Stats 150, Fall 2019
Themes in the Course

Simple and quick review sheet summarizing some of the broader concepts.

1 Markov Chains

Defined with

$$\mathbb{P}(X_{n+1} = y | X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = P(x, y)$$

and m-step transition probability

$$P^m(x, y) = \mathbb{P}(X_{n+m} = y | X_n = x).$$

Define $T_y := \min\{n \geq 1 : X_n = y\}$ and

$$\rho_{xy} = \mathbb{P}_x(T_y < \infty)$$

as the probability X_n visits y (not including current state).

Theorem 1.1. If $\rho_{xy} > 0$ but $\rho_{yx} < 1$, then state x is transient.

Theorem 1.2. Finite closed and irreducible implies recurrent.

If $N(y)$ is the number of visits to y at times $n \geq 1$, then

$$\sum_{n=1}^{\infty} P^n(x, y) = \mathbb{E}_x N(y) = \frac{\rho_{xy}}{1 - \rho_{yy}}.$$

So state y is recurrent if and only if $\mathbb{E}_y N(y) = \infty$. This is simply saying that at y, we return an infinite number of times, otherwise obvious this isn't a recurrent state.

2 Stationary Distributions

if P irreducible and recurrent then define $Y_x := \inf\{n \geq 1 : X_n = x\}$.
State space S finite and irreducible implies this stationary distribution is unique.
$\mathbb{E}_x T_x < \infty$ means positive recurrent, in which case $\mu_x(x) = 1$ so

$$\boxed{\pi(x) = \frac{1}{\mathbb{E}_x T_x}}$$

3 Convergence Theorems

Theorem 3.1. Suppose P irreducible, aperiodic, and has stationary distributio π.

Then $P^n(x, y) \xrightarrow{n \to \infty} \pi(y)$.

Theorem 3.2. Suppose P is irreducible and recurrent and let $N_n(y)$ be the number of visits to y up to time n. Then

$$\frac{N_n(y)}{n} \to \frac{1}{\mathbb{E}_y T_y}.$$

Theorem 3.3. Suppose P irreducible and $\sum_x |f(x)|\pi(x) < \infty$. Then

$$\frac{1}{n}\sum_{m=1}^{n} f(X_m) \to \sum_x f(x)\pi(x)$$

4 Exit Distributions

Let $F \subset S$ and define $V_F := \min\{n \geq 0 : X_n \in F\}$.

Theorem 4.1. Let $A, B \subset S$ and $C = S - \{a, b\}$ is finite. Suppose for all $x \in C$, $\mathbb{P}_x(V_A \wedge V_B < \infty) > 0$.
If $h(a) = 1$ for $a \in A$ and $h(b) = 0$ for $b \in B$, and

$$h(x) = \sum_y P(x, y)h(y), \forall_{x \in C},$$

then $\boxed{h(x) = \mathbb{P}_x(V_A < V_B)}$.

Let $r(x, y)$ be the part of the matrix $P(x, y)$ with $x, y \in C$. For $x \in C$, let $v(x) := \sum_{y \in A} P(x, y)$ which we can think of as a column vector.
Because $h(a) = 1$ for $a \in A$ and $h(b) = 0$ for $b \in B$, the equation for h can be written for $x \in C$ as

$$\begin{aligned} h(x) &= v(x) + \sum_y r(x, y)h(y) \\ \implies h &= (I - r)^{-1} v. \end{aligned}$$

5 Exit Times

Theorem 5.1. Suppose $\mathbb{P}_x(V_A < \infty) > 0$ for $x \in C$. If $g(a) = 0, \forall_{a \in A}$ and $g(x) = 1 + \sum_y P(x,y)g(y)$ for all $x \in C$, then

$$g(x) = \mathbb{E}_x(V_A).$$

Because $g(x) = 0$ for $x \in A$, the equaiton for g can be written for $x \in C$ as

$$g(x) = 1 + \sum_y r(x,y)g(y).$$

Let $\mathbb{1}$ be a column vector consisting of all 1s. Then

$$g = (I - r)^{-1}\mathbb{1}.$$

Remark: The expected number of visits to y starting from x is

$$(I - r)^{-1}(x,y).$$

So if we multiply by $\mathbb{1}$, then we sum over all $y \in C$, so that the result is the expected exit time.

6 Exponential with rate λ

$f_T(t) = \lambda e^{-\lambda t}$. CDF is $\mathbb{P}(T \le t) = 1 - e^{-\lambda t}$. Sum of n iid exponentials(λ) has gamma($n, \lambda 0$) density: $\lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$.
Has lack of memory property.

7 Exponential Races

Let $T_1, \ldots, T_n$ be independent with $S = \min\{T_1, \ldots, T_n\}$.
Then $S = \text{Exponential}\,(\lambda_1 + \cdots + \lambda_n)$. Also,

$$\mathbb{P}(T_i = \min\{T_1, \ldots, T_n\}) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}.$$

8 Poisson Thinning

Ex 1. Customers arrive at a sporting goods store at rate 10/hr. 60% of customers are men and 40% are women. Men stay at the store for a duration that is exponential with mean 1/2 (recall $\mathbb{E}(X) = \frac{1}{\lambda}$ for exponential). Women stay for a duration that is unformly distributed.
By poisson thinning, **arrivals** of men and women are independent Poisson with rate 6 and 4.
$\mathbb{E}[T_M] = \frac{1}{2}$ and $\mathbb{E}[T_W] = \frac{1}{4}$.
So equillibrium number of men and women are independent Poissons with means 3 and 1.

$$\mathbb{P}(M = 4, W = 2) = e^{-3}\frac{3^4}{4!}e^{-1}\frac{1^2}{2!}$$

Ex 2. People arrive at a puzzle exhibit according to a Poisson process with rate 2/min. The exhibit has enough copies of the puzzle so everyone at the

exhibit can have one to play with upon arrival. Suppose the puzzle takes an amount of time to solve that is uniform on $(0, 10)$ minutes.
a) What is the distribution of the number of people working on the puzzle in equilibrium?
The probability a customer who arrived x mins ago is still working on the puzzle is $\frac{x}{10}$, so by Poisson thinning, the number is Poisson with mean

$$2\int_0^{10} \frac{x}{10} dx = 10.$$

b) What is the probability that there are three people at the exhibit working on puzzles, one that has been working mre than four mintues, and two less than four minutes?
The number of people working for more than 4 minutes is Poisson with mean

$$2\int_0^{6} \frac{x}{10} dx = \frac{36}{10} = 3.6.$$

So the number working for less than 4 minutes is Poisson with mean $10 - 3.6 = 6.4$. Then the desired probability of 2 working less than 4 mins and 1 working for longer is:

$$\mathbb{P}(\text{event}) = \frac{e^{-3.6}(3.6)}{1!} \frac{e^{-6.4}(6.4)^2}{2!}$$

9 Superposition

Poisson thinning is taking a Poisson process and splitting it into multiple. Superposition is the opposite direction.

Theorem 9.1. Suppose $N_1(t), \ldots, N_k(t)$ are independent Poisson processes with rates $\lambda_1, \ldots, \lambda_k$.
Then $N_1(t) + \cdots + N_k(t)$ is a Poisson process with rate $\lambda + \cdots + \lambda_k$.

10 Poisson Race

Given a Poisson process of red arrivals (rate λ) and an independent Poisson process of green arrivals (rate μ), what is the probability that we get 6 red arrivals before a total of 4 green ones?

Solution. Notice that this is equivalent to getting 6 red in the first 9.
Then the probability of 6 arrivals before total of 4 green ones is

$$\sum_{k=6}^{9} \binom{9}{k} p^k (1-p)^{9-k},$$

where $p := \frac{\lambda}{\lambda+\mu}$.
We think of this as one $\lambda + \mu$ Poisson process, where points are marked different colors by flipping that coin w.p. p. ☐

11 Conditioning on $N(t) = n$

Theorem 11.1. If we condition on $N(t) = n$, then the vector $(T_1, T_2, \dots, T_n)$ has the same distribution as $(V_1, V_2, \dots, V_n)$.
Hence the set of arrival times $\{T_1, T_2, \dots, T_n\}$ has the same distribution as $\{U_1, U_2, \dots, U_n\}$.

This theorem implies:

Remark:
If we condition on having n arrivals at time t, then the locations of the arrivals are the same as the location of n points thrown uniformly on $[0, t]$.

Theorem 11.2. The conditional distribution of $N(s)$ given $N(t) = n$ is Binomial $\left(n, \frac{s}{t}\right)$.

If times $0 < s < t$ and counts $0 \leq m \leq n$, then

$$\mathbb{P}[N(s) = m | N(t) = n] = \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}$$

12 Random Sums

Let $Y_1, Y_2, \dots$ be iid and N an independent nonnegative integer-valued random variable.
Set $S := Y_1 + \cdots + Y_N$ (with convention $S = 0$ when $N = 0$).

i) If $\mathbb{E}|Y_i|, \mathbb{E}|N| < \infty$, then $\mathbb{E}(S) = \mathbb{E}(N) \cdot \mathbb{E}(Y_i)$.

ii) If $\mathbb{E}(Y_i)^2, \mathbb{E}(N^2) < \infty$, then $\mathrm{Var}(S) = \mathbb{E}N\mathrm{Var}(Y_i) + \mathrm{Var}(N)(\mathbb{E}Y_i)^2$.

13 Renewal Reward Processes

Theorem 13.1. Let $\mu = \mathbb{E}(t_i)$ be the mean inter-arrival time.
If $\mathbb{P}(t_i > 0) > 0$, then with probability 1,

$$\frac{N(t)}{t} \xrightarrow{t \to \infty} \frac{1}{\mu}$$

In words, if our lightbulb lasts μ years on average, then in t years we will use up about t/μ light bulbs.

Theorem 13.2. Strong Law of Large Numbers:
Let $X_1, X_2, \dots$ be iid with $\mathbb{E}X_i = \mu$. Let $S_n := X_1 + \cdots + X_n$.
With probability 1,

$$\frac{S_n}{n} \to \mu$$

as $n \to \infty$.

Theorem 13.3. Suppose at the time of the ith renewal we get a reward r_i.
Let $R(t) := \sum_{i=1}^{N(t)} r_i$ be the total amount of rewards accrued by time t.
Then with probability 1,

$$\frac{R(t)}{t} \to \frac{\mathbb{E}r_i}{\mathbb{E}t_i}$$

assuming each (r_i, t_i) is iid.

Remark: In words, this is

$$\text{limiting reward per time} = \frac{\text{expected reward/cycle}}{\text{expected time/cycle}}$$

14 CTMC

Continuous time Markov chains are defined by giving their transition probabilities $P_t(i,j)$. These satisfy the Chapman-Komogorov equation

$$\sum_k P_s(i,k)P_t(k,j) = P_{s+t}(i,j).$$

In practice, the basic data to describe the chain are the rate $Q(i,j)$ at which jumps occur from i to $j \neq i$.
Letting $\lambda_i := \sum_{j \neq i} Q(i,j)$ be the total rate of jumps OUT OF state i, then

$$Q(i,j) = \begin{cases} Q(i,j), & i \neq j \\ -\lambda_i, & i = j. \end{cases}$$

Then we have the differential equations:

$$P_t'(i,j) = \sum_k Q(i,k)P_t(k,j) = \sum_k P_t(i,k)Q(k,j).$$

These are only solved in a few select examples.

15 Stationary Distributions - CTMC

A stationary distribution satisfies $\pi P_t = \pi, \forall_{t>0}$.
Equivalent to $\pi Q = 0$.
To solve, replace last column of Q with 1's and define into matrix A. Then π is the last row of A^{-1}.

Remark: If X_t is irreducible and has stationary distribution π then

$$P_t(i,j) \to \pi(j)$$

16 Detailed Balance Condition

Sufficient condition to be stationary (if exists) is

$$\pi(i)Q(i,j) = \pi(j)Q(j,i).$$

Ex: Birth/Death chain where $Q(i,j) = 0$ when $|i-j| \neq 1$. In this case,

$$\pi(n) = \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} \cdot \pi(0).$$

17 Martingales

We say $(M_n, n = 0, 1, 2, \dots)$ is a **martingale** relative to $X_0, X_1, \dots)$ if and only if for $\mathbb{E}|M_n| < \infty$ and M_n is a function of $X_0, X_1, \dots, X_n$ for all n, with

$$\mathbb{E}(M_{n+1} - M_n | X_n, \dots, X_0) = 0.$$