

Assignment 3

TU Delft, 3ME, DCSC, Spring 2021

- The assignments are individual. You may consult and discuss with your colleagues, but the answers you provide, the code you use, and the report you hand in must be independent.
- Provide detailed answers, describing the steps you followed. Only end results of calculations are not sufficient.
- Provide a clear report as a PDF file, typed, preferably using L^AT_EX.
- Submit the report digitally via Brightspace before the deadline: 9:00, June 30, 2021.

2p Problem 1

- 1p** (a) Consider the following convex optimization problem with a complicating constraint:

$$\begin{aligned} & \underset{\theta_1, \theta_2}{\text{minimize}} && f_1(\theta_1) + f_2(\theta_2) \\ & \text{subject to} && \theta_1 \in \Theta_1, \quad \theta_2 \in \Theta_2 \\ & && h_1(\theta_1) + h_2(\theta_2) \leq 0 \end{aligned}$$

where Θ_1, Θ_2 are convex sets and all the functions are convex.

Apply primal decomposition to the problem and show the resulting two subproblems. What is the role of the master problem?

- 1p** (b) In order to solve the master problem in question (a), the two subproblems are solved independently to obtain subgradients. It turns out that we can find a subgradient for the optimal value of each subproblem from an optimal dual variable associated with the coupling constraint. This leads to the second question:

Let $p(z)$ be the optimal value of the (possibly non-smooth) convex optimization problem:

$$\begin{aligned} & \underset{\theta}{\text{minimize}} && f(\theta) \\ & \text{subject to} && \theta \in \Theta \\ & && h(\theta) \leq z \end{aligned}$$

Let λ^* be an optimal dual variable associated with the constraint $h(\theta) \leq z$. Show that $-\lambda^*$ is a subgradient of p at z .

2p

Problem 2

Consider the combined consensus/projected incremental subgradient method for N agents shown in the lecture slides:

$$\theta_{k+1}^i = \mathcal{P}_\Theta \left[\sum_{j=1}^N [W^\varphi]_{ij} (\theta_k^j - \alpha_k g^j(\theta_k^j)) \right], \quad i = 1, \dots, N$$

Assume that the communication graph used for consensus is strongly connected, balanced, and the matrix W is doubly stochastic. Show that as $\varphi \rightarrow \infty$ (i.e., the agents reach consensus in each iteration of the algorithm), the combined consensus / projected incremental subgradient method becomes a standard subgradient method.

3p

Problem 3

Consider the positive definite quadratic function partitioned into two sets of variables

$$V(u) = \frac{1}{2} u^T H u + c^T u + d$$

$$V(u_1, u_2) = \frac{1}{2} \begin{pmatrix} u_1^T & u_2^T \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1^T & c_2^T \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + d$$

in which $H > 0$. Imagine we wish to optimize this function by first optimizing over the u_1 variables holding u_2 fixed and then optimizing over the u_2 variables holding u_1 fixed as shown in Figure 1 for a scalar case.

Let us see if this procedure, while not necessarily efficient, is guaranteed to converge to the optimum.

1p

(a) Given an initial point (u_1^p, u_2^p) , show that the next iteration is

$$u_1^{p+1} = -H_{11}^{-1} (H_{12} u_2^p + c_1)$$

$$u_2^{p+1} = -H_{22}^{-1} (H_{21} u_1^p + c_2)$$

The procedure can be summarized as

$$u^{p+1} = A u^p + b$$

in which the iteration matrix A and constant b are given by

$$A = \begin{pmatrix} 0 & -H_{11}^{-1} H_{12} \\ -H_{22}^{-1} H_{21} & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -H_{11}^{-1} c_1 \\ -H_{22}^{-1} c_2 \end{pmatrix}$$

1p

(b) Establish that the optimization procedure converges by showing the iteration matrix is stable

$$|\text{eig}(A)| < 1$$

1p

(c) Given that the iteration converges, show that it produces the same solution as

$$u^* = -H^{-1} c$$

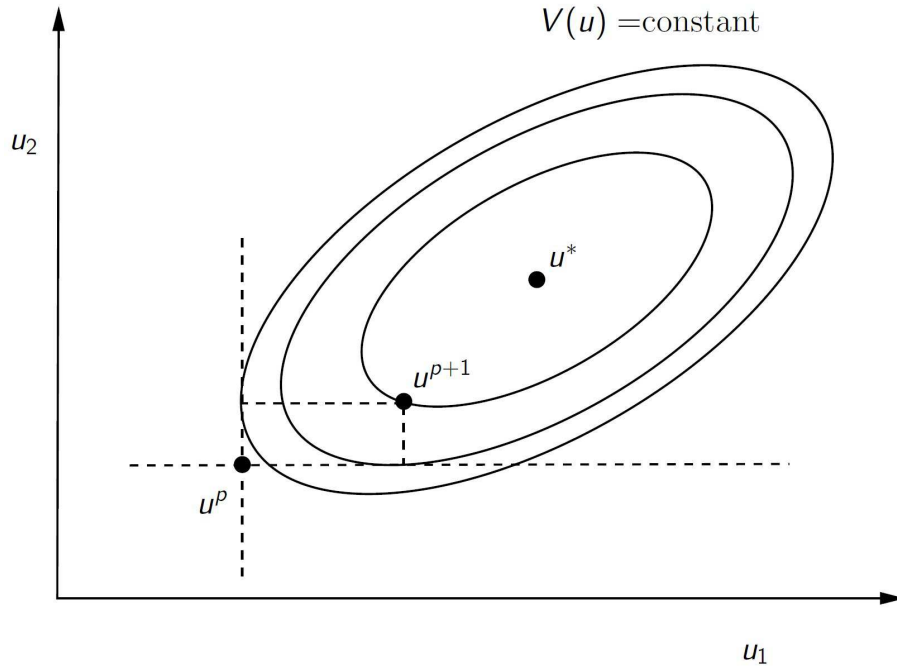


Figure 1: Optimizing a quadratic function in one set of variables at a time.

2p Problem 4

Consider again the iteration defined in the previous problem.

- 1p (a) Prove that the cost function is monotonically decreasing when optimizing one variable at a time

$$V(u^{p+1}) < V(u^p) \quad \forall u^p \neq -H^{-1}c$$

- 1p (b) Show that the following expression gives the size of the decrease

$$V(u^{p+1}) - V(u^p) = -\frac{1}{2}(u^p - u^*)^T P (u^p - u^*)$$

in which

$$P = HD^{-1}\tilde{H}D^{-1}H, \quad \tilde{H} = D - N, \quad D = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & H_{12} \\ H_{21} & 0 \end{pmatrix}$$

and $u^* = -H^{-1}c$ is the optimum.

Hint: to simplify the algebra, first change coordinates and move the origin of the coordinate system to u^* .

Useful facts:

- If H is a positive definite, symmetric matrix partitioned in the following way, then

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} > 0 \quad \Rightarrow \quad \bar{H} = \begin{pmatrix} H_{11} & -H_{12} \\ -H_{21} & H_{22} \end{pmatrix} > 0.$$

- For any Q real symmetric and R real matrices:

$$Q > 0 \text{ and } R \text{ nonsingular} \quad \Rightarrow \quad R^T Q R > 0.$$