Math 113, Fall 2019
Lecture 11, Thursday, 10/3/2019

1 Clicker Questions

1. Which of the groups

$$G_1 = \mathbb{Z}_6 \times \mathbb{Z}_{20} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5$$
$$G_2 = \mathbb{Z}_{15} \times \mathbb{Z}_8 \cong \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_8$$
$$G_3 = \mathbb{Z}_{12} \times \mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_5$$

Answer: $G_3 \cong G_1 \ncong G_2$.

2. Let $G := \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_{27}$. Let m be the number of elements of G of order ≤ 3.
Choose the following correct answer:
(A) $m \leq 3$
(B) $3 < m \leq 27$
(C) $27 < m \leq 81$
(D) $81 < m < |G|$
(E) $m \geq |G|$
Answer: Notice $|G| = 3^8$ (which is an odd order), so no elements are of order 2. Then $a \in G$ has order ≤ 3 if and only if its order divides 3.

2 Review

Last time we defined the (finite) product of groups

$$G_1 \times \cdots \times G_n = \prod_{i=1}^{n} G_i.$$

Also, if $G_1, \ldots, G_n$ are abelian, then so is their product. In this case, we may also call $\prod G_i$ the "direct sum", and write it as

$$G_1 \oplus \cdots \oplus G_n \quad \text{or} \quad \bigoplus_{i=1}^{n} G_i.$$

A word of caution: One can also define an infinite product $\prod_{i \in I} G_i$ with $|I| = \infty$. However, in this case, if all G_i are abelian, the direct sum is different.

3 Examples of Products:

(0) The additive group of a product of vector spaces.
(1) The Klein V group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (other isomorphisms are possible)

$$e \leftrightarrow (0,0)$$
$$a \leftrightarrow (0,1)$$
$$b \leftrightarrow (1,0)$$
$$c \leftrightarrow (1,1)$$

(2) How about $\mathbb{Z}_2 \times \mathbb{Z}_3$? What is the order of $(1,1)$? We note that the order of this must divide the order of the group, which is 6. The fact that $(1,1), 2(1,1), 3(1,1)$ are all not $(0,0)$ shows that the order is not 1,2,3 respectively. Hence it must have order 6.
This is a special case of the following theorem:

Theorem 3.1. Let $(a_1, \ldots, a_n) \in G_1 \times \cdots \times G_n$. If a_i is of finite order r_i in G_i, then for all i, the order of $(a_1, \ldots, a_n)$ is equal to the least common multiples of each of their orders. That is,

$$\text{order}(a_1, \ldots, a_n) = \text{lcm}(r_1, \ldots, r_n).$$

Proof. The proof in the text is the same as we've done in the previous example. $\square$

Corollary 3.1.1. Let $m_1, \ldots, m_n \in \mathbb{Z}^+$. Then the largest order of an element of $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ is $\text{lcm}(m_1, \ldots, m_n)$, and it occurs with the element $(1, 1, 1, \ldots, 1)$, where 1 means 0 if $m_i = 1$).

Proof. Let $(a_1, \ldots, a_n) \in \prod \mathbb{Z}_{m_i}$, and let $r_i := |a_i|$. Then $r_i | m_i \forall_i$, so

$$|(a_1, \ldots, a_n)| = \text{lcm}(r_1, \ldots, r_n) \mid \text{lcm}(m_1, \ldots, m_n),$$

so it's less than or equal to $(\leq)$.
If $a_1 = 1$ for all i, then $r_i = m_i, \forall_i$, and we have equality. $\square$

Corollary 3.1.2. Let $m, n \in \mathbb{Z}^+$. Then $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if $\gcd(m, n) = 1$.
If so, then $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$, and $\mathbb{Z}_m \times \mathbb{Z}_n$ is generated by $(1, 1)$.

Proof. Notice $\gcd(m, n) = 1$ implies that $\text{lcm}(m, n) = \frac{m}{\gcd(m,n)} = mn$, which then implies that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic, generated by $(1, 1)$.

On the other hand, $\gcd(m, n) \neq 1$ implies $\gcd(m, n) > 1$, which then implies $\text{lcm}(m, n) < mn$, and so $\mathbb{Z}_m \times \mathbb{Z}_n$ has no element of order mn. Hence $\mathbb{Z}_m \times \mathbb{Z}_n$ is NOT cyclic. $\square$

4 Facts about Products

4.1

$G_1 \times G_2 \cong G_2 \times G_1$, and

$$(G_1 \times G_2) \times G_3 \cong G_1 \times (G_2 \times G_3) \cong G_1 \times G_2 \times G_3.$$

Similarly, products of higher numbers of groups also commute and associate (up to isomorphism).

5 Finitely Generated Abelian Groups

Definition: Finitely generated -

A group is **finitely generated** if it can be generated by a finite subset.

Example: Any finite group G with $G = \langle G \rangle$, the group $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{mn}$, and

$$\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n} \times \mathbb{Z} \times \cdots \times \mathbb{Z}, (m_1, \ldots, m_n \in \mathbb{Z}^+).$$

Theorem 5.1. (Fundamental Theorem of Finitely Generated Abelian Groups.)

Let G be a finitely generated abelian group. Then

$$G \cong \mathbb{Z}_{p_1^{r_1}} \times \cdot \times \mathbb{Z}_{p_n^{r_n}} \times \mathbb{Z} \times \cdots \times \mathbb{Z},$$

where $n \in \mathbb{N}$, where primes $p_1, \ldots, p_n$ may be repeated, and $r_1, \ldots, r_n \in \mathbb{Z}^+$. Moreover, these factors are unique, up to reordering them.

We won't prove this result. The converse states that all groups of the form

$$\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$$

are finitely generated abelian groups. Additionally, we can make this representation unique by requiring that

$$p_1 \leq p_2 \leq \cdots \leq p_n,$$

and $r_i \geq r_{i+1}$ whenever $p_i = p_{i+1}$.
However, the isomorphism need not be unique. For example, recall that $V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ in many ways.

5.1 Examples

(1) $\mathbb{Z}_4 \times \mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_5$.
(2) Find (up to isomorphism) all groups of order $96 = 2^5 \cdot 3$.

Solution. For $p = 3$, we can only have one $p_i = 3$, and its exponent must be 1: $p_n = 3, r_n = 1$.
For $p = 2$, we have $p_1 \cdots p_{n-1} = 2$ (where $n > 1$) and $R_1 + \cdots + r_{n-1} = 5$.
How many ways are there to write 5 as $r_1 + \cdots + r_{n-1}$ with integers $r_1 \geq r_2 \geq \cdots \geq r_{n-1} > 0$? We find 7 possibilities:

$$\begin{aligned}
r_1 = 5 &\implies n = 2, 5 = 5 \\
r_1 = 4 &\implies n = 3, 5 = 4 + 1 \\
r_1 = 3 &\implies 5 = 3 + 2 = 3 + 1 + 1 \\
r_1 = 2 &\implies 5 = 2 + 2 + 1 = 2 + 1 + 1 + 1 \\
r_1 = 1 &\implies 5 = 1 + 1 + 1 + 1 + 1.
\end{aligned}$$

□

(3) Find all abelian groups of order 10.

Solution. This is much easier: it's just $\mathbb{Z}_2 \times \mathbb{Z}_5$, because $10 = 2 \times 5$. This is the same for all products of distinct primes (Them 11.17). □

6 Clicker Question

3. Which of the following **must be** homomorphisms?
(A) $\varphi : H \to G$ given by $\varphi(x) = x, \forall_{x \in H}$, where $H \leq G$, for all G, H.
(B) $\varphi : G \to G$ given by $\varphi(x) = x^{-1}, \forall_G$.
(C) $\psi \circ \varphi : G \to G''$, where $\varphi : G \to G'$ and $\psi : G' \to G$" are homomorphisms, for all G, G', G", φ, ψ.
Answer: (A) and (C) are true.

4. How many of the following are true:
(1) A_n is a normal subgroup of S_n (for all $n \in \mathbb{N}$) (true)
(2) For all groups G, G', there exists a homomorphism from G to G' (true, just take the trivial homomorphism)
(3) A homomorphism is one-to-one if and only if its kernel is the trivial subgroup (true)
(4) The image of a group of 6 elements under a homomorphism may have 12 elements (false)
(5) It is not possible to have a nontrivial homomorphism from some finite group to some infinite group (false; an example would be $U_3 \to \mathbb{C}^*$.)

Lecture ends here.