Stats 135, Fall 2019
Lecture 14, Wednesday, 10/2/2019

1 Review

Last time, we went through §9.2 and defined two types of errors:

$$\alpha = \mathbb{P}_0(d(x) = 1), \text{significance level, often .05}$$
$$\beta = \mathbb{P}_1(d(x) = 0),$$

and we set that LRT (likelihood ratio test) gives: reject H_0 if

$$\Lambda = \frac{f_0(x)}{f_1(x)} < c,$$

for some c determined by (fixed, pre-determined) α.
We defined the **power** (which we want to maximize) as

$$1 - \beta = \mathbb{P}_1(d(x) = 1)$$

Then the Neyman Pearson Lemma (NPL) just says that we'll reject half the line. It says that LRT gives the most powerful test between simple hypothesis (taking the rejection region of $\Lambda < c$).

Topics Today:

- Review LRT and Power
- "Uniformly most power test" is a LRT, most powerful, even for non-simple hypothesis. This usually doesn't exist, but there are cases where it does.
- Power curve: For simple hypotheses, we can compute the power (for the alternative). However, for nonsimple hypotheses, there will be a power curve instead.

2 Review Likelihood Ratio Test (LRT)

Example: Let $X \sim \text{Exponential}(\lambda)$ (like waiting times for something to happen), with

$$f(x) = \lambda e^{-\lambda x}, x > 0.$$

Then take $H_0 : \lambda = 1$ and $H_1 : \lambda = 2$, with $\alpha := .05$.
Our cutoff will look something like $x < c$ by our sketch of the graphs. We can tell that the rejection region will be .

(a) Lucas tasks us to find the LRT.

LRT says to reject H_0 if $X < k$. By drawing the picture and knowing the cut-off is on the left, take

$$\alpha = \mathbb{P}_0(X < k) = \int_0^k e^{-x}\, dx = 1 - e^{-k}$$
$$\implies e^{-k} = 1 - \alpha = 0.95$$
$$\implies k = -\log(.95) = 0.0513$$

Equivalently, by definition, in terms of $\Lambda = \frac{f_1(x)}{f_0(x)}$, we have:

$$\begin{aligned}
\alpha &= \mathbb{P}_0(\Lambda < c) \\
&= \mathbb{P}_0\left(\frac{1}{2}e^{-x} < c\right) \\
&= \mathbb{P}_0\left(x < \underbrace{\log(2c)}_{=:k}\right) \\
&= \int_0^k e^{-x}\ dx \\
&= -e^{-x}\big|_0^k \\
&= 1 - e^{-k} \\
\implies e^{-k} &= 1 - \alpha = .95 \\
\implies k = -\log(.95) = 0.0513
\end{aligned}$$

(b) For $\alpha = .04$, what is the maximum power that we can achieve, among all hypothesis tests?

LRT is most powerful by NPL (Neyman Pearson Lemma), with

$$\text{Power} = \mathbb{P}_1(d(x) = 1) = \int_0^{.0513} 2e^{-2x}\ dx = -e^{-2x}\big|_0^{0.0513} = 1 - e^{-2(.053)}$$

3 §9.2.3: Uniformly Most Powerful Test (UMPT)

Definition: Uniformly Most Powerful Test -

The NPL requires H_0, H_1 to be simple. If the null is simple and the alternative is composite (for example, say $H_0 : \theta = 2, H_1 : \theta > 2$), then a likelihood ratio test (LRT) that is most powerful for **every** simple alternative is called **uniformly most powerful test**.

This test must have the same rejection region for every single possible alternative. We ask, does this exist often? Lucas says this is a start to considering composite alternatives.
Otherwise, we would use a generalized likelihood ratio test, which is much more complicated.

Example: Let X be a single observation from a probability density function

$$f(x|\theta) = \frac{1}{2}\theta \exp(-\theta|x|),$$

which we call the Laplace distribution. Find the most powerful test for $\alpha = .05$, testing $H_0 : \theta = 1$ and $H_1 : \theta = \frac{1}{2}$.
From the plot, we gain insight to reject the null if $|x| > k$. We need to find the appropriate k. So we have

$$\begin{aligned}
\alpha = \mathbb{P}_0(|x| > k) = 2\mathbb{P}_0(X > k) &= \cancel{2}\int_k^\infty \frac{1}{\cancel{2}}e^{-x}\ dx \\
&= -\exp(-x)\big|_k^\infty = \exp(-k),
\end{aligned}$$

so

$$e^{-k} = .05 \implies k = \log(.05) = 3,$$

where log denotes the natural log. Hence the likelihood ratio test tells us to reject H_0 if $|x| > 3$.

Next, let's see if this test is UMP (uniformly most powerful) for composite alternative, say:

$$H_0 : \theta = 1$$
$$H_1 : \theta = b, 0 < b < 1$$

We want to reject H_0 if $|x| > k$. The question is if this is a function of b? Consider:

$$\begin{aligned}
\alpha &= \mathbb{P}_0(|x| > k) = 2\mathbb{P}_0(x > k) \\
&= 2\int_k^\infty \frac{1}{2}\exp(-x)\ dx \\
&= -\exp(-x)\Big|_k^\infty \\
&= \exp(-k) = .5,
\end{aligned}$$

where $k = 3$ and is not a function of b. Hence the rejection region is the same for all b in $0 < b < 1$. So the likelihood ratio test that we have uniformly most powerful (UMP) over this range of values of b!.
Now we may ask, what happens when $b > 1$? We'll check this quickly. When $b > 1$, our rejection region changes:

$$\begin{aligned}
\alpha &= \mathbb{P}(|x| < k) = 2\mathbb{P}_0(x < k) \\
&= 2\int_0^k \frac{1}{2}\exp(-x)\ dx \\
&= -\exp(-x)\Big|_0^k \\
\implies e^{-k} &= .95 \\
\implies k = -\log(.95) &= \boxed{.0513}.
\end{aligned}$$

Now we're going to reject the null H_0 if $|x| < .0513$, and we conclude that the test is uniformly most powerful for $0 < b < 1$ as well as $b > 1$, but not for $b < 0$ or $b = 1$.

4 Power Curve

Example: Let's find the power curve for the Laplace example. Recall that

$$\text{Power} = P_1(d(x) = 1),$$

and take $H_0 : \theta = 1$ and $H_1 :$ any $\theta > 0$.
Then for $0 < \theta < 1$,

$$\text{Power} = \mathbb{P}_1(|x| > 3) = 2\int_3^\infty \frac{\theta}{2}\exp(-\theta x)\ dx = \exp(-3\theta),$$

once we've solved this.
Consider then $\theta > 1$, so that the rejection region is now inside. Then

$$\text{Power} = \mathbb{P}_1(|x| < .051) = 2\int_0^{.051} \frac{\theta}{2}\exp(-\theta x)\ dx = 1 - \exp(-.051\theta)$$

Now these are both functions of θ. This gives the power curve. We'll analyze this more in coming days.

Lecture ends here.