Stats 135, Fall 2019
Lecture 13, Monday, 9/30/2019

1 Review

Last time, we worked through §9.2, the Neyman-Pearson Paradigm. We have the decision function

$$d(x) = \begin{cases} 1, & \text{if reject } H_0 \\ 0, & \text{if accept } H_0 \end{cases}$$

A simple hypothesis has null and alternative distribution is specified. There are two types of error:

type 1 error: $\alpha = \mathbb{P}_0(d(x) = 1)$, level of significance
type 2 error: $\beta = \mathbb{P}_1(d_(x) = 0)$.

We set a cap for α and want to minimize β, given that α.
Many believe that a type 1 error is worse than a type 2 error. We will see that α and β are negatively correlated so lowering one type of error makes the other type larger.
Hence we fit α at a tolerable level (say $\alpha = 0.05$) and design our experiment to minimize β with that fixed value of α. We call α the **significance level** of the test.

Topics Today:

- Likelihood Ratio Test
- Power of a test
- p-value

2 Likelihood Ratio Test (LR)

A **test statistic (TS)** is a function of your sample that leads you to a decision whether to accept or reject the null (hypothesis). Usually we choose a TS that has a distribution that we know.
Typically we'll work with the χ-squared distribution, but we'll first work with some basic examples.

Example: Suppose there are 2 coins, where coin 0 has $p = .5$ of landing heads, and coin 1 has $p = .7$ of landing heads. We don't know which coin we have.
Let H_0 (our null hypothesis) is that we have a fair coin (coin 0). Our alternative H_1 is that we have coin 1 (biased with $p = .7$). To test this, we flip our coin 3 times. Let X be the number of heads in 3 tosses. Then consider the following table:

x	$f_0(x)$	$f_1(x)$	$\Lambda(x) = \frac{f_0(x)}{f_1(x)}$
3	$\binom{3}{3}(.5)^3 = .125$	.343	.364
2	$\binom{3}{2}(.5)^3 = .375$	.411	.850
1	.375	.189	1.984
0	.125	.027	4.630

Our acceptance region is in the bottom half (high LR), and the rejection region is in the top half (low LR). We reject the the null H_0 for small Λ.

Definition: -

For a fixed α, the LRT (likelihood ratio test) says to reject H_0 if $\Lambda < c$ (for some c we need to specify).
The cutoff c is a function of α since

$$\alpha = \mathbb{P}_0(d(x) = 1) = \mathbb{P}_0(\Lambda < c).$$

For $\alpha = .5$,

$$.5 = \mathbb{P}_0(\Lambda < c).$$

Now from the table above, $\mathbb{P}_0(\Lambda < 1.984) = .125 + .575 = .5$ implies $c = 1.984$.
Then the LRT is to reject H_0 if $\Lambda < 1.984$, or (because we know $x \sim$ Binomial), we reject H_0 if $x > 1$.

Definition: Rejection Region, Acceptance Region -

The region of values of Λ in which we reject H_0 is called the **rejection region**.

Definition: Power of a test -

We define

$$\text{Power} = 1 - \beta = 1 - \mathbb{P}_1(d(x) = 0) = \mathbb{P}_1(d(x) = 1).$$

Lucas notes that for powers, the subscript of $\mathbb{P}$ on will be the same (1) as the equality. In the previous example,

$$\begin{aligned} \text{Power} = \mathbb{P}_1(d(x) = 1) &= \mathbb{P}_1(\Lambda < 1.984) \\ &= .343 + .441 \\ &= .784 \end{aligned}$$

To summarize the flow, we fixed α and found the cutoff using the null distribution. Then, we find the power using the alternative distribution. Let's do another example.

Example: (Finding the rejection and acceptance region of an LRT).
Suppose under the null that a random variable X has a uniform distribution on $[0, 1]$, and under the alternative, it has density $f(x) = 2x$, for $0 < x < 1$.

(a) What is the LRT at $\alpha = .1$ level of significance? Doing so, we get the cutoff, which we can use to solve the next problem.

We draw a picture (plot) of the densities, where H_1 takes on a line with slope 2, and H_0 is the uniform density on $[0, 1]$. Then

$$\alpha = \mathbb{P}_0(d(x) = 1) = .1,$$

so LRT rejects H_0 if $\Lambda = \frac{f_0(x)}{f_1(x)} < c$. That is,

$$\begin{aligned} \alpha = .1 = \mathbb{P}_0\left(\frac{1}{2x} < c\right) \\ .1 = \mathbb{P}_0\left(x > \frac{1}{2c}\right) \implies \frac{1}{2c} = .9 \implies c = \frac{1}{1.8} = \boxed{\frac{5}{9}} \end{aligned}$$

Then we conclude that LRT rejects H_0 if $\Gamma < \frac{5}{9}$ or reject H_0 if $x > .9$.

Later in the course, we'll draw the distribution with respect to Λ, instead of the distribution with respect to x.

(b) What is the power of the test?

We found that .9 is the magical cutoff, and so the power, $1 - \beta$ is the area from .9 to 1 under the alternative distribution. That is,

$$\begin{aligned} \text{Power} &= \mathbb{P}_1(d(x) = 1) \\ &= \mathbb{P}(X > .9) \\ &= \int_{.9}^{1} 2x \; dx = x^2\big|_{.}9^1 = 1 - .81 = .19. \end{aligned}$$

3 Neyman Pearson Lemma

Lucas notes this is an important lemma.

Lemma 3.1. Suppose that H_0, H_1 are simple hypothesis (that is, we know the distributions for both of them). Suppose that the LRT rejects H_0 with significance level α. Then any other test for which the significance level is α has less power.

This is a great result. This means that the likelihood ratio test that we have (that we will reject when $\Lambda < c$) is as good as we can get. We may be able to cook up other tests, but they will not be as good. So for simple hypothesis, likelihood ratio tests are "the bomb," as given by Lucas. In other words, LRT is the best we can do for hypothesis testing (HT) of simple hypothesis.

4 p-values

We want to quickly talk about p-values for a test statistic, as it's quite easy.

Definition: p-value -

A test statistic's p-value is the probability that it is as or more extreme than what we observe.

Let's look at a tricky example. Suppose a TS (test statistic), T, has the standard normal distribution. If the test rejects for large $|T|$, what is the p-value if we observe -1.5? (In other words, we reject if $|T| < c$ for $c > 0$). This means that the p-value is

$$\begin{aligned} p\text{- value} &= \mathbb{P}_0(T < -1.5 \text{ or } T > 1.5) \\ &= 2(1 - \Phi(1.5)) \\ &= 2(.068) \\ &= .1336. \end{aligned}$$

The important thing is to note that if the likelihood ratio test says to reject for large negative values, then the p value is the probability that T is less than -1.5 (in other words, we would have one tail). That is, p-value= $\mathbb{P}_0(T < -1.5)$. But in our situation, we consider both tails and double this value.

Lecture ends here.