Review We have constructed the Hermite interpolant $H(t)$ that matches given function values $f(t_j)$ and derivatives $f'(t_j)$ at $n+1$ distinct points t_j. In terms of Lagrange-type basis functions $H_{jk}(t)$, we have

$$H(t) = \sum_{j=0}^{n} f(t_j)H_{j0}(t) + f'(t_j)H_{j1}(t)$$

where

$$H_{j0}(t) = (1 - 2L_j'(t_j)(t - t_j))L_j(t)^2$$

and

$$H_{j1}(t) = (t - t_j)L_j(t)^2$$

in terms of the original Lagrange basis functions

$$L_j(t) = \lambda_j \prod_{k \neq j}(t - t_k), \qquad \lambda_j = 1/\prod_{k \neq j}(t_j - t_k).$$

We use Hermite interpolation to build integration rules with degree of precision $2n+1$ and points t_j. (It is impossible to do better because a rule with degree of precision $2n+2$ or greater would integrate the nonnegative polynomial $\omega(t)^2$ exactly, leading to a contradiction:

$$0 < \int_{-1}^{1} (t - t_0)^2(t - t_1)^2 \cdots (t - t_n)^2 dt = \sum_{j=0}^{n} w_j \cdot 0 = 0.)$$

Such a rule takes the form

$$\int_{-1}^{1} f(t)dt = \int_{-1}^{1} H(t)dt + \int_{-1}^{1} E(t)dt$$

where E is the error in Hermite interpolation of f. Thus

$$\int_{-1}^{1} H(t)dt = \sum_{j=0}^{n} w_{j0}f(t_j) + w_{j1}f'(t_j)$$

where

$$w_{j0} = \int_{-1}^{1} H_{j0}(t)dt = \int_{-1}^{1} (1 - 2L_j'(t_j))(t - t_j))L_j(t)^2 dt$$

and

$$w_{j1} = \int_{-1}^{1} H_{j1}(t)dt = \int_{-1}^{1} (t - t_j)L_j(t)^2.$$

This rule has degree of precision $2n+1$ because Hermite interpolation reproduces polynomials of degree $2n+1$ exactly. However, it uses derivative values $f'(t_j)$ which may not be available, as well as the usual function values $f(t_j)$.

The key idea of *Gaussian* integration is to choose the points t_j so that all the derivative weights w_{j1} vanish. Then derivative values become unnecessary and Gaussian rules give high-order accuracy without derivatives. Since the weights w_{j1} are complicated nonlinear functions of the points, the implementation of this idea is nontrivial.

Since

$$H_{j1}(t) = (t - t_j)L_j(t)^2 = \lambda_j \omega(t) L_j(t),$$

the derivative weights will all vanish iff the monic degree $n+1$ polynomial $\omega(t)$ satisfies $n+1$ linear equations

$$\int_{-1}^{1} \omega(t) L_j(t) dt = 0$$

for $0 \le j \le n$. Since the degree-n polynomials$L_j(t)$ form a basis for all degree-n polynomials, $\omega(t)$ must satisfy

$$\int_{-1}^{1} \omega(t) p(t) dt = 0$$

for all degree-n polynomials $p(t)$. This property is called orthogonality by analogy with the orthogonality property

$$x^T y = \sum_{j=1}^{n} x_j y_j = 0$$

of n-vectors $x = (x_1, \ldots, x_n) \in R^n$ and $y = (y_1, \ldots, y_n) \in R^n$. The values of the polynomials play the role of the components of the vectors, and integration replaces summation.

Since $\omega(t) = (t - t_0)(t - t_1) \cdots (t - t_n)$, it has $n+1$ parameters. So orthogonality to the $n+1$ degree-n polynomials $L_j(t)$ is a square $n+1 \times n+1$ nonlinear system of equations for the t_js. In terms of the coefficients of $\omega(t)$, it is a square *linear* system. So the game plan is:

(a) Find a degree $n+1$ polynomial

$$\omega(t) = t^{n+1} + \cdots$$

satisfying the orthogonality conditions,

(b) Find its roots t_0 through t_n (e.g. by bisection), and

(c) Find the nonzero weights $w_j = w_{j0}$ by

$$w_j = \int_{-1}^{1} H_{j0}(t) dt = \int_{-1}^{1} (1 - 2L_j'(t_j))(t - t_j)) L_j(t)^2 dt = \int_{-1}^{1} L_j^2(t) dt = \int_{-1}^{1} L_j(t) dt.$$

Example For $n = 0$ the degree-1 polynomial $\omega(t) = t$ is orthogonal to $L_0(t) = 1$. Its zero $t_0 = 0$ gives the 1-point Gauss rule

$$\int_{-1}^{1} f(t) dt \approx 2f(0)$$

known as the midpoint rule. It is exact for linear polynomials.

Example For $n = 1$ the quadratic polynomial $\omega(t) = t^2 - c$ will be orthogonal to all linear polynomials iff

$$\int_{-1}^{1} (t^2 - c) 1 dt = 0$$

or

$$c = 1/3,$$

since any odd function integrates to 0 over a symmetric interval. Then its roots $t_0, t_1 = \pm 1/\sqrt{3}$ will give the 2-point Gauss rule

$$\int_{-1}^{1} f(t) dt \approx f(-1/\sqrt{3}) + f(1/\sqrt{3}).$$

Computing $\omega(t)$ for arbitrary n is difficult to do directly as in the above examples. A special choice of basis makes it much simpler. Suppose we have computed the desired polynomial $P_m(t) = \omega(t)$ for every $0 \le m \le n$ and want to compute the $(n+1)$-point Gauss rule. Since $P_m(t)$ is a monic polynomial of degree m the set $\{P_0(t) = 1, P_1(t) = t, P_2(t) = t^2 - 1/3, \ldots, P_n(t)\}$ forms a basis for degree-n polynomials. Hence the polynomial $P_{n+1}(t) = \omega(t)$ which determines the points for the $(n+1)$-point Gauss rule is determined by the linear system

$$\int_{-1}^{1} P_{n+1}(t) P_j(t) dt = 0$$

for $0 \le j \le n$. We can seek it in the form

$$P_{n+1}(t) = t P_n(t) - c_n P_{n-1}(t) - d_n P_{n-2}(t) - \cdots$$

which ensures that $P_{n+1}(t)$ is a monic polynomial of degree $n+1$. Orthogonality requires

$$0 = \int_{-1}^{1} P_{n+1}(t) P_j(t) dt = \int_{-1}^{1} P_n(t) t P_j(t) dt - c_n \int_{-1}^{1} P_{n-1}(t) P_j(t) dt - d_n \int_{-1}^{1} P_{n-2}(t) P_j(t) dt - \cdots ,$$

which simplifies to $d_n = 0$ if $j = n-2$ (because $tP_j(t)$ is then a degree $n-1$ polynomial and $P_n(t)$ is orthogonal to degree $n-1$ polynomials). Similarly $e_n = f_n = \cdots = 0$. So actually the constant c_n in

$$P_{n+1}(t) = t P_n(t) - c_n P_{n-1}(t)$$

is determined by orthogonality of P_{n+1} and P_{n-1}:

$$\int_{-1}^{1} P_{n+1}(t) P_{n-1}(t) dt = \int_{-1}^{1} t P_n(t) P_{n-1}(t) - c_n P_{n-1}^2(t) dt$$

or

$$c_n = \frac{\int_{-1}^{1} t P_n(t) P_{n-1}(t)}{\int_{-1}^{1} P_{n-1}^2(t) dt}.$$

Orthogonality of P_{n+1} and P_n is guaranteed by parity.

Example Try $P_3(t) = tP_2(t) - c_2 P_1(t)$ where

$$c_2 = \frac{\int_{-1}^{1} t(t^2 - 1/3)tdt}{\int_{-1}^{1} t^2} = \frac{4}{15}$$

or

$$P_3(t) = t^3 - \frac{3}{5}t.$$

Thus the 3-point Gauss rule has points $t_0 = -\sqrt{3/5}$, $t_1 = 0$ and $t_2 = \sqrt{3/5}$. The weights are given by

$$w_0 = \frac{5}{9}, \qquad w_1 = \frac{8}{9}, \qquad w_2 = \frac{5}{9}.$$

It turns out that in general

$$c_n = \frac{n^2}{4n^2 - 1}.$$

Cleanup: We need to check that the integration points t_j are inside the interval $[-1, 1]$. Suppose for example that P_4 has only three zeroes t_0, t_1, t_2 with $|t_j| \le 1$. Then the cubic polynomial $z(t) = (t - t_0)(t - t_1)(t - t_2)$ changes sign whenever $P_4(t)$ does on the interval $[-1, 1]$. Hence we can assume $z(t)P_4(t) \ge 0$ for $|t| \le 1$ and therefore

$$0 < \int_{-1}^{1} z(t)P_4(t)dt = 0$$

contradicts the orthogonality of P_4 with all cubic polynomials.

All integration weights are positive because

$$w_j = \int_{-1}^{1} L_j(t)^2 dt.$$

Positive weights are usually desirable in numerical integration because they guarantee that a function which can be approximated within ϵ by a polynomial of degree $2n + 1$ can be integrated by the n-point Gauss rule with error bounded by 2ϵ.

Algebraic theory: One can also derive Gaussian integration points by algebra. Let $p(t)$ be a degree $2n+1$ polynomial, t_j be arbitrary for now, and $w_j = \int L_j(t)dt$ so the rule already has degree of precision n. Divide $p(t)$ by a degree $n + 1$ polynomial $\omega(t)$:

$$p(t) = q(t)\omega(t) + r(t)$$

where

$$\deg q = n \qquad \deg r \le n.$$

Then

$$\int_{-1}^{1} p(t)dt = \int_{-1}^{1} q(t)\omega(t) + r(t)dt.$$

If $\omega(t)$ is chosen so that

$$\int_{-1}^{1} q(t)\omega(t)dt = 0$$

for all degree n polynomials q, then

$$\int_{-1}^{1} p(t)dt = \int_{-1}^{1} r(t)dt.$$

Since w_j have degree of precision n,

$$\int_{-1}^{1} p(t)dt = \sum_{j=0}^{n} w_j r(t_j)$$

If we now choose t_0 through t_n to be the zeroes of ω, then

$$p(t_j) = r(t_j)$$

for $0 \le j \le n$ and

$$\int_{-1}^{1} p(t)dt = \sum_{j=0}^{n} w_j p(t_j)$$

so the rule is Gaussian.