Characterisation of Linear-Optical Devices via PhaseLift

R. Kueng, **D. Suess**, and D. Gross

University of Freiburg

Source code of numerical simulations and these slides? www.github.com/dseuss/optics-phaselift

The Question is…

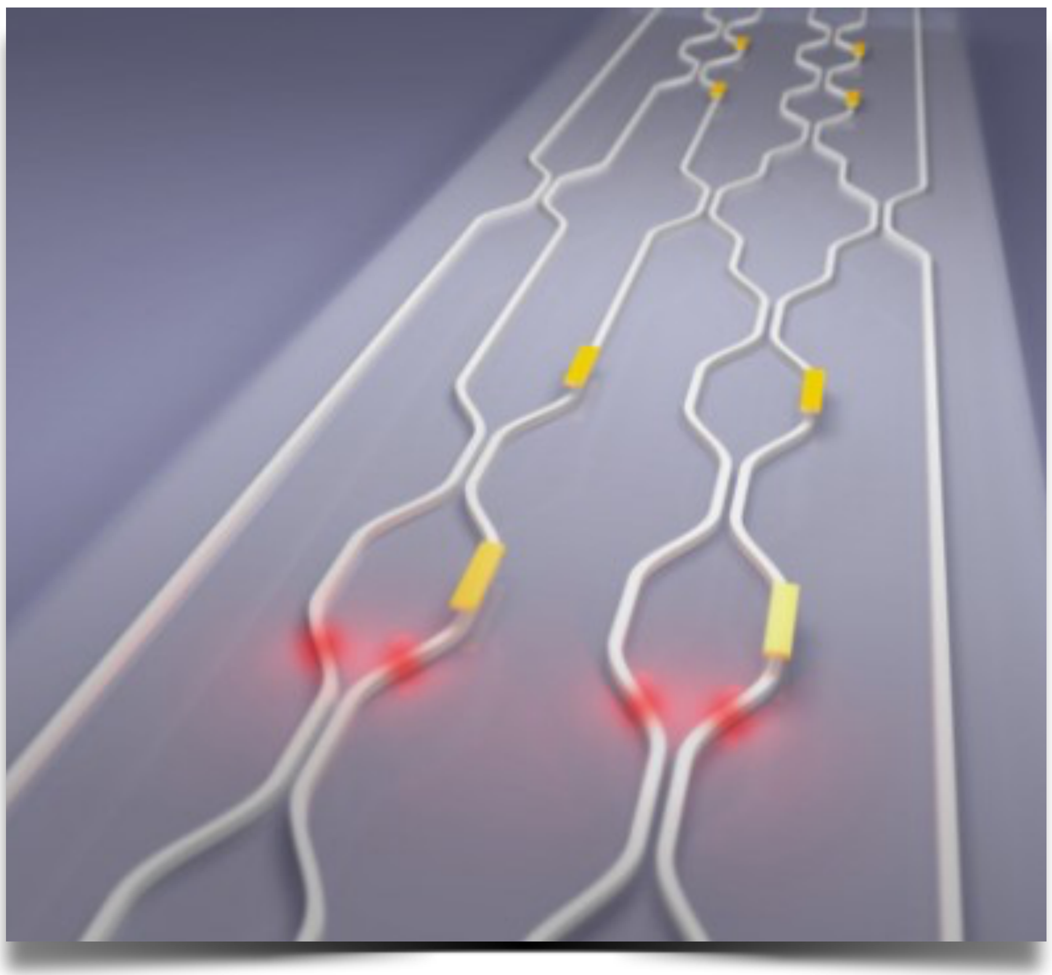

Image: http://www.bris.ac.uk/news/2011/8109.html

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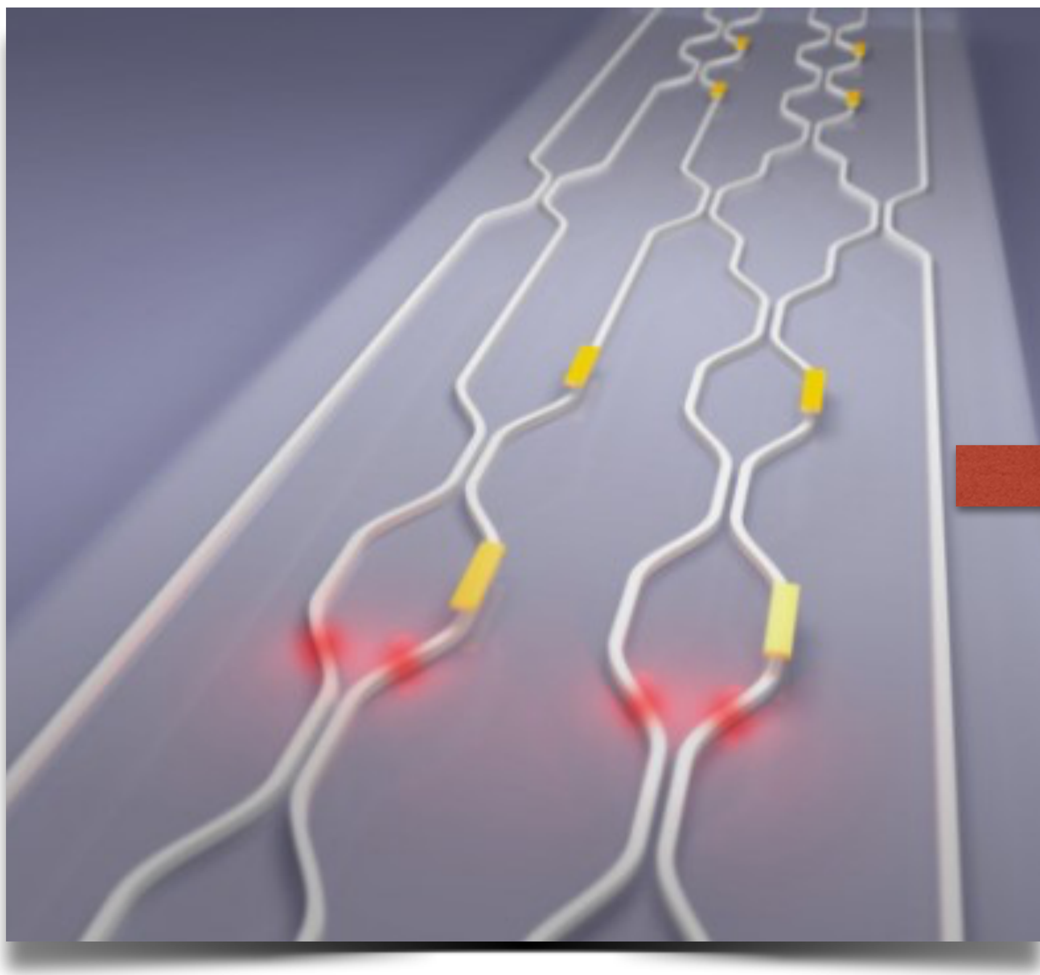

$$\begin{pmatrix} M_{1,1} & \cdots & M_{1,n} \\ M_{2,1} & \cdots & M_{2,n} \\ \vdots & \ddots & \vdots \\ M_{n,1} & \cdots & M_{n,n} \end{pmatrix}$$

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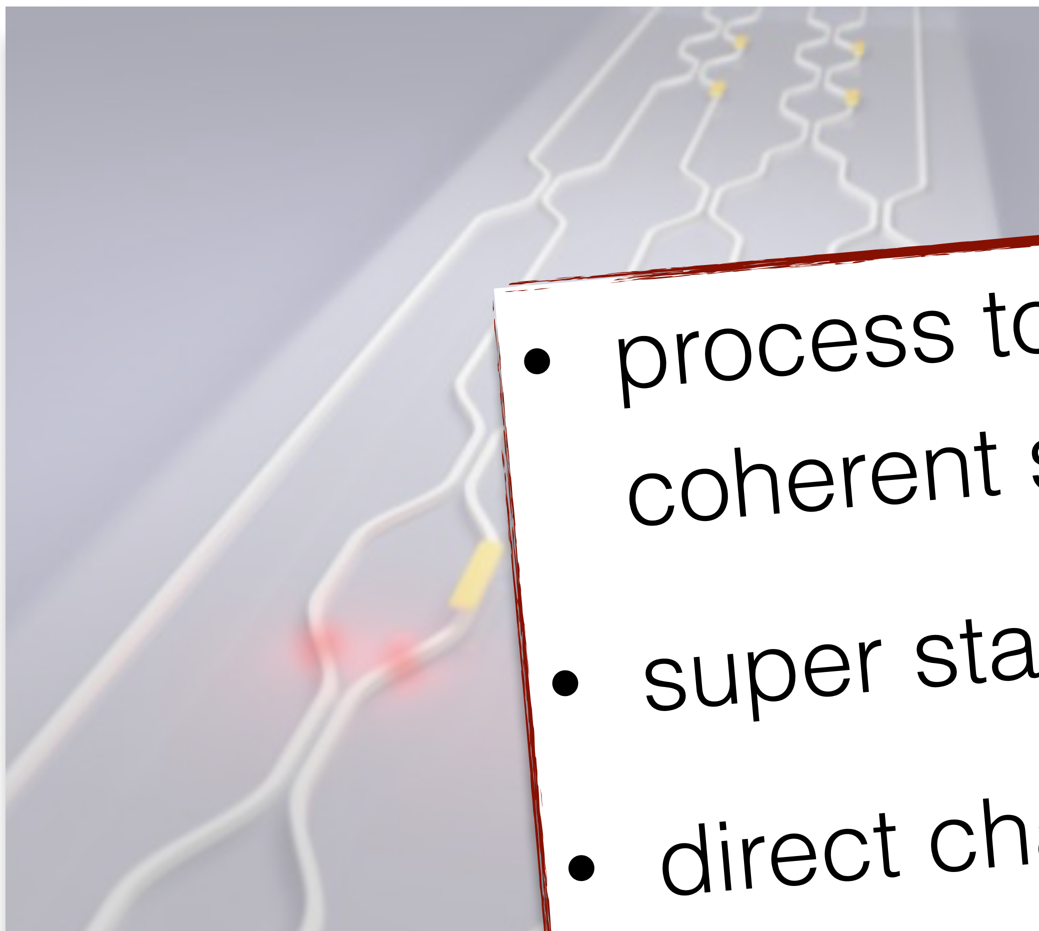

- process tomography with coherent states
- super stable tomography
- direct characterisation
- …

$$\begin{matrix} \cdots & M_{1,n} \\ \cdots & M_{2,n} \\ \ddots & \vdots \\ \cdots & M_{n,n} \end{matrix}\Bigg)$$

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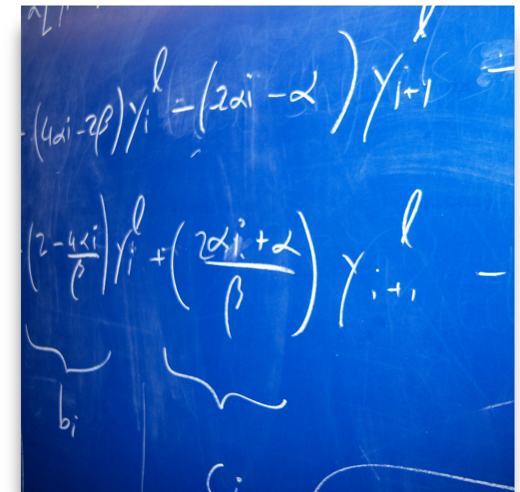
PhaseLift Characterisation

laser source & photo diodes



rigorous recovery guarantees

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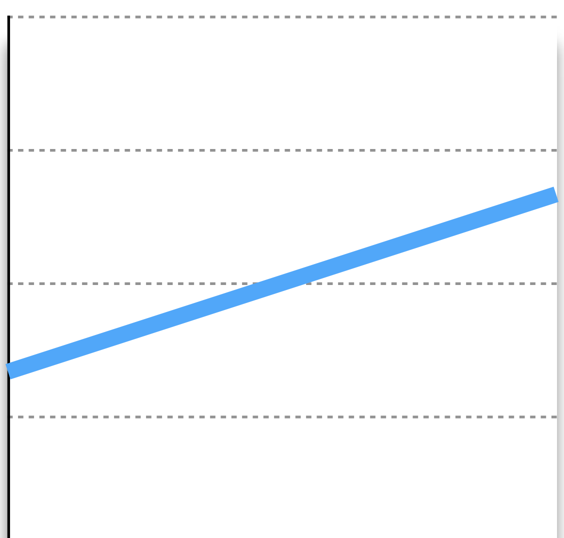

nr. of measurements scale linear in size

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Outline

- What is PhaseLift?
- Setup for Device Characterisation
- Characterisation via PhaseLift
- Numerical Results
- Conclusions & Outlook

What is PhaseLift?

PhaseLift: Exact and Stable Signal Recovery from Magnitude Measurements via Convex Programming

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$$\boxed{I(\boldsymbol{a}) = |\langle \boldsymbol{x}, \boldsymbol{a} \rangle|^2 \quad \longrightarrow \quad \boldsymbol{x} = ?}$$

What is PhaseLift?

$$y_i = |\langle \boldsymbol{x}, \boldsymbol{a}_i \rangle|^2 = \mathrm{Tr}\left(|\boldsymbol{x}\rangle\langle\boldsymbol{x}|\,|\boldsymbol{a}_i\rangle\langle\boldsymbol{a}_i|\right) \qquad (\ast)$$

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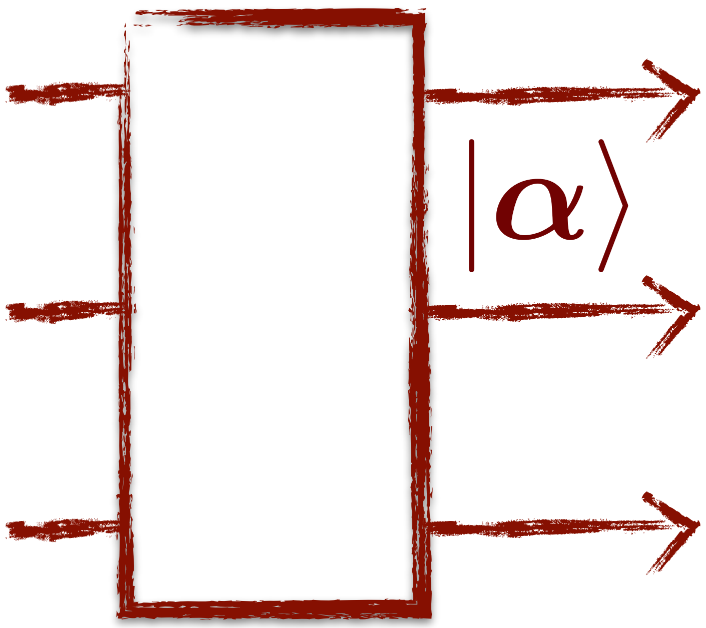
- up to a global phase of $\boldsymbol{x}$ (lost in measurement)

Device Characterisation

Preparation



Device Characterisation

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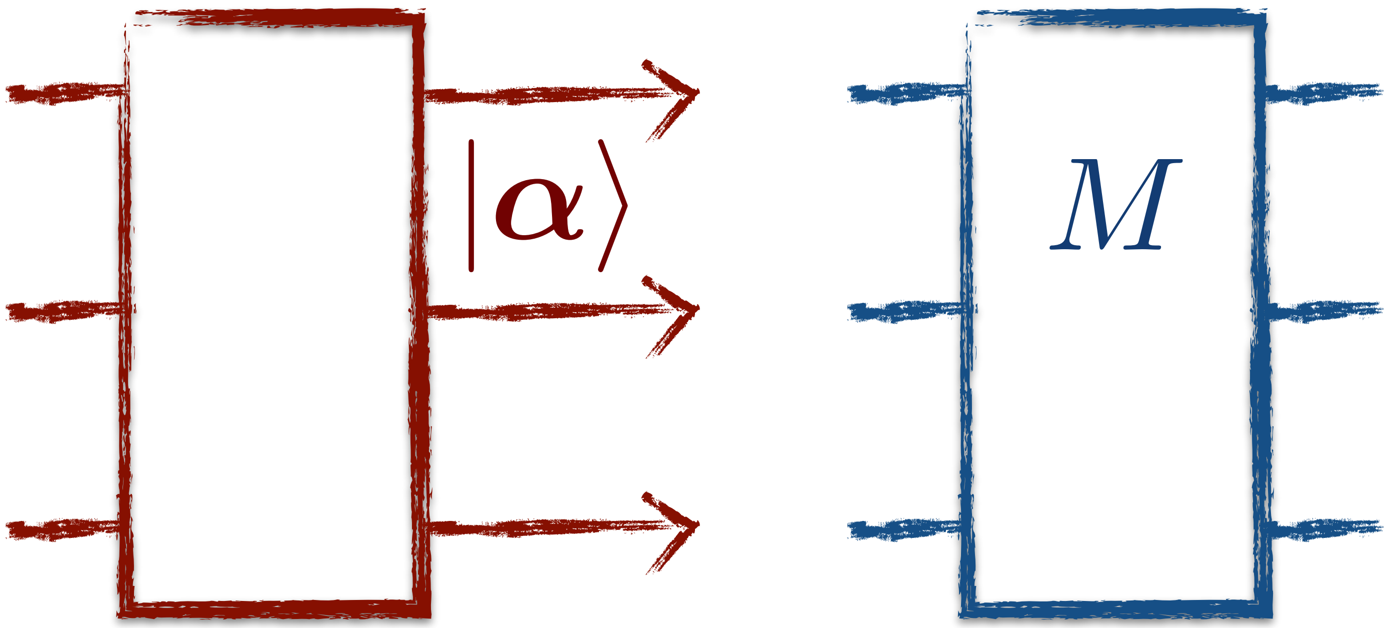
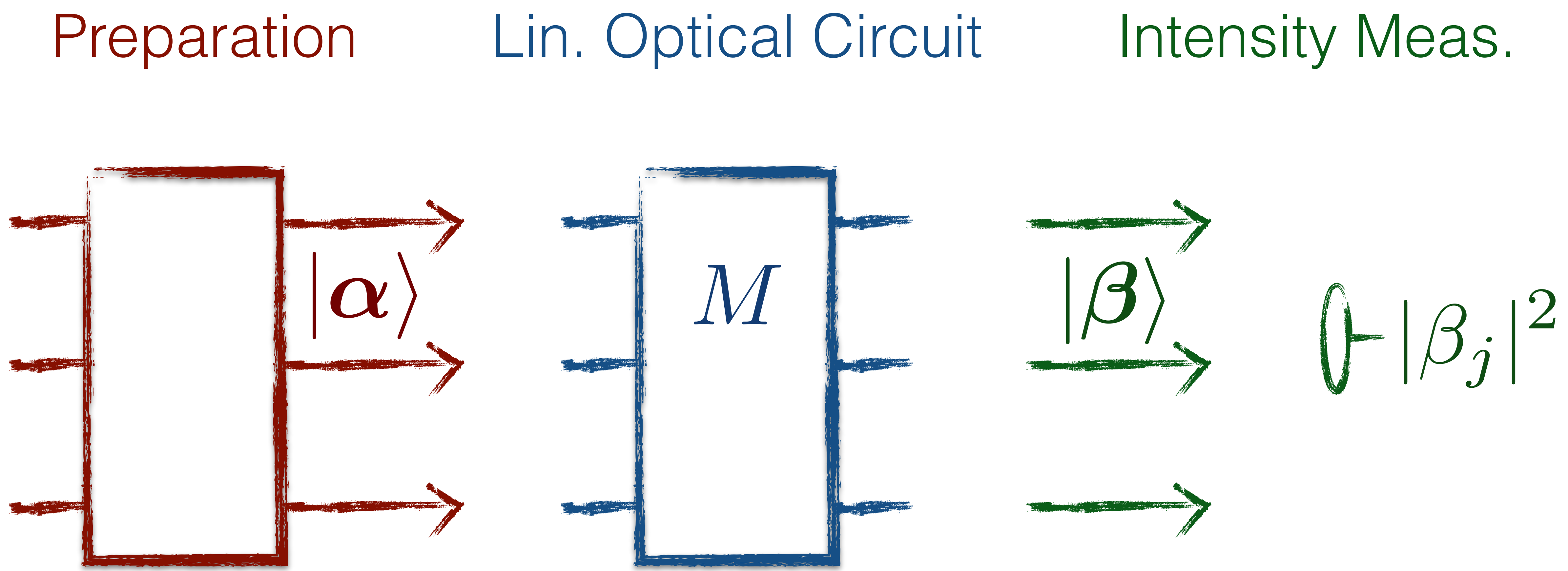
Lin. Optical Circuit





Preparation
Lin. Optical Circuit
Intensity Meas.
$|\boldsymbol{\alpha}\rangle$
M
$|\boldsymbol{\beta}\rangle$
$|\beta_j|^2$

Characterisation via PL

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$$I_j(\boldsymbol{\alpha}) = \left|\sum_k M_{j,k}\,\alpha_k\right|^2 = |\langle \boldsymbol{M}_j, \boldsymbol{\alpha}\rangle|^2,$$

where $\boldsymbol{M}_j = (M^*_{j,1}, \ldots, M^*_{j,n})$

1. input coherent states sampled uniformly from complex unit sphere

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1. input coherent states sampled uniformly from complex unit sphere
2. measure intensities on each output mode
3. run PhaseLift to recover rows of $\boldsymbol{M}$ up to global phases

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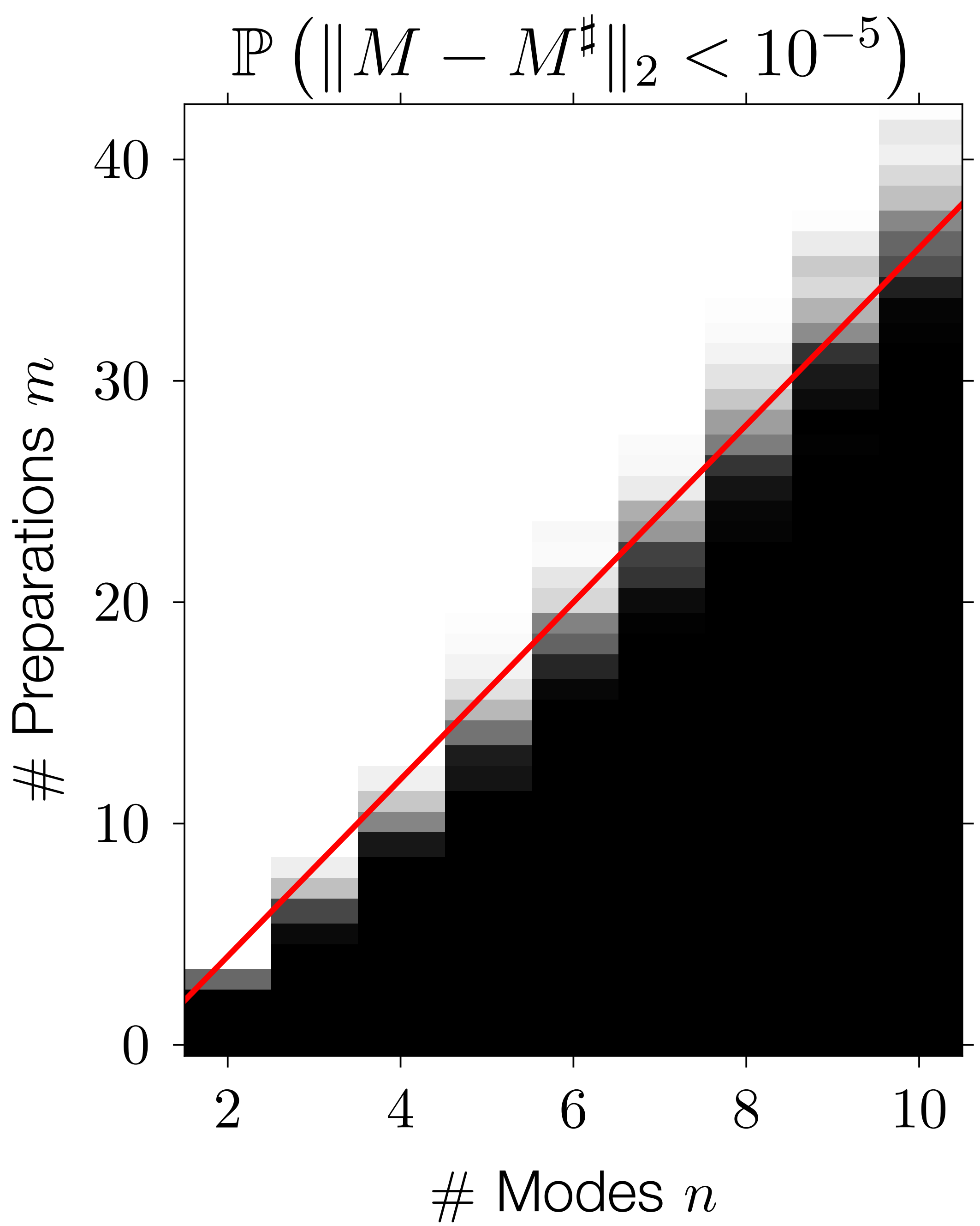
1. input
   from
2. r
3. ru
   g

Perfect reconstruction – i.e. $\boldsymbol{M}^\sharp = \boldsymbol{M}$ – by means of this Protocol holds with probability at least $1 - \mathcal{O}\left(\mathrm{e}^{-\gamma m}\right)$, provided that the number of randomly chosen input states obeys $m \geq Cn$. Here, C, γ are absolute constants.

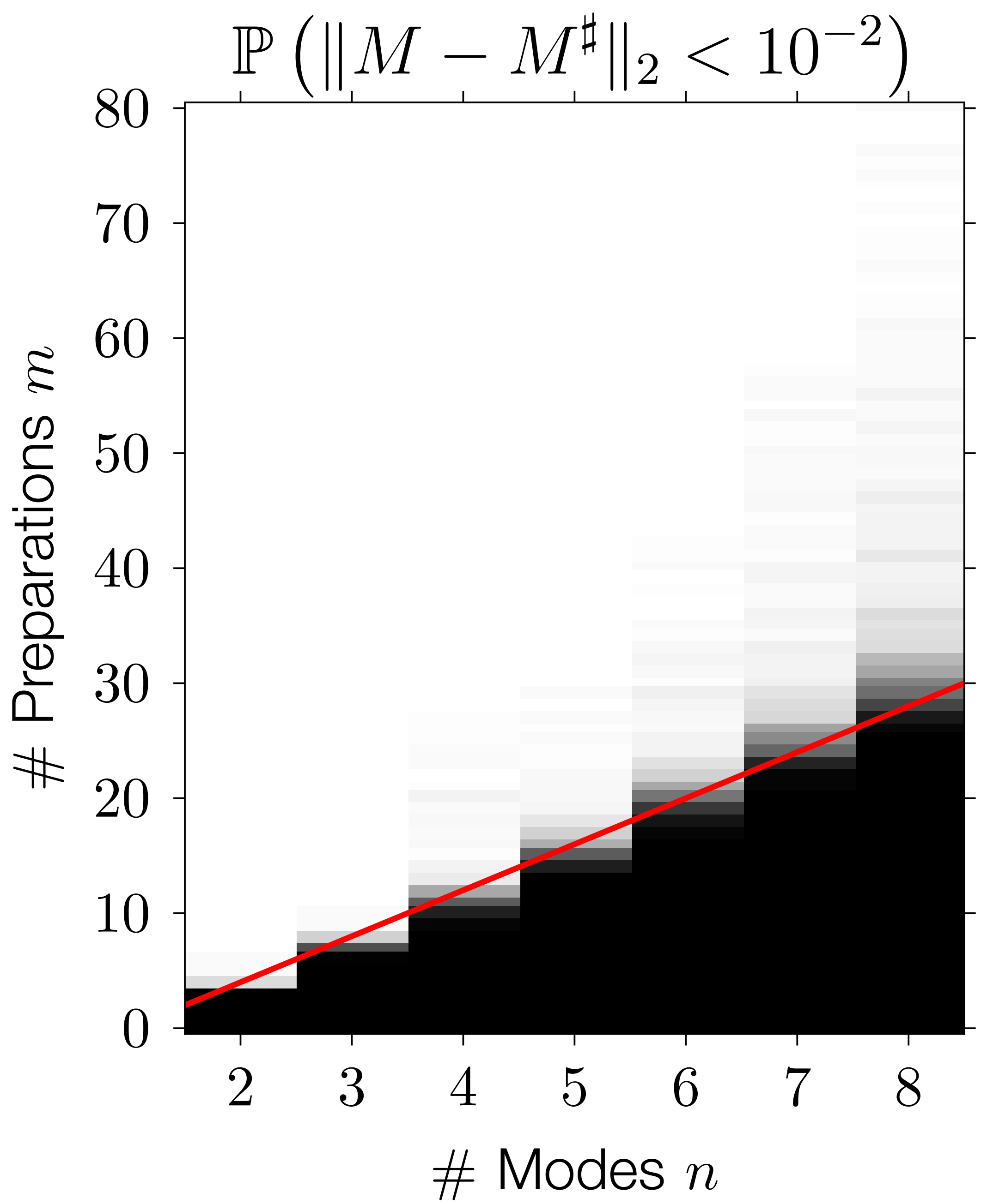
Numerical Results



- 100 samples from GUE
- finite precision due to numerical accuracy
- red line: conjectured phase transition at *m = 4n - 4*

Implementation based on S. Diamond, E. Chu, and S. P. Boyd: "CVXPY: A Python-Embedded Modeling Language for Convex Optimization"

Numerical Results



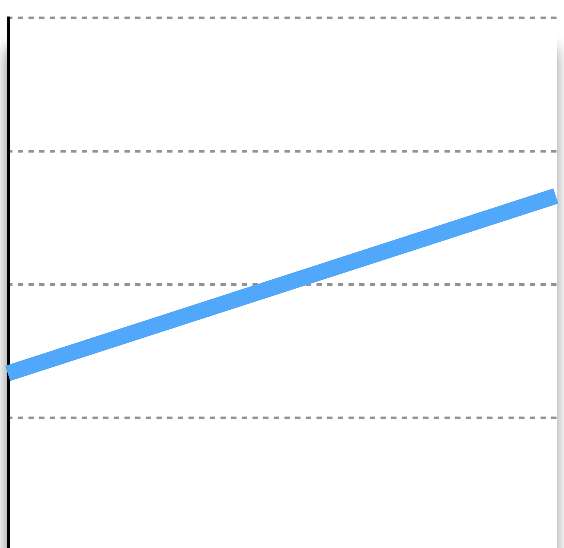
- noisy measurements:

$$I_j(\boldsymbol{\alpha}) = |\langle \boldsymbol{M}_j, \boldsymbol{\alpha} \rangle|^2 + \epsilon_j$$

- several convex programs robust to additive noise
- recovery guarantees depend on *n* and strength of noise

Conclusion

laser source & photo diodes



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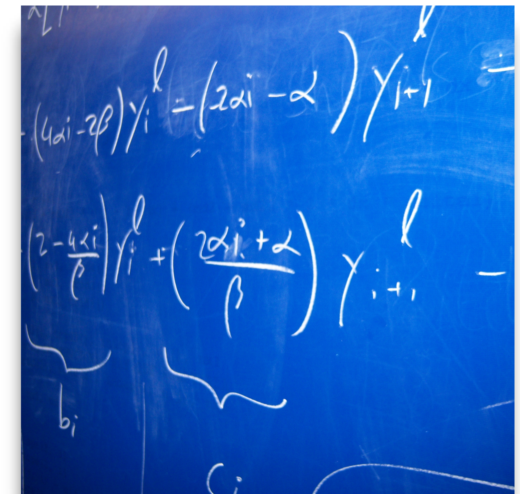

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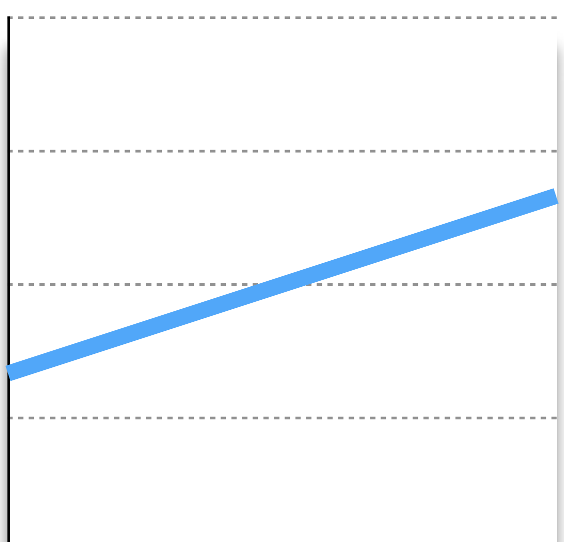

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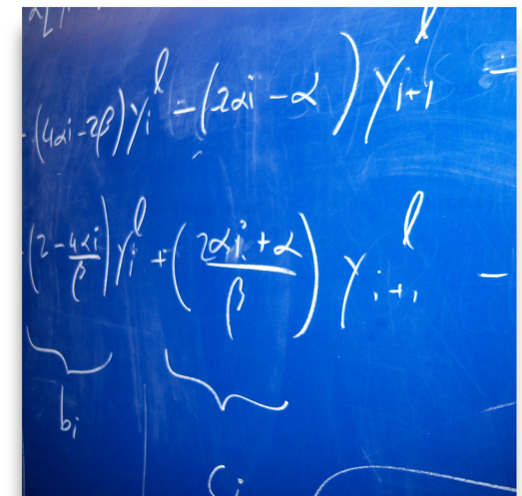

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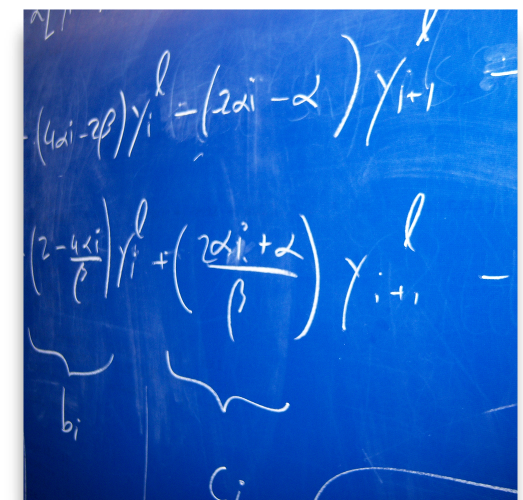

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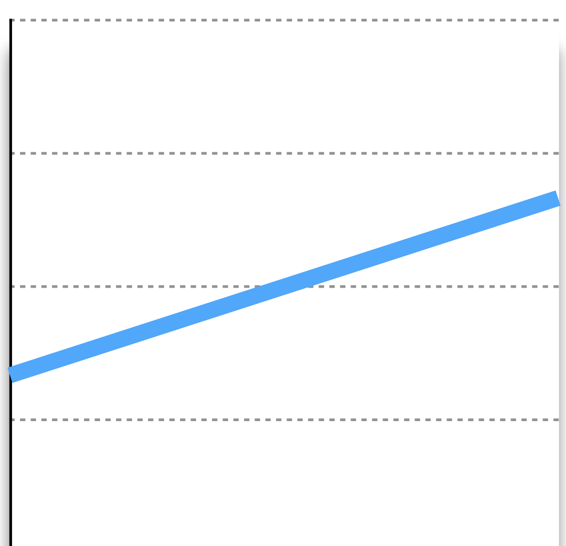

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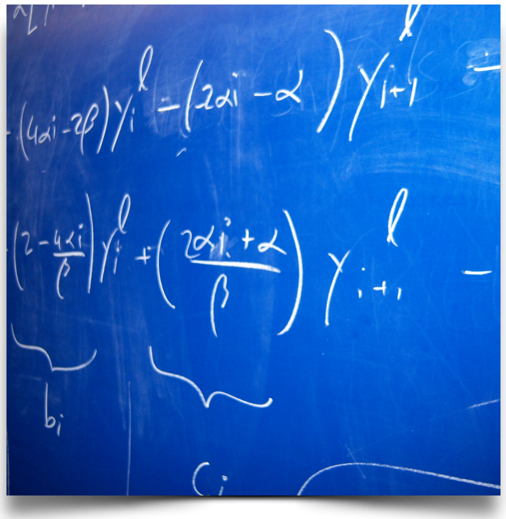

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Outlook & Open Questions

- Experimental Implementation
- Comparison to established protocols
- Which program works best for the particular experimental setting?
- How scalable is the PhaseLift approach?

Thank you!

Richard Kueng

David Gross

$$\|\boldsymbol{A}\|_p := \left(\sum_k \sigma_k(\boldsymbol{A})^p\right)^{\frac{1}{p}}$$