TUTORIAL - I

Q.no:01 - If two random variables have the joint density function

$$f(x,y) = \begin{cases} xy, & 0<x<2,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

Find the probabilities that

a). both random variables will take on values less than 1.

b). the sum of the values taken by the two random variables will be less than 1.

Soln:

Given $f(x,y) = xy$ within the specified limits:

a). $P(X<1, Y<1) = \int_0^1\int_0^1 xy\,dx\,dy$

Integrate w.r.t x,

$$\int_0^1 xy\,dx = y\int_0^1 x\,dx = y\left[\frac{x^2}{2}\right]_0^1 = y/2$$

Integrate w.r.t y,

$$\int_0^1 \frac{1}{2}y\,dy = \frac{1}{2}\int_0^1 y\,dy = \frac{1}{2}\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$$

$\therefore P(X<1, Y<1) = 1/4$

b). $P(X+Y<1) = \int_0^1\int_0^{1-y} xy\,dx\,dy$

Integrate w.r.t x :

$$\int_0^{1-y} xy\,dx = y\int_0^{1-y} x\,dx = y\left[\frac{x^2}{2}\right]_0^{1-y} = \frac{y.(1-y)^2}{2} = \frac{y(1-y)^2}{2}$$

Integrate w.r.t y :

$$\int_0^1 \frac{y(1-y)^2}{2}\,dy = \frac{1}{2}\int_0^1 y(1-2y+y^2)\,dy$$

$$= \frac{1}{2}\int_0^1 (y-2y^2+y^3)\,dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{6}{12} - \frac{8}{12} + \frac{3}{12}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{12} = \frac{1}{24}$$

$$\therefore P(X+Y<1) = \frac{1}{24}$$

Q.no:02 - Two scanners are needed for an experiment. Of the five available, two have electronic defects, another one has a defect in the memory, and two are in good working order. Two units are selected at random.

a). Find the joint probability distribution of X = the number of electronic defects and Y = the number with a defect in memory

b) Find the probability of 0 or 1 total defects among the two selected.

c). Find the marginal probability distribution of X & Y.

soln:

Given,

2 have electronic defects (E).

1 has a defect in memory (M).

2 are in good working order (G).

We are selecting 2 scanners at random. Define the random variables:

X = the no. of electronic defects.

Y = the no. with a defect in memory.

a). Possible values of (X, Y) are :

1. $(0,0)$: No defects

2. $(0,1)$: One defect in memory.

3. $(1,0)$: One electronic defect.

4. $(1,1)$: One electronic defect and one memory defect

5. $(2,0)$: Two electronic defects

i) $P(X=0, Y=0)$: Both scanners are in good working order

$\Rightarrow {}^2C_2 = 1$

$P(X=0, Y=0) = 1/10$

ii) $P(X=0, Y=1)$: One scanner is memory defect and the other is in good working order.

$\Rightarrow {}^1C_1 \times {}^2C_1 = 1 \times 2 = 2$

$P(X=0, Y=1) = 2/10 = 1/5$

iii) $P(X=1, Y=0)$: One scanner has an electronic defect and the other is in good working order.

$\Rightarrow {}^2C_1 \times {}^2C_1 = 2 \times 2 = 4$

$P(X=1, Y=0) = 4/10 = 2/5$

iv) $P(X=1, Y=1)$: One scanner has an electronic defect and the other has a memory defect.

$\Rightarrow {}^2C_1 \times {}^1C_1 = 2 \times 1 = 2$

$P(X=1, Y=1) = 2/10 = 1/5$

v). $P(X=2, Y=0)$: Both scanners have electronic defects

$\Rightarrow {}^2C_2 = 1$

$P(X=2, Y=0) = 1/10$

summarizing the joint probability distribution,

	Y = 0	Y = 1
X = 0	$\frac{1}{10}$	$\frac{1}{5}$
X = 1	$\frac{2}{5}$	$\frac{1}{5}$
X = 2	$\frac{1}{10}$	0

b). Probability of 0 or 1 total defects

$$P(X+Y=0) = P(X=0, Y=0) = \frac{1}{10}$$

$$P(X+Y=1) = P(X=0, Y=1) + P(X=1, Y=0) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$$

$\therefore$, $P(X+Y = 0 \text{ or } 1) = P(X+Y=0) + P(X+Y=1)$

$$= \frac{1}{10} + \frac{3}{5}$$

$$= \frac{1+2\times3}{10} = \frac{7}{10}$$

c). Marginal distribution of X, Y:

X	P(X)
0	$\frac{1}{10} + \frac{2}{10} = \frac{3}{10}$
1	$\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$
2	$\frac{1}{10} + 0 = \frac{1}{10}$

Y	P(Y)
0	$\frac{1}{10} + \frac{2}{5} + \frac{1}{10} = \frac{3}{5}$
1	$\frac{1}{5} + \frac{1}{5} + 0 = \frac{2}{5}$

Q.no:03- If the joint probability distribution of a two dimensional random variables X, Y given by

$$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x>0;\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

, find the marginal densities of X and Y. Also Find $[P(X<1, Y<1)]$.

Soln:

$$\frac{\partial^2}{\partial x \partial y}[F(x,y)] = f(x,y)$$

$$\frac{\partial^2}{\partial x \partial y}[1-e^{-y}-e^{-x}+e^{-(x+y)}]$$

$$\frac{\partial}{\partial x}[0-e^{-y}(-1)+e^{-(x+y)}(-1)] = \partial/\partial x \, [e^{-y}-e^{-(x+y)}]$$

$$= 0 - e^{-(x+y)}$$

$$f(x,y) = e^{-(x+y)}$$

Marginal densities of x and y :

$$f(x) = \int_0^\infty f(x,y)\,dy \qquad f(y) = \int_0^\infty f(x,y)\,dx$$

$$= \int_0^\infty e^{-(x+y)}\,dy \qquad = \int_0^\infty f(x,y)\,dx$$

$$= \int_0^\infty e^{-x}.e^{-y}\,dy \qquad = e^{-y}\left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$= e^{-x}\left[\frac{e^{-y}}{-1}\right]_0^\infty \qquad = -e^{-y}[e^{-\infty}-e^{-0}]$$

$$= -e^{-y}[0-1]$$

$$= -e^{-x}[e^{-\infty}-e^0] \qquad = e^{-y}$$

$$= -e^{-x}[0-1] = e^{-x}$$

$$\therefore f(x) = e^{-x}$$

$$\therefore f(y) = e^{-y}$$

$P(X<1, Y<1)$

$$= \int_0^1\int_0^1 e^{-(x+y)}\,dx\,dy = \int_0^1 e^{-y}\left[-e^{-1}+1\right]$$

$$= [-e^{-1}+1][-e^{-1}+1]$$

$$= (1-e^{-1})^2 \qquad \therefore P(X<1, Y<1) = (1-e^{-1})^2.$$

MODULE - III PAIR OF RANDOM VARIABLES

TUTORIAL - II

Q.no:01- If X and Y are random variables having the joint density function $f(x,y) = \begin{cases} xy/96, & 0 \le x \le 4,\ 1 \le y \le 5 \\ 0, & \text{otherwise} \end{cases}$ Find correlation coefficient of ((X, Y)).

Soln:

Compute the marginal densities:

Marginal Density of X

$$f_X(x) = \int_1^5 \frac{xy}{96}\,dy = \frac{x}{96}\int_1^5 y\,dy = \frac{x}{96}\left[\frac{y^2}{2}\right]_1^5 = \frac{x}{96}\left(\frac{25}{2} - \frac{1}{2}\right)$$

$$= \frac{x}{96}\cdot 12 = \frac{x}{8}$$

$$f_X(x) = \begin{cases} x/8, & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Marginal Density of Y

$$f_Y(y) = \int_0^4 \frac{xy}{96}\,dx = \frac{y}{96}\int_0^4 x\,dx = \frac{y}{96}\left[\frac{x^2}{2}\right]_0^4 = \frac{y}{26}\cdot\frac{16}{2}$$

$$= \frac{y}{96}\cdot 8 = \frac{y}{12}$$

$$f_Y(y) = \begin{cases} y/12, & 1 \le y \le 5 \\ 0, & \text{otherwise} \end{cases}$$

Compute the Expected value

$$E[X] = \int_0^4 x\,f_X(x)\,dx = \int_0^4 x\cdot\frac{x}{8}\,dx = \frac{1}{8}\int_0^4 x^2\,dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{8}\cdot\frac{64}{3} = \frac{8}{3}$$

$$E[Y] = \int_1^5 y\,f_Y(y)\,dy = \int_1^5 y\cdot\frac{y}{12}\,dy = \frac{1}{12}\int_1^5 y^2\,dy = \frac{1}{12}\left[\frac{y^3}{3}\right]_1^5$$

$$= \frac{1}{12}\left(\frac{125}{3} - \frac{1}{3}\right) = \frac{31}{9}$$

Compute the variances

var(X)

$$E[X^2] = \int_0^4 x^2 f_X(x)\,dx = \int_0^4 x^2\cdot\frac{x}{8}\,dx = \frac{1}{8}\int_0^4 x^3\,dx = \frac{1}{8}\left[\frac{x^4}{4}\right]_0^4$$

$$= \frac{1}{8}\cdot 64 = 8$$

$$var(X) = E[X^2] - (E[X])^2 = 8 - (8/3)^2 = 8 - \frac{64}{9}$$

$$= \frac{72 - 64}{9} = 8/9$$

var (Y)

$$E[Y^2] = \int_1^5 y^2 f_Y(y)\, dy = \int_1^5 y^2 \cdot y/_{12}\, dy = 1/_{12}\left[\frac{y^4}{4}\right] = 1/_{12} \cdot \frac{624}{4}$$

$$= \frac{52}{3}$$

$$var(Y) = E[Y^2] - (E[Y])^2 = \frac{52}{3} - \left(\frac{31}{9}\right)^2 = \frac{52}{3} - \frac{961}{81} = \frac{443}{81}$$

compute $E[XY]$

$$E[XY] = \int_0^4\int_1^5 xy \cdot \frac{xy}{96}\, dy\, dx = \int_0^4\int_1^5 \frac{x^2y^2}{96}\, dy\, dx$$

Integrate w.r.t y :

$$\int_1^5 \frac{x^2y^2}{96}\, dy = \frac{x^2}{96}\int_1^5 y^2\, dy = \frac{x^2}{96}\left[y^3/_3\right]_1^5 = \frac{x^2}{96}\left(\frac{125}{3} - 1/_3\right) = \frac{31x^2}{72}$$

Integrate w.r.t x :

$$\int_0^4 \frac{31x^2}{72}\, dx = \frac{31}{72}\int_0^4 x^2\, dx = 31/_{72}\left[\frac{x^3}{3}\right]_0^4 = 31/_{72} \cdot 64/_3 = \frac{1984}{216} = \frac{248}{27}$$

compute the covariance

$$Cov(X, Y) = E[XY] - E[X]E[Y] = \frac{248}{27} - \left(8/_3 \cdot 31/_9\right) = \frac{248}{27} - \frac{248}{27} = 0$$

compute the correlation coefficient

$$r_{(X,Y)} = \frac{Cov(X, Y)}{\sqrt{var(X)}\sqrt{var(Y)}} = \frac{0}{\sqrt{8/9} \cdot \sqrt{\frac{443}{81}}} = 0$$

∴ , the correlation coefficient of X and Y is 0.

Q.no: 02 - Find the coefficient of correlation between X and Y from the data given below :

x:	10	14	18	22	26	30
y:	18	12	24	6	30	36

soln:

compute the means :

$$E[x] = \bar{x} = \frac{1}{6}(10+14+18+22+26+30) = 20$$

$$E[y] = \bar{y} = \frac{1}{6}(18+12+24+6+30+36) = 21$$

compute $E[xy]$

$$E[xy] = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$$

$$E[xy] = \frac{1}{6}(10*18 + 14.12 + 18.24 + 22.6 + 26.30 + 30.36)$$

$$= \frac{1}{6}(180+168+432+132+780+1080)$$

$$= \frac{1}{6}(2772) = 462$$

compute the variance

$$var(x) = \frac{1}{n}\sum_{i=1}^{n}(x_i - E[x])^2$$

$$= \frac{1}{6}((10-20)^2 + (14-20)^2 + (18-20)^2 + (22-20)^2 + (26-20)^2 + (30-20)^2)$$

$$= \frac{1}{6}(100+36+4+4+36+100) = \frac{280}{6} = 46.6$$

$$var(Y) = \frac{1}{n}\sum_{i=1}^{n}(y_i - E[y])^2$$

$$= \frac{1}{6}((18-21)^2 + (12-21)^2 + (24-21)^2 + (6-21)^2 + (30-21)^2 + (36-21)^2)$$

$$= \frac{1}{6}(9+81+9+225+81+225) = \frac{630}{6} = 105$$

compute correlation coefficient

$$r(x,y) = \frac{E[xy] - E[x]E[y]}{\sqrt{var\,x}\,\sqrt{var\,y}}$$

$$E[x]E[y] = 20 \times 21 = 420$$

$$r = \frac{462 - 420}{\sqrt{46.6}\,\sqrt{105}} \approx 0.599$$

$\therefore r \approx 0.599$

Q.no:03 - Marks obtained by 10 students in Mathematics (x) and statistics (Y) are given below: Find the two regression lines. Also find y when x = 55.

X :	25	28	35	32	31	36	29	38	34	32
Y :	43	46	49	41	36	32	31	30	33	39

soln:

$\Sigma x = 320$ $\qquad \bar{x} = \frac{1}{n} \Sigma x = \frac{320}{10} = 32$

$\Sigma y = 380$ $\qquad \bar{y} = \frac{1}{n} \Sigma y = \frac{380}{10} = 38$

The line regression eqn y on x

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) \quad - ①$$

The line regression eqn x on y

$$(x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y}) \quad - ②$$

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
25	43	-7	5	49	25	-35
28	46	-4	8	16	64	-32
35	49	3	11	9	121	33
32	41	0	3	0	9	0
31	36	-1	-2	1	4	2
36	32	4	-6	16	36	-24
29	31	-3	-7	9	49	21
38	30	6	-8	36	64	-48
34	33	2	-5	4	25	-10
32	39	0	-1	0	1	0
		$\Sigma(x-\bar{x})=0$	$\Sigma(y-\bar{y})=0$	$\Sigma(x-\bar{x})^2=140$	$\Sigma(y-\bar{y})^2=398$	$\Sigma(x-\bar{x})(y-\bar{y}) = -93$

$$r \times \frac{\sigma_y}{\sigma_x} = b_{yx} = \frac{\Sigma(y-\bar{y})(x-\bar{x})}{\Sigma(x-\bar{x})^2} = \frac{-93}{140} = -0.664 \quad - (3)$$

$$r \times \frac{\sigma_x}{\sigma_y} = b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2} = \frac{-93}{398} = -0.233 \quad - (4)$$

Line regression y on x :

$(y-\bar{y}) = b_{yx}(x-\bar{x})$

$y-38 = (-0.664)(x-32)$

$y = -0.664 + 59.248$

Line regression x on y :

$(x-\bar{x}) = b_{xy}(y-\bar{y})$

$x - 32 = b_{xy}(y-38)$

$x = -0.233y + 40.854$

$y = ?$ when $x = 55$

$y = -0.664(55) + 59.243$

$y = -36.52 + 59.243$

$y = 22.723.$