Math 128A, Summer 2019
Lecture 14, Tuesday 7/16/2019

1 Review

Half-hour spent on talking over nilpotent matrices and linear algebra (Jordan Canonical Form), tangential from the course material.

2 Gaussian Integration

Yesterday we talked about Euler-Maclaurin. One of the particularly useful applications of the Euler-Maclaurin integration formula is to estimate discrete infinite sums.
Today we'll talk about Gaussian Integration, which we use to optimize numerical integration rules. Consider

$$\int_{-1}^{1} f(t)\, dt = \sum_{j=0}^{n} w_j f(t_j) + \underbrace{c f^{(2n+2)}(\xi)}_{\text{error}}$$

Normally we choose the weights w_j to get high accuracy on equispaced point. In Gaussian integration, we also increase the accuracy via choosing where to place points t_j.
This is exact for:

$$\begin{aligned} f(t) = 1; \sum w_j &= 2 \\ f(t) = t; \sum w_j t_j &= 0 \\ f(t) = t^2; \sum w_j t_j^2 &= \frac{2}{3} \end{aligned}$$

The idea here is to choose $(n+1)$ weights w_j and $(n+1)$ points t_j. This gives a horribly non-linear system of equations. We ought to be able to solve this for up to $(2n+2)$ non-linear equations and $(2n+2)$ variables $t_0, \ldots, t_n$. That means this will be exact for polynomials of degree $\leq 2n+1$. We know this because testing on $f(t) = t^{2n+1}$,

Remark: If we scale down to intervals of length h, we have sums of this form:

$$\int_{-h}^{h} f(x)\, dx = h \sum_{j=0}^{n} w_j f(t_j h) + c h^{2n+3} f^{(2n+2)}(\xi)$$

We just solved for the 'primal' example of Gaussian Integration. We chose a symmetric interval $[-1, 1]$ just so that by parity, half of our results equal to 0.
Now we try Hermite Interpolation to optimize the numertcal integration rule.

We'll build some polynomial $p(t)$ with:

$$p(t) = \sum_{j=0}^{n} A_j(t) f_j + B_j(t) f_j'$$

Recall that Hermite interpolation takes the function and its derivative and matches those values at our interpolation points (which we haven't decided yet).
Last time (Lec 11), we found:

$$\begin{aligned} A_j(t) &= \left(1 - 2L_j'(t_j)(t - t_j)\right)^2 L_j(t) \\ B_j(t) &= (t - t_j) L_j(t)^2 \end{aligned}$$

Thus:

$$\begin{aligned} \int_{-1}^{1} p(t)\ dt &= \sum_{j=0}^{n} \int_{-1}^{1} A_j(t)\ dt f_j + \int_{-1}^{1} B_j(t)\ dt f_j' \\ &= \sum_{j=0}^{n} \omega_j f_j + \underbrace{b_j}_{0} f_j' \end{aligned}$$

Gauss's idea is to set $b_j = 0$. That is,

$$b_j = \int_{-1}^{1} B_j(t)\ dt = \lambda_j \underbrace{\int_{-1}^{1} \omega(t) L_j(t) = 0}_{\forall 0 \leq j \leq n}$$

we have this property for all j if and only if for degree $p \leq n$:

$$\iff \int_{-1}^{1} \underbrace{p(t)}_{\sum_{j=0}^{n} p(t_j) L_j(t)} \quad dt = 0$$

This property above is called 'orthogonality' (inner product). That is, if we take:

$$x^T y = x_1 y_1 + \cdots + x_n y_n = 0,$$

then we say x is perpendicular to y (orthogonal). We like orthogonal functions because this is essentially linear independence 'to the max'; that is, projections to each other net 0.
We choose $t_0, \ldots, t_n$ such that

$$\int_{-1}^{1} \omega(t) p(t)\ dt = 0$$

That is, our $\omega(t)$ we construct is orthogonal to the degree $p \leq n$ polynomial. Once we achieve this goal, we'll achieve the Gaussian integration rule, because the weights $w_0, \ldots, w_n$ are given by the exact same formula as A:

$$\begin{aligned} w_j &= \int_{-1}^{1} A(t)\ dt \\ &= \int_{-1}^{1} \left(1 - 2L_j'(t_j) \underbrace{(t - t_j)}\right) \underbrace{L_j(t)^2}\ dt \\ &= \int_{-1}^{1} L_j(t)^2\ dt > 0 \end{aligned}$$

We can also ignore the power 2 to get the weights, but we leave this here because this guarantees nonnegativity. Getting to the last line, we ignore the $-2L_j'(t_j)$ because that will not contribute to $B_j(t) = 0$.

Step 1: Find a representation of $\omega(t)$ in our favorite basis.
Step 2: Then our desired $t_0, \dots, t_n$ are simply the roots (zeros) to $\omega(t)$.

Newton's method does not work well. The standard monomial basis is never good. The Chebyshev basis could be good, but we might not know what to do (in order to achieve our goal).

Gauss suggests that we use ω for fewer points as a basis for degree n polynomials.

For example, consider the simple example of $n = 0$.

$$\begin{aligned}
\omega(t) &= t - t_0 =: P_1(t) \\
&\int_{-1}^{1} (t - t_0)p(t)\ dt = 0, \quad \deg p \leq 0 \\
&\implies t_0 = 0; \\
w_0 &= \int_{-1}^{1} 1^2\ dt = 2
\end{aligned}$$

This nets us

$$\begin{aligned}
\frac{1}{2}(t - t_0)^2 \Big|_{-1}^{1} &= 0 \\
(1 - t_0)^2 &= (-1 - t_0)^2 \\
1 - 2t_0 + t_0^2 &= 1 + 2t_0 + t_0^2 \\
t_0 &= 0.
\end{aligned}$$

We write that $P_j(t)$ above is the (monic) "Legendre polynomial" of degree j. And we define $P_0(t) := 1$ (zero-degree) polynomial.
We will construct the $P_j(t)$ for 1 point, 2 points, and onwards. Then $P_{j+1}(t)$ will be easy just using orthogonality.
Now solve:

$$\int_{-1}^{1} \underbrace{P_{n+1}(t)}_{\omega(t) := 1 \cdot t^{n+1} + \cdots ;\deg n+1; n+1 \text{ pts}} P_j(t)\ dt = 0; \quad 0 \leq j \leq n$$

We try

$$\begin{aligned}
&\overbrace{P_{n+1}(t) = \underbrace{[t]}\, P_n(t) + c_n P_{n-1}(t)}^{\text{will be our recurrence relation}} + d_n P_{n-2}(t) + \cdots \\
&= t^{n+1} + \cdots
\end{aligned}$$

We need to boost the degree up by 1 somehow, so we insert this t. Each of these terms is orthogonal to each of the lower-degree polynomials. That is, all $j < k$, P_j, P_k are orthogonal. The coefficient for $P_n(t)$ has no numeric factor because we built this as a monic polynomial.
Then, we have:

$$\int_{-1}^{1} P_{n-1}(t)P_j(t)\ dt = 0; \quad j < n-1$$

and

$$\int_{-1}^{1} \left[tP_n(t) + c_nP_{n-1}(t) + d_nP_{n-2}(t) + \cdots\right] P_j = 0; \qquad 0 \le j \le n$$

And for all $j < n-1$, by orthogonality and linearity, we have:

$$\int_{-1}^{1} P_n(t)tP_j(t)\ dt = 0.$$

The only interesting combination (multiplication) is:

$$\int_{-1}^{1} \left[tP_n + c_nP_{n-1}\right] P_{n-1} = 0; \qquad j = n-1$$

Consider

$$\int_{-1}^{1} tP_nP_n = 0 \implies \int_{-1}^{1} tP_n^2(t) = 0.$$

This is because $P_n(t)$ has even degree, and thus $tP_n^2(t)$ has odd degree, hence the symmetric integration equals 0. Then this suggests to us a parity between evens and odds, which brings us back to :

$$\begin{aligned} P_{n+1}(t) &= \underbrace{[t]} P_n(t) + c_nP_{n-1}(t) + \overbrace{d_nP_{n-2}(t) + \cdots}^{0} \\ &= t^{n+1} + \cdots \end{aligned}$$

and we know that $d_n = 0$ and onwards are all zero. To see this explicitly, see:

$$\begin{aligned} \int_{-1}^{1} \left[tP_n(t) + c_nP_{n-1}(t) + d_nP_{n-2}(t)\right] P_{n-2}(t)\ dt = 0 \\ \int_{-1}^{1} P_n(t) \left[\underbrace{tP_{n-2}(t)}_{\deg\ n-1}\right]\ dt = 0 \\ \int_{-1}^{1} c_nP_{n-1}(t)P_{n-2}(t)\ dt = 0 \\ \int_{-1}^{1} d_nP_{n-2}^2(t)\ dt = 0 \\ \implies d_n = 0 \implies e_n = 0 \implies f_n = 0 \implies \cdots \end{aligned}$$

We get this result particularly from our strong induction construction of Legendre polynomials on orthogonality.
Consider, for $P_0(t) := 1, P_1(t) := t$, the

Recurrence relation:

$$P_{n+1}(t) = tP_n(t) - c_nP_{n-1}(t)$$

This recurrence relation

$$\int_{-1}^{1} tP_n(t)P_{n-1}(t)\ dt = c_n \int_{-1}^{1} P_{n-1}^2(t)\ dt$$

$$\implies c_n := \frac{\int_{-1}^{1} tP_n(t)P_{n-1}(t)\ dt}{\int_{-1}^{1} P_{n-1}^2(t)\ dt}$$

So

$$c_1 = \frac{\int_{-1}^{1} tP_1(t)P_0(t)}{\int_{-1}^{1} P_0^2} = \frac{2/3}{2} = \frac{1}{3}$$

Hence $P_2(t) = t^2 - \frac{1}{3}$.
And thus our **Two-Point Gauss Rule** gives us:

$$\begin{aligned} t_0 &= \frac{1}{\sqrt{3}}; \qquad w_0 = 1 \\ t_1 &= \frac{-1}{\sqrt{3}}; \qquad w_1 = 1. \end{aligned}$$

It turns out, with Legendre polynomials:

$$c_n = \frac{n^2}{4n^2 - 1}$$

Then testing this, we see $c_2 = \frac{4}{15}$, and

$$\begin{aligned} P_3(t) &= tP_2(t) - \frac{4}{15}P_1(t) \\ &= t^3 - \frac{1}{3}t - \frac{4}{15}(t) = t^3 - \frac{3}{5}t \\ &= t\left(t - \sqrt{\frac{3}{5}}\right)\left(t + \sqrt{\frac{3}{5}}\right) \end{aligned}$$

And if we want a 3-point Gaussian quadrature, we can use

$$t_0 := 0 \qquad t_1 := \sqrt{\frac{3}{5}} \qquad t_2 := -\sqrt{\frac{3}{5}}$$

We would expect: $w_1 = w_2$. Then

$$\begin{aligned} w_0 &= \int_{-1}^{1} L_0(t)\ dt \\ &= \int_{-1}^{1} \frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)}\ dt \\ &= \frac{-5}{3}\int_{-1}^{1} t^2 - \underbrace{t_1 + t_2}_{0} + t_1 t_2\ dt \\ &= -\frac{5}{3}\left(\frac{2}{3} - \frac{3}{5}2\right) = \frac{8}{9} \end{aligned}$$

We have $w_0 + w_1 + w_2 = 2$ (from earlier), so $w_1 = w_2 = \frac{5}{9}$.
Thus we say that we built the inductive proof of a theorem:

Theorem 2.1. For any $n \geq 0$, there are points $t_0, \ldots, t_n \in [-1, 1]$ and weights $w_0, \ldots, w_n > 0$ with:

$$\int_{-1}^{1} f(t)\ dt = \sum_{j=0}^{n} w_j f(t_j),$$

when the degree of $f \leq (2n+1)$.

Recall that we can only find roots explicitly for degree up to 5, but we have one at zero, so secretly this becomes just a quadratic; two positive, two negative.
It turns out, for us we can take up to 9; past this, for higher degree polynomials, we solve numerically.

Midterm topics: fixed point iteration, big-O, floating point; interpolation ; no questions on numerical integration ; dont need integral IVT.
New problems! Not just old mt problems, but re-using 1.