Math 128A, Summer 2019
Lecture 17, Monday 7/22/2019

CLASS ANNOUNCEMENTS: Topics today:
- Review
- Weights in numerical integration (quadrature)

1 Review

From the previous homework, recall the question on Differential coefficients.

$$\delta_{n,j}^{m} = \cdots \delta_{n-1,j}^{m} + \cdots \delta_{n-1,j}^{m-1}$$

This gives:

$$f^{(m)}(a) = \sum_{j=0}^{N} \delta_{N,j}^{m} f(t_j).$$

We start from one 'corner' and build successive columns (or triangles), successively larger and larger by 1.

1.1 Homework 5

Let's say we want to calculate the Taylor expansion of some horrible function, say

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} a_m t^m.$$

The way a human-being should do this is to cross-multiply and say:

$$\begin{aligned} t &= (e^t - 1) \sum_{m=0}^{\infty} a_m t^m \\ &= \left(\sum_{k=0}^{\infty} \frac{t^k}{(k+1)!} \right) \left(\sum_{m=0}^{\infty} a_m t^m \right) \\ &= \left(\frac{1}{1!} + \frac{t}{2!} + \frac{t^2}{3!} + \cdots \right) \left(a_0 + a_1 t + a_2 t^2 + \cdots \right) \end{aligned}$$

This gives an infinite system of linear equations for the a's, so this would be very scary, except that this is a **triangular** system of linear equations. That is, a_0 is determined right away, and a_1 is given by looking at the linear terms. Hence we can write this neatly as a recurrence relation (although it doesn't look like a normal recurrence relation). Each element in the solution depends on every previous element.
As applied to our homework, this is how we calculate our Bernoulli coefficients, $b_0, \ldots, b_{10}$. Else, for our homeworks, we can plug this into Mathematica and tell it to Taylor expand symbolically.
Without this 'trick' of multiplying by $(e^t - 1)$, we can taylor expand the entire $\frac{t}{e^t-1}$. These Bernoulli numbers have been proven to not have 'nice'

recurrence relation forms but rather requires all previous terms (combinatorics).
In our previous example, this gives a relation something like:

$$\sum_{j=0}^{k} \frac{a_j}{(k+1-j)!} = 0.$$

The triangular system of equations arises like:

$$\begin{aligned} 1 &= \frac{a_0}{1!} \\ 0 &= \frac{a_1}{1!} + \frac{a_0}{2!} \\ 0 &= \frac{a_2}{1!} + \frac{a_1}{2!} + \frac{a_0}{3!} = 0 \\ &\vdots \end{aligned}$$

1.2 Question 1: a,b

To find the recursive formula for Euler-Maclaurin,

$$\int_0^n d(x)\ dx = \left[\sum_{j=0}^{n} f(j)\right] - \sum_{m=1}^{n} b_{2m} f^{(2m-1)}(0).$$

We take $f(x) := e^{-\lambda x}$ or $f(x) := x^p$ to get b_{2m} if we know the power sums.
In our problem set, we want to show that, in terms of the Bernoulli numbers, $\sum_{n=0}^{N} n^p$ is equal to the polynomial $P_{n+1}(n)$.

1.3 Problem 5

Recall the famous **monic Legendre polynomials**:

$$\begin{aligned} P_0(t) &:= 1 \\ P_1(t) &:= t \\ P_{n+1}(t) &:= tP_n(t) - c_n P_{n-1}(t), \qquad c_n := \frac{n^2}{4n^2-1}. \end{aligned}$$

Then $P_2(t) = t^2 - \frac{1}{3} \cdot 1 = t^2 - \frac{1}{3}$, and $P_3(t) = t^3 - \frac{3}{5}t$. Then later,

$$P_5(t) = t^5 + \cdots t^3 + \cdots t.$$

The idea for this problem is that by Galois, we know we cannot explicitly find zeroes for this quintic polynomial; however, our function is odd, so factoring out a t, then we have:

$$P_5(t) = tP_2(t^2),$$

and we know the formula for the quadratic polynomial. Hence we can find explicit forms of our roots.

2 Integration Weights

Strain shares with us here what he's been toying with over the weekend.
Let $t_0, \ldots, t_n$ be the zeroes of $P_{n+1}(t) = t^n + \cdots$, and $L_j(t_k) = \delta_{j,k}$, where

$$L_j(t) = \frac{\prod_{k \neq j}(t - t_k)}{\prod_{k \neq j}(t_j - t_k)} = \lambda_j \frac{P_{n+1}(t)}{t - t_j},$$

and

$$P_n(t) = (t - t_0)(t - t_1) \cdots (t - t_n).$$

By chain rule, each time we differentiate, we differentiate one factor and leave the the rest, and we sum all of these results. Hence we write:

$$P_{n+1}(t) = \sum_{j=0}^{n} \prod_{k \neq j} (t_j - t_k).$$

Particularly,

$$\begin{aligned} P'_{n+1}(t_i) &= \prod_{k \neq i} (t_i - t_k) \\ &= \frac{1}{\lambda_i}. \end{aligned}$$

Then

$$L_j(t) = \frac{1}{P'_{n+1}(t_j)} \frac{P_{n+1}(t)}{t - t_j}.$$

Let

$$\begin{aligned} w_j &:= \int_{-1}^{1} L_j(t)\ dt \\ &= \frac{1}{P'_{n+1}(t_j)} \int_{-1}^{1} \frac{P_{n+1}(s)}{s - t_j}\ ds. \end{aligned}$$

If we can perform this integration (we may not want to do this via quadrature), then this integral is simply a function evaluation, $Q_{n+1}(t_j)$, where we write:

$$\begin{aligned} Q(t) &:= \int_{-1}^{1} \frac{P_{n+1}(s)}{s - t}\ ds - c_n Q_{n-1}(s) \\ &= \int_{-1}^{1} \frac{sP(s) - c_n P_{n-1}(s)}{s - t}\ ds. \end{aligned}$$

So it looks like a recurrence relation.
This is the integral-free formula for the integration weights of Gaussian Integration (Quadrature.)
So we write:

$$\begin{aligned} Q_{n+1}(t) &= \overbrace{\int_{-1}^{1} P_0(s) \cdot P_n(s)\ ds}^{0} + t \int_{-1}^{1} \frac{P_n(s)}{s - t}\ ds - c_n Q_{n-1}(t) \\ &= t_j Q_n(t_j) - c_n Q_{n-1}(t_j) =: w_j \end{aligned}$$

Then explicitly,

$$Q_0(t) = \int_{-1}^{1} \frac{1}{s-t}\ ds = \ln\left(\frac{1-t}{1+t}\right)$$
$$Q_1(t) = \int_{-1}^{1} \frac{s}{s-t}\ ds = 2 + Q_0(t).$$

Once we get these expressions,
While we are recurring to P_{n+1}, we can differentiate to get:

$$P_{n+1} = tP_n - c_n P_{n-1}$$
$$P'_{n+1} = tP'_n + P_n - c_n P'_{n-1}$$

Then

$$P'_0 = 0; \qquad P'_1 = 1.$$

5 Minute Break.

3 Ordinary Differential Equations (ODEs)

Let T be some (current) time.

$$y' = f(t, y)$$
$$g : [0, T] \to \mathbb{R}$$
$$f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$$

Our y is a column vector:

$$y'(t) = \begin{bmatrix} y_1'(t) & \cdots & y_n'(t) \end{bmatrix}^T$$

Today we'll talk about the theory behind ODEs.

3.1 Three (3) Canonical Examples:

1. Linear, Homogeneous real function

$$y' = a(t)y + g(t)$$

This is linear in y, not necessarily linear in t. The solution always exists and is always unique, and we can simply write them down in terms of integrals, due to the the work of Euler-Maclaurin from last week.
One way to solve this equation is to remember the **integrating factor**: $e^{-A(t)}$, where $A'(t) := a(t)$.

$$\begin{aligned} e^{-A(t)}\left(y' - a(t)y\right) &= e^{A(t)} g(t) \\ \left(e^{-A(t)} y(t)\right)' &= e^{-A(t)} g(t), \end{aligned}$$

where we recognize the perfect derivative in the second line. We write:

$$\begin{aligned} \int_0^t \left(e^{-A(s)} y(s)\right)' \, ds &= e^{-A(t)} y(t) - e^{-A(0)} y(0) \\ &= \int_0^t e^{-A(s)} g(s) \, ds \end{aligned}$$

When we integrate 0 to t, do not call our variable of integration also t (here we use s). If we can't solve this analytically (exactly), then we know how to do it numerically (i.e. via Gaussian Integration).
So we have:

$$y(t) = e^{A(t)-A(0)} y_0 + \int_0^t e^{A(t)-A(s)} g(s) \, ds.$$

We conclude that **if a solution exists**, then it **must** satisfy this formula (provided our integrals above make sense). Hence this proves **existence** and **uniqueness** of our solution.
Existence and uniqueness is how we justify initial value problems like $y' = f(t, y)$ with $y(0) = y_0$.

(2) Cautionary Example of Nonexistence (of solution at a finite time).
Let

$$y' = y^2$$
$$y(0) = y_0.$$

We separate variables, to write: $\frac{y'}{y^2} = 1 =$ independent.
Hence

$$\int_0^1 \frac{y'(s)\ ds}{[y(s)]^2} = \int_0^t 1\ ds,$$

so if y is monotone, then we can use it perfectly well as a variable of integration. That is, $dy = y'(s)ds$. So we write:

$$\int_{y_0}^{y(t)} \frac{du}{u^2} = \int_0^1 \frac{y'(s)\ ds}{[y(s)]^2} = \int_0^t 1\ ds = \frac{-1}{u}\Big|_{u=y(0)}^{u=y(t)},$$

Hence this gives us:

$$\frac{-1}{y(t)} + \frac{1}{y_0} = t$$
$$\implies y(t) = \frac{1}{\frac{1}{y_0} - t}.$$

This tells us to watch out in the denominator, that this expression can blow up at $t = \frac{1}{y_0}$, if $y_0 > 0$.

(3) Non-uniqueness $\implies$ We Cannot Predict the Future in this Model
Let $y' := 2\sqrt{y}, \qquad y(0) = y_0 \geq 0$. We can proceed similarly as in (2). We can write:

$$\frac{y'}{2\sqrt{y}} = \sqrt{y}' = 1,$$

which gives us:

$$\sqrt{y(t)} - \sqrt{y_0} = t \implies y(t) = (t + \sqrt{y_0})^2$$

For example, $y_0 = 0$, so $y(t) = t^2$. But also, $y'(t) = 2t = 2\sqrt{y(t)}$ gives $y(t) = 0$.
We have two solutions, so which is correct?
Also consider:

$$y(t) = \begin{cases} (t-c)^2, & 0 < c \leq t \\ 0, & 0 \leq t \leq c. \end{cases},$$

where we have any $c \geq 0$. This is also a solution!!! Everything is zero, and $y' = 2\sqrt{y}$. We don't know if it stays 0 forever, or until an arbitrary time where it starts increasing.
Notice that is function is differentiable at c.

4 How to Rule Out Bad Behavior (Non-existence, Non-uniqueness)

Our answer is to try to prove existence and uniqueness and to see what assumptions we need to make about $f(t, y)$.
To prove existence and uniqueness, look at:

$$y'(t) := f(t, y), \qquad y(0) = y_0.$$

A useful thing to do here is to integrate this, to get:

$$\int_0^t y'(s)\ ds = \int_0^t f(s, y(s))\ ds,$$

and by FTC (fundamental theorem of calculus) on the LHS, we have:

$$y(t) = \underbrace{y_0 + \int_0^t f(s, y(s))\ ds}_{F}.$$

If we know y up to (and barely including t), then we have a solution that is equivalent to our given initial condition. We can see it works in both directions. We call this the Picard integral equation.
The single equation looks like a fixed-point equation. The idea is that we stick $y(s)$ into the integral and get the same thing. That is,

$$y = F[y]$$

Notice f takes on point values and gives a value. On the other hand, F takes a whole function and gives us back a whole function. So we can write something like:

$$y(t) = F[y](t),$$

as a fixed-point equation in the vector space of functions.
Explicitly, we define:

$$\begin{aligned} &F : y \to y \\ &F[y](t) := y_0 + \int_0^t f(s, y(s))\ ds \end{aligned}$$

Before we try Newton's method, we are legally obligated to try fixed point iteration. We guess something and plug it into the RHS. So fixed point iteration would say:

$$y_{n+1}(t) = y_0 \int_0^t f(s, y_n(s))\ ds,$$

with $y_0(t) := y_0$.

Example: $y' = y^2, \qquad y(0) = 1$. Notice from earlier, we have this is exactly equivalent (in both directions) to

$$y(t) = 1 + \int_0^t y(s)^2\ ds$$

Then $y_0(t) = 1$, and

$$y_1(t) = 1 + \int_0^t y_0(s)^2 \ ds = 1 + \int_0^t 1 \ ds = 1 + t.$$

Hence

$$\begin{aligned} y_2(t) &= 1 + \int_0^t (1+s)^2 \ \ ds \\ &= 1 + \int_0^t [1 + 2s + s^2] \ ds \\ &= 1 + t + t^2 + \frac{t^3}{3} \end{aligned}$$

We guess: $y_3(t) = [1 + t + t^2 + t^3] + \cdots$, where we don't know what $\cdots$ is. But we can guess: $\to \frac{1}{1-t}$.

Lecture ends here.

Next time, we'll see how long we can guarantee good behavior, and how to do so.