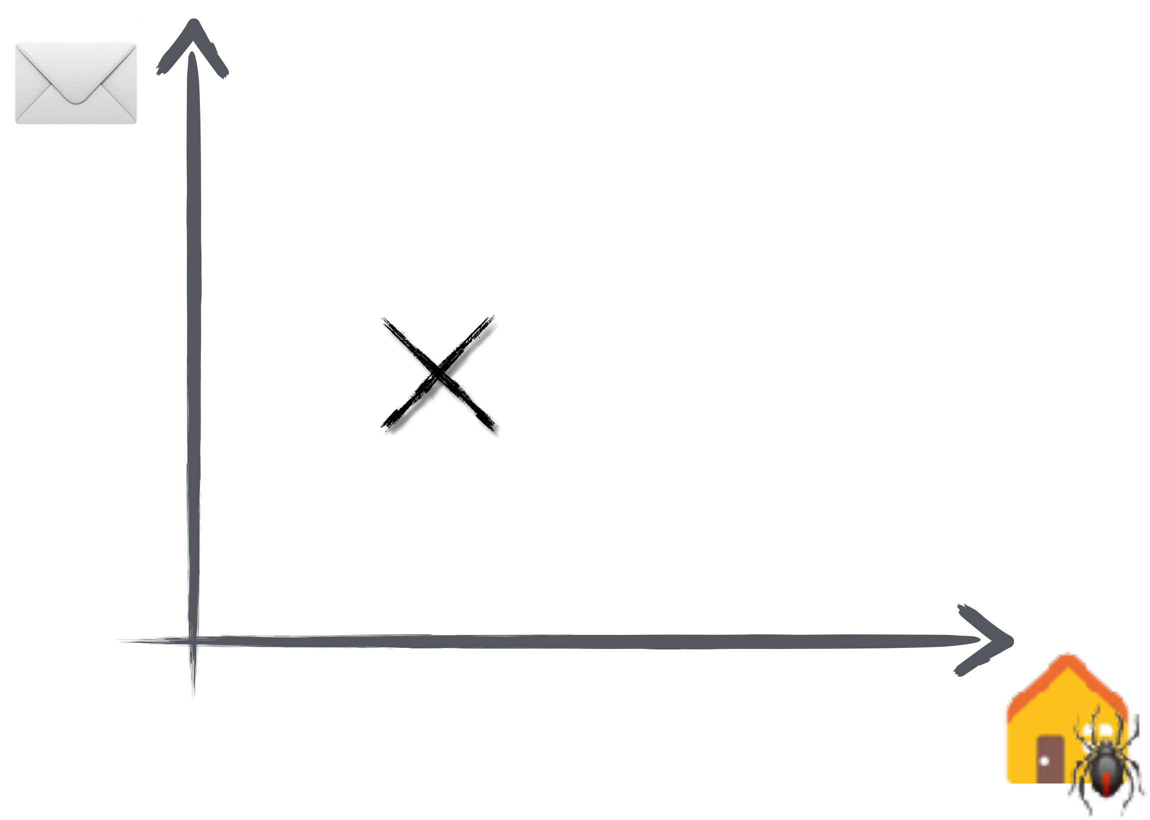

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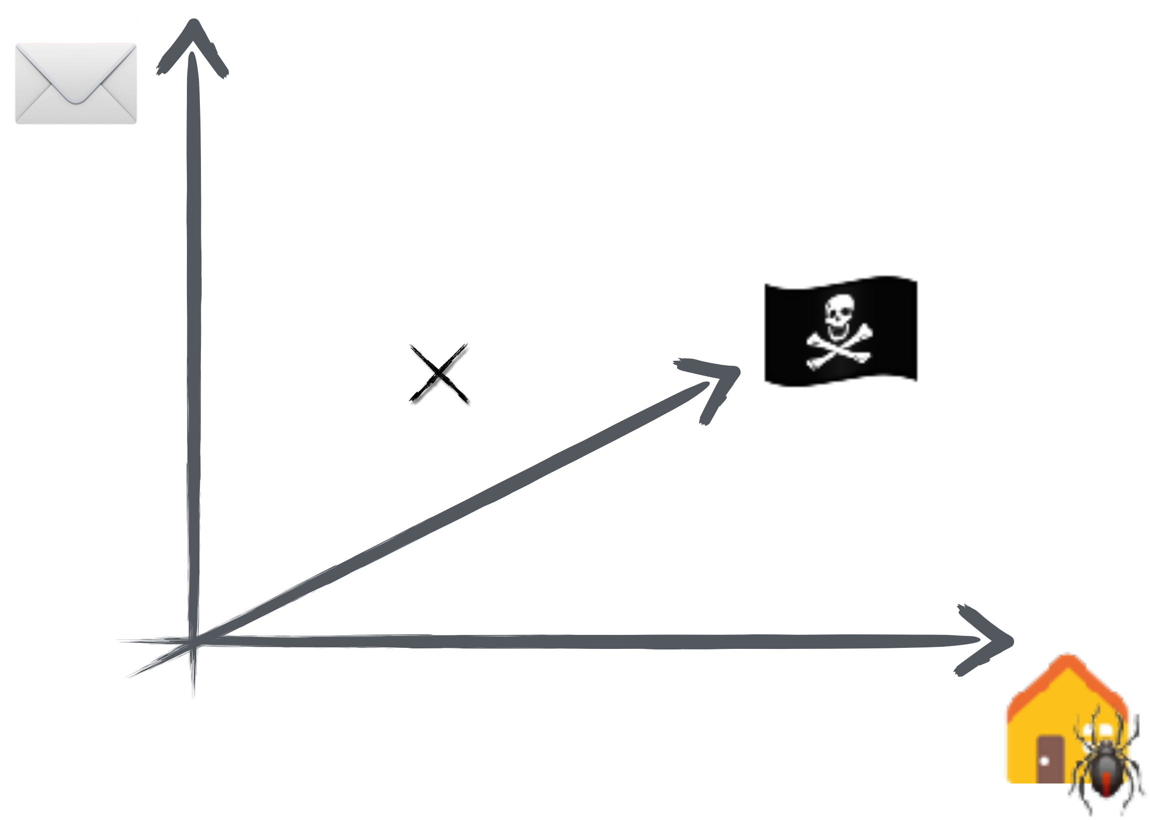

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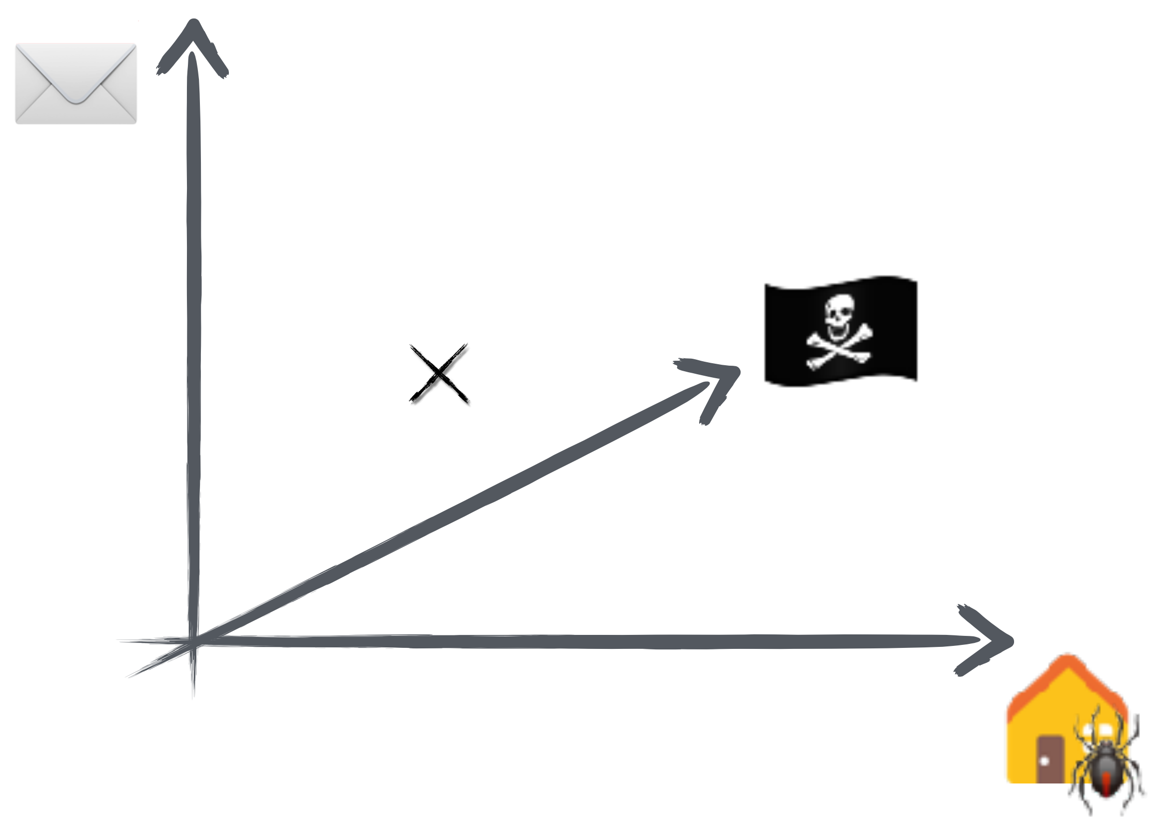

https://en.wikipedia.org/wiki/Stein%27s_example

Conclusion & Outlook

- non-trivial trade-off between optimality and computational efficiency for error regions
  - determining minimal volume credible region for Gaussian posterior computationally hard
  - application to Frequentists' confidence regions: ¯_(ツ)_/¯
- for practical applications, relaxations ("almost optimal", probabilistic, etc.) often suffice

Frequentist POV

- special case: Pauli measurements $\implies \rho = \frac{1}{d}\,\mathrm{I} + \sum_i \langle \rho \rangle_i\, \sigma_i$
- estimate $\langle \rho \rangle_i$ by empirical mean

$$\langle \hat{\rho} \rangle_i := \frac{1}{N_i} \sum_{k=1}^{N_i} y_{i,k}$$

where $y_{i,k} \in \{\pm 1\}$…outcome of the k-th measurement of σ_i

- Gaussian assumption $\langle \hat{\rho} \rangle_i \sim \mathcal{N}(\langle \hat{\rho} \rangle_i, \Sigma_i) \implies$ “natural” confidence region

$$\mathcal{E} = \left\{ \hat{\rho} + \sum_i \Sigma_i\, u_i\, \sigma_i : \; \sum u_i^2 \leq R^2 \right\}$$