Stats 150, Fall 2019
Lecture 25, Tuesday, 12/3/2019

We'll continue our discussion on Brownian Motion.

1 Brownian Motion

We denote this stochastic process $B = (B_t, t \geq 0)$, and note that it has some nice properties:

- Gaussian process
- Markov process
- Stationary independent increments
- $B_0 := 0, B_t \sim \mathcal{N}(\mu = 0, \text{Var} = t)$
- Has continuous paths

At least on the highest level of abstraction, we should understand these terms and their meanings.

A Gaussian process of $X = (X_t, t \in I)$ is the joint distribution of $X_{t_1}, X_{t_2}, \dots, X_{t_n}$ in multivariate normal (Gaussian) on $\mathbb{R}^n$. This is equivalent to

$$\begin{aligned}(X_{t_1}, X_{t_2}, \dots, X_{t_n}) &= (\mu_1 + \sum_{j=1}^n a_{1i} Z_i,\ \mu_2 + \sum_{i=1}^n a_{2i} Z_i,\ \dots,\ \mu_n + \sum_{i=1}^n a_{ni} Z_i) \\ &= A\vec{Z},\end{aligned}$$

for $\vec{Z} = (Z_1, Z_2, \dots, Z_n)$ iid $\mathcal{N}(0, 1)$ and some mean vector $\vec{\mu} = (\mu_1, \dots, \mu_n)$. The covariances are

$$\mathbb{E}(X_s - \mu_s)(X_t - \mu_t) = \text{ some positive definite function of } (s, t).$$

In multivariate normals, all calculations reduce down to the standardized case of standard bivariate normal (X, Y) with correlation ρ. This gives us an important picture:

We draw the z axis and take a point in the data cloud, and project onto the axes. Take $Z = X\cos(\theta) + Y\sin(\theta)$, and $Z \overset{d}{=} X \overset{d}{=} Y$ which is obvious by symmetry of this cloud because we have arbitrarily labelled the X, Y, Z axes. The symmetry here extends to higher dimensions, and we see that with $\mathbb{E}X = \mathbb{E}Y = 0$ and $\mathbb{E}X^2 = \mathbb{E}Y^2 = 1$, we have:

$$\begin{aligned}\text{Cov}(X, Z) &= \mathbb{E}(X\cos(\theta) + Y\sin(\theta))X \\ &= \mathbb{E}X^2\cos(\theta) \\ &= \cos(\theta) \\ &= \rho.\end{aligned}$$

Now we know that for $0 < s < s+t$, we think of $(B_s, B_{s+y}) = (B_s, B_s + B'_t)$, where $B'_t := B_{s+t} - B_s$, the independent increment, is a copy of B_t ($B'_t \overset{d}{=} B_t$). Recall that fundamental property of Brownian motion is $B_s \overset{d}{=} \sqrt{s}B_1$ and $B'_t \overset{d}{=} \sqrt{t}B_1$ and B_s, B'_t independent increments.

Today we'll assume (and defer proof to graduate-level statistics courses) that Brownian motion (BM) exists and is a continuous random walk. We'll look into the distribution of variables.

We can look at the running maximum:

$$M_t := \max_{0 \leq s \leq t} B_s \quad \text{and} \quad T_x := \min\{t \geq 0 : B_t = x\}.$$

These two are related, and recall that a critical property of continuity gives that if we are above a value x, at some point we must have hit x. That is,

$$(M_t \geq x) = (T_x \leq t),$$

via the intermediate value theorem for continuous paths. This is a useful duality relation because if we are interested in one of these events, we can look at the other. Recall the familiar duality between an increasing process and its inverse, such as a Poisson Point Process or Renewal Theory.

Now there's a key idea to find the distribution of M_t.

2 Finding Distribution of M_t

Notice that $B \stackrel{d}{=} -B$ via symmetry (we can take the bijection of flipping the path vertically). THat is, $B_t \stackrel{d}{=} \sqrt{t}Z$.
Additionally, we have the Strong Markov Property (SMP) at time T_x:

$$(B(T_x + t) - x, t \geq 0) \stackrel{d}{=} (B_t, t \geq 0) = B.$$

We create a new Brownian motion in the same manner that we have in Markov chains. This is a famous result that we've used for many years, and mathematicians have realized that there are some Markov chains without the Strong Markov Property (but that's not the focus of today's discussion).

3 Reflection Principle

The key idea here is to reflect (flip up and down) once we've arrived at x. Let B be the original path and let $\hat{B}$ be the reflected path. Then $B \stackrel{d}{=} \hat{B}$. Pitman notes that the simple random walk version of this is obvious (± 1 increments with $\frac{1}{2}$ probability).
Applying the reflection principle, we learn the following:

$$\boxed{\mathbb{P}(B_t \geq x) = \frac{1}{2}\mathbb{P}\underbrace{(M_t \geq x)}_{\equiv (T_x \leq t)}}$$

We can see this intuitively from the following diagram:
Even if we condition on T_x, we have:

$$\mathbb{P}(B_t \geq x | T_x = s) = \frac{1}{2}, \forall_{0 \leq s \leq t}.$$

We have

$$\begin{aligned} \mathbb{P}(M_t \geq x) &= 2\mathbb{P}(B_t \geq x) \\ &= \mathbb{P}(|B_t| \geq x), \end{aligned}$$

so that

$$\boxed{M_t \stackrel{d}{=} |B_t|}$$

This gives:

$$\begin{aligned}
\mathbb{P}(T_x \le t) &= \mathbb{P}(M_t \ge x) \quad \text{(duality from earlier)} \\
&= \mathbb{P}(|B_t| \ge x) \\
&= \mathbb{P}(\sqrt{t}|B_1| \ge x) \quad \text{(by scaling)} \\
&= \mathbb{P}(tB_1^2 \ge x^2) \quad \text{(squaring, as all quantities are nonnegative)} \\
&= \mathbb{P}\left(\frac{1}{tB_1^2} \le \frac{1}{x^2}\right) \\
&= \mathbb{P}\left(\frac{x^2}{B_1^2} \le t\right),
\end{aligned}$$

and so

$$\boxed{T_x \stackrel{d}{=} \frac{x^2}{B_1^2}}$$

which holds for both $x > 0$ and $x < 0$ from simply flipping the sign of the process (or looking at the equality in distribution to x^2). This has some alarming implications. In particular,

$$T_x \stackrel{d}{=} x^2 T_1 \stackrel{d}{=} x^2 \frac{1}{B_1},$$

which says that the time to hit 2 is equal to 4 times the time to hit 1, or in other words, $T_2 \stackrel{d}{=} 4T_1$.
Looking at

$$ET_1 = 2\int_0^\infty \frac{1}{\sqrt{2\pi}} \frac{1}{z^2} e^{-\frac{1}{2}z^2}\ dz \ge c\int_0^1 \frac{1}{x^2}\ dz = \infty,$$

so $\mathbb{E}T_x = x^2 \mathbb{E}T_1 = \infty$, and note that a simple random walk (SRW) has this property as well, by Wald's identity. The central limit theorem does **not** apply, and the scaling is very weird.
For the integral above, we used

$$\mathbb{E}g(B_1) = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} g(z) e^{-\frac{1}{2}z^2}\ dz$$

Notice that in performing these calculations, we try to reduce the problems via simpel manipulations to rarely talk about the multivariate densities.

Take B to be a standard Brownian Motion and we can make some variations:

- start at x and look at $(x + B_t, t \ge 0)$
- start at x with a drift μ and look at $(x + B_t + \mu t, t \ge 0)$
- start at x with drift μ and σ^2, $(x + \sigma B_t + \mu t, t \ge 0)$.

Consider the usual Markov
We have the simple (but reasonable) guess

$$\mathbb{P}_a(B \text{ hits } b \text{ before } 0) = \mathbb{P}_a(T_b < T_0) = \frac{a}{b}, \forall_{0 \le a \le b}.$$

We wait until the Brownian motion hits either an upper barrier or lower barrier. We can rationalize this by looking at

$$T := T_0 \wedge T_b = \text{time to escape from strip}$$

and so $(B_{t\wedge T}, t \ge 0)$ under $\mathbb{P}_a$ with $B_0 = a$, which is surely not a Gaussian process, however is a martingale.
We do need to do a little work and prove the Martingale stopping theorem for right-continuous paths.

Theorem 3.1. (Martingale Stopping Theorem) If M is a martingale, then $(M_{t\wedge T}, t \geq 0)$ is a martingale for any T.

Proof. Consider $\mathbb{P}(T < \infty) = 1$ and

$$\begin{aligned} a &= \mathbb{E}M_0 \\ &= \mathbb{E}M_\infty \\ &= b\mathbb{P}(\text{hit } 1) + 0\mathbb{P}(\text{hit } 0) \end{aligned}$$

□

Then for constant D via the Dominated Convergence Theorem, in a bounded process, $0 \leq B_{T\wedge t} \leq b$ implies

$$\mathbb{E}B_{T\wedge t} \to \mathbb{E}B_t \text{ as } t \to \infty$$

For modelling stock prices using Brownian motion, notice that stock prices can never be negative, so we'll want to use a multiplicative model via exponentiation:

$$\exp\left(x + \mu t + \sigma B_t\right),$$

which is commonly known as "Geometric Brownian Motion", as is used in financial modeling and Black-Scholes, etc.
We agree the $T_x \overset{d}{=} \frac{x^2}{B_1^2}$, and additionally that

$$T_x + T_y' \overset{d}{=} T_{x+y},$$

via the Strong Markov Property. Now this is very tempting to take Laplace Transforms (as moment generating functions are problematic as we don't have moments here). Laplace Transforms (LTs) give us:

$$\mathbb{E}e^{-\lambda T_{x+y}} = \mathbb{E}e^{-\lambda T_x} \cdot \mathbb{E}e^{-\lambda T_y},$$

which is true for all $x > 0$, and if we play around with multiplicative forms of this, we have:

$$\mathbb{E}e^{-\lambda T_x} = e^{x\psi(\lambda)},$$

where all we know is $\mathbb{E}e^{-\lambda T_1} = e^{\psi(\lambda)}$. Then

$$\begin{aligned} e^{\psi(\lambda)} &= \mathbb{E}e^{-\lambda T_1} \\ &= \mathbb{E}e^{-\lambda/B_1^2} \\ &= 2\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\lambda/2^2} e^{-\frac{1}{2}z^2}\, dz \\ &= \sqrt{2\lambda} \quad \text{(by numerical integration)} \end{aligned}$$

This implies

$$\mathbb{E}e^{-\lambda T_x} = e^{-x\sqrt{2\lambda}}.$$

3.1 A Famous Problem

Suppose that (X, Y) is a 2-dimensional Brownian Motion, with $X \overset{d}{=} B$, $Y \overset{d}{=} B$, and X, Y are independent.
This presents a challenge: find the distribution $Y(T_x)$ of the vertical level when the horizontal first hits x. We need a technique because our distribution has such a fat tail. We've used moments, LTs, MGFs, but this is very dangerous territory because we don't have finite moments. As technique, we'll use Fourier transforms.
Start from the Gaussian MGF, and we know that we can compute

$$\mathbb{E}e^{\theta Z} = e^{\frac{1}{2}\theta^2},$$

which is true by computation. This is true for all real or complex θ, and replace $0 \to i\theta$, where $i = \sqrt{-1}$, and so we get:

$$\mathbb{E}e^{i\theta Z} = e^{\frac{1}{2}(i\theta)^2} = e^{-\frac{1}{2}\theta^2}$$

as the Fourier transform of the Gaussian. This implies

$$\begin{aligned}
\mathbb{E}e^{i\theta B_t} &= \mathbb{E}e^{i\theta\sqrt{t}B_1} \\
&= e^{-\frac{1}{2}\theta^2(\sqrt{t})^2} \\
&= e^{-\frac{1}{2}\theta^2 t} \\
&= e^{-t\theta^2/2}\psi(\theta)
\end{aligned}$$

Now we want to compute the Fourier transform of this $Y(T_x)$, which is to compute $\mathbb{E}e^{i\theta Y(T_x)}$ to help us find the distribution of $Y(T_x)$. We'll condition on T_x (because we don't know this quantity):

$$\begin{aligned}
\mathbb{E}e^{i\theta Y(T_x)} &= \int_0^\infty \mathbb{E}\left[e^{i\theta Y_t}\right]\mathbb{P}(T_x \in dt) \\
&= \int_0^\infty e^{-t\theta^2/2}\mathbb{P}(T_x \in dt) \\
&= \mathbb{E}e^{-\frac{1}{2}\theta^2 T_x} \\
&= e^{-x\sqrt{2\frac{1}{2}\theta^2}} \quad \text{(by our numerical integration above)} \\
&= \boxed{e^{-x|\theta|}}
\end{aligned}$$

It remains to invert the fourier transform (and in the interest of time we'll just skip to the conclusion for the lecture). Let $x = 1$ so that $Y(T_1)$ has a density $f(y)$ and say

$$\int_{-\infty}^{\infty} e^{i\theta y} f(y)dy = e^{-|\theta|}.$$

Now we do **Fourier inversion** (which may require further Analysis courses) to get

$$\boxed{f(y) = \frac{1}{\pi(1+y^2)}}$$

which is known as the **standard Cauchy density**. Equivalently, we can write

$$Y(T_1) \overset{d}{=} \frac{X_1^2}{Y_1^2}, \text{ for } X_1, Y_1 \text{ iid } \mathcal{N}(0,1).$$

This is a beautiful example that the Cauchy distribution gives the hitting time in two variables as described earlier.

Lecture ends here.