# Math 128A, Summer 2019

## PSET #2 (due Wednesday 7/10/2019)

**Problem 1.** The Fibonacci numbers $f_n$ are defined with $f_0 := 0$ and $f_1 := 1$ by

$$f_{n+2} = f_{n+1} + f_n.$$

(a) Show that for $n \to \infty$,

$$\frac{f_{n+1}}{f_n} \to \varphi = \frac{1+\sqrt{5}}{2},$$

(b) Determine the rate of convergence of $\frac{f_{n+1}}{f_n}$ to $\varphi$.

**Solution.** (a) For a formal solution see the next solution included below. A quick and naive way to show this limit is to assume that at sufficiently large $n$, we we define the ratio $r := \frac{f_{n+1}}{f_n}$ and have

$$\frac{f_{n+2}}{f_{n+1}} \approx \left(\frac{f_{n+1}}{f_n}\right)^2 = r^2.$$

Consider:

$$\begin{aligned} f_{n+2} = f_{n+1} + f_n \implies \frac{f_{n+2}}{f_n} &= \frac{f_{n+1}}{f_n} + \frac{f_n}{f_n} \quad \text{(dividing } f_n \text{ across, as } f_n \neq 0, \forall n \geq 1) \\ r^2 &= r + 1 \implies r^2 - r - 1 = 0 \quad \text{(substituting in } r \text{ as defined above)} \\ r &= \frac{1 \pm \sqrt{1+4}}{2} \quad \text{(solutions to quadratic)} \end{aligned}$$

and because $f_n$ is strictly increasing for $n \geq 1$, we know $r > 0$ and conclude $r = \frac{1+\sqrt{5}}{2} = \varphi$, as desired. $\square$

**Solution.** Let $r_n := \frac{f_{n+1}}{f_n}$, with $f_0 := 0, f_1 := 1$. Then our function $r_n$ is defined for all $n \geq 1$. We show that $\lim r_n = \frac{1+\sqrt{5}}{2} =: \varphi$.
Similar to above (now without making unproven assumptions), the recursive definition of fibonacci numbers gives:

$$\begin{aligned} f_{n+1} &:= f_n + f_{n-1} \quad \text{(recursive definition of fibonacci sequence)} \\ \frac{f_{n+1}}{f_n} &= 1 + \frac{f_{n-1}}{f_n} \quad (\text{for all } n \geq 1,\ f_n \neq 0) \\ r_n &= 1 + \frac{1}{r_{n-1}} \end{aligned}$$

Substituting in $r_n$ and $\varphi$, for some large $n$, we have:

$$\begin{aligned} |r_n - \varphi| &= \left| \left(1 + \frac{1}{r_{n-1}}\right) - \left(1 + \frac{1}{\varphi}\right) \right| \\ &= \left| \frac{1}{r_{n-1}} - \frac{1}{\varphi} \right| \\ &= \left| \frac{\varphi - r_{n-1}}{r_{n-1} \cdot \varphi} \right| \\ &= \frac{1}{|r_{n-1|} \cdot \varphi} \cdot |r_{n-1} - \varphi| \quad \text{(drop abs values, as } 1 < \varphi = \frac{1+\sqrt{5}}{2} < 2) \end{aligned}$$

Because $r_{n-1} \geq 1 \quad \forall n \geq 1$, we have:

$$|r_n - \varphi| \leq \left(\frac{1}{\varphi}\right) |r_{n-1} - \varphi|$$

And equivalently, repeating this procedure starting with $|r_{n-1} - \varphi|$, we get:

$$|r_{n-1} - \varphi| \leq \left(\frac{1}{\varphi}\right) |r_{n-2} - \varphi|$$

Repeating down to $r_1$ (1 is the lowest index for which $s_n \neq 0$ and thus $r_n$ is defined), we have:

$$\begin{aligned}
|r_n - \varphi| &\leq \left(\frac{1}{\varphi}\right) |r_{n-1} - \varphi| \\
&\leq \left(\frac{1}{\varphi}\right) \left[\ \frac{1}{\varphi} |r_{n-2} - \varphi|\ \right] \\
&\leq \left(\frac{1}{\varphi^2}\right) \left[\ \frac{1}{\varphi} |r_{n-3} - \varphi|\ \right] \\
&\ \ \vdots \\
&\leq \frac{1}{\varphi^{n-1}} |r_1 - \varphi| \qquad \text{(we use this result later)}
\end{aligned}$$

We know $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$ provided $a_n$ and $b_n$ converge finitely, so because $1 < \varphi = \frac{1+\sqrt{5}}{2} < 2$, we have $0 < \frac{1}{\varphi} < 1$. Thus by radius of convergence for the sequence $a^n$, we have:

$$\lim_{n\to\infty} \left(\frac{1}{\varphi}\right)^{n-1} = 0 \implies \lim_{n\to\infty} \left[\left(\frac{1}{\varphi}\right)^{n-1} \cdot |r_1 - \varphi|\right] = 0$$

So by the squeeze lemma for limits,

$$|r_n - \varphi| \leq \frac{1}{\varphi^{n-1}} |r_1 - \varphi| \quad \text{(above)}$$
$$\implies \lim_{n\to\infty} |r_n - \varphi| \leq \lim_{n\to\infty} \frac{1}{\varphi^{n-1}} |r_1 - \varphi| = 0$$

The nonnegativity property of absolute values provides a lower bound and suggests

$$\left(0 \leq \lim_{n\to\infty} |r_n - \varphi| \leq 0\right) \implies \left(\lim_{n\to\infty} |r_n - \varphi| = 0\right).$$

Hence we conclude $r_n \xrightarrow{\infty} \varphi$, precisely as desired.

$\square$

(b)

**Solution.** Consider the generating function for all fibonacci numbers, in addition to the recursive definition. That is,

$$f_n = \left(\frac{1}{\sqrt{5}}\right) [\varphi^n - (1 - \varphi)]$$

To see this in action, we explicitly write out a few terms of the sequence:

$$\begin{aligned}
&f_0 = 0; \\
&f_1 = \frac{1}{\sqrt{5}}\left(2\varphi^1 - 1\right) = \frac{1}{\sqrt{5}}\left(-1 + 2 \cdot \frac{1+\sqrt{5}}{2}\right) = 1 \\
&f_2 = \frac{1}{\sqrt{5}}\left[\varphi^2 - (1-\varphi)^2\right] = \frac{1}{\sqrt{5}}\left[2\left(\frac{1+\sqrt{5}}{2}\right) - 1\right] = 1 \qquad \vdots
\end{aligned}$$

So we have, for order of convergence,

$$\begin{aligned}
|a_n - \varphi| &= \frac{\varphi^{n+1} - (1-\varphi)^{n+1}}{\varphi^n - (1-\varphi)^n} - \varphi \\
&= \frac{\varphi^{n+1} - (1-\varphi)^{n+1} - \varphi^{n-1} + \varphi(1-\varphi)^n}{\varphi^n - (1-\varphi)^n} \\
&= \frac{(2\varphi - 1)(1-\varphi)^n}{\varphi^n - (1-\varphi)^n} \\
&= \frac{(1-\varphi)^n}{\varphi^n} \cdot \overbrace{\left[\frac{(2\varphi-1)}{1 - \underbrace{\left(\frac{1}{\varphi} - 1\right)^n}_{\text{tends to } 0}}\right]}^{\text{constant } c} \\
&= c\left(\frac{1-\varphi}{\varphi}\right)^n,
\end{aligned}$$

but we know $\varphi^2 - \varphi - 1 = 0$ from earlier , which is equivalent to $\varphi - 1 = \frac{1}{\varphi}$ (see part a above or verify for $\varphi := \frac{1+\sqrt{5}}{2}$). This gives

$$|a_n - \varphi| = c\left(\frac{1}{\varphi^2}\right)^n = c\left(\frac{1}{\varphi}\right)^{2n},$$

so we conclude

$$r_n = \varphi + O\left(\frac{1}{\varphi^{2n}}\right).$$

$\square$

**Problem 2.** Consider the fixed point iteration

$$x_{n+1} := \frac{-x_n^2 - c}{2b}, \tag{1}$$

where $b, c$ are fixed real number parameters.
(a) If $x_n \to x$, what does $x$ solve?
(b) Analyze and sketch the region of $(b, c)$ values, where (1) above converges at a rate $O(2^{-n})$ or better, from an interval of starting values $x_0$ near $x$.

**Solution.** **(a)** We have $b, c \in \mathbb{R}$. Taking the limit as $x_n \to x$, then we have that our fixed-point iteration $x_{n+1}$ is equivalent to:

$$\lim_{x_n \to x} x_{n+1} = \lim_{x_n \to x} \left[\frac{-x_n^2 - c}{2b}\right] \implies x = \frac{-x_n^2 - c}{2b} \implies x_n^2 + 2bx + c = 0,$$

a monic quadratic polynomial. □

**Solution.** **(b)** If we let

$$x = -\frac{x^2 + c}{2b} =: g(x),$$

then the derivative gives:

$$g'(x) = -\frac{x}{b},$$

and thus if we want to guarantee $|g'(x)| \leq \frac{1}{2}$, we can impose the condition $|x| \leq \frac{|b|}{2}$. Thus we consider the interval $x \in [-\frac{b}{2}, \frac{b}{2}]$. Because we know local extrema occur at endpoints and where the derivative $g'(x) = 0$, we check: $g\left(-\frac{b}{2}\right), g(0)$, and $g\left(\frac{b}{2}\right)$.

$$\begin{aligned} g\left(-\frac{b}{2}\right) &= \frac{-\left(\frac{b}{2}\right)^2 - c}{2b} \\ &= \frac{-b^2 - 4c}{8b} = g\left(\frac{b}{2}\right), \end{aligned}$$

and

$$g(0) = \frac{-c}{2b}.$$

Hence in plotting our $(b, c)$-space, we exclude $b = 0$ where $g(0)$ is undefined. Recall from (b) above, $g'(x) = -\frac{x}{b}$. So $g$ is either strictly increasing or decreasing (on our interval), depending on the sign of $b$. For convergence at $O(2^{-n})$ or better, we need the following invariance conditions:

$$\begin{aligned} -\frac{|b|}{2} &\leq g(0) \leq \frac{|b|}{2} \\ -\frac{|b|}{2} &\leq g\left(\frac{-b}{2}\right) = g\left(\frac{|b|}{2}\right) \leq \frac{|b|}{2}. \end{aligned}$$

Substituting our expressions for $g$, this gives

$$\begin{aligned} \frac{-|b|}{2} \leq \frac{-c}{2|b|} \leq \frac{|b|}{2} &\implies -b^2 \leq c \leq b^2 \\ -\frac{|b|}{2} \leq \frac{-b^2 - 4c}{8|b|} \leq \frac{|b|}{2} &\implies \frac{-5b^2}{4} \leq c \leq \frac{3b^2}{4} \end{aligned}$$

And because $b^2 > 0$ via the trivial inequality and $b \neq 0$, we notice $[b > 0 \implies b^2 > 0] \implies [\frac{3b^2}{4} < b^2]$ and $[\frac{-5b^2}{4} < -b^2]$. To satisfy the two compound inequalities, we take the intersection of both sets to get:

$$-b^2 \leq c \leq \frac{3b^2}{4},$$

and given by Wolfram Alpha,

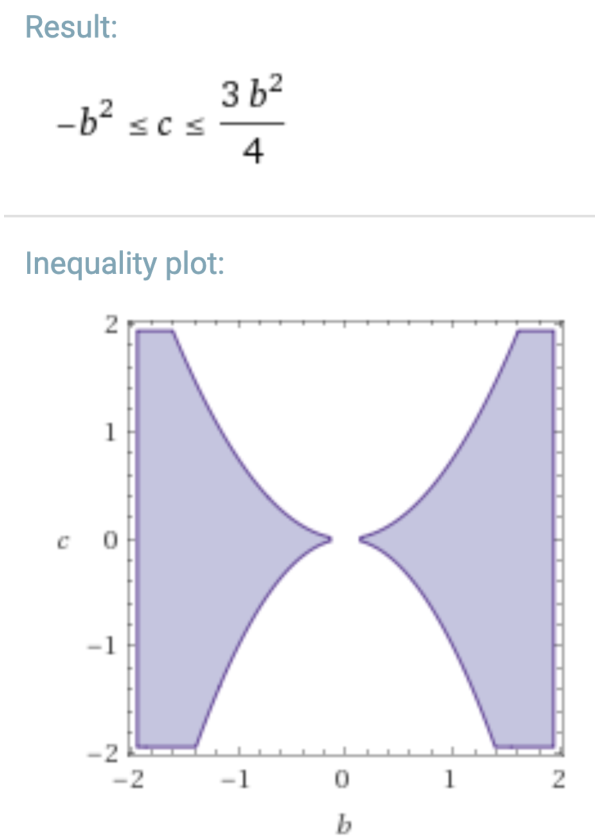


Figure 1: qualifying $(b, c)$-space shaded satisfying $-b^2 \leq c \leq \frac{3b^2}{4}$ and $b \neq 0$

Notice that $(b, c) = (0, c)$ is not included in the plot as this is undefined for our fixed-point iteration.

□

**Problem 3.** Consider the fixed point iteration

$$x_{n+1} := -b - \frac{c}{x_n} = g(x_n) \tag{2}$$

(a) Show that $|g'(x)| \leq \frac{1}{2}$ whenever $x^2 \geq 2|c|$.
(b) Show that $g(x)^2 \geq 2|c|$ whenever $x^2 \geq 2|c|$ **and** $b^2 \geq \frac{9}{2}|c|$.
(c) Draw the region of $(b,c)$-space where (1) above converges at a rate of at least $O(2^{-n})$ from any starting point $x_0$ with $x_0^2 \geq 2|c|$.

**Solution.** **(a)** We want to show $|g'(x)| \leq \frac{1}{2}$ whenever $x^2 \geq 2|c|$. Logically this is equivalent to showing

$$(x^2 \geq 2|c|) \implies (|g'(x)| \leq \frac{1}{2}).$$

Consider $g(x) = -b - \frac{c}{x_n} \implies g'(x) = \frac{c}{x_n{}^2}$.
We are given $x^2 \geq 2|c|$. If $c = 0$, then $g(x_n) = -b$, a constant function which has slope $g'(x_n) = 0 \leq \frac{1}{2}$ as desired.
Else if $c > 0$, then $(x^2 \geq 2|c| = 2c) \iff (-\sqrt{2c} \leq x \leq \sqrt{2c})$ implies that

$$\begin{aligned} g'(x) &= \frac{c}{x_n{}^2} \\ \implies 0 \leq g'(x) &\leq \frac{c}{2c} = \frac{1}{2}, \end{aligned}$$

precisely as required.
Else if $c < 0$, then we have the same $x^2 \geq 2|c| \iff -\sqrt{2c} \leq x^2 \leq \sqrt{2c}$ as above. Then

$$\begin{aligned} g'(x) &= \frac{c}{x_n{}^2} \\ \implies 0 \leq |g'(x)| &= \left|\frac{c}{x_n{}^2}\right| \\ &= \frac{|c|}{x_n^2} \leq \frac{|c|}{2|c|} = \frac{1}{2}, \end{aligned}$$

again precisely as required.

□

**Solution.** **(b)** Now we want to show that imposing the additional condition $(b^2 \geq \frac{9}{2}|c|)$ implies $[g(x)]^2 \geq 2|c|$.
Consider that $b^2 \geq \frac{9}{2}|c| \implies |b| \geq \sqrt{\frac{9}{2}|c|}$.

$$\begin{aligned} |g(x)| &= \left|(-1)\left(b + \frac{c}{x}\right)\right| = \left|b + \frac{c}{x}\right| \\ &= \left|b + \frac{c}{x}\right| + \left|-\frac{c}{x}\right| - \left|-\frac{c}{x}\right| \\ &\geq \left|b + \frac{c}{x} + \left(\frac{-c}{x}\right)\right| - \left|\frac{-c}{x}\right| \qquad \text{(by triangle inequality)} \\ &= |b| - \left|\frac{c}{x}\right| \\ &\geq \left|\sqrt{\frac{9}{2}|c|}\right| - \frac{|c|}{\sqrt{2|c|}} \qquad (x^2 \geq 2|c|, b^2 \geq \frac{9}{2}|c|) \\ &= \sqrt{2|c|}, \end{aligned}$$

and squaring both sides of the inequality, we have $[g(x)]^2 \geq 2|c|$, which was to be shown.

□

**Solution.** **(c)** Here we want to sketch the region of $(b, c)$-space where the fixed point iteration converges at a rate of **at least** $O(2^{-n})$ from any $x_0$ with ${x_0}^2 \geq 2|c|$. We showed in (a) above that $x^2 \geq 2|c|$ implies $|g'(x)| \leq \frac{1}{2}$.
If $c = 0$, this simply become the trivial inequality which is true for all $x_0 \in \mathbb{R}\backslash\{0\}$, where we exclude $x_0 = 0$ because our iteration $x_{n+1} := -b - \frac{c}{x_n} = g(x_n)$ involves division by $x_0$.
On the other hand, if $c \neq 0$, then we restrict to $x_0^2 \geq 2|c|$, and we exclude the open interval $\left(-\sqrt{2|c|}, \sqrt{2|c|}\right)$.
Let $x$ be the true (fixed point) solution, which we try to approximate via iterated fixed point. Then we can express convergence by looking at the distance between the true solution $x$ and $x_n$ as $n \to \infty$:

$$|x - x_n| = \left|\left(-b - \frac{c}{x}\right) - \left(-b - \frac{c}{x_{n-1}}\right)\right| = \left|\frac{c}{x_{n-1}} - \frac{c}{x}\right| = |c| \cdot \frac{|x - x_{n-1}|}{|x||x_{n-1}|}$$

We showed in part (b) above that if $x_0 \geq 2|c|$ and $b^2 \geq \frac{9}{2}|c|$, then we have $x_n \geq \sqrt{2|c|}$ Likewise, we also have $x \geq \sqrt{2|c|}$. Hence

$$\begin{aligned} |x - x_n| &= |c| \cdot \frac{|x - x_{n-1}|}{|x||x_{n-1}|} \\ &\leq \frac{|c|}{\sqrt{2|c|} \cdot \sqrt{2|c|}} |x - x_{n-1}| \\ &= \frac{1}{2} |x - x_{n-1}| \end{aligned}$$

And applying $|x - x_n| \leq \frac{1}{2} |x - x_{n-1}|, \quad |x - x_{n-1}| \leq \frac{1}{2} |x - x_{n-2}|, \quad \ldots$ recursively down to $x_0$, we have:

$$|x - x_n| \leq \left(\frac{1}{2}\right)^n |x - x_0|,$$

which shows our fixed point iteration converges $x_n \to x$ with $O(2^{-n})$, as desired.
Plotting our $(b, c)$-space, where our fixed point iteration converges with our desired rate, we thus need to satisfy $b^2 \geq \frac{9}{2}|c|$. That is, if we plot $b$ on the horizontal axis and $c$ on the vertical, then our region given by Wolfram Alpha is:

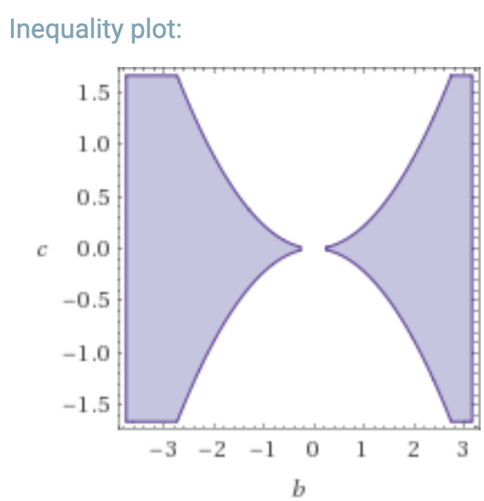


Figure 2: qualifying $(b, c)$ space shaded where $b^2 \geq \frac{9}{2}|c|$

Notice that $(b, c) = (0, 0)$ is not included because $x_0 = 0$ in our first case $c = 0$ leads to division by 0 in our algorithm. $\square$

**Problem 4.** Fix $a > 0$ and consider the fixed point iteration

$$x_{n+1} := x_n(2 - ax_n). \tag{3}$$

(a) Show that $(x_n \to x)$ implies $x = \frac{1}{a}$ or $x = 0$.
(b) Find an interval $(\alpha, \beta)$ containing $\frac{1}{a}$ such that (3) above converges to $\frac{1}{a}$, whenever $x_0 \in (\alpha, \beta)$.
(c) Find the rate of convergence in (b) above.

**Solution.** **(a)** Taking the $\lim_{x_n \to x}$ of our fixed point iteration definition, we have:

$$\begin{aligned} \lim_{x_n \to x} x_{n+1} &= \lim_{x_n \to x} x_n(2 - ax_n) \\ x &= x(2 - ax) \\ 0 &= x - ax^2 = x(1 - ax) \\ \implies x &= \left\{0, \frac{1}{a}\right\}, \qquad a > 0, \end{aligned}$$

are the only solutions, as desired. □

**Solution.** **(b)** To find the interval $(\alpha, \beta)$ containing $\frac{1}{a}$ such that our fixed point iteration defined here converges to $\frac{1}{a}$ for any starting point $x_0 \in (\alpha, \beta)$. Thus to form conditions for convergence, we want to express $x_{n+1}$ in some way relative to $x_n$. Notice we almost have this already via $x_{n+1} = x_n(2 - ax_n)$, so we try to slightly modify the expression via substitution to obtain something more elegant and explicit. Consider the substitution:

$$\begin{aligned} u_n &:= bx_n + c \\ u_n &= bx_n + c = b[x_{n-1}(2 - ax_{n-1})] + c \\ &= -abx_{n-1}^2 + 2bx_{n-1} + c \end{aligned}$$

If we define $b := -a, c := 1$ in our $u$-substitution, this is equivalent to:

$$\begin{aligned} u_n &= a^2x_{n-1}^2 - 2ax_{n-1} + 1 \\ &= (ax_{n-1} - 1)^2 \\ &= (-bx_{n-1} - c)^2 = (bx_{n-1} + c)^2 \\ &= {u_{n-1}}^2. \end{aligned}$$

But we have, $u_n = u_{n-1}^2, \quad u_{n-1} = u_{n-2}^2, \quad$ and onwards, so combining these, we have:

$$u_n = {u_{n-1}}^2 = \left[{u_{n-2}}^2\right]^2 = \cdots = [u_0]^{2^n}$$

To be within radius of convergence, we need

$$|u_0| = |bx_0 + c| = |-ax_0 + 1| < 1.$$

Because $a > 0$, and we know we are solving (converging to) $\frac{1}{a}$, this inequality is equivalent to:

$$\begin{aligned} \left|x_0 - \frac{1}{a}\right| &< \frac{1}{a} \\ \implies 0 < x_0 &< \frac{2}{a}, \end{aligned}$$

and we conclude that our desired interval is $(\alpha, \beta) := \left(0, \frac{2}{a}\right)$. □

**Solution.** **(c)** To find the rate of convergence on the interval $x_0 \in \left(0, \frac{2}{a}\right)$, recall our finding

$$\begin{aligned} u_n &= [u_0]^{2^n} \\ [bx_n + c] &= [bx_0 + c]^{2^n} \quad (u_n := bx_n + c) \\ [-ax_n + 1] &= [-ax_0 + 1]^{2^n} \quad (b := -a, c := 1) \\ \left|x_n - \frac{1}{a}\right| &= \frac{1}{a} \cdot |ax_0 - 1|^{2^n} \end{aligned}$$

and we have

$$\begin{aligned} \frac{|x_{n+1} - \frac{1}{a}|}{|x_n - \frac{1}{a}|^k} &= \lambda \\ \frac{\frac{1}{a} \cdot |ax_0 - 1|^{2^{n+1}}}{\left[\frac{1}{a} \cdot |ax_0 - 1|^{2^n}\right]^k} &= \lambda \\ \frac{\frac{1}{a} \cdot |ax_0 - 1|^{2^{n+1}}}{\left[\frac{1}{a} \cdot |ax_0 - 1|^{2^n}\right]^k} &= \lambda \end{aligned}$$

and if $k := 2$, then we have the expression equal to a constant $\lambda = a$ (and vice-versa), and $k = 2$ gives **quadratic converence**. So, to summarize convergence and the rate at which our iteration converes, we can also write:

$$x_n = \frac{1}{a} + O\left(\frac{1}{a}|ax_0 - 1|^{2^n}\right).$$

$\square$

**Problem 5.** (a) Write down Newton's method in the form:

$$x_{k+1} = g(x_k)$$

for solving:

$$f(x) = x^2 - 2bx + b^2 - d^2 = 0,$$

where $b > 0$ and $d > 0$ are parameters.
(b) Show that $|g'(x)| \leq \frac{1}{2}$ whenever $|x - b| \geq \frac{d}{\sqrt{2}}$
(c) Show that $|g(x) - b| \geq \frac{d}{\sqrt{2}}$ whenever $|x - b| \geq \frac{d}{\sqrt{2}}$.
(d) Sketch the graph of $f(x)$ with the roots of $f(x) = 0$ and the intervals of $x$ where Newton's method is **guaranteed** to converge.

**Solution.** **(a)** We explicitly write $f'(x) = 2x - 2b$. Newton's method gives us

$$\begin{aligned} g(x) = x - \frac{f(x)}{f'(x)} &= x - \frac{x^2 - 2bx + b^2 - d^2}{2x - 2b} \\ &= \frac{x(2x - 2b)}{2x - 2b} - \frac{x^2 - 2bx + b^2 - d^2}{2x - 2b} \\ &= \frac{x^2 + d^2 - b^2}{2x - 2b} \end{aligned}$$

□

**Solution.** **(b)** We wish to show $|x - b| \geq \frac{d}{\sqrt{2}} \implies |g'(x)| \leq \frac{1}{2}$. Consider

$$\begin{aligned} g'(x) &= \left[\frac{x^2 + d^2 - b^2}{2x - 2b}\right]' = \left[(x^2 + d^2 - b^2)(2x - 2b)^{-1}\right]' \\ &= \left[(x^2 + d^2 - b^2)'(2x - 2b)^{-1}\right] + \left[(x^2 + d^2 - b^2)\left((2x - 2b)^{-1}\right)'\right] \\ &= \left[(2x)(2x - 2b)^{-1}\right] - \left[(x^2 + d^2 - b^2)(2x - 2b)^{-2}(2)\right] \\ &= \frac{(2x)(2x - 2b) - 2(x^2 + d^2 - b^2)}{(2x - 2b)^2} = \frac{4x^2 - 4bx - 2x^2 - 2d^2 + 2b^2}{(2x - 2b)^2} \\ &= \frac{2x^2 - 4bx + 2b^2 - 2d^2}{(2x - 2b)^2} = \frac{2(x - b)^2 - 2d^2}{4(x - b)^2} \\ &= \frac{1}{2} - \frac{d^2}{2(x - b)^2} \end{aligned}$$

We are given $b, d > 0$. If $|x - b| \geq \frac{d}{\sqrt{2}}$, then $2(x - b)^2 \geq d^2$. So $|x - b| \geq \frac{d}{\sqrt{2}}$ implies:

$$\min\left[2(x - b)^2\right] = d^2 \implies \max\left[\frac{d^2}{2(x - b)^2}\right] = 1 \implies \min\left[g'(x)\right] = -\frac{1}{2},$$

and it's easy to see $g'(x) \leq \frac{1}{2}$ because $b, d > 0$. Hence we conclude $|g'(x)| \leq \frac{1}{2}$, as desired. □

**Solution.** **(c)** Again consider $|x-b| \geq \frac{d}{\sqrt{2}} \implies 2(x-b)^2 \geq d^2$. We want to show $|g(x)-b| \geq \frac{d}{\sqrt{2}}$. Recall from (a) that our $g(x)$ for Newton's Method is

$$g(x) = \frac{x^2+d^2-b^2}{2(x-b)}.$$

Hence

$$\begin{aligned}
|g(x)-b| &= \left|\frac{x^2+d^2-b^2}{2(x-b)} - b\right| \\
&= \left|\frac{x^2-b^2+d^2-2b(x-b)}{2(x-b)}\right| \\
&= \left|\frac{x^2-2bx+b^2}{2(x-b)} + \frac{d^2}{2(x-b)}\right| \\
&= \left|\frac{x-b}{2} + \frac{d^2}{2(x-b)}\right| \qquad \text{(both terms share same sign if } x-b<0 \text{ or } x-b>0) \\
&= \frac{1}{2}\left[|x-b| + \frac{d^2}{|x-b|}\right] \\
&\geq \sqrt{|x-b| \cdot \frac{d^2}{|x-b|}} \qquad \text{(AM} \geq \text{GM inequality for 2 positive quantities)} \\
&= \sqrt{d^2} = d > \frac{d}{\sqrt{2}} \qquad \text{(Because } d>0, \text{ then } |x-b| \geq d/\sqrt{2} \text{ means } |x-b| \neq 0.\text{)},
\end{aligned}$$

even better than required.

$\square$

**Solution.** **(d)** We use Newton's method to find roots of $f(x) = x^2-2bx+b^2-d^2 = 0$. Notice

$$\begin{aligned}
f(x) &= x^2-2bx+b^2-d^2 \\
&= (x-b)^2-d^2 \\
&= [(x-b)+d]\,[(x-b)-d]\,,
\end{aligned}$$

which then for $b, d > 0$ gives

$$x = \{b-d, b+d\}.$$

Piecing together the above parts (a), (b), and (c), to have

$$|g(x)-b| \geq \frac{d}{\sqrt{2}}$$

and

$$|g'(x)| \leq \frac{1}{2},$$

we only need

$$|x-b| \geq \frac{d}{\sqrt{2}}.$$

Explicitly, these invariance properties give that $x_0 \in \left(-\infty, b+\frac{d}{\sqrt{2}}\right]$ converges to $b-d$, and $x_0 \in \left[b+\frac{d}{\sqrt{2}}, +\infty\right)$ converges to $b+d$. Because this holds for all $d>0$, we conclude that Newton's method is guaranteed to converge for all $x_0 \neq b$, where otherwise Newton's method is undefined for $x_0 := b$ (division by zero). $\square$

**Problem 6.** Implement Newton's method in a Matlab or Octave program `newton.m` of the form:
Implementing newton's method is simple here, since the user provides the function and derivative handles.

```
function r = newton(x0, f, p, n)
% x0 : initial estimate of the root
% f : function and derivative handle [ y, yp ] = f(x,p)
% p : parameters to pass through to f
% n : number of steps of iteration
    r = x0; % assign starting guess to r then feed thru iteration
    for j = 1:n
        [y, yp] = f(r,p); % function and derivative handle:
        r = r - y/yp; % iterate newton's, given the symbolic expressions for f(x), f'(x)
    end
end
```

(a) Use `newton.m` to find an approximation to within $\varepsilon$ to the first positive value of $x$ with $x = 2\sin x$. Report the number of steps, the final result, and the absolute and relative errors. Characterize the convergence as **linear** or **quadratic** by tabulating the number of correct bits at each step of the iteration.

Command Window

```
>> tablNewton(1.5, @(x, p) deal(x - 2 * sin(x), 1 - 2 * cos(x)), 1, 7)
```

| numSteps | newtonSoln | abs_error | rel_error | precisionBits |
|---|---|---|---|---|
| 0 | 1.5 | 0.39549 | 0.20865 | 1 |
| 1 | 2.0766 | 0.18106 | 0.095523 | 0 |
| 2 | 1.9105 | 0.015012 | 0.00792 | 6 |
| 3 | 1.8956 | 0.00012774 | 6.7389e-05 | 12 |
| 4 | 1.8955 | 9.4388e-09 | 4.9796e-09 | 26 |
| 5 | 1.8955 | 2.2204e-16 | 1.1714e-16 | 52 |
| 6 | 1.8955 | 0 | 0 | 52 |
| 7 | 1.8955 | 0 | 0 | 52 |

```
>> tablNewton(2.5, @(x, p) deal(x - 2 * sin(x), 1 - 2 * cos(x)), 1, 7)
```

| numSteps | newtonSoln | abs_error | rel_error | precisionBits |
|---|---|---|---|---|
| 0 | 2.5 | 0.60451 | 0.31892 | 0 |
| 1 | 1.9993 | 0.10377 | 0.054746 | 3 |
| 2 | 1.9009 | 0.0054293 | 0.0028643 | 7 |
| 3 | 1.8955 | 1.6928e-05 | 8.9307e-06 | 15 |
| 4 | 1.8955 | 1.658e-10 | 8.7469e-11 | 32 |
| 5 | 1.8955 | 0 | 0 | 52 |
| 6 | 1.8955 | 0 | 0 | 52 |
| 7 | 1.8955 | 0 | 0 | 52 |

Figure 3: Precision seems to double at each step, hence quadratic convergence.

(b) Use `newton.m` as many times as necessary to find all solutions $x > 0$ of the equation:

$$f(x) = \frac{1}{x} + \ln x - 2 = 0$$

Report the number of steps, the final result, and the absolute and relative errors. Characterize the convergence as linear or quadratic by tabulating the number of correct bits at each step of the iteration.

Command Window

```
>> tablNewton(0.4, @(x, p) deal(1./x + log(x) - 2, -1./x.^2 + 1./x), 1, 7)
```

| numSteps | newtonSoln | abs_error | rel_error | precisionBits |
|---|---|---|---|---|
| 0 | 0.4 | 0.082156 | 0.25848 | 1 |
| 1 | 0.28899 | 0.028855 | 0.090784 | 3 |
| 2 | 0.31471 | 0.0031351 | 0.0098636 | 6 |
| 3 | 0.31781 | 3.8e-05 | 0.00011956 | 12 |
| 4 | 0.31784 | 5.6014e-09 | 1.7623e-08 | 25 |
| 5 | 0.31784 | 2.2204e-16 | 6.986e-16 | 50 |
| 6 | 0.31784 | 1.1102e-16 | 3.493e-16 | 51 |
| 7 | 0.31784 | 1.1102e-16 | 3.493e-16 | 51 |

```
>> tablNewton(0.25, @(x, p) deal(1./x + log(x) - 2, -1./x.^2 + 1./x), 1, 7)
```

| numSteps | newtonSoln | abs_error | rel_error | precisionBits |
|---|---|---|---|---|
| 0 | 0.25 | 0.067844 | 0.21345 | 1 |
| 1 | 0.30114 | 0.016702 | 0.052549 | 3 |
| 2 | 0.31678 | 0.0010634 | 0.0033456 | 7 |
| 3 | 0.31784 | 4.3815e-06 | 1.3785e-05 | 15 |
| 4 | 0.31784 | 7.447e-11 | 2.343e-10 | 31 |
| 5 | 0.31784 | 0 | 0 | 52 |
| 6 | 0.31784 | 0 | 0 | 52 |
| 7 | 0.31784 | 0 | 0 | 52 |

```
>>
```

Figure 4: We see precision of bits seems to double at each step: 1,36,12,25,50,51, hence quadratic convergence. We note that sometimes we can't get rid of absolute error.

```
Command Window
        4        -4.4398e+05     4.4398e+05     3.9998e+06         -2
        5         3.0964e+11     3.0964e+11     2.7895e+12          0

>> tablNewton(8, @(x, p) deal(1./x + log(x) - 2, -1./x.^2 + 1./x), 1, 7)
    numSteps    newtonSoln    abs_error     rel_error     precisionBits
    ________    __________    __________    __________    _____________

        0               8         1.6946       0.26875          0
        1          6.1308        0.17458      0.027687          4
        2          6.3034      0.0019727    0.00031286         10
        3          6.3054     2.5044e-07    3.9719e-08         23
        4          6.3054     6.2172e-15    9.8602e-16         49
        5          6.3054     2.6645e-15    4.2258e-16         50
        6          6.3054     8.8818e-16    1.4086e-16         52
        7          6.3054     8.8818e-16    1.4086e-16         52

fx >>
```

Figure 5: To get the second root, we can set `x0 = 8`.

(c) Use `newton.m` to solve the equation:

$$f(x) = \left(x - \varepsilon^3\right)^3 = 0$$

Report the number of steps, the final result, and the absolute and relative errors. Characterize the convergence as linear or quadratic by tabulating the number of correct bits at each step of the iteration. **Explain your results.**

Command Window

```
>> tablNewton(eps^3-eps, @(x, p) deal( (x - eps^3).^3, 3.*(x - eps^3).^2 ), 1,
```

| numSteps | newtonSoln | abs_error | rel_error | precisionBits |
|---|---|---|---|---|
| 0 | -2.2204e-16 | 2.8422e-16 | 4.5714 | -2 |
| 1 | -1.4803e-16 | 2.102e-16 | 3.381 | -2 |
| 2 | -9.8686e-17 | 1.6086e-16 | 2.5873 | -2 |
| 3 | -6.5791e-17 | 1.2796e-16 | 2.0582 | -2 |
| 4 | -4.3861e-17 | 1.0603e-16 | 1.7055 | -2 |
| 5 | -2.924e-17 | 9.1413e-17 | 1.4703 | -2 |
| 6 | -1.9494e-17 | 8.1666e-17 | 1.3135 | -2 |
| 7 | -1.2996e-17 | 7.5168e-17 | 1.209 | -2 |
| 8 | -8.6638e-18 | 7.0836e-17 | 1.1394 | -2 |
| 9 | -5.7759e-18 | 6.7948e-17 | 1.0929 | -2 |
| 10 | -3.8506e-18 | 6.6023e-17 | 1.0619 | -2 |
| 11 | -2.5671e-18 | 6.474e-17 | 1.0413 | -2 |
| 12 | -1.7114e-18 | 6.3884e-17 | 1.0275 | -2 |
| 13 | -1.1409e-18 | 6.3313e-17 | 1.0184 | -2 |
| 14 | -7.6061e-19 | 6.2933e-17 | 1.0122 | -2 |
| 15 | -5.0707e-19 | 6.268e-17 | 1.0082 | -2 |
| 16 | -3.3805e-19 | 6.2511e-17 | 1.0054 | -2 |
| 17 | -2.2537e-19 | 6.2398e-17 | 1.0036 | -2 |
| 18 | -1.5024e-19 | 6.2323e-17 | 1.0024 | -2 |
| 19 | -1.0016e-19 | 6.2273e-17 | 1.0016 | -2 |
| 20 | -6.6775e-20 | 6.2239e-17 | 1.0011 | -2 |
| 21 | -4.4517e-20 | 6.2217e-17 | 1.0007 | -2 |
| 22 | -2.9678e-20 | 6.2202e-17 | 1.0005 | -2 |
| 23 | -1.9785e-20 | 6.2192e-17 | 1.0003 | -2 |
| 24 | -1.319e-20 | 6.2186e-17 | 1.0002 | -2 |
| 25 | -8.7934e-21 | 6.2181e-17 | 1.0001 | -2 |

```
>>
```

Figure 6: The relative error goes down quickly but then very slowly approaches 1. Seems something like linear convergence.

Command Window

| | | | | |
|---|---|---|---|---|
| 24 | 1.0947e-47 | 3.066e-48 | 0.2188 | 0 |
| 25 | 1.0947e-47 | 3.0658e-48 | 0.21878 | 0 |

```
>> tablNewton(2^128*realmin, @(x, p) deal( (x - eps^3).^3, 3.*(x - eps^3).^2 ),
```

| numSteps | newtonSoln | abs_error | rel_error | precisionBits |
|---|---|---|---|---|
| 0 | 7.5715e-270 | 1.4013e-47 | 1 | 1 |
| 1 | 3.6492e-48 | 1.0364e-47 | 0.73958 | 4 |
| 2 | 6.082e-48 | 7.931e-48 | 0.56597 | 2 |
| 3 | 7.7039e-48 | 6.3091e-48 | 0.45023 | 2 |
| 4 | 8.7851e-48 | 5.2278e-48 | 0.37307 | 1 |
| 5 | 9.506e-48 | 4.507e-48 | 0.32163 | 1 |
| 6 | 9.9865e-48 | 4.0265e-48 | 0.28734 | 0 |
| 7 | 1.0307e-47 | 3.7061e-48 | 0.26447 | 0 |

```
>>
```

Figure 7: We can also get very whacky divergence.

(d) Use `newton.m` to solve the equation

$$f(x) = \arctan(x - \varepsilon^2) = 0$$

for a diverse selection of starting values. Find starting values that lead to convergence, divergence, and oscillation. Report the number of steps, the final result, and the absoute and relative errors.

```
Command Window
>> tablNewton(eps^2/2, @(x,p) deal( atan(x - eps^2), 1./(1 + (x - eps^2).^2) ), 1, 20)
    numSteps    newtonSoln    abs_error     rel_error    precisionBits
    ________    __________    __________    _________    _____________

       0        2.4652e-32    3.1554e-32      0.5614           2
       1        4.9304e-32    6.9025e-33     0.12281           2
       2        4.9304e-32    6.9025e-33     0.12281           2
       3        4.9304e-32    6.9025e-33     0.12281           2
       4        4.9304e-32    6.9025e-33     0.12281           2
       5        4.9304e-32    6.9025e-33     0.12281           2
       6        4.9304e-32    6.9025e-33     0.12281           2
       7        4.9304e-32    6.9025e-33     0.12281           2
       8        4.9304e-32    6.9025e-33     0.12281           2
       9        4.9304e-32    6.9025e-33     0.12281           2
      10        4.9304e-32    6.9025e-33     0.12281           2
      11        4.9304e-32    6.9025e-33     0.12281           2
      12        4.9304e-32    6.9025e-33     0.12281           2
      13        4.9304e-32    6.9025e-33     0.12281           2
      14        4.9304e-32    6.9025e-33     0.12281           2
      15        4.9304e-32    6.9025e-33     0.12281           2
      16        4.9304e-32    6.9025e-33     0.12281           2
      17        4.9304e-32    6.9025e-33     0.12281           2
      18        4.9304e-32    6.9025e-33     0.12281           2
      19        4.9304e-32    6.9025e-33     0.12281           2
      20        4.9304e-32    6.9025e-33     0.12281           2

fx >> |
```

Figure 8: Converges to `eps^ 2 = 4.9304e-32`.

```
Command Window
>> tablNewton(eps^2 + pi/4, @(x,p) deal( atan(x - eps^2), 1./(1 + (x - eps^2).^2) ), 1
    numSteps    newtonSoln    abs_error     rel_error     precisionBits
    ________    __________    __________    __________    _____________

       0            0.7854        0.7854    8.8519e+23          1
       1          -0.29106       0.29106    3.2804e+23         -2
       2          0.016169      0.016169    1.8224e+22          4
       3        -2.818e-06     2.818e-06    3.1761e+18         -2
       4        1.4919e-17    1.4919e-17    1.6815e+07          8
       5        4.9304e-32    8.8726e-25             1          3
       6        4.9304e-32    8.8726e-25             1          3
       7        4.9304e-32    8.8726e-25             1          3
       8        4.9304e-32    8.8726e-25             1          3
       9        4.9304e-32    8.8726e-25             1          3
      10        4.9304e-32    8.8726e-25             1          3
      11        4.9304e-32    8.8726e-25             1          3
      12        4.9304e-32    8.8726e-25             1          3
      13        4.9304e-32    8.8726e-25             1          3
      14        4.9304e-32    8.8726e-25             1          3
      15        4.9304e-32    8.8726e-25             1          3
      16        4.9304e-32    8.8726e-25             1          3
      17        4.9304e-32    8.8726e-25             1          3
      18        4.9304e-32    8.8726e-25             1          3
      19        4.9304e-32    8.8726e-25             1          3
      20        4.9304e-32    8.8726e-25             1          3

fx >>
```

Figure 9: Converges to within $\varepsilon^2$, so within machine number accuracy.

```
Command Window
>> tablNewton(eps^2 + pi/2, @(x,p) deal( atan(x - eps^2), 1./(1 + (x - eps^2).^2) ), 1
    numSteps     newtonSoln      abs_error       rel_error      precisionBits
    ________    ____________    ___________    ____________    _____________

        0             1.5708         1.5708       2.654e+19            1
        1            -1.9101         1.9101      3.2272e+19           -2
        2             3.1497         3.1497      5.3216e+19            1
        3            -10.647         10.647      1.7989e+20           -2
        4             158.27         158.27      2.6742e+21            2
        5             -39034          39034      6.5951e+23           -2
        6         2.3933e+09     2.3933e+09      4.0436e+28            5
        7         -8.997e+18      8.997e+18      1.5201e+38           -2
        8         1.2715e+38     1.2715e+38      2.1483e+57            1
        9        -2.5395e+76     2.5395e+76      4.2907e+95           -2
       10         1.013e+153     1.013e+153     1.7116e+172            3
       11        -1.612e+306     1.612e+306             Inf           -2
       12                Inf            Inf             Inf         -Inf
       13                NaN            NaN             NaN           52
       14                NaN            NaN             NaN           52
       15                NaN            NaN             NaN           52
       16                NaN            NaN             NaN           52
       17                NaN            NaN             NaN           52
       18                NaN            NaN             NaN           52
       19                NaN            NaN             NaN           52
       20                NaN            NaN             NaN           52

fx >>
```

Figure 10: Alternating divergence.

```
Command Window
>> tablNewton(1.4, @(x,p) deal( atan(x - eps^2), 1./(1 + (x - eps^2).^2) ), 1, 7)
    numSteps    newtonSoln    abs_error     rel_error     precisionBits
    ________    __________    _________    __________    _____________

       0             1.4          1.4      3.5629e+18          1
       1         -1.4136       1.4136      3.5975e+18         -2
       2          1.4501       1.4501      3.6905e+18          1
       3         -1.5506       1.5506      3.9462e+18         -2
       4          1.8471       1.8471      4.7006e+18          4
       5         -2.8936       2.8936      7.3639e+18         -2
       6          8.7103       8.7103      2.2167e+19          0
       7         -103.25       103.25      2.6276e+20         -2

fx >>
```

Figure 11: Close to oscillation.

```
Command Window
>> tablNewton(1.39, @(x,p) deal( atan(x - eps^2), 1./(1 + (x - eps^2).^2) ), 1, 7)
    numSteps    newtonSoln    abs_error    rel_error     precisionBits
    ________    __________    _________    __________    _____________

       0            1.39          1.39     3.6333e+18          1
       1         -1.3871        1.3871     3.6258e+18         -2
       2          1.3796        1.3796     3.6062e+18          1
       3           -1.36          1.36     3.5549e+18         -2
       4          1.3095        1.3095     3.4228e+18          1
       5         -1.1843        1.1843     3.0957e+18         -2
       6         0.90495       0.90495     2.3654e+18          4
       7        -0.43296       0.43296     1.1317e+18         -2

fx >>
```

Figure 12: Better, but we can do better.

```
Command Window
>> tablNewton(1.3917451995, @(x,p) deal( atan(x - eps^2), 1./(1 + (x - eps^2).^2) ), 1
    numSteps    newtonSoln    abs_error    rel_error     precisionBits
    ________    __________    _________    __________    _____________

        0          1.3917       1.3917     3.6185e+18          1
        1         -1.3917       1.3917     3.6185e+18         -2
        2          1.3917       1.3917     3.6185e+18          1
        3         -1.3917       1.3917     3.6185e+18         -2
        4          1.3917       1.3917     3.6185e+18          1
        5         -1.3917       1.3917     3.6185e+18         -2
        6          1.3917       1.3917     3.6185e+18          1
        7         -1.3917       1.3917     3.6185e+18         -2
        8          1.3917       1.3917     3.6185e+18          1
        9         -1.3917       1.3917     3.6185e+18         -2
       10          1.3917       1.3917     3.6185e+18          1
       11         -1.3917       1.3917     3.6184e+18         -2
       12          1.3917       1.3917     3.6183e+18          1
       13         -1.3915       1.3915     3.6179e+18         -2
       14          1.3911       1.3911     3.6169e+18          1
       15         -1.3901       1.3901     3.6144e+18         -2

fx >>
```

Figure 13: About 13 steps of oscillation, so we're close to the true value.