Stats 150, Fall 2019
Lecture 9, Thursday, 9/26/2019

The Tuesday of the midterm week will be focused on midterm review. Pitman wants to make one comment that midterm-related questions should be asked in course. Jake (our GSI) has no idea what will be on the midterm. Pitman motivates us to visit Piazza as there are hints for this week's homework.

1 Potential Theory (Green Matrices)

Green was an English mathematician who mainly worked on differential equations. This is a concept borrowed from differential equations, applied to our present context.
Let P be a transition matrix on a countable state space S. Define

$$\begin{aligned} G(x,y) &:= \sum_{n=0}^{\infty} P^n(x,y) \\ &= \mathbb{E}_x \sum_{n=0}^{\infty} \mathbb{1}(X_n = y) \\ &= \mathbb{E}_x N_y \end{aligned}$$

As a bit of book-keeping, we define:

$$N_y := \text{ total \# visits to } y \text{ including a visit at time } n = 0.$$

We can tell if states are transient or recurrent by looking at the Green matrix. Recall that we saw before that

$$\begin{aligned} G(x,x) < \infty &\iff x \text{ is transient} \\ G(x,x) = \infty &\iff x \text{ is recurrent.} \end{aligned}$$

Remark: If S is finite, then it is obvious that $G(x,x) = \infty$ for some x. Particularly, some state must be recurrent.

We're interested in a central case where we have infinite state space and a transient chain.
Recall that we defined

$$\begin{aligned} T_y &:= \min\{n \geq 1 : X_1 = y\} \\ V_y &:= \min\{n \geq 0 : X_n = y\}. \end{aligned}$$

The following is generally true with no assumptions. There is a relation between $G(x,y)$ and $G(y,y)$. Surely, $G(x,y) < G(y,y)$.

Key Fact: For all x, y (including $x = y$),

$$G(x,y) = \mathbb{P}_x(V_y < \infty) G(y,y)$$

We ask: why is this formula true? Informally, we reason that once we get to y, it starts over as if from y. To be more formal, we would cite the Strong Markov Property.

Now let's look at a key example.

1.1 Example

Consider a simple random walk on the integers $\mathbb{Z} := \{\dots, -1, 0, 1, \dots\}$, where from any integer, we go left one with probability q and right one with probability p.
Pitman notes that we have something similar to this on homework 4, where we are moving on a circle. However, we could very easily unwrap the circle into the integer line.
Let's compute the potential kernel. First of all, we ask if we really need the first parameter x, where $x, y \in \mathbb{Z}$. The definition of subtraction (translation invariance of the transition matrix) gives us:

$$G(x, y) = G(0, y - x).$$

So it's enough to discuss $G(0, y)$. Because of the key fact above, most of the action is looking at $G(y, y) = G(0, 0)$, and hence

$$G(x, y) = \underbrace{G(0,0)}\,\underbrace{\mathbb{P}_x(\text{event of hitting } y)}.$$

These two are our key ingredients. Let's first compute $G(0, 0)$. Notice that this random walk can only come back on even n (this is a periodic walk).

$$\begin{aligned} G(0,0) &:= \sum_{n=0}^{\infty} P^n(0,0) \\ &= \sum_{m=0}^{\infty} P^{2m}(0,0) \\ &= \sum_{m=0}^{\infty} \binom{2m}{m} p^m q^m, \end{aligned}$$

and comparing this against the case where $p = q = \frac{1}{2}$ and adjusting, we have:

$$G(0,0) = \sum_{m=0}^{\infty} \binom{2m}{m} 2^{-2m} (4pq)^m$$

Now Pitman states the following fact (and requires that we perform this tedious computation once in our life):

$$\binom{2m}{m} 2^{-2m} = \frac{\left(\frac{1}{2}\right) m_{\uparrow}}{m!} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+m-1\right)}{m(m-1)\cdots 1},$$

and we know (from the previous lecture) that

$$\begin{aligned} (1+x)^r &= \sum_{m=0}^{\infty} \binom{r}{m} x^r \\ \implies (1-x)^{-r} &= \sum_{m=0}^{\infty} \binom{-r}{m} (-x)^r \\ &= \boxed{\sum_{m=0}^{\infty} \frac{(r)_{m\uparrow}}{m!} x^m}, \end{aligned}$$

where we call this the negative binomial expansion. Bringing this back to the problem at hand (recognizing the negative binomial coefficient), we have:

$$G(0,0) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{m\uparrow}}{m!} (4pq)^m,$$

by negative binomial expansion with $r := \frac{1}{2}$ and $x := 4pq$. Then this gives

$$G(0,0) = (1 - 4pq)^{-\frac{1}{2}}.$$

Pitman reminds that in using this expansion, we should always be cautious for convergence in ensuring $|x| < 1$. Hence in this problem, provided $4pq < 1$ (equivalent to $p \neq \frac{1}{2}$),
Notice that if $p = \frac{1}{2}$, then in our formula we have $(1 - 1) = 0$ to a negative power, which gives us ∞. We can easily check:

$$\binom{2m}{m}\left(\frac{1}{2}\right)^{2m} \sim \frac{c}{\sqrt{m}},$$

as $m \to \infty$. This precisely gives $G(0,0) = \infty$ in the case $p = \frac{1}{2}$.
Hence:

$$G(0,0) = \begin{cases} \infty, & p = \frac{1}{2} \\ (1 - 4pq)^{-\frac{1}{2}}, & p \neq \frac{1}{2} \end{cases}.$$

Pitman says we can be a bit cuter about this. Notice:

$$\begin{aligned} 1 - 4pq &= 1 - 4p + 4p^2 \\ &= (2p - 1)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} G(0,0) &= \left[(2p-1)^2\right]^{-\frac{1}{2}} \\ &= \frac{1}{|2p - 1|} \\ &= \frac{1}{2\left|p - \frac{1}{2}\right|}. \end{aligned}$$

2 Escape Probability

We'll continue the p, q walk. This is the probability, starting at 0, that we never come back, which we write as $\mathbb{P}_0(T_0 = \infty)$, and set this equal to w. Now we ask, what is w in terms of $G(0,0)$? Recall that $G(0,0) = \mathbb{E}_0 N_0$, the expected number of hits on 0. Again, our convention is to count the starting position at time 0. Then

$$\begin{aligned} \mathbb{P}_0(N_0 = 1) &= \mathbb{P}_0(T_0 = \infty) \\ &= w, \end{aligned}$$

which is familiar from a past discussion. That is, under P_0, starting at 0, N_0 has a very friendly distribution.

Review of geometric distribution:
Recall that for $N \sim$ Geometric(p) on $\{1, 2, 3, \ldots\}$, where the probability of success is p. Then

$$\mathbb{P}(N = n) = (1 - p)^{n-1} p$$

Hence for our problem,

$$\mathbb{P}_0(N_0 = n) = (1-w)^{n-1}w,$$

so

$$N_0 \sim \text{Geometric(w)}.$$

Hence we can do away with our placeholder w, so that $w = \frac{1}{G(0,0)}$, so that

$$\mathbb{P}_0(T_0 = \infty) = \frac{1}{G(0,0)}$$

Break time.

3 More Formulas for Simple Random Walks (SRW)

3.1

We consider the same context as our example. Let $S_n = \Delta_1 + \cdots + \Delta_n$, where each Δ_i is either +1 or -1 with probability p or q, respectively.

(1) What is the probability that S_n tends to $+\infty$? This depends on p. In the recurrent case, this probability will be zero. Now if $p > q$, then we have a "drift" for our expectation that carries us off to ∞. That tells us that:

$$\mathbb{P}(S_n \to +\infty) = \begin{cases} 0, & p < q \\ 1, & p > 1 \end{cases},$$

or more neatly, using indicator notation,

$$\mathbb{P}(S_n \to +\infty) = \mathbb{1}(p > \frac{1}{2})$$
$$\mathbb{P}(S_n \to -\infty) = \mathbb{1}(p < \frac{1}{2}).$$

3.2

Now assume $p > q$. Then for $x \geq 1$, we have:

$$\mathbb{P}_x(T_0 < \infty) = \left(\frac{q}{p}\right)^x,$$

by Gambler's ruin probability (in the limit) from a previous lecture. Now $\frac{q}{p} < 1$, so

$$\mathbb{P}_x(T_0 = \infty) = 1 - \left(\frac{q}{p}\right)^x$$
$$\mathbb{P}_{-x}(T_0 < \infty) = 1,$$

because the drift up ($\uparrow$) takes us to $+\infty$ with probability 1 and must hit 0 along the way.

4 Green's Matrix for Finite State Space S

This is only interesting in if we are looking at a transient state. Take the example where the set A is an absorbing state, and let $S - A$ (or $S \setminus A$) be interior states, and

$$\mathbb{P}_x(T_A < \infty) = 1, \forall_{x \in S}.$$

In the Gambler's Ruin example, take equal probability ($\frac{1}{2} \uparrow, \frac{1}{2} \downarrow$). Let $S := \{0, 1, \dots, N\}$ and $A = \{0, N\}$.
Notice

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix},$$

where Q is a $(S - A) \times (S - A)$ matrix.
We claim that for $x, y \in S - A$,

$$\begin{aligned} G(x,y) &= \sum_{n=0}^{\infty} P^n(x,y) \quad \text{(from earlier)} \\ &= \sum_{n=0}^{\infty} Q^n(x,y), \end{aligned}$$

because the entries we would extract.
Notice that Q is NOT stochastic; in fact, we say that Q is "sub-stochastic". We can see:

$$(Q\mathbb{1})(x) = \mathbb{P}_x(X_1 \notin A),$$

which will sometimes be absorbed and will sometimes be less than 1. In any case, it's ≤ 1. We want to focus only on the non-degenerate (interesting) parts.
So, assuming $\mathbb{P}_x(V_a < \infty) > 0$ (there is some positive probability), then $G(x, a) = \infty$ for every a. This follows from

$$G(x,a) = \mathbb{P}_x(V_a < \infty) \underbrace{G(a,a)}_{=\sum_{n=0}^{\infty} 1 = \infty}.$$

Remark: Now we'll throw away all the absorbing states for our discussion, so that all our matrices are indexed by $S - A$. We're shrinking our matrix to focus on the interesting portion of our potential kernel.

Hence:

$$\begin{aligned} G &= \sum_{n=0}^{\infty} Q^n, \text{ on } S - A \\ &= I + Q + Q^2 + Q^2 + \cdots, \end{aligned}$$

and compare against:

$$QG = Q + Q^2 + Q^3 + \cdots,$$

and subtracting these gives:

$$G - QG = G - GQ = G(I - Q) = I,$$

which implies

$$\boxed{G = (I - Q)^{-1}}.$$

Computationally, this boils down to simply using our computers to crunch the inverse. Of course, for large matrix powers, we may run into underflow, overflow, or computationally singular matrices. However, there are ways to treat this issue within numerical linear algebra.

4.1 Returning to the Gambler's Ruin example:

We'd like to find $G(x, \cdot)$, which is the row x of the Green matrix for Gambler's Ruin.
Take

$$G - GQ = I \implies G = I + GQ,$$

so take $\frac{1}{2}$ from our Q matrix to get:

$$G(x, x) = 1 + G(x-1, x)\frac{1}{2} + G(x+1, x)\frac{1}{2},$$

and recall that our Q matrix has no column for n, so we take the convention that if $x-1 = 0$, then $G(x-1, x) = 0$. On the other hand, if $x+1 = N$, then $G(N, x) = 0$. This is because we're restricting to the interior state space.
Now we'd like to graph $G(x, y)$. We would like the following equation to be true:

$$G(x, y) = \mathbb{1}(x = y) + \frac{1}{2}G(x, y+1) + \frac{1}{2}G(x, y-1),$$

under the same conventions we've just established. This is certainly true, as we should check.
Now if we ignore the indicator term, the right two terms gives a straight line from where we start, to the boundary.
Graphically, this equation tells us that comparing points with abscissae differing by 2, their values will differ by exactly 1.
Now there is exactly one value of $G(x, x)$ for which this is true. This implies a formula for $G(x, y)$

4.2 Conclusion

We should check that once we've calculated this $G(x, y)$, then

$$\sum_{y \in S-A} G(x, y) = \mathbb{E}_x \left(\sum_{y \in S-A} N_y \right) = \mathbb{E}_x V_A = \boxed{x(N - x)},$$

which we found from a previous lecture. Every step is allocated to a state, so $\sum_{y \in S-A} N_y$ is simply V_A.

Lecture ends here.