Stats 150, Fall 2019
Lecture 18, Tuesday, 11/5/2019

We're currently on Durett's chapter 4, looking at continuous-time Markov chains.
We have the following topics to cover (we have covered the first two):
- §4.1: Definitions and Examples
- §4.2: Transition Probability Functions (TPF) with $P_t = \exp(Qt)$
- §4.2.1: Branching Processes
- §4.3: Limit Behavior
- §4.3.1: Detailed balance
- §4.4: Exit distributions
- §4.5: Queues
- §4.6: Queueing Networks

Today we'll cover §4.3 and 4.3.1, and on Thursday we'll cover 4.4 and 4.5. We won't go over 4.6 explicitly as part of the syllabus.

1 Limit Behavior

We'll work with the theory for continuous-time Markov chains (CTMC). We've already done this for discrete time. For simplicity, today let's assume that the state space S is finite.

1.1 Review of the Discrete Case

In the discrete case, for a transition probability matrix P, wassuming that P is irreducible, which is equivalently

$$\forall_{x,y} \exists_n : P^n(x,y) > 0,$$

then

$$\exists! \text{invariant } \pi : \pi P = \pi, \quad \pi \mathbb{1} = 1.$$

Also, we always have that

$$\frac{1}{n} \sum_{k=1}^{n} P^k \to \Pi = \begin{pmatrix} \pi \\ \pi \\ \vdots \end{pmatrix}, \Pi = \mathbb{1}\pi$$

Moreover, we have

$$\pi_x = \frac{1}{\mathbb{E}_x T_x},$$

which we call discrete renewal theory. We've generalized this very well in the continuous case via renewal theory. It is not immediately obvious that π is the solution to the equations given by the quantifiers above, but this is proved in the text. Note this fails in the periodic case.

1.2 Extending to a Continuous Parameter Markov Chain

On a finite state space S, consider the following:
1. **irreducible** means $\forall_{x,y} : P_t(x,y) > 0, \forall_t$.

To see why this is true, recall that we know $P_t = \exp(tQ)$ and that $X_t = Y_{N(t)}$ where Y is a suitable jumping chain in discrete time. Also, $N(t)$, the number of attempted jumps by time t, is a Poisson(λ) process with some rate $\lambda \geq \max_i \lambda_x$, where λ_x is the holding rate of state $x \in S$. To find the holding rates from the Q matrix, we note

$$\lambda_x = -Q(x,x)$$

and simply pick this off the diagonal. The holding rate is the rate of loss of probability from x.
If we are starting at 1 at time 0, the most common situation is to decrease to a limit. Alternatively, we may have a dampening oscillation, but then again we will converge to a limit. Then

$$\boxed{\lambda_x = -\frac{d}{dx} P_t(x,x)\Big|_{t\to 0}}$$

This is essentially the **rate of exponential** hold at x.
From the Poisson construction, we have $P_t = \exp(Qt)$ for a suitable Q, and more or less by definition (Taylor's theorem), we have

$$\begin{aligned} \exp(Qt) &:= I + Qt + \frac{Q^2t^2}{2!} + \frac{Q^3t^3}{3!} + \cdots \\ \implies \frac{d}{dt}\exp(Qt) &= Q\exp(Qt) \\ \implies \frac{d}{dt}\exp(Qt)\Big|_{t\to 0} &= Q. \end{aligned}$$

To verify the negative sign, we start with probability 1 at $t = 0$ and we can only go down (hence a negative slope).

1.3 An Old Example

Recall the silly matrix example

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which simply alternates between two states. Then it cannot be that $P = P_t$ for a continuous time transition probability function P_t. Take U to be the transition probability matrix for $Y_0, Y_1, Y_2, \ldots$. Then from $X_t = Y_{N(t)}$, we have

$$\mathbb{P}_x(X_t = y) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}U^n(x,y). \tag{1}$$

Suppose $P_t(x,y) > 0$ for some $t > 0$, depending on x and y. Then in the representation (1), we have that if the sum is positive, then there exists some n so that the term is strictly positive. Then $U^n(x,y) > 0$, all terms are positive, and U is irreducible.
Essentially, being able to transfer between states in some amount of time implies that it is possible for any other amount of time.
In summary, we learned that (P_t) irreducible implies $P_t(x,y) > 0$ for all x, y. Then this implies that

$$P_{n\delta}(x,y) \to \pi(y)$$

for some $\pi(y)$ that does not depend on x. We can check this for $\delta = 1, \frac{1}{2}, 2, \frac{1}{4}, 4, \ldots$. With real analysis, we can generalize this further.

Now there remains the issue of how to find the limit. Pitman would like to emphasize one point. We do not have to have our hands on a formula or differential equations. We have a theory to show that these limits exist and can be found even if we do not have the differential equations.
That is, we know that for a finite irreducible chain, there is a stationary probability π with

$$\pi P_t = \pi, \qquad \pi \mathbb{1} = 1. \tag{2}$$

In practice, usually we don't have a useful formula for P_t. So the expression for π in terms of Q is very useful. Noticing that (2) has no t on the RHS, we differentiate this vector equation. Then we have

$$\frac{d}{dt}\pi P_t = \vec{0}.$$

We can push the differential operator through the rows of the matrix:

$$\begin{aligned} \implies \pi \frac{d}{dt} P_t &= 0 \\ \pi Q P_t &= 0, \forall_{t \geq 0}. \end{aligned}$$

Now letting $t \downarrow 0$, we see that $P_t \to I$ (where almost no time has elapsed). Then this implies

$$\pi Q = 0,$$

and conversely.

2 Detailed Balance

Recall that solving $\pi P = \pi$ in discrete case is often simplified greatly by solving the detailed balance equations (which often related with reversible equilibrium). This is

$$\pi(x)P(x,y) = \pi(y)P(y,x), \forall_{x,y}.$$

Detailed balance implies $\pi P = \pi$ in discrete time. The same notion works in continuous time.
For the analog, look at

$$\pi(x)P_t(x,y) = \pi(y)P_t(y,x), \forall_{t \geq 0}.$$

If we repeat the previous argument, this is true if and only if

$$\pi(x)Q(x,y) = \pi(y)Q(y,x), \forall_{x \neq y}.$$

This has a very intuitive meaning. Take $\pi(x)Q(x,y)$ to be the long-run rate of transitions from $x \to y$.
Pitman notes that there is an alternation: every upstep is followed by an upstep, and vice versa. This is not always true. For example, consider a degenerate chain on a circle $x \xmapsto{y}\xmapsto{z}\xmapsto{x}$.
In our present case, $x \to y$ transitions are perfectly balancing $y \to x$ transitions. That is, $\pi(y)Q(y,x)$ is the long-run rate of transition from $y \to x$.

3 Example: Birth/Death Chain

Consider the birth/death chain on $S = \{0, 1, 2, \ldots, N\}$ for some N. As definition of a birth/death chain if

$$P(x, y) > 0 \iff y = x \pm 1.$$

We have a key fact.

Remark: The only possible stationary distributions for a birth/death (B/D) chain are reversible.

We can seek a reversible solution. Take the set of equations

$$\pi_0 \lambda_0 = \pi_1 \mu_1$$
$$\pi_1 \lambda_1 = \pi_2 \mu_2$$
$$\vdots$$

Quite trivially, we have $\pi_1 = \pi_0 \frac{\lambda_0}{\mu_1}$. Similarly,

$$\pi_2$$

We can ask: under which conditions on λ_i and μ_i does there exist a stationary distribution? We seek π_0, so take:

$$\pi_0 \underbrace{\left(\frac{\lambda_0}{\mu_1} + \frac{\lambda_0}{\mu_1} \frac{\lambda_1}{\mu_2} + \cdots \right)}_{\text{must be } < \infty} = 1,$$

by the normalization condition. Now $\pi_0 > 0$, so the underbraced portion must be finite.
As a fact, if the sum is strictly finite, then the chain is **positive recurrent and irreducible**, and

$$P_t(x, y) \xrightarrow{t \to \infty} \pi(y)$$

from the above equations. The earlier discussion is the case with $\lambda_N := 0$ and $\lambda_0, \ldots, \lambda_{N-1} > 0$ and $\mu_1, \ldots, \mu_N > 0$.

4 Example: M/M/1 Queue

Consider customers arriving at a rate λ via a Poisson point process. Service times are iid exponential(μ). Now $\lambda_n \equiv \lambda$ and $\mu_n \equiv \mu$. Then with λ as arrival rate and μ service rate, we have

$$\lambda < \mu \iff \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \cdots < \infty$$

as the condition for stable equilibrium.
This implies that $X_t \xrightarrow{d}$ Geometric $\left(1 - \frac{\lambda}{\mu}\right)$ on $\{0, 1, 2, \ldots\}$ as $t \to \infty$. Then with $p := 1 - \frac{\lambda}{\mu}$, we have:

$$\mathbb{E} X_t \to \frac{q}{p} = \frac{\lambda/\mu}{1 - \lambda/\mu} = \boxed{\frac{\lambda}{\mu - \lambda}},$$

which is the **limit mean queue length**. Recall that $L = \lambda W$ as discussed in the text. Pitman notes there are slight modifications to this example in the text including networks of queues, where we can largely apply detailed balance and usually get the solution.

5 Summary

We have essentially developed the core theory on discrete and continuous Markov chains, and any further lectures will be working through examples. We can work through more examples until we are comfortable with the material.
Following, we will do a bit of Brownian motion and Martingales to finish off the semester.

Lecture ends here.