

Math 113, Fall 2019

Lecture 8, Tuesday, 9/24/2019

Topics Today:

- Finish cyclic groups
- Generating sets
- Cayley digraphs (not really)

Assigned readings: §8 and 9.

We open with two clicker questions on Cayley digraphs.

1 Review from last lecture

Let G be a cyclic group generated by a , and let $n := |G|$, $s \in \mathbb{Z}$, $b := a^s$, and let $H = \langle b \rangle$. Then

$$\{k \in \mathbb{Z} \mid a^k \in H\} = \{in + js \mid i, j \in \mathbb{Z}\}$$

This is a nontrivial subgroup of \mathbb{Z} , equal to $d\mathbb{Z}$, where d is its smallest positive element. Then

$$d := \gcd(s, n),$$

and H contains $\{a^d, a^{2d}, \dots, a^{\frac{n}{d}d} = 1\}$, so $H \supseteq \langle a^d \rangle$, which has $\frac{n}{d}$ elements. Now from an earlier proof (of Theorem 6.6), $H = \langle a^d \rangle$, and $|H| = \frac{n}{d}$. So we've shown part of the following theorem:

Theorem 1.1. Let G, a , and n be as above. Let $b \in G$, and pick $s \in \mathbb{Z}$ such that $b = a^s$.

Let $d := \gcd(s, n)$. Then $|B| = \frac{n}{d}$ and $\langle b \rangle = \langle a^d \rangle$. Also, for all $t \in \mathbb{Z}$,

$$\langle a^s \rangle = \langle a^t \rangle \iff \gcd(s, n) = \gcd(t, n)$$

Proof. We've computed $|b| = \frac{n}{d}$, and showed $\langle b \rangle = \langle a^d \rangle$. Also, $\gcd(s, n) = \gcd(t, n) \implies \langle a^s \rangle = \langle a^t \rangle$, since both equal $\langle a^d \rangle$, where $d = \gcd(s, n) = \gcd(t, n)$.

Conversely, if $\langle a^s \rangle = \langle a^t \rangle$, then $\gcd(s, n) = \frac{n}{|\langle a^s \rangle|} = \frac{n}{|\langle a^t \rangle|} = \gcd(t, n)$. \square

As a corollary, we have the additional homework problem (on HW4).

Example: We showed that the subgroups of U_6 are U_1, U_2, U_3 , and U_6 , and $1, 2, 3, 6$ are the positive divisors of 6.

As another corollary, the cyclic generators of G are

$$\{a^s \mid \gcd(s, n) = 1\} = \{b \in G \mid \langle b \rangle = G\}.$$

2 Generating Sets

Recall that for all $a \in G$, we have:

$$\langle a \rangle := \{a^n \mid n \in \mathbb{Z}\}.$$

We also said (or Vojtá notes perhaps the book said) that $\langle a \rangle$ is the smallest subgroup of G containing a .

Let's look in more detail. Suppose we have a nonempty collection of subgroups of a group G , and let

$$K := \bigcap_{i \in I} H_i.$$

Then K is a subgroup of G .

Proof. Notice $e \in H_i$ for all i , so $e \in \bigcap H_i = K$. In particular, $K \neq \{\}$. Let x, y be arbitrary elements of K . Then $x, y \in H_i, \forall i$. Therefore, $xy^{-1} \in H_i, \forall i$, which implies $xy^{-1} \in K$, and hence $K \leq G$. \square

Further, consider that $\langle a \rangle$ is the **intersection of all subgroups** of G containing a .

Proof. Let $\{H_i\}_{i \in I}$ be all subgroups of G containing a . This is a nonempty collection because it contains G . Let $K := \bigcap_{i \in I} H_i$. Then $\langle a \rangle \subseteq K$ because all of the H_i contain a . Hence they all contain $\langle a \rangle$. Also, $K \subseteq \langle a \rangle$ because $\langle a \rangle$ is one of the H_i . \square

Vojta notes that we now have two characterizations of cyclic groups. That is, $\langle a \rangle$ is

- (1) $\{a^n \mid n \in \mathbb{Z}\}$, and
- (2) The intersection of all subgroups containing a .

Vojta gives the analogy that the first is the '3d printing method', building it up piece by piece, whereas the second is the 'sculpture method', chipping away piece by piece.

Now, instead of a single element a , consider a subset of G .

3 Generating Sets

We now consider generating sets as opposed to the cyclic group generated by a single cyclic generator. We emphasize here that we take a generic subset (in fact, it's most interesting when our subset is NOT a subgroup).

Let G be a group, and let $S \subseteq G$ be a subset. Let $\{H_i\}_{i \in I}$ be the set of all subgroups of G containing S .

Again, G itself is one of these subgroups, so $I \neq \{\}$.

Then we can let $K := \bigcap_{i \in I} H_i$. This is a subgroup of G , and it contains S .

In fact, it is the smallest subgroup of G containing S . That is,

$$K \subseteq H_i, \forall i,$$

and $K \in \{H_i\}_{i \in I}$.

Definition: Subgroup generated by a set -

$K := \bigcap_{i \in I} H_i$ as above is the **subgroup of G generated by S** , and is denoted $\langle S \rangle$.

Remark: If $S = \{a_1, \dots, a_n\}$, then we also write:

$$\langle a_1, \dots, a_n \rangle = \langle S \rangle.$$

and if $S = \{a_1, a_2, \dots\}$, we write:

$$\langle a_1, a_2, \dots \rangle = \langle S \rangle.$$

On the other hand, if we take the 3d-printing method, we have:

$$\langle S \rangle = \{s_1^{n_1}, \dots, s_m^{n_m} \mid m \in \mathbb{N}, s_1, \dots, s_m \in S, n_1, \dots, n_m \in \mathbb{Z}\}$$

The right hand side (rhs) is a subgroup of G , because take $m = 0$ and $e \in rhs$. Then rhs is closed under the group operation (concatenate the formulas). Also, the rhs is closed under the inverse, namely

$$(s_1^{n_1} \dots s_m^{n_m})^{-1} = s_m^{-n_m} \dots s_1^{-n_1}.$$

Also, the rhs contains S , because $\forall s \in S$, take $m = n_1 = 1$ and $s_1 = s$. Also, all H_i contain the rhs (the H_i contains S and are closed under taking integer powers and taking products). Then we get the desired equality.

Examples:

- Let G be any group. Then $S = \{e\}$ implies $\langle S \rangle = \{e\}$.
- $G = \mathbb{R}$ (or \mathbb{Q} or \mathbb{C}) and $S = \{1\}$ imply $\langle S \rangle = \langle 1 \rangle = \mathbb{Z}$.
- $G = \mathbb{R}$ (or \mathbb{C}), and $S = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. These imply that $\langle S \rangle = \mathbb{Q}$.

Definition: Generating set -

A **generating set** for a group G is a subset S of G such that $\langle S \rangle = G$. If so, then we also say that S **generates** G .

Often we'll use the word "generator" to mean an element of some **specific** generating set (otherwise it has no meaning, because always $G = \langle G \rangle$).

4 Cayley Digraphs

Vojta notes that he will hold us responsible for Cayley digraphs as discussed in the text, except the examples 7.10 and 7.12. All others are fair game.

5 Clicker Questions

3. For which of these pairs G, S is $G \neq \langle S \rangle$?

- $G := \mathbb{Z}, S = \{-1\}$
- $G = \mathbb{R}, S = (0, 1) \cup \mathbb{Z}$
- $G = \mathbb{C}^*, S = \mathbb{R}^* \cup U$
- (d) None of the above (all S above generate the given G)**
- $G \neq \langle S \rangle$ for more than one of the above three.

Lecture ends here.