Stats 135, Fall 2019

Lecture 15, Friday, 10/4/2019

1 Review

Last time we worked through the Likelihood Ratio Test (LRT) for a simple hypothesis where

$$\Lambda = \frac{f_0(x|\theta)}{f_1(x|\theta)}$$

,
We consider an alternative version: the generalized likelihood ratio test:
Let $\Omega = \{\theta_0, \theta_1\}$ and let

$$\tilde{\Lambda} = \frac{f_0(x|\theta)}{\max_{\theta\in\Omega} f(x|\theta)},$$

where the denominator is just the likelihood at the MLE. Then we would reject if $\tilde{\Lambda} < c$. Notice that these two are equivalent, where small values of Λ corresponds to small values of $\tilde{\Lambda}$.

Topics Today:

- Duality of Confidence Interval and Hypothesis Test
- Generalized Likelihood Ratio Test (GLRT)

2 Duality of CI and HT

In words, the test statistics (TS) for a lever α hypothesis test (HT) with null $H_0 : \theta = \theta_0$ lies in the acceptance region if and only if the null value θ_0 lies in a $100(1-\alpha)$ confidence interval for θ. Hence we see this connection between confidence intervals and hypothesis tests. This is the duality, and we'll demonstrate this through an example.

Example: Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$, where σ^2 is known. Take the null $H_0 : \mu = \mu_0$ and the alternative $\mu \neq \mu_2$. Now fix α level of significant, and we sill see later today that the acceptance region of GLRT for this hypothesis test is

$$\overline{X} = \mu_0 \pm z\left(\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n}}$$

We draw a picture of x bar under the null, centered at mu_0.
Notice that mathematically, the following two inequalities are equivalent.

$$\mu_0 - z\left(\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu_0 + z\left(\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n}},$$

which defines the acceptance region, and

$$\overline{X} - z\left(\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n}} < \mu_0 < \overline{X} + z\left(\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n}},$$

which gives the $100(1-\alpha)\%$ confidence interval for μ.
It follows that we accept for the null when $\overline{X}$ is in the acceptance region or equivalently when the null parameter μ_0 is in the $100(1-\alpha)\%$ confidence interval for μ. This is certainly obvious for our example, and for generalizing, see proof of A,B of §9.3.

3 Generalized Likelihood Ratio Test (GLRT)

Most hypothesis tests are not simple, and the UMPT often doesn't work in the range of alternative values that we need. Then in this case, we use a GLRT.
Let $\Omega := \omega_0 \oplus \omega_1$, the direct sum (disjoint union). Consider the null $H_0 : \theta \in \omega_0$ and the alternative $H_1 : \theta \in \omega_1$.
Then define

$$\Lambda := \frac{\max_{\theta\in\omega_0}(\text{lik}(\theta))}{\max_{\theta\in\Omega}(\text{lik}(\theta))}.$$

Notice the denominator is just simply $\text{lik}(\hat{\theta}_{ML})$.
Then the GLRT tells us to reject the null H_0 if $\Lambda < c$, and otherwise accept.
Example:
We would like to test $H_0 : \mu = \mu_0$, where μ_0 is fixed. Take the alternative to be $H_0 : \mu \neq \mu_0$. Here, $\omega_0 = \{\mu_0\}$ and $\omega_1 = \mathbb{R} - \{\mu_0\}$. This is so that $\Omega = \mathbb{R}$. Then there is just one element, so

$$\begin{aligned}\max_{\theta\in\omega_0}(\text{lik}(\theta)) &= \text{lik}(\mu_0)\\ &= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2\right),\end{aligned}$$

and so we write

$$\begin{aligned}\max_{\theta\in\Omega}(\text{lik}(\theta)) &= \text{lik}(\overbrace{\hat{\theta}_{ML}}^{\overline{X}})\\ &= \cancel{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n} \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^n (X_i-\overline{X})^2\right)\end{aligned}$$

Then we have

$$\Lambda = \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^n \left[(X_i-\mu_0)^2 - (X_i-\overline{X})^2\right]\right)$$

so then

$$\Lambda < c \iff -2\log\Lambda > -2\log c,$$

which gives

$$-2\log\Lambda = \frac{1}{\sigma^2}\sum_{i=1}^n \left[(X_i-\mu_0)^2 - (X_i-\overline{X})^2\right],$$

which after some algebra gives

$$\frac{n}{\sigma^2}(\overline{X}-\mu_0)^2.$$

Hence $-2\log\Lambda > k$, which gives

$$\frac{n}{\sigma^2}(\overline{X}-\mu_0)^2 > k,$$

which is a chi square with 1 degree of freedom: $\left(\frac{\overline{X}-\mu_0}{\frac{\sigma}{\sqrt{n}}}\right)^2 = z^2 \sim \chi^2_1$.

It turns out that asymptotically, for large n, we will have a χ-square distribution. So the Generalized Likelihood Ratio Test says to reject H_0 if

$$\left(\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}\right)^2 > \chi_1^2(\alpha).$$

This converts a two-sided test into a one-sided test.
We find that the acceptance region is

$$\mu_0 - z\left(\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu_0 + z\left(\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n}}.$$

However, this only works for this particular example with $X_1, \ldots, X_n \sim N(\mu, \sigma)$.
Now more generally,

Theorem 3.1. (Thm A Rice p. 341):
If the likelihood function of X is smooth, then the null distribution of $-2\log\Lambda$ approaches $\chi^2_{\dim\Omega - \dim\omega_0}$.

In our last example, we had $\Omega = \mathbb{R} - \dim 1$ and $\omega_0 = \{\mu_0\} - \dim 0$.

4 Applications

Suppose a pollster wants to know the fraction of the population θ that supports a particular legislative bill. They want to test if $\theta = \frac{1}{2}$ versus $\theta \neq \frac{1}{2}$. What is the GLRT at a 5% level of significance?
We have $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(\theta)$. Then the likelihood is given by
Recall that $\hat{\theta}_{ML} = \overline{X}$. Then $H_0 : \theta = \frac{1}{2}$ and $H_1 : \theta \neq \frac{1}{2}$ gives:

$$\boxed{\Gamma = \frac{\left(\frac{1}{2}\right)^n}{\overline{X}^{n\overline{X}}(1-\overline{X})^{n-n\overline{X}}}},$$

which we can find from data.
The GLRT tells us to reject the null for $-2\log\Lambda > k$ and accept otherwise.
For large n,

$$-2\log\Lambda \sim \chi^2_{1-0},$$

under the null distribution. Then

$$k = \chi_1^2(.05) = \text{qchisq}(.95,\ \text{df} = 1) = 3.84.$$

Hence the GLRT tells us to reject H_0 if $-2\log\Lambda > 3.84$, and accept otherwise.
The bias then is $\mathbb{E}(\hat{})$