Stats 150, Fall 2019
Lecture 14, Tuesday, 10/22/2019

CLASS ANNOUNCEMENTS:
The midterm had quite a large spread, and it did require that we have some manipulative skill to perform actions on the material (not simply true/false checks).
To prepare for the exams, we should practice examples to really get familiar with the material. The purpose of this course is to acquire concepts and skills and be able to translate them into applications.
This week's homework is an overhang of Poisson processes that we've covered in lecture and section.

1 Introduction to Renewal Theory

Renewal Theory serves as an introduction into continuous-time parameter models. This is an important technique.

We have a basic counting process in continuous time (Poisson Process with λ constant rate). We can make $\lambda(t)$ vary (as we do in the homework).

In this model, we have that

$$\boxed{N_t = \sum_{n=1}^{\infty} \mathbb{1}(T_n \leq t)}$$

where $0 < T_1 < T_2, \dots$ is a sequence of arrival times. We have a picture that we've drawn before
This picture follows logically from our formula. Logically (understanding what the variables mean), we have the 'inversion' or duality relation that we have found before:

$$(N_t \leq n) \iff (T_n \geq t),$$

relating $(T_n, n = 1, 2, \dots)$ with continuous values at discrete points and $(N_t, t \geq 0)$ which has discrete values over continuous time.

Notice that if we have a model for the distribution of $(T_n, n = 1, 2, \dots)$, it determines the distribution of the process $(N_t, t \geq 0)$ and vice versa.
We have seen this in a first model

$$\text{PPP}(\lambda) : T_n = \text{sum of } n \text{ iid Exponential}(\lambda)$$

which is logically equivalent with $(N_t, t \geq 0)$ has stationary independent Poisson

$$N_t \sim \text{Poisson}(\lambda t).$$

2 Renewal Theory

Now, we consider a model for Renewal Theory. The text sets

$$T_n = X_1 + X_2 + \cdots + X_n,$$

where the X_i are iid with the same distribution ≥ 0.
This is a new model, which is to consider small changes are iid and are not exponential.

Remark: Duality relation holds in renewal theory.

We say that a renewal occurs at time T_n and say that $(N_t, t \geq 0)$ is the renewal counting process.
Often, we say that such a sequence of random times T_n and such a renewal process are embedded in (functions of) some more complicated stochastic processes. We have seen this before in a familiar discrete time setup (Strong Markov Property).
Take a Markov chain with some reference state 0 (start at state 0). Let T_n be the time of the nth return to 0. Then $X_1 = T_1, X_2, X_3, \dots$ are iid copies of $X_1 = T_1$ by the Strong Markov Property. This is essentially discrete renewal theory. We've worked with this in a past homework with a generating function problem with tail probabilities.
To consider the more interesting (continuous) scenario, in general, we may imagine T_n as an nth "regeneration time" in some stochastic process.

2.1 Example: Queueing Model

Consider a queueing model with the following diagram of a set of 'busy cycles.' We start at $Q(t) = 0$, and the value goes up as more people are being attended to, and with more time, those are 'finished' and replaced with new items of the queue.
We'll see in the text that several queues are analyzed in terms of their busy cycles via renewal theory.
We notice that if we just do our renewal counting process, there is a lot less going on than otherwise.
This renewal process is a component of a much more difficult stochastic process. This is motivation as to how we can think in terms of renewal theory before looking closer into queueing models.

2.2 Example: A Janitor Replacing Lightbulbs

Take $X_1, X_2, \dots$ as the lifetimes of lightbulbs, which are iid (from the same factory with the same level of quality control). Suppose that new bulbs are installed at time 0, and as each bulb burns out, it is replaced by a new one. We want the Poisson process to be a special case, so we take the convention of defining

$$N(t) := \# \text{ of replacements by time } t.$$

Of course, this can be more complicated than a Poisson process, because there is no reason to believe that the lifetimes of the lightbulbs follow an exponential distribution. We want the abstraction and the additional flexibility to accomodate things like busy cycles of queues.

Now we ask, what can we say about the **renewal counting process** $(N(t), t \geq 0)$? If the X_i are iid Exponential(λ), then $N(t), t \geq 0$ has the very special property of independent increments.

In general, this is in fact characteristic of the exponential distribution, at least in contiuous time. If we did this in discrete time, we have the geometric distribution, which corresponds to binomial counts.

Here, we take spacings to be generic and counts are no longer independent increments. However, for $N_t = N(t)$, we still have the duality

$$(T_n > t) \iff (N_t < n).$$

If we can find the CDF of T_n, then we get the CDF of N_t. The difficulty is that there are very few models for which we have an explicit formula. We can use transforms (Laplace, MGF's), but then we have to invert, which is a difficult task. The main parts of renewal theory get away from explicit formulas and deal with much more general things when we cannot get an explicit handle on the details.
The main idea is that we know a lot about the sums of random variables $T_n = X_1 + \cdots + X_n$, especially the Law of Large numbers:

$$\frac{T_n}{n} \to \mathbb{E}X =: \mu = \frac{1}{\lambda}$$

Remark on inversion:
Pitman warns us that often the expected time μ gets **inverted** to $\frac{1}{\mu}$, which is inevitable as the mean of an Exponential(λ) is $\frac{1}{\lambda}$. When we look at renewal processes, we often want to have a language for the **rate** of renewals.

3 Strong Sense of Convergence

First of all, we'll define a weak sense of convergence. Fix $x > 0$ and look at the probability that T_n differs from its expected value by some value ϵ (which we may set to 10^{-6} in practice):

$$\underbrace{\mathbb{P}\left(\left|\frac{T_n}{n} - \mathbb{E}X\right| > \epsilon\right)}_{\leq \frac{\sigma^2}{n\epsilon^2}} \to 0.$$

We have an honest proof to say that this probability in the case of a finite variance is bounded by $\frac{\sigma^2}{n}$, by Chebyshev's inequality. This is the logic for the l
The convergence promised by this weak sense of convergence can be very slow. Typically, if we look at more moments, we may have a faster sense (rate) of convergence.
Pitman explains in a heuristic way what this strong sense of convergence is (Durett does not make this mathematically explicit as to not enter details of Analysis).

3.1 Strong Law of Large Numbers

This was used in the theory of Markov chains and will continue to be used explicitly many times. This is:

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{T_n}{n} = \mathbb{E}x\right) = 1$$

We think of this as the probability of an event. The difficulty is this is an infinite time horizon event. We cannot say if the event has happened at any given (finite) point in time, but this does not make us stop thinking about it.

For a full explanation, take Math 104, Math 202, and Stat 205.
As a quick explanation, we consider some axioms of probability. Essentially, we define $\mathbb{P}(F)$ for a collection of events F that is closed under finite and **countable** set operations. That means that we can look at things like

$$\mathbb{P}\left(\cup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(F_n), \tag{1}$$

for disjoint F_n. We call this the **infinite sum rule** that we use for mathematical convenience. We'll assume that $0 \leq \mathbb{P}(F) \leq 1$ and $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ for $A \cap B \neq \emptyset$, and $\mathbb{P}(\Omega) = 1 \implies \mathbb{P}(\emptyset) = 0$.
We note that (1) is equivalent to the set of familiar assumptions if:

- $F_1 \supseteq F_2 \supseteq \ldots$ so that
$$\mathbb{P}(\cap_n F_n) = \lim_{n\to\infty} \mathbb{P}(F_n)$$
- $F_1 \subseteq F_2 \subseteq \ldots$ so that
$$\mathbb{P}(\cup_n F_n) = \lim_{n\to\infty} \mathbb{P}(F_n).$$

So we simply look at the event that $\left(\frac{T_n}{n} \to \mu\right)$

$$\begin{aligned}\left(\frac{T_n}{n} \to \mu\right) &= \left(\forall_{\epsilon>0}\exists_N : \forall_{n\geq N}\left|\frac{T_n}{n} - \mu\right| \leq \epsilon\right)\\ &= \bigcap_{\epsilon>0}\bigcup_{N>0}\bigcap_{n\geq N}\left(\left|\frac{T_n}{n} - \mu\right| \leq \epsilon\right)\end{aligned}$$

Mathematicians discovered that doing this gives a probability 1 (the limit exists with probability 1). In this way, we combine the axiom of probability and limits to prove the Strong Law of Large Numbers. We will not rigorously go over the SLLN in this course. Pitman reminds that we take this detour simply to elaborate upon the convergence that Durett simply assumes (where it is stated that there is convergence without discussion or motivation).

Now we consider a picture of counts over time. We perform a very radical scaling:
Consider a Poisson Process with rate $\lambda = \frac{1}{2}$ with $t \to N_t$.
If we look only at the points of expectation, the weak law of large numbers only gives us that with very overwhelming probability, we will be in the bands. The mere fact that we have these bands tells us almost nothing about the behavior in between (we can only use Boole's inequality to sum a million inequalities to take a union bound).
The strong law of large numbers tells us that we can essentially look a uniform band around a linear approximation. Consider two lines as follows: Hence the Strong Law of Large Numbers gives this as a bound that as we follow along time, the Poisson process will always stay within the bounds.
Recall that the T_n increments satisfy the Strong Law. That is,

$$\frac{T_n}{n} \to \mathbb{E}X \text{ with probability 1.}$$

However, via the duality argument, we have:

$$\frac{N_t}{t} \to \frac{1}{\mathbb{E}X} \quad \text{(by duality or inversion)}$$

For a more careful treatment and discussion, see Durett.

Theorem 3.1. Strong Law of Large Numbers (SLLN) in Renewal Theory (RT):
If $\mathbb{E}X < \infty$ and $N(t) = \sum_{n=1}^{\infty} \mathbb{1}(T_n \leq t)$ gives a renewal counting process, then

$$\frac{N(t)}{t} \to \frac{1}{\mathbb{E}X}.$$

The intuition here is best seen via the duality argument above. Pitman says this is clever because it is often not practical to compute $N(t)$ but we can get at this via inversion.

4 Renewal Reward Theorem

This is a simple theorem which is a variation on the Strong Law of Large Numbers. Suppose we have a renewal process, and in each cycle of length X_i, we get a reward R_i. Let $R(t) :=$ cumulative reward up to time t. Then this is

$$R(t) = \sum_{i=1}^{N(t)} R_i + \underbrace{\text{something from the current cycle}}.$$

One of the main ideas is that whatever this value is, it will be negligible versus the left term under reasonable assumptions.
Although we have a random number of cycles, this is almost deterministic as:

$$\frac{\mathbb{R}(t)}{t} \to \frac{\mathbb{E}(R)}{\mathbb{E}(X)}.$$

We arrive at this intuitively, but this is a mathematical fact that can be proven. This holds in the sense of the Strong Law of Large numbers, with minimal assumptions that $\mathbb{E}|R| < \infty$ and $\mathbb{E}X < \infty$.

Lecture ends here.