Math 113, Fall 2019
Lecture 13, Tuesday, 10/22/2019

1 Clicker Questions

1. Let G be a group.
A. The commutator subgroup of G is the set

$$\{xyx^{-1}y^{-1} : x, y \in G\}$$

B. The center of G is the set

$$Z(G) := \{g \in G : gx = xg, \forall_{x \in G}\}$$

Solution. Only B is correct, because we would need to take the subgroup generated by the set in A.

□

2. How many elements are there in the group

$$(\mathbb{Z}_4 \times \mathbb{Z}_4)/\left\langle (2,2) \right\rangle?$$

Solution. 8 elements. □

2 Review

Last time, we found:

$$S_3/A_3 \cong \mathbb{Z}_2, \text{ with } A_3 \cong \mathbb{Z}_3$$

and

$$\mathbb{Z}_5/\left\langle 2 \right\rangle \cong \mathbb{Z}_2 \text{ with } \left\langle 2 \right\rangle \cong \mathbb{Z}_3,$$

both of which groups are "built out of $\mathbb{Z}_2$ and $\mathbb{Z}_3$. " Additionally, we found:

$$\mathbb{Z}_9/\left\langle 3 \right\rangle \cong \mathbb{Z}_3 \text{ with } \left\langle 3 \right\rangle \cong \mathbb{Z}_3.$$

In our iClicker question, we had

$$(\mathbb{Z}_4 \times \mathbb{Z}_4)/\left\langle (2,2) \right\rangle,$$

which had 8 elements. We look at $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
We say that these "building blocks" do provide some information on the structure of the group. This brings us to a definition:

3 Simple groups

Definition: simple -

A group is **simple** if it is nontrivial and its only normal subgroups are the trivial subgroup and the whole group.

We should think of these as a counterpart to prime numbers. Notice that we do not let 1 be prime otherwise we would use uniqueness of factorization.

3.1 Examples

1. $\mathbb{Z}_p$ is simple for all primes p
2. A_n is simple for all $n \geq 5$ (this requires a proof but we'll leave that as an exercise)
3. A_4 is NOT simple (see the current homework)

4 Factor Groups

Theorem 4.1. A **factor group** of a cyclic group is cyclic.

Proof. Let G be a cyclic group generated by a_1 and let $H \leq G$. Then $H \triangleleft G$ (G is abelian). Then G/H is cyclic, generated by aH because $\forall_{xH \in G/H}, x \in G$ so $x = a^n$ for some n, and then $xH = (aH)^n$.
More generally, if $\varphi : G \to G'$ is a homomorphism and $S \subseteq G$ generates G, then $\varphi[S]$ generates $\varphi[G] : \varphi[G] = \langle \varphi[S] \rangle$. $\square$

Definition: Center -

The **center** $Z(G)$ of a group G is the subset

$$\{g \in G : gx = xg, \forall_{x \in G}\}.$$

This is closed under the group operation (Ex 5.52), is nonempty (contains e), and closed under inverse (left as an exercise). Hence $Z(G)$ is an abelian subgroup. Additionally, it is also normal because

$$xZ = Zx, \forall_{x \in G}.$$

Definition: Commutator -

A **commutator** in a group G is an element of the form $xyx^{-1}y^{-1}$ with

$$(xyx^{-1}y^{-1} = e \iff xy = yx)$$

As in the clicker question, the commutator subgroup of G is the subgroup generated by ALL commutators of G. It is the trivial subgroup if and only if G is abelian.

Proposition 4.1.1. The commutator subgroup of a group is a normal subgroup.

Proof. Let G be a group, and let C_0 be the set of commutators in G, and let $C := \langle C_0 \rangle$ be the commutator subgroup. Let $\sigma : G \to G$ be an automorphism of G. Then $\sigma(C_0) \subseteq C_0$, because

$$\sigma(xyx^{-1}y^{-1}) = \sigma(x)\sigma(y)\sigma(x)^{-1}\sigma(y)^{-1} \in C_0.$$

Similarly, $\sigma^{-1}(C_0) \subseteq C_0$, so $C_0 \subseteq \sigma(C_0)$. Therefore $\sigma(C_0) = C_0$, and hence $\sigma(C) = c$.
So c is invariant under ALL automorphisms of G, and hence it is invariant under all **inner** automorphisms, and hence normal. $\square$

Alternatively, as an easier way (for both $Z(G)$ and the commutator subgroup), both subgroups are structurally defined, therefore they are invariant under all automorphisms.

4.1 Example

If G is a group and C is its commutator subgroup, then what is the commutator subgroup of G/C?

Notice that a commutator in G/C is

$$(xC)(yC)(xC)^{-1}(yC)^{-1} = \underbrace{(xyx^{-1}y^{-1})}\, C = C,$$

where the underbraced expression is a commutator in G and hence $xyx^{-1}y^{-1} \in C$.
Then the answer is that the commutator subgroup of G/C is the trivial subgroup. In particular, G/C is abelian. More generally, for all normal subgroups N of G,

$$G/N \text{ abelian} \iff N \geq C,$$

which Vojta leaves as an exercise.

5 Example

Find the commutator subgroup of S_3.

Let's take the commutator, which is

$$(3,2)(2,1)(3,2)^{-1}(2,1)^{-1} = \underbrace{(3,2)(2,1)}\,\underbrace{(3,2)(2,1)} = (3,2,1)^2 = (1,2,3)$$

Then the commutator subgroup is

$$C \geq \langle (1,2,3) \rangle = A_3.$$

Also, $C \subseteq A_3$ because all commutators $xyx^{-1}y^{-1}$ in S_3 are even.

6 Example computation

This is example 5 on p. 151.

Classify $\underbrace{(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_8)}_{G} / \underbrace{\langle 1,2,4 \rangle}_{H}$.

To start, we find the order of each. Notice

$$H = \{(0,0,0), (1,2,4), (2,0,0), (3,2,4)\},$$

and hence $|H| = 4$ and $|G/H| = \frac{4\cdot 4\cdot 8}{4} = 32$. Then we have the possibilities:

$$\mathbb{Z}_{32}, \mathbb{Z}_{16} \times \mathbb{Z}_2, \mathbb{Z}_8 \times \mathbb{Z}_4, \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2^5.$$

Now all elements of G have order ≤ 8, so the same is true for G/H. To check if G/H has an element of order 8, look at $(0,0,1)$. This has order $\neq 1$. Then

$$\begin{aligned}
&(0,0,1) \notin H \\
&(0,0,2) \notin H \implies \text{order} \neq 2 \\
&(0,0,4) \notin H \implies \text{order} \not| 4.
\end{aligned}$$

Therefore, $(0,0,1)+H$ has order 8 in G/H.
Now how do we tell whether it's $\mathbb{Z}_8 \times \mathbb{Z}_4$ or $\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$? We look at the number of elements that have order ≤ 2. We see that

$$(a,b,c) \text{ has order } \leq 2 \quad \iff \quad (2a,2b,2c) \in H$$

which implies that a,b,c equals $0,1,2$, or 3. Let's look at the last factor.
$(a,b,c)+H$ has order $\leq 2 \iff (2a,2b,2c) \in H \iff (2a,2b,2c) = (0,0,0)$ or $(2,0,0)$.
Notice that $(2,0,0) \in H$ so $(a,b,c)+H = (a+2,b,c)+H$, so we have $a=0$ or 1.
If $a=1$, then $(1,2,4) \in H$ also, so that eliminates $a=1$. Hence the elements of order ≤ 2 are

$$\begin{aligned}
&(0,0,0)+H \\
&(0,0,4)+H \\
&(0,2,0)+H \\
&(0,2,4)+H,
\end{aligned}$$

so $G/H \cong \mathbb{Z}_8 \times \mathbb{Z}_4$.

7 Group Actions

Definition: action -

An **action** of a group G on a set X is a function $G \times X \to X$, defined by

$$(g,x) \mapsto g * x,$$

such that:
(1) $e * x = x, \forall_x \in X$ and
(2) $g_1 * (g_2 * x) = (g_1 g_2) * x, \forall_{g_1,g_2 \in G}, \forall_{x \in X}$. where $*$ is the group action and $g_1 g_2$ specifies a group operation in G.

From the clicker questions, we have some examples:
(1) Any group G acts on itself by left translation: $g * x = gx, \forall_x \in G$.
(2) S_n acts on $\{1,2,\ldots,n\}$ by $\sigma * j = \sigma(j)$.
(3) If G acts on X and $H \leq G$, then H acts on X (by the induced operation).
(4) The **trivial action** $g * x = x, \forall_{g \in G, x \in X}$ and any G and X.
(5) If V is a real vector space, then $\mathbb{R}^*$ acts on V by scalar multiplication.

Lecture ends here.