Euler-Maclaurin formula Integrate by parts as in Taylor expansion:

$$\begin{aligned}
&\int_0^1 f(x)dx = \int_0^1 \frac{d}{dx}(x-1/2)f(x)dx \\
= \ & (1/2)(f(0)+f(1)) - \int_0^1 (x-1/2)f'(x)dx \\
= \ & (1/2)(f(0)+f(1)) - \int_0^1 \frac{d}{dx}\frac{1}{2}(x-1/2)^2 f'(x)dx \\
= \ & (1/2)(f(0)+f(1)) - \frac{1}{2}(1/2)^2(f'(1)-f'(0)) + \int_0^1 \frac{d}{dx}\frac{1}{3!}(x-1/2)^3 f''(x)dx \\
= \ & (1/2)(f(0)+f(1)) - \frac{1}{2!}(1/2)^2(f'(1)-f'(0)) + \frac{1}{3!}(1/2)^3(f''(1)+f''(0)) - \int_0^1 \frac{d}{dx}\frac{1}{4!}(x-1/2)^4 f'''(x)dx
\end{aligned}$$

and so forth. Rearrange to get an error formula for the trapezoidal rule:

$$(1/2)(f(0)+f(1)) = \int_0^1 f(x)dx + \frac{1}{2!}(1/2)^2(f'(1)-f'(0)) - \frac{1}{3!}(1/2)^3(f''(1)+f''(0)) + \int_0^1 \frac{d}{dx}\frac{1}{4!}(x-1/2)^4 f'''(x)dx.$$

Apply the formula to f'' in place of f:

$$\begin{aligned}
&(1/2)(f''(0)+f''(1)) = \int_0^1 f''(x)dx + \frac{1}{2!}(1/2)^2(f'''(1)-f'''(0)) - \frac{1}{3!}(1/2)^3(f''''(1)+f''''(0)) + \cdots \\
= \ & f'(1)-f'(0) + \frac{1}{2!}(1/2)^2(f'''(1)-f'''(0)) - \frac{1}{3!}(1/2)^3(f''''(1)+f''''(0)) + \int_0^1 \frac{d}{dx}\frac{1}{4!}(x-1/2)^4 f'''''(x)dx.
\end{aligned}$$

Key step: Use the result to eliminate the term involving $f''(1)+f''(0)$ from the previous formula:

$$(1/2)(f(0)+f(1)) = \int_0^1 f(x)dx + (1/12)(f'(1)-f'(0)) + \frac{1}{2!}(1/2)^2(f'''(1)-f'''(0)) + \int_0^1 \frac{d}{dx}\frac{1}{4!}(x-1/2)^4 f'''(x)dx.$$

Now imagine repeating the elimination infinitely often. The result would be to eliminate all the terms with plus signs between even derivatives of f and leave an infinite series of the form

$$(1/2)(f(0)+f(1)) = \int_0^1 f(x)dx + b_1(f'(1)-f'(0)) + b_2(f'''(1)-f'''(0)) + b_3(f'''''(1)-f'''''(0)) + \cdots$$

with some unknown constants $b_1 = 1/12$, b_2, b_3, ..., multiplying differences of odd-numbered derivatives of f. The Euler-Maclaurin summation formula follows by compounding:

$$\frac{1}{2}f(0)+f(1)+f(2)+\cdots+f(n-1)+\frac{1}{2}f(n) = \int_0^n f(x)dx + b_1(f'(n)-f'(0)) + b_2(f'''(n)-f'''(0)) + \cdots$$

because the differences of derivatives all telescope, canceling the interior terms. Conclusion: The error in the trapezoidal rule depends only on the derivatives of the integrand at the endpoints of the domain of integration. For example, the trapezoidal rule integrates a smooth periodic function over a full period with great accuracy.

ECTR It follows that the order of accuracy (degree of precision) of the trapezoidal rule can be increased by *endpoint corrections* which change the weights only near the endpoints of the interval. Such corrections can be derived by coupling the Euler-Maclaurin formula with finite difference approximations to the derivatives, or by polynomial interpolation as follows.

Let's use cubic interpolation to derive a fourth-order endpoint corrected trapezoidal rule. To do this, we interpolate four successive function values f_0, f_1, f_2, f_3 to integrate over the interval $[1, 2]$. Since the Lagrange basis functions are

$$L_0(x) = (x-1)(x-2)(x-3)/(0-1)(0-2)(0-3) = (x-1)(x-2)(x-3)/(-6)$$

$$L_1(x) = (x-0)(x-2)(x-3)/(1-0)(1-2)(1-3) = (x-0)(x-2)(x-3)/2$$
$$L_2(x) = (x-0)(x-1)(x-3)/(2-0)(2-1)(2-3) = (x-0)(x-1)(x-3)/(-2)$$
$$L_3(x) = (x-0)(x-1)(x-2)/(3-0)(3-1)(3-2) = (x-0)(x-1)(x-2)/6$$

the resulting rule is

$$\int_1^2 f(x)dx = w_0 f(0) + w_1 f(1) + w_2 f(2) + w_3 f(3)$$

where

$$w_0 = \int_1^2 L_0(x)dx = -1/24 = w_3$$

and

$$w_1 = \int_1^2 L_1(x)dx = 13/24 = w_2.$$

At the end intervals such as $[0,1]$ we do not have $f(-1)$ so we drop to quadratic interpolation with

$$L_0(x) = (x-1)(x-2)/(0-1)(0-2) = (x-1)(x-2)/2$$
$$L_1(x) = (x-0)(x-2)/(1-0)(1-2) = (x-0)(x-2)/(-1)$$
$$L_2(x) = (x-0)(x-1)/(2-0)(2-1) = (x-0)(x-1)/2$$

The resulting rule is

$$\int_0^1 f(x)dx = w_0 f(0) + w_1 f(1) + w_2 f(2)$$

where

$$w_0 = 10/24, \qquad w_1 = 16/24, \qquad w_2 = -2/24.$$

Putting it all together gives the fourth-order endpoint corrected trapezoidal rule

$$\int_0^1 f(x)dx = \frac{h}{24}\left(9f(0) + 23f(h) + 28f(2h) + 24f(3h) + 24f(4h) + \cdots + 9f(nh)\right).$$