Stats 150, Fall 2019
Lecture 11, Thursday, 10/3/2019

1 Introduction: Poisson Processes:

We'll do a quick review of basic properties of the Poisson distribution.
Recall that we have the $\binom{(}{n}, p)$ distribution of $X_1(p) + X_2(p) + \cdots + X_n(p)$, where the $X_i(p)$ are independent Bernoulli (p) variables.
We have a nice construction. Take $U_1, U_2, \ldots$ iid over interval $[0, 1]$. Let $X_n(p) := \mathbb{1}(U_n \leq p)$.
Now we look at when $n \to \infty$, with $p \downarrow 0$, so that $np \cong \mu$ is fixed. Then

$$\mathbb{E}S_n(p) = np \cong \mu,$$

so that

$$\mathbb{P}(S_(p) = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Notice then that this converges to $e^{-\mu}\frac{\mu^k}{k!}$ as $n \to \infty$ and $p \downarrow 0$.
This is the way that the poisson distribution arises from a binomial distribution.
Now when $\mu > 0$, let N_μ denote a random variable with this Poisson(μ) limit law:

$$\mathbb{P}(N_\mu = k) = e^{-\mu}\frac{\mu^k}{k!}, k = 0, 1, 2, \ldots$$

Some basics:

$$\mathbb{E}N_\mu = \mu$$
$$\text{Var}(N_\mu) = \mu$$

Notice that

$$\text{Var}(S_n(p)) = np(1-p) = \mu(1-p) \to \mu,$$

as $p \downarrow 0$.
Pitman notes that we should check these by summation. Additionally, by probability generating functions, we have

$$\begin{aligned} G_{N(\mu)}(z) &:= \mathbb{E}z^{N(\mu)} \\ &= \sum_{n=0}^{\infty} z^n e^{-\mu}\frac{\mu^n}{n!} \\ &= e^{-\mu}e^{\mu z} \\ &= e^{-\mu(1-z)}. \end{aligned}$$

Take the derivatives $\frac{d}{dz}, \frac{d^2}{dz^2}$ at $z = 1$. This gives us the formulas for expectation and variance.

2 Sum of Poissons

Take $N-1, N_2, \ldots, N_m$ independent Poissons with means $\mu_1, \mu_2, \ldots, \mu_m$. We know that this implies that $N_1 + \cdots + N_m \sim \text{Poisson}(\mu_1 + \cdots + \mu_m)$.

The proof is very obvious from binomials (before we pass into the limit). Alternatively, we can easily just use probability generating functions.

3 Key Fact

What is the distribution of the following random vector?

$$N_1, N_2, \dots, N_m \mid N_1 + N_2 + \cdots + N_m = n)$$

We compute this with Bayes rule. We should perform this computation once in our lives, as this is the beginning of the entire theory of Poisson processes. Consider:

$$\begin{aligned} &\mathbb{P}(N_1 = n_1, N_2 = n_2, \dots, N_m = n_m \mid N_1 + N_2 + \cdots + N_m = n) \\ &= \frac{\mathbb{P}(N-1 = n_1, N_2 = n_2, \dots, N_m = n_m)}{\mathbb{P}(N_1 + \cdots + N_m = n)}, \end{aligned}$$

where in the numerator $n_1 + n_2 + \cdots + n_m = n$.
Then this probability is equal to

$$= \frac{\frac{e^{-\mu_1}\mu_1^{n_1}}{n_1!} \cdots \frac{e^{-\mu_m}\mu_m^{n_m}}{n_m!}}{\frac{e^{-(\mu_1+\cdots+\mu_m)}(\mu_1+\cdots+\mu_m)^{n_1+\cdots+n_m}}{(n_1+\cdots+n_m)!}},$$

and notice that these exponentials cancel across the numerator and denominator. Hence this equals

$$= \frac{n!}{n_1! \cdots n_m!} p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m},$$

where $n = n_1 + \cdots + n_m$, with $\mu = \mu_1 + \mu_m$ and $p_k = \frac{\mu_1}{\mu}$. We recognize this as the familiar **multinomial** distribution.
Hence for $N = N_1 + \cdots + N_m$ as before, we have:

$$(N_1, \dots, N_m | N = n) \stackrel{d}{=} \text{multinomial}(n, p_1, p_2, \dots, p_m).$$

In words, $N_1, \dots, N_m$ is like the counts of values with probabilities $(p_1, \dots, p_m)$ in n multinomial trials.
Now we'd like to reverse this derivation.

Theorem 3.1. (**Poissonization of the Multinomial**) The following are true for a vector of counts $N_1, \dots, N_m$
(1) $N_1, \dots, N_m$ are independent Poisson$(\mu_1, \dots, \mu_m)$.
(2) $N_1 + \cdots + N_m$ is Poisson$(\mu_1 + \cdots + \mu_m)$ and given $n = N_1 + \cdots + N_m$, the $(N_1, \dots, N_m)$ are multinomial$(n, p_1, \dots, p_n)$ with $p_m = \frac{\mu_m}{\mu_1 + \cdots + \mu_m}$.

Pitman notes that these formal statements are important but easy to check. Nevertheless, these were clever ideas when this was discovered.

Corollary 3.1.1. Suppose we have $Y_1, Y_2, \dots$ iid with probability

$$\mathbb{P}(Y_i = k) = p_k,$$

for some probability $(p_1, \dots, p_m)$ on $\{1, \dots, m\}$. Assume that N is independent of $Y_1, Y_2, \dots$ and $N \sim \text{Poisson}(\mu)$.
Define

$$N_k := \sum_{i=1}^{N} \mathbb{1}(Y_i = k),$$

which in words is the number of Y values equal to k in the N trials.
Notice that by design (definition), $N_1 + N_2 + \cdots + N_m = N$. Additionally, given $N = n$, the $(N_1, \dots, N_m)$ are multi$(n, p_1, \dots, p_k)$. Then we can plug into the theorem with $\mu_i := p_i \mu$, so that

$$\mu_1 + \cdots + \mu_m = \mu.$$

Then $N_1, N_2, \dots, N_m$ are independent Poisson$(\mu_1, \dots, \mu_m)$.

If we randomize N, we somehow make the counts independent. This is highly nonobvious and this fact is key and will be exploited heavily for Poisson Processes.

Break time.

4 Poisson Point Processes (PPP)

We'll presume that we have seen Poisson processes on a line. Then we'll look at a PPP in a strip of a plane.
In each square, pick $N_i \sim \text{Poisson}(\lambda)$, where λ is some fixed rate or unit area. Take $\lambda := 2$ for this example. Given $N_i = n$, throw down n iid points with uniform probability on space. For example, take the independent uniform X, y
The Poisson probability function has highest probability at n and $n - 1$.
Now Pitman asks what happens if we project this strip down onto a line. Notice that the probability of projecting onto the same point on the line is 0. Or in other words, there are no multiple points (repeated values) in the strip.
Let $W_1, W_2, W_3, \dots$ be the spacings between points along the X-axis.

4.1 Case: $0 < t < 1$

Notice $\mathbb{P}(W_1 > t)$ is simply the probability that there there are no points to the left of t.

$$\mathbb{P}(W_1 > t) = \mathbb{P}(N_{\lambda t} = 0) = e^{-\lambda t},$$

by design and the Poissonization of the binomial.

4.2 Case: $1 < t < 2$

Here, we use independence to have:

$$\begin{aligned}
\mathbb{P}(W_i > t) &= \mathbb{P}(N_1 = 0 \text{ and count in } [1,t] \times [0,1] = 0) \\
&= \mathbb{P}(N_i = 0)\mathbb{P}(\text{count in } [1,t] \times [0,1] = 0) \\
&= e^{-\lambda e^{-\lambda(t-1)}} \\
&= e^{-\lambda t}.
\end{aligned}$$

This result makes us very happy, and we claim this can be continued inductively.

5 Repeating this discussion in a variant example

Let N_t be the number of points to the left or equal to t. Then we simply have:

$$N_t \sim \text{Poisson}(\lambda t).$$

The independent throwdown makes it so this works even when t is not an integer. Consider if $0 \le s \le t$. If s, t are integers, this surely gives the same result. Now if they are not integers, this still works via the poisson of the binomial for their fractional parts. That is, $N_t - N_s$ is the number of points in $(s, t]$. We have that

$$N_t - N_s \sim \text{Poisson}(\lambda(t-s)),$$

and notice that this count is **independent** of the Ns. Continuing this, we can generalize the usual definition of the Poisson point distribution on the half-line from 0 to ∞.
Let $0 \le t_1 < t_2 < t_3 \le \cdots \le t_n$. Then

$$N(t_1), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$$

are independent Poisson with parameters

$$\lambda t_1, \lambda(t_2 - t_1), \ldots, \lambda(t_n - t_{n-1}).$$

Theorem 5.1. The index of a counting process $(N(t), t \ge 0)$ is equivalent to $W_1, W_2, \ldots$ iid exponential (λ), where $W_1, W_1 + W_2, \ldots$ are the arrival times.

This picture has a continuous time axis and a discrete count (vertical) axis. There are formulas that correspond to the picture. Because this is a count, we write it as a sum of indicators:

$$N_t = \sum_{n=1}^{\infty} \mathbb{1}(T_n \le t),$$

which is simply counting the T_i less than or equal to t. Then for the inverse direction, we have the attained min:

$$T_n = \min\{t : N_t = n\} = \min\{t : N_t \ge n\},$$

where the second expression is generally true even if we jump over a value. Pitman reminds that these are very important formulas. Now we want to notice a key duality.

A Key Duality:

$$(T_n > t) = (N_t < n)$$

Equivalently, $(T_n \le t) = (N_t \ge n)$. To check this, we logically check implication in both ways. In fact, $t \to N_t$ by definition is increasing ($\uparrow$) and right-continuous, and no missing value (that is to say no repeated points).

6 Applications

$$\begin{aligned}\mathbb{P}(T_n \le t) &= \mathbb{P}(N_t \ge n)\\ &= 1 - \mathbb{P}(N_t < n)\\ &= 1 - \sum_{k=0}^{n-1} e^{-\lambda t}\frac{(\lambda t)^k}{k!}.\end{aligned}$$

Pitman asks us to find the density, given the CDF. To do so, we differentiate. This gives

$$\begin{aligned}f_{T_n}(t) &= \frac{d}{dt}\mathbb{P}(T_n \le t)\\ &= \frac{d}{dt}\left(1 - \sum_{k=0}^{n-1} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\right),\end{aligned}$$

which gives a telescoping sum. After evaluating, we get a gamma distribution, $\Gamma(n, \lambda)$.

7 Secret Method

Pitman gives a secret method so that we are sure to get this right (no book will tell us this). This is much better than performing the telescoping sum. Take $dt := [t, t + dt]$, so that

$$\begin{aligned}\mathbb{P}(t \le T_n \le t + dt) &= \mathbb{P}(n - 1 \text{ points in } [0, t] \text{ and } \ge 1 \text{ points in } [t, t + dt]) + O(dt)\\ &= \left[\frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}\right]\left[\frac{\cancel{e^{-\lambda dt}}(\lambda dt)^1}{1!} + \text{small}\right]\\ &= e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}\lambda\ dt\end{aligned}$$

and so this gives the answer that

$$f_{T_n}(t) = \Gamma(n, \lambda) = \frac{e^{-\lambda t}\lambda^n t^{n-1}}{(n-1)!}$$

Lecture ends here.