Stats 150, Fall 2019
Lecture 26, Thursday, 12/5/2019

In preparation for the Final exam, a compilation of about 50 problems will

1 Notes for Homework

We'll look at a few points relevant to the last homework.
1) How do we show that something is a Brownian Motion (BM) or Brownian Bridge (BB)?
This is relatively easy by looking at Gaussian Processess. To identify a Gaussian Process, we only need to:
a) Show its joint distributions are multivariate normal (MVN)
b) To see which Multivariate Normal distributions, check means and covariances.
Recall that if two Gaussian Processes have the same $\mu(s)$ and $\sigma^2(s,t)$, they are the same Gaussian Processes. For BB, we check

$$\mathbb{E}b_n = 0, \quad \mathbb{E}b_t b_n = t(1-n), \text{ for } 0 < t < u < 1.$$

For Brownian motion, we check that:

$$\mathbb{E}B_t = 0, \quad \mathbb{E}(B_t B_n) = t \text{ for}$$

2 Lévy Processes

This unifies several discussions. We more or less know the definition for this:

Definition: Lévy Processes -

A Lévy process is a stochastic process with Stationary Independent Increments. $(X_t, t \geq 0)$ has Stationary Independent Increments (SII) if:
- increments of X over disjoint intervals are independent
- increments of X over $(s, s+t)$ has distribution dependent on t only:

$$X_{s+t} - X_s \overset{d}{=} X_t.$$

Let's look at a few basic examples.
1) Poisson Process $X_t = N_t \sim \text{Poisson}\left(()\, \lambda t\right)$, for some $\lambda > 0$.
2) Compound Poisson process $X_t = \Delta_1 + \Delta_2 + \cdots + \Delta_{N(t)}$, where $(N(t), t \geq 0)$ is a PPP(λ) independent of $\Delta_1, \Delta_2, \ldots$ IID.
We know that if the Δ_i are discrete, we have a bag of Poisson tricks in that we can take:

$$X_t = v_1 N_1(t) + v_2 N_2(t) + \cdots,$$

where $v_1, v_2, \ldots$ are the quantities of Δ_i and

$$\begin{aligned} N_i(t) &:= \#\text{ of } \Delta \text{ values equal to } v_i \text{ up to time } t \\ &= \sum_{j=1}^{N(t)} \mathbb{1}(\Delta_j = v_i). \end{aligned}$$

Then $N_1(t), N_2(t), \dots$ are independent PPPs with $\lambda_i = \mathbb{P}(\Delta = v_i)\lambda$.
Recall this is the Poissonization of the Multinomial. On the final exam, we should have this tool in our belts.
3) Brownian Motion $X_t = (B_t, t \geq 0)$.
4) Easy transforms of such Lévy Processes (assuming right-continuous paths). If X, Y are independent Lévy processes, not necessarily with the same law, then $X + Y$ is also a Lévy process, by checking the definitions. Similarly, $X - Y$ and $aX + bY$ are SIIs.
Notice that XY and X^2 and similar multiplicative operations do not preserve the SII property.
Let's build a bit of theory. To make sense of a Lévy Process, we nee da suitable family of 1-dimensional distributions. This will play the role of Poisson, Compound-Poisson, Normal, etc distributions.
Call this $F_t(\cdot) = \mathbb{P}(X_t \in \cdot)$. Then the key property is, in random-variable notation, that if we take X_s and X_t' an independent copy of X_s, then we have:

$$X_s + X_t' \overset{d}{=} X_{s+t},$$

by familiar properties of the Poisson Process and Brownian motion, and this works in the setting of Compound Poisson processes. Similarly, we have:

$$F_s * F_t = F_{s+t},$$

where $*$ denotes **convolution**. Notice that $F_s * F_t$ is by definition the distribution of $X_s + X_t'$ independent with $X_s \sim F_s$ and $X_t \sim F_t$, and so in the discrete case, we have:

$$(F_s * F_t)(z) = \sum_x F_s(x) F_t(z - x).$$

Then in the continuous, density case, we have:

$$F_t(dx) = f_t(x)dx$$
$$(f_s * f_t)(z) = \int_{x=-\infty}^{\infty} f_s(x) f_t(z - s) dx.$$

We can convolve any two distributions. Let's look at an example of convolution of a Poisson Process and Brownian Motion:

3 Gamma Processes

We'll look at gamma processes $(\gamma_r, r \geq 0)$. Recall that Gamma (r, λ) is the distribution of γ_r/λ where

$$\mathbb{P}(\gamma_r \in dt) = \frac{1}{\Gamma(r)} t^{r-1} e^{-t} dt, \quad t > 0$$
$$\mathbb{P}\left(\frac{\gamma_r}{\lambda} \in ds\right) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} dt, \quad t > 0.$$

We know (or we can perform the integral using the fact that $\Gamma(r+1) = r\Gamma(r)$) that:

$$\mathbb{E}\left(\frac{\gamma r}{\lambda}\right) = \frac{\mathbb{E}\gamma r}{\lambda} = \frac{r}{\lambda}.$$

From probability, we know that

$$\gamma_r + \gamma_s' \overset{d}{=} \gamma_{r+s}, \text{ and } \frac{\gamma_r}{\gamma_r + \gamma_s'} \overset{d}{=} \beta_{r,s}.$$

Also, the function is independent of the sum:

Beta-Gamma Algebra:

$$\left(\frac{\gamma_r}{\gamma_r+\gamma'_s}, \gamma_r+\gamma'_s\right) \stackrel{d}{=} (\beta_{r,s}, \gamma_{r+s}).$$

Notice that the 2-dimensional equality in distribution is easier, and it turns out to be easier to prove at once than each part individually.

3.1 An Old Example

Recall a pet problem that Pitman has teased before. Take a Poisson Point Process and draw arcs, and draw darts into random directions, and project downwards.
Look at the combined Poisson Process with original "true" points x and the constructed "fake" points o. Let T_i be the true points and F_i be the fake points.
Then we have: $0 < F_1 < T_1 < F_2 < T_2 < \cdots$, and notice that $F_1, T_1 - F_1$ are iid $\gamma_{1/2}$ variables by $\beta - \gamma$ algebra. Look at the location on $[0,1]$ of the single F with $0 < F < 1$. Then we find that

$$F \stackrel{d}{=} \beta_{\frac{1}{2},\frac{1}{2}}.$$

Pitman reminds us that if we take a point on a semicircle and project downwards, we have a distribution exactly $\beta_{\frac{1}{2},\frac{1}{2}}$. Recall:

$$\mathbb{P}(\beta_{r,s} \in du) = \frac{1}{B(r,s)} u^{r-1}(1-u)^{s-1} du \mathbb{1}(0 < u < 1).$$

Multiplying together the coordinates from the $\beta - \gamma$ algebra, we have a variation:

$$(\beta_{r,s}\gamma_{r+s}, (1-\beta_{r,s})\gamma_{r+s}) \stackrel{d}{=} (\gamma_r, \gamma'_s),$$

To conclude the present problem, the merge of true and fake points forms a **Renewal Process** with iid gamma$(\frac{1}{2})$ spacings.
In a variant problem, we need to check that the sum of $\beta_{r,s}$ are γ_{r+s}. If we think in terms of firing a particle, we draw a different picture (not semicircles). Consider our friendly old Poisson process.

4 Jumps in the Gamma (Γ) Distribution

We'll conclude the lecture series by proving that the Γ process must have jumps. Look at the discrete version of a γ-process:

$$\left(\gamma_{\frac{k}{2^n}}, 0 \le k \le 2^n\right).$$

Let's count the jumps of this process that are larger than some set level x. That is,

$$\sum_{k=1}^{2^n} \mathbb{1}(\underbrace{\gamma_{\frac{k}{2^n}} - \gamma_{\frac{k-1}{2^n}}}_{2^n \text{ iid } \gamma_{\frac{1}{2^n}}} > x) = \text{Binomial}\left(2^n, p = \mathbb{P}(\gamma_{\frac{1}{2^n}} > x)\right)$$

Notice that the expectation of this is simple to compute. The expected number of jumps that are larger than x is:

$$
\begin{aligned}
&= 2^n \mathbb{P}(\gamma_{\frac{1}{2^n}} > x) \\
&= \frac{1}{r} \int_x^\infty \frac{1}{\Gamma(r)} t^{r-1} e^{-t} dt \text{ at } r = \frac{1}{2^n} \\
&= \frac{1}{r\Gamma(r)} \int_x^\infty t^{r-1} e^{-t} dt \\
&= \frac{1}{\Gamma(r+1)} \int_x^\infty t^{r-1} e^{-t} dt \to \int_x^\infty t^{-1} e^{-t} dt > 0.
\end{aligned}
$$

This implies that the number of jumps of our γ process larger than x converges in distribution to a Poisson process. Put neatly, that is,

$$(\#\text{ of jumps of } \gamma \text{ process} > x) \xrightarrow{d} \mathbb{P}(\lambda(x)).$$

This implies that $(\gamma_r, r \geq 0)$ can be constructed with right-continuous paths and $\{(r, \Delta_r), \Delta_r > 0)\}$ is the set of points of a PPP$(dr\ t^{-t} e^{-t}\ dt)$.
The Gamma process is full of jumps, and the jumps are simply one massive Poisson process.

Lecture ends here.