Stats 150, Fall 2019
Lecture 15, Thursday, 10/24/2019

CLASS ANNOUNCEMENTS: Today we'll be finishing up our discussion on renewal theory. There are four examples in the text, and Pitman urges that we look at these. Our homework next week will encompass renewal theory.

1 Age and Residual Life

Let's look at an application of renewal theory.
Recall that we've looked at points on a line according to a Poisson process. We allow a more interesting structure with $X_1, X_2, \dots$ iid with $\mathbb{P}(X_i > 0) = 1$. Then let $T_n = X_1 + \cdots + X_n$ be the time for the nth renewal. Then via indicators, we have:

$$N(t) := \sum_{n=1}^{\infty} \mathbb{1}(T_n \le t).$$

We look at various associated stochastic processes, for example an **Age process**, where we define A_t as the age of component in use at time t.
This yields us a sort of 'sawtooth' process. We look for a closed-form formula for this process. Set the convention $T_0 := 0$ and notice we have the formula

$$A_t = t - T_{N(t)}.$$

In some sense, we have a renewal at time 0, but by convention, we do not count it. For today's discussion (for convenience and to not enter the study of null-recurrent processes), we assume $\mathbb{E}X_1 < \infty$.

The exact distribution of A_t for a fixed time t is easy in the Poisson case, but it's a lot more difficult in general. In the Poisson(λ) case, consider the probability $\mathbb{P}(A_t > a)$, looking back a units of time from time t. Then in terms of renewals, we have no Poisson points in the interval a, which gives (provided $a < t$):

$$\begin{aligned}\mathbb{P}(A_t > a) &= \mathbb{P}(\text{no renewals in } (t-a, t)) \\ &= \mathbb{P}(N(t-a, t) = 0) \\ &= e^{-\lambda a}.\end{aligned}$$

Now in general,

$$\mathbb{P}(A_t = t) = e^{-\lambda t}$$

where $0 \le A_t \le t$. Notice via right-tails that

$$\begin{aligned}\mathbb{P}(A_t > a) &\xrightarrow{t\to\infty} e^{-\lambda a} \\ \mathbb{P}(A_t \le a) &\xrightarrow{t\to\infty} 1 - e^{-\lambda a},\end{aligned}$$

via the convergence of CDFs. That is, the limit distribution of A_t as $t \to \infty$ is exponential with rate λ and mean $\frac{1}{\lambda}$.

1.1 General Lifetime Distribution of X

We may ask, what happens for a general lifetime distribution of X? Previously, we took $X \sim \text{Exponential}(\lambda)$, the same as the limit distribution of A_t as $t \to \infty$.
To look at this, we approach this problem indirectly. Let's suppose that the limit distribution of A_t can be understood in terms of long run averages. That is, look at A_∞, the limit distribution. Then define, in the context of renewal theory:

$$\mathbb{P}(A_\infty > a) := \text{long run fraction of time } t \text{ that } A_t > a.$$

Sometimes, the age will be above a, and sometimes below a. This yields an indicator process, where we have 0 when the age is ≤ 0 and 1 when age is > 0. This is the indicator process:

$$(\mathbb{1}(A_t > a), t \geq 0).$$

Then we can apply the Renewal Reward Theorem from last lecture. Suppose that we get a reward R_n from each renewal interval $[T_{n-1}, T_n]$ of length X_n. Then define:

$$\begin{aligned} R(t) &:= \text{accumulated reward up to time } t \\ &= \sum_{n=1}^{N(t)} R_n + (\text{things in last cycle}) \end{aligned}$$

We discuss the reward per unit time and have via the Strong Law of Large numbers:

$$\frac{R(t)}{t} \to \frac{\mathbb{E}(R)}{\mathbb{E}(X)},$$

with probability 1 of convergence.
Here in the present problem, define

$$R_n = (X_n - a)\mathbb{1}(X_n > a) =: (X_n - a)_+$$

Then we see that $R(t)$ is the total length of times $s \leq t : A_s > a$. This implies

$$\frac{R(t)}{t} \to \frac{\mathbb{E}R_1}{\mathbb{E}X_1},$$

by the Renewal Reward Theorem. So we argue this is the limit probability

$$\mathbb{P}(A_\infty > a) = \lim_{t \to \infty} \mathbb{P}(A_t > a).$$

This limit exists via an argument of Ergodic theory. Then we have the formula:

$$\mathbb{P}(A_\infty > a) = \frac{\mathbb{E}(X_1 - a)_+}{\mathbb{E}X_1}.$$

However, we can do a little better than this. We claim that A_∞ has probability density:

$$\mathbb{P}(A_\infty \in da) = \frac{\mathbb{P}(X_1 > a)}{\mathbb{E}X_1}.$$

To see why this is true, this is basically the tail-integral formula for $\mathbb{E}X_1$ and $\mathbb{E}(X_1 - a)_+$. Pitman draws us a picture to convice us of this fact.
To see where the mean is, Pitman reminds us of the tail integral formula (or integration by parts) for the mean:

$$\begin{aligned}\mathbb{E}(X_1) &= \int_0^\infty (1 - F(t))\ dt \\ &= \int_0^\infty f(t)t\ dt.\end{aligned}$$

Now we introduce a level a and we ask what is $\mathbb{E}X_1$. By another application of the tail integral formula,

$$\begin{aligned}\mathbb{E}(X_1 - a)_+ &= \text{area to right of } a \text{ and above the CDF} \\ &\leq \mathbb{E}X_1.\end{aligned}$$

Bringing this into our previous formula, this gives:

$$\mathbb{P}(A_\infty \in da) = \frac{\mathbb{P}(X_1 > a)}{\mathbb{E}X_1} = \frac{\int_a^\infty (1 - F(t))\ dt}{\mathbb{E}X_1},$$

so we take $-\frac{d}{da}$ with

$$\frac{\mathbb{P}(A_\infty \in da)}{da} = -\frac{d}{da} = \frac{1 - F(a)}{\mathbb{E}X_1}.$$

Pitman notices that conclusion required 'tools of the trade' in terms of thinking about tail expectations and approaching the problem in a different way.

1.2 A Quick Check

Take $X_1 \sim \text{Exponential}(\lambda)$. Then $1 - F(t) = e^{-\lambda t}$ and $\mathbb{E}X_1 = \frac{1}{\lambda}$. Then our formula above gives

$$\frac{\mathbb{P}(A_\lambda \in da)}{da} = \frac{e^{-\lambda a}}{1/\lambda} = \lambda e^{-\lambda a},$$

which we know as the familiar density of Exponential(λ). We can do this in a general setting, and it specializes well as we see here where we already know the result.

2 Exercise

Pitman gives the following exercise for us to work with. Let Z_t be the residual life at t, or equivalently the remaining lifetime of component in use at t.
(1) Find a formula for Z_t
(2) Describe the limit distribution of Z_t as $t \to \infty$.

Solution. To arrive at the formula, we draw a picture of the process:
Then we should expect

$$\boxed{Z_t = T_{N(t)+1} - t}$$

and in the long-run, we know that the limiting probability should be represented by the fraction of time that the process is above a level.

Over a long stretch of time, the difference only comes from the current cycle. So we try:

$$\frac{\mathbb{P}(Z_\infty \in dz)}{dz} = \frac{\mathbb{P}(X_1 > z)}{\mathbb{E}X_1}$$

and argue that limit distributions are the same. This is nontrivial but is true by application of the Renewal Reward Theorem. Pitman notes that we should also see Durett 3.3, which looks into the joint distribution:

$$\lim_{t\to\infty} \mathbb{P}(A_t > a, Z_t > b) = \frac{\mathbb{P}(X_1 > a+b)}{\mathbb{E}X_1},$$

which gives a generalization for two variables by the same token of the Renewal Reward Theorem.

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3 Queueing Theory: Terminology

This discussion is due to David Kendall's notation from 1953.
Our formalism is the notation of the form

$$A/B/C,$$

where A is the input, B is the service, and C is the number of servers. That is,

$$A \in \{M, G\}$$

where M is Markov or Poisson(λ) and G or GI is a General Input or Revival Process Input, and λ is the arrival rate, or equivalently the inverse mean into arrival.

$$B \in \{M, G\}$$

where M is Markov or Exponential(μ) service and G is the general service time distribution. Note that μ is the rate of service.

$$C \in \{1, 2, 3, \dots, \infty\}$$

is the number of servers.

3.1 Main Examples

(1) We've seen before in the satellite problem "$M/G/\infty$". The sattelites are put up at Poisson(λ) times, where each satellite lives some lifetime (G), and there is no limit to the number of satellites in orbit. In the text, we looked into a very simple analysis by Poisson trials.
(2) We can also consider $M/G/1$ Queues, with Poisson(λ) input, 1 server, any service time distribution, and assume that service times are iid, independent of arrival times.
Let Q_t be the queue length process, where S_n is the nth service time and T_n is the nth arrival time.
There's a notion of 'Queue-Discipline'. We consider FIFO = First In First Out, and LIFO = Last In First Out. Considering queues, we are typically

interested in stability of the queue, average wait times, or questions that do not depend on queue discipline, so we will not pursue this in detail.
All the 1 server queues are stable if and only if the arrival time λ is less than the service rate μ (that is, $\lambda < \mu$). This gives positive recurrence. The case of equality $\lambda = \mu$ gives null recurrence. This means that the queue clears out eventually, but the expected time of this event is infinite.

4 Little's Formula

This is a famous formula, relating mean rates. Take λ to be the arrival rate in a stable queue ()$\lambda < \mu$), and let L be the long-run average queue length, and let W be the stationary state mean waiting time (not a random variable, in queue + service) per customer.

$$\boxed{L = \lambda W}$$

The text presents a long-run cost argument (Durett p. 131), and we give a similar treatment.
In this picture, we have:

$$L = \lim_{t\to\infty} \frac{1}{t} \int_0^t Q_s \; ds,$$

where Q_s is the number of customers in the system at time s. This limit exists by a law of large numbers argument.
Our original picture is the composition of this sum of bars. The integral is the sum of lengths of these bars (size 1 for each customer) over time. Then these bars are being initiated at rate λ.

Lecture ends here.