Math 113, Fall 2019
Lecture 9, Thursday, 9/26/2019

1 Clicker Questions

1. Consider the following statements:
   - α - The symmetric group S_{10} has 10 elements
   - β - The symmetric group S_3 is cyclic
   - γ - S_n is not cyclic for any n

Which of these are true?

It turns out, all cyclic groups are abelian, so none of the above is true.

2. Let G be a be a **finite** group.
Cayley's theorem constructs an isomorphism of G with a subgroup H of S_G.
We have $H = S_G$ if and only if:
(a) G is trivial
(b) $|G| \leq 2$
(c) $0 = 1$ (never)
(d) $G \cong S_n$ (for some $n \in \mathbb{N}$)
(e) None of the above
Vojta says this boils down to counting, and $|S_n| = n!$, so this is true precisely for $n = 1, 2$.

2 Permutation Groups

Theorem 2.1. Let A be a set. Then S_A (the set of permutations of A) is a group under composition of functions.

Proof. Composition is a well defined operation on S_A. Let $f, g \in S_A$. Then $f \circ g$ is a function from A to A, and it is bijective because it's a composition of bijections. Hence $f \circ g \in S_A$.
We check the three requirements (axioms). Associativity is proved already for composition of functions. The identity function $\mathrm{id_A : A \to A}$ is bijective, so it's in S_A. Also,

$$\mathrm{id_A \circ f = f \circ id_A = f}, \forall_{\mathrm{f \in S_A}}.$$

Finally, to check the existence of the inverse element, consider that for all $f \in S_A$, because f is a bijection, it has a (unique) inverse function $f^{-1} A \to A$ characterized by

$$f \circ f^{-1} = f^{-1} \circ f = \mathrm{id_A},$$

so f^{-1} is an inverse **element** of f in S_A, the set of permutations (bijections). □

Vojta reminds us that we've seen this in a Clicker question, but:

Definition: Permutation group $S_{\{1,2,\ldots,n\}}$ -

For all $n \in \mathbb{N}$, S_n is the permutation group $S_{\{1,2,\ldots,n\}}$.

Notice that $|S_n| = n!$ for all $n \in \mathbb{N}$ because choosing $\sigma \in S_n$ involves n choices for $\sigma(1)$, $n-1$ choices for $\sigma(2)$, and so on until 2 choices for $\sigma(n-1)$, and 1 choice for $\sigma(n)$, where these can be in any order.

Example: One such example is to consider permutations (shuffling orders) of a deck of cards: S_{52}. So S_{52} is the set of possible rearrangements of a 52-card deck.

Example: A simpler example is S_3, which we can write as:

$$S_3 = \{\rho_0, \rho_1, \rho_2, \sigma_1, \sigma_2, \sigma_3\},$$

where $\sigma_1 := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ and $\rho_1 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Vojta reminds that there's a full group table on page 79 of our text.
From this table, we see that $\rho_i = \rho_1^i, \forall_{i=0,1,2}$ and $\mu_1 = \mu_1 \rho_{i-1}$ (where $\mu = \sigma$).
Hence $S_3 = \langle \rho_1, \mu_1 \rangle$.
Permutations tell us something about all finite groups due to the following theorem:

Theorem 2.2. If the sets A and B have the same cardinality, then $S_A \cong S_B$.

Proof. Because they have the same cardinality, that means there exists some bijection $f : A \to B$. The idea for the rest of the proof is to use f to relabel the elements of A. $\square$

3 Orbits and Cycles

4 Even and Odd Permutations