Math 128A, Summer 2019
Midterm Study Guide

1 Topics

conditions for convergence (fixed point iteration) - contractive - invariance
added conditions for quadratic convergence ; other big-O practice
- bounding error of a step of fixed point iteration by max of derivative on that interval (examples!!)
ieee delta/epsilon error bounds
[fixed point, iteration, big-O, floating point]

2 Key Results

2.1 Convergence of Fixed Point Iteration

Theorem 2.1. If $a \leq x \leq b$ implies:
1. $a \leq g(x) \leq b$ [Invariant]
2. $|g'(x)| \leq \frac{1}{2}$ [Contractive],

then for any $x_n \in [a, b]$, we have:

$$x_{n+1} = g(x_n); x = g(x) \implies |x_n - x| \leq 2^{-x}|x_0 - x|$$

Moreover, this gives a **unique** solution.

2.2 Rounding in Floating Point

Remark: Rounding is "monotonic"; that is, it preserves order (inequalities), with possible equality.

$$a < b \implies fl(a) \leq fl(b)$$

The 'Mantra' of FP arithmetic is it delivers the exact result, correctly rounded (for binary operations). Note that we write $|\delta| < \frac{\varepsilon}{2}$ as relative error, completely independent of the dimensions of x.

2.3 Quadratic Convergence: Fixed Point

Theorem 2.2. If $x \in [a, b]$ implies:
1. $g(x) \in [a, b]$
2. $|g'(x)| \leq \frac{1}{2}$
3. $|g''(x)| < C; C|b - a| \leq 1$

If $p \in [a, b]$ has the property $|g''(x)| < M$ for $x \in [a, b]$ and $g'(p) = 0$ (derivative is zero at 2nd derivative), we have convergence at least order of 2 (quadratic).
For derivation of the error bound here, look at the equations in this text:

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

$$g(x) = g(p) + g'(p)(x - p) + \frac{g''(\xi)}{2}(x - p)^2,$$

;en x and p. The hypotheses $g(p) = p$ and $g'(p) =$

$$g(x) = p + \frac{g''(\xi)}{2}(x - p)^2.$$

ı $x = p_n$,

$$p_{n+1} = g(p_n) = p + \frac{g''(\xi_n)}{2}(p_n - p)^2,$$

$_n$ and p. Thus,

$$p_{n+1} - p = \frac{g''(\xi_n)}{2}(p_n - p)^2.$$

< 1 on $[p - \delta, p + \delta]$ and g maps $[p - \delta, p + \delta]$ i
int Theorem that $\{p_n\}_{n=0}^{\infty}$ converges to p. But ξ_n is b
also converges to p, and

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{|g''(p)|}{2}.$$

ies that the sequence $\{p_n\}_{n=0}^{\infty}$ is quadratically conve
convergence if $g''(p) = 0$.
' is continuous and strictly bounded by M on the in
; that, for sufficiently large values of n,

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

3 Lagrange Interpolation and Error

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $\min\{x_0, x_1, \ldots, x_n\}$, and the $\max\{x_0, x_1, \ldots, x_n\}$and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n), \tag{3.3}$$

where $P(x)$ is the interpolating polynomial given in Eq. (3.1).

4 Hermite Interpolation

If $f \in C^1[a, b]$ and $x_0, \ldots, x_n \in [a, b]$ are distinct, the unique polynomial of least degree agreeing with f and f' at $x_0, \ldots, x_n$ is the Hermite polynomial of degree at most $2n + 1$ given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x),$$

where, for $L_{n,j}(x)$ denoting the jth Lagrange coefficient polynomial of degree n, we have

$$H_{n,j}(x) = [1 - 2(x - x_j)L'_{n,j}(x_j)]L^2_{n,j}(x) \quad \text{and} \quad \hat{H}_{n,j}(x) = (x - x_j)L^2_{n,j}(x).$$

Moreover, if $f \in C^{2n+2}[a, b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \ldots (x - x_n)^2}{(2n + 2)!} f^{(2n+2)}(\xi(x)),$$

for some (generally unknown) $\xi(x)$ in the interval (a, b). ■

z	$f(z)$	First divided differences	Second divided differences
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		$f[z_0, z_1, z_2] = \dfrac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0}$
		$f[z_1, z_2] = \dfrac{f[z_2] - f[z_1]}{z_2 - z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1, z_2, z_3] = \dfrac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1}$
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(x_1)$		$f[z_2, z_3, z_4] = \dfrac{f[z_3, z_4] - f[z_2, z_3]}{z_4 - z_2}$
		$f[z_3, z_4] = \dfrac{f[z_4] - f[z_3]}{z_4 - z_3}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		$f[z_3, z_4, z_5] = \dfrac{f[z_4, z_5] - f[z_3, z_4]}{z_5 - z_3}$
		$f[z_4, z_5] = f'(x_2)$	
$z_5 = x_2$	$f[z_5] = f(x_2)$		

5 Other Practice Exams

5.1 Hare F2003

: **Problem 1.** (a) Let $f(x)$ be a continuous function on $[2,3]$. Further, assume that for all $x \in [2,3]$ that $f(x) \in [2,3]$. Prove that $f(x)$ has a fixed point $p \in [2,3]$. (b) Further assume that $|f'(x)| \leq \frac{1}{11}$ for all $x \in [2,3]$. How many steps of fixed point iteration are needed to guarantee accuracy of within 10^{-3}?

Write $g(x) := f(x) - x$. Because $f(2) \in [2,3]$, we have

$$g(2) = f(2) - 2 \geq 0.$$

Further, because $f(3) \in [2,3]$, we have

$$g(3) = f(3) - 3 \leq 0.$$

Hence by the IVT, $g(x)$ has a root $p \in [2,3]$. So $0 = g(p) = f(p) - p$, which gives us $f(p) = p$ as desired as a fixed point.
Let $p_0 \in [2,3]$ be our initial guess and let p be the true unique fixed point. Then we know, for all n:

$$|p_n - p| < \left(\frac{1}{11}\right)^n |p_0 - p| < \left(\frac{1}{11}\right)^n$$

So to ensure precision within 10^{-3}, $n \geq 3$ should suffice.

Remark: The right inequality follows from $|p_0 - p| < 3 - 2$, but why do we have the left inequality?

Give an example of a, b, c where three-digit rounding fails associativity; that is,

$$a + (b + c) \neq (a + b) + c$$

Solution. Define $a := 1.00, b := 0.004, c := 0.004$. □

Let $g(x) := ax + b$ for some fixed nonzero a, b. How many steps of Newton's method are needed to find a root of $g(x)$ to within 10^{-6}?

Solution. One step guarantees the exact solution. Intuitively, Newton's method finds the intercept (root) of the tangent line of a function. Because our function itself is a line, for any starting point, we find the actual root to the line $g(x)$.
To see this precisely, consider:

$$\begin{aligned} x - \frac{g(x)}{g'(x)} &= x - \frac{ax+b}{a} \\ &= x - x - \frac{b}{a} \\ &= -\frac{b}{a}, \end{aligned}$$

which is the exact solution (root). □

Define what it means for a sequence $\{p_n\}_{n=0}^{\infty}$ to converge **linearly** to p.

Solution. We say the sequence p_n converges linearly to p if there exists some $\lambda \in (0, 1)$ such that:

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^1} = \lambda.$$

□