Math 113, Fall 2019

Lecture 2, Tuesday, 9/3/2019

Topics Today:

- Relations, Equivalence Relations, and Partitions
- Multiplication on the Unit Circle in $\mathbb{C}$.
- Binary Algebraic Structures (time permitting)

Readings for Thursday: up to and including page 40.

1 iClicker Question

In simplest terms, $(1+i)(2+3i)$ is:
(a) $2+5i+3i^2$
(b) $5+5i$
$\boxed{\text{(c)}}$ $-1+5i$
(d) $3+4i$
(e) I don't have an iclicker yet.
And for the second question,

$$4 +_{2\pi} 5 = \boxed{9-2\pi},$$

where we omit the other answer selections.

2 Relations

Definition: Relation -

A relation $\mathcal{R}$ on a set S is a subset $\mathcal{R}$ of $S \times S$. For $x, y \in S$, we write:

$$x \ \mathcal{R} \ y$$

if the pair $(x, y) \in S$.

Of course, 'if' in a definition means 'if and only if'. As examples, for any set S, $x \ \mathcal{R} \ y$ if $x = y$ is a relation. If $S = \mathbb{R}$, then "$<$" is a relation on $\mathbb{R}$: $x \ \mathcal{R} \ y$ if $x < y$.

Let σ be a collection of sets (set of sets). Then "$\subseteq$" is a relation on σ. We say that $A \ \mathcal{R} \ B$ if $A \subseteq B$.

2.1 Same Cardinality as a Relation

For the same collection Σ "having the same cardinality" is a relation on Σ. That is,

$$A \ \mathcal{R} \ B \text{ if there is a bijection } A \to B.$$

There are certain kinds of relations that play a special role in this class, and let's start with:

Definition: Reflexive, Symmetric, Transitive -

Let $\mathcal{R}$ be a relation on set S. Then $\mathcal{R}$ is:
(1) **reflexive** if $x \mathcal{R} x, \ \forall_{x \in S}$
(2) **symmetric** if $x \mathcal{R} y \implies y \mathcal{R} x, \ \forall_{x,y \in S}$
(3) **transitive** if the **conjunction** of $x \mathcal{R} y$ and $y \mathcal{R} z$ implies $x \mathcal{R} z, \ \forall_{x,y,z \in S}$.

Example: Consider $<$ on $\mathbb{R}$. Surely this is not reflexive because no number is less than itself. Additionally, it is not symmetric; however,it is transitive. Now taking $\leq$ on $\mathbb{R}$, we have reflexivity and transitivity; however, symmetry fails.

Example: The relation 'have the same cardinality' (where $S := \Sigma$ is a collection of sets). We go back to the definition:

$$A \mathcal{R} B \text{ if there is a bijection } A \to B,$$

and to see reflexivity, for all $A \in \Sigma$, the **identity map** is a bijection $A \to A$ with:

$$\mathrm{id_A(x) = x, \ \forall_{x \in A}}.$$

Now for symmetry, of course the wording implies yes; however, we need to revisit the definition. If A, B have the same cardinality, do B, A have the same cardinality? If A, B have the same cardinality, then this implies that there exists a bijection $f : A \to B$ which implies $f^{-1} : B \to A$ is also a bijection (left as an exercise), which implies that B, A have the same cardinality.
To see transitivity, if A, B have the same cardinality, and B, C have the same cardinality, then there exist the following:

$$f : A \to B, \quad g : B \to C,$$

and $g \circ f : A \to C$ is a bijection (left as an exercise). Therefore A, C have the same cardinality.

Definition: Equivalence Relation -

We say that an **equivalence relation** on a set S is a relation (on S) which satisfies (1) reflexivity, (2) symmetry, and (3) transitivity.

Example: (1) 'Have the same cardinality' is an equivalence relation (on any collection of sets).
(2) '$<$' on $\mathbb{R}$ is **not** an equivalence relation.
(3) For any set S, '$=$' is an equivalence relation.
(4) And slightly more generally, let S be a set and take $f : S \to T$ be a function, and define a relation $\sim$ (pronounced 'twiddle') on S by

$$x \sim y \text{ if } f(x) = f(y).$$

Then $\sim$ is an equivalence relation on S.

Remark: In fact, we will see fairly soon that all equivalence relations can be made to look like this. In other words, given any equivalence relation, there is a function for which the relation is given by this function.

Before we get to this, consider:

Definition: Partition -

A **partition** of a set S is a collection of nonempty subsets of S called **cells**, such that each element of S lies in **exactly one cell**.

Equivalently, cells are all nonempty, mutually disjoint ($C_1 \cap C_2 = \{\}, \ \forall_{\text{ cells } C_1 \neq C_2}$), and the union of all cells is S.

The intuition here is sorting everything into piles, not including any empty piles. Recall that in homework 1, we have an example of a partition with an infinite number of cells, so a partition need not have a finite number of cells.

Theorem 2.1. Let S be a set. Then there is a natural bijection:

$$\{\text{partition of } S\} \to \text{equivalence relations on } S\}$$
$$p \mapsto x \sim y \text{ if } x, y \text{ lie in the same cell,}$$

where $p \in$ partition of S. The inverse is:

$$\text{the collection of all equivalence classes for } \sim \quad \hookleftarrow \quad \sim$$

For proof, see the handout.

Definition: Equivalence Class -

The equivalence class of $x \in S$ is

$$\overline{x} := \{y \in S \mid y \sim x\}.$$

$x \in \overline{x}, \quad \forall_x$ (follows from reflexivity)
$\overline{x} = \overline{y} \iff x \sim y$ (follows from symmetry and transitivity)

Now, given an equivalence relation $\sim$ on a set S, let T be the set of equivalence classes and define:

$$f : S \to T$$
$$f(x) = \overline{x}$$

where $\overline{x}$ is the equivalence class containing x. Then

$$f(x) = f(y) \iff \overline{x} = \overline{y} \iff x \sim y,$$

so $\sim$ is of the type (4) in our examples above.

Example: For an important family of examples of equivalence relations, let $n \in \mathbb{Z}^+ := \{1, 2, 3, \dots\}$. For $x, y \in \mathbb{Z}$, define $x \cong_n y$ if $x - y = qn$ for some $q \in \mathbb{Z}$. This is called a **congruence modulo** n and is usually written:

$$x \cong y \pmod{n}.$$

To see reflexiity, consider that

$$x \cong x \pmod{n}$$

because $x - x = 0 \cdot n (0 \in \mathbb{Z})$. To see symmetry, consider:

$$\begin{aligned} x \cong y \pmod{n} &\implies x - y = qn, \text{ with } q \in \mathbb{Z} \\ &\implies y - x = qn, \text{ with } -q \in \mathbb{Z} \\ &\implies y \cong x \pmod{n}. \end{aligned}$$

Now to see transitivity, consider:

$$x \cong y, y \cong z \pmod{n} \implies x \cong z \pmod{n}$$

because if $x - y = q_1 n$ and $y - z = q_2 n$, then $x - z = (q_1 + q_2)n$ with $q_1 + q_2 \in \mathbb{Z}$.
Now, the equivalence classes are: $\overline{0}, \overline{1}, \ldots, \overline{n-1}$ (by the 'division algorithm'). For example,

$$\overline{2} = \{\ldots, 2 - 2n, 2 - n, 2, 2 + n, 2 + 2n, \ldots\}$$

2.2 iClicker Question.

Which of the following is a partition of $\{1, 2, 3, 4\}$? Choices include:

$$\begin{aligned} P &: \{\{1, 2, 3\}, \{2, 4\}\} \\ Q &: \{\{1, 2\}, \{3, 4\}, \{\}\ \} \end{aligned}$$

Neither are, because 2 is present in two cells of P and Q has an empty cell.

3 Complex Numbers

C is a vector space over $\mathbb{R}$ with the basis $\{1, i\}$. Multiply by simplifying and applying $i^2 := -1$. To divide,

$$\frac{z}{w} = zw^{-1}, \text{ where } w^{-1} = \underbrace{\frac{1}{|w|^2}}_{\mathbb{R}^+} \overline{w},$$

where $w \neq 0$. Further, we have:

$$e^{x+iy} = e^x (\cos y + i \sin y),$$

so then

$$\left|e^{x+iy}\right| = |e^x| \cdot |\cos y + i \sin y| = e^x \sqrt{\cos^2 y + \sin^2 y} = e^x.$$

Also, note that

$$\left|e^{i\theta}\right| = 1, \quad \forall_{\theta \in \mathbb{R}}.$$

Now let $U := \{z \in \mathbb{C} \mid |z| = 1\}$, where:

$$\begin{aligned} \mathbb{R} &\xrightarrow{\text{onto}} U \\ \theta &\mapsto e^{i\theta}, \end{aligned}$$

and taking $[0, 2\pi)$ gives:

$$\{x \in \mathbb{R} \mid 0 \leq x \leq 2\pi\} = [0, 2\pi) \xrightarrow{\text{bijective}} U.$$

The set U is closed under multiplication in that:

$$z, w \in U \implies |z| = |w| = 1 \implies |zw| = |z| \cdot |w| = 1 \cdot 1 = 1 \implies zw \in U.$$

However, it is easier to think of multiplication on U in terms of angles. If $z = e^{i\theta}$ and $w = e^{i\phi}$ then $zw = e^{i\theta} \cdot e^{i\phi} = e^{i(\theta+\phi)}$.

So if $\theta, \phi \in [0, 2\pi)$, then we would like to write $e^{i\theta} \cdot e^{i\phi}$ as $e^{i\psi}$ for some $\psi \in [0, 2\pi)$. Sometimes, we have: $\psi = \theta + \phi$, but not if $\theta + \phi \geq 2\pi$.

$$\theta +_{2\pi} \phi = \begin{cases} \theta + \phi, & \theta + \phi < 2\pi \\ \theta + \phi - 2\pi, & \theta + \phi \geq 2\pi. \end{cases}$$

Definition: Binary Operation -

A **binary operation** on a set S is a function from $S \times S \to S$. For each $a, b \in S$, we often use a symbol to denote the value of this function at $(a, b) \in S \times S$. For example,

$$\begin{aligned} S \times S &\longrightarrow S \\ (a, b) &\mapsto a * b \end{aligned}$$

where $(a, b) \in S \times S$ and $a * b \in S$.

Example: The binary operations $+, \times$ on $\mathbb{Z}$.

Definition: Closed under $*$, Induced Operation -

Let $*$ be a binary operation on a set S and let $H \subseteq S$. We say that H is **closed under** $*$ if

$$x * y \in H, \quad \forall_{x,y \in H}.$$

If so, then restricting $*$ (as a function $S \times S \to S$) to $H \times H$ (note $H \times H \subseteq S \times S$) gives a function $H \times H \to H$; i.e. a binary operation on H. This is called the **binary operation on H induced by $*$,** or simply the **induced operation** on H.

As an example of this, $\mathbb{Z}$ is closed under $+$ on $\mathbb{Q}$, and $+$ on $\mathbb{Z}$ is the induced operation, and etcetera. Then $+_{2\pi}$ is a binary operation on $R_{2\pi} = [0, 2\pi)$.

3.1 Last iClicker Question:

$$7 +_{2\pi} 3 = \boxed{10 - 2\pi}$$

Lecture ends here.