PROJETO MECÂNICO

Sistema de Posicionamento de Válvula de Injeção de Gás para LWFA

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1 Task 2

$$\frac{dT_b(t)}{dt} = a_1\dot{q}_{solar}(t) + a_2[\dot{q}_{occ}(t) + \dot{q}_{ac}(t) - \dot{q}_{vent}(t)] + 3a_3[T_{amb}(t) - T_b(t)]$$

$$\frac{dT_b(t)}{dt} \approx \frac{T_b(k+1) - T_{b,k}}{\Delta t}$$

$$\frac{T_b(k+1) - T_{b,k}}{\Delta t} \approx a_1\dot{q}_{solar}(t) + a_2[\dot{q}_{occ}(t) + \dot{q}_{ac}(t) - \dot{q}_{vent}(t)] + a_3[T_{amb}(t) - T_b(t)]$$

$$T_b(k+1) \approx (1 - a_3\Delta t)T_b(k) + \Delta t \begin{bmatrix} a_1 & a_2 & a_2 & -a_2 & a_3 \end{bmatrix} \begin{bmatrix} \dot{q}_{solar}(k) \\ \dot{q}_{occ}(k) \\ \dot{q}_{ac}(k) \\ \dot{q}_{vent}(k) \\ T_{amb}(k) \end{bmatrix} \tag{1}$$

$$A = 1 - a_3\Delta t \qquad B = \Delta t \begin{bmatrix} a_1 & a_2 & a_2 & -a_2 & a_3 \end{bmatrix} \tag{2}$$

2 Task 3

We are given the task of:

$$\min_{a_1,a_2,a_3} \sum_{k=1}^{2159} \left(T_b(k+1) - \left(AT_b(k) + B \begin{bmatrix} \dot{q}_{solar}(k) & \dot{q}_{occ}(k) & \dot{q}_{ac}(k) & \dot{q}_{vent}(k) & T_{amb}(k) \end{bmatrix}^\intercal\right)\right)^2 \tag{3}$$

One can start by noticing this is an mean square error minimization problem. In fact, [1] as:

$$T_b(k+1) = AT_b(k) + B \begin{bmatrix} \dot{q}_{solar}(k) & \dot{q}_{occ}(k) & \dot{q}_{ac}(k) & \dot{q}_{vent}(k) & T_{amb}(k) \end{bmatrix}^\intercal + e(k) \tag{4}$$

$$\Leftrightarrow$$

$$e(k) = T_b(k+1) - \left(AT_b(k) + B \begin{bmatrix} \dot{q}_{solar}(k) & \dot{q}_{occ}(k) & \dot{q}_{ac}(k) & \dot{q}_{vent}(k) & T_{amb}(k) \end{bmatrix}^\intercal\right) \tag{5}$$

What we are trying to do is find the best parameters, $\{a_1, a_2, a_3\}$ that minimize e_k in a square sense.

If we take into account [2], we can rewrite the minimization problem [3] with respect to our minimization parameters:

$$\min_{a_1,a_2,a_3} \sum_{k=1}^{2159} \left((T_b(k+1) - T_b(k)) - \Delta t \begin{bmatrix} \dot{q}_{solar}(k) & \dot{q}_{occ}(k) + \dot{q}_{ac}(k) - \dot{q}_{vent}(k) & T_{amb}(k) - T_b(k) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right)^2$$

This problem can further be viewed as follows:

$$\min_{a_1,a_2,a_3} \sum_{k=1}^{2159} \hat{e}^2 = \min_{a_1,a_2,a_3} E^\intercal E$$

Where E is a $k \times 1$ matrix, with each individual sum of our minimization problem being a different column of E. This matrix takes the following form:

$$E = Y - \phi x$$

$$Y = \begin{bmatrix} T_b(2) - T_b(1) \\ (\dots) \\ T_b(k+1) - T_b(k) \end{bmatrix}; \phi = \Delta t \begin{bmatrix} \dot{q}_{solar}(1) & \dot{q}_{occ}(1) + \dot{q}_{ac}(1) - \dot{q}_{vent}(1) & T_{amb}(1) - T_b(1) \\ (\dots) & (\dots) & (\dots) \\ \dot{q}_{solar}(k) & \dot{q}_{occ}(k) + \dot{q}_{ac}(k) - \dot{q}_{vent}(k) & T_{amb}(k) - T_b(k) \end{bmatrix}; x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Plugging this back into our minimization problem, we obtain:

$$\begin{aligned} &\min_{a_1,a_2,a_3} E^\intercal E \Leftrightarrow \min_x (Y - \phi x)^\intercal (Y - \phi x) \Leftrightarrow \\ &\min_x Y^\intercal Y + x^\intercal \phi^\intercal \phi x - x^\intercal \phi^\intercal Y - Y^\intercal \phi x \Leftrightarrow \\ &\min_x Y^\intercal Y + \frac{1}{2} x^\intercal 2 \phi^\intercal \phi x - 2 Y^\intercal \phi x \Leftrightarrow \\ &\min_x d + \frac{1}{2} x^\intercal H x + c^\intercal x \end{aligned} \tag{6}$$

Note: Because $x \in \mathbb{R}^{1\times 3}$, $\phi \in \mathbb{R}^{3\times K}$, $Y \in \mathbb{R}^{1\times K}$, the product $(x^\intercal \phi^\intercal Y)$ yields a scalar expression (denoted as N below). Therefore, the following applies:

$$x^\intercal \phi^\intercal Y = N \Leftrightarrow (x^\intercal \phi^\intercal Y)^\intercal = N^\intercal \Leftrightarrow Y^\intercal \phi x = N \Longrightarrow x^\intercal \phi^\intercal Y = Y^\intercal \phi x$$

In this form, it's easy to see this as an unconstrained optimization problem Where:

$$d = Y^\intercal Y = \begin{bmatrix} T_b(2) - T_b(1) \\ (\dots) \\ T_b(k+1) - T_b(k) \end{bmatrix}^\intercal \begin{bmatrix} T_b(2) - T_b(1) \\ (\dots) \\ T_b(k+1) - T_b(k) \end{bmatrix} \tag{7}$$

$$c = -2Y^{\intercal}\phi = \begin{bmatrix} T_b(2) - T_b(1) \\ (\dots) \\ T_b(k+1) - T_b(k) \end{bmatrix}^{\intercal} \begin{bmatrix} \dot{q}_{solar}(1) & \dot{q}_{occ}(1) + \dot{q}_{ac}(1) - \dot{q}_{vent}(1) & T_{amb}(1) - T_b(1) \\ (\dots) & (\dots) & (\dots) \\ \dot{q}_{solar}(k) & \dot{q}_{occ}(k) + \dot{q}_{ac}(k) - \dot{q}_{vent}(k) & T_{amb}(k) - T_b(k) \end{bmatrix} \tag{8}$$

$$H = 2\phi^{\intercal}\phi \qquad\qquad x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{9}$$

Further, since d is constant, it plays no role in the minimization problem, merely shifting the solution up or down. Therefore, we obtain, finally, the Unconstrained Quadratic Programming Minimization problem:

$$\min_x \frac{1}{2}x^{\intercal}Hx + c^{\intercal}x \Leftrightarrow \min_{a_1,a_2,a_3} \frac{1}{2} \begin{bmatrix} a1 & a2 & a3 \end{bmatrix} 2\phi^{\intercal}\phi \begin{bmatrix} a1 \\ a2 \\ a3 \end{bmatrix} - 2Y^{\intercal}\phi \begin{bmatrix} a1 \\ a2 \\ a3 \end{bmatrix} \tag{10}$$

3 Task 4

The minimization problem is presented as:

$$\min_{\substack{T_b(2),(\dots),T_b(N+1),\\ \dot{q}_{ac}(1),(\dots),\dot{q}_{ac}(N)}} \quad \sum_{k=1}^{N} \left(\Phi(k)\dot{q}_{ac}(k)\Delta t + \frac{E_2+1}{10}(T_b(k) - T_{ref})^2 \right) \tag{11}$$

$$J = \sum_{k=1}^{N} \left(\Phi(k)\dot{q}_{ac}(k)\Delta t + \frac{E_2+1}{10}(T_b(k) - T_{ref})^2 \right) \tag{12}$$

One starts by noticing that (12) isn't a function of the variables we are trying to minimize. To get an expression in the correct terms, we need to replace the term $T_b(k)$ with $T_b(k+1)$. We can do this by using the relationship 1 found in section (1), and reproduced here:

$$\begin{aligned} &T_b(k+1) = AT_b(k) + B \begin{bmatrix} \dot{q}_{solar}(k) & \dot{q}_{occ}(k) & \dot{q}_{ac}(k) & \dot{q}_{vent}(k) & T_{amb}(k) \end{bmatrix}^{\intercal} \Leftrightarrow \\ &T_b(k) = \frac{1}{1 - a_3\Delta t} \left(T_b(k+1) - \Delta t(a_1\dot{q}_{solar}(k) + a_2\dot{q}_{occ}(k) + a_2\dot{q}_{ac}(k) - a_2\dot{q}_{vent}(k) + a_3T_{amb}(k)) \right) \end{aligned} \tag{13}$$

Replacing the rearranged expression for $T_b(k)$, found in (13), into (12), we get the following result:

$$\begin{aligned} J(k) = \Phi(k)\dot{q}_{ac}(k)\Delta t + \frac{E_2+1}{10(1-a_3\Delta t)^2}(T_b(k+1) - \Delta t(a_1\dot{q}_{solar}(k) + \\ a_2\dot{q}_{occ}(k) + a_2\dot{q}_{ac}(k) - a_2\dot{q}_{vent}(k) + a_3T_{amb}(k)) - (1-a_3\Delta t)T_{ref})^2 \end{aligned} \tag{14}$$

This is a rather unwieldy expression. Let's define the following auxiliary variables:

$$D = \frac{E_2 + 1}{10(1 - a_3\Delta t)^2}$$

$$F(k) = -(1 - a_3\Delta t)T_{ref} - \Delta t(a_1\dot{q}_{solar}(k) + a_2\dot{q}_{occ}(k) - a_2\dot{q}_{vent}(k) + a_3T_{amb}(k))$$

$$G(k) = \Phi(k) - 2DF(k)a_2)\Delta t$$

Replacing these new variables in 14, we get a nicer expression we can keep working with:

$$\begin{gathered} J(k) = \Phi(k)\dot{q}_{ac}(k)\Delta t + D(T_b(k+1) - \Delta t a_2\dot{q}_{ac}(k) + F(k))^2 = \\ G(k)\dot{q}_{ac}(k) + 2DF(k)T_b(k+1) + D(T_b(k+1) - \Delta t\dot{q}_{ac}(k))^2 + F(k)^2 \end{gathered} \tag{15}$$

The result is now a function of the correct variables, and we can define our optimization vector:

$$x = \begin{bmatrix} T_b(2) & (\dots) & T_b(N+1) & \dot{q}_{ac}(1) & (\dots) & \dot{q}_{ac}(N) \end{bmatrix}^\intercal \tag{16}$$

Further, we can identify this as a Quadratic Optimization problem. To make this clear, we can start by getting rid of the constant $F(k)^2$ because, similarly to our last task, it'll only work to move our function up or down, having no impact in the solution itself. As for the other terms in the expression, we can look at the as two different parts.

Start by re-writing the linear components of our expression:

$$\begin{gathered} J_{lin} = \sum_{k=1}^{N} ((G(k)\dot{q}_{ac}(k) + 2DF(k)T_b(k+1)) \Leftrightarrow \\ J_{lin} = \begin{bmatrix} 2DF(1) & (\dots) & 2DF(N) & G(1) & (\dots) & G(N) \end{bmatrix} x \end{gathered} \tag{17}$$

As for the quadratic components, one obtains:

$$\begin{gathered} J_{quad} = D(T_b(k+1) - \Delta t\dot{q}_{ac}(k))^2 \Leftrightarrow \\ J_{quad} = x^\intercal D \left[\begin{array}{c|c} I_N & -\Delta t a_2 I_n \\ \hline -\Delta t a_2 I_n & \Delta t^2 a_2^2 I_N \end{array}\right] x \end{gathered} \tag{18}$$

We can use the two matrices found above to find the matrices c and H of the standard form of the optimization problem:

$$c = \begin{bmatrix} 2DF(1) & (\dots) & 2DF(N) & G(1) & (\dots) & G(N) \end{bmatrix}^\intercal, H = 2D \left[\begin{array}{c|c} I_N & -\Delta t a_2 I_n \\ \hline -\Delta t a_2 I_n & \Delta t^2 a_2^2 I_N \end{array}\right] \tag{19}$$

This way, we have re-wrote our problem into the standard form for Quadratic Optimization:

$$\min_x \frac{1}{2} x^\intercal H x + c^\intercal x \tag{20}$$

We must now identify the constraints on our problem.

3.1 Equality Constraints

Our equality constraints are imposed on the problem by the dynamic model of the system:

$$T_b(k+1) = AT_b(k) + B\begin{bmatrix}\dot{q}_{solar}(k) & \dot{q}_{occ}(k) & \dot{q}_{ac}(k) & \dot{q}_{vent}(k) & T_{amb}(k)\end{bmatrix}^\intercal \tag{21}$$

Further, the problem imposes $T_b(1) = 22.43°C$. All in all, these can be re-written as:

$$\begin{aligned}
&x(k+1) = Ax_k + B\begin{bmatrix}\dot{q}_{solar}(k) & \dot{q}_{occ}(k) & x_{N+k} & \dot{q}_{vent}(k) & T_{amb}(k)\end{bmatrix}^\intercal \Leftrightarrow \\
&x(k+1) + (a_3\Delta t - 1)x(k) - \Delta t a_2 x(N+k) = \\
&\qquad\qquad\qquad \Delta t\left(a_1\dot{q}_{solar}(k) + a_2(\dot{q}_{occ}(k) - \dot{q}_{vent}(k)) + a_3T_{amb}(k)\right)
\end{aligned} \tag{22}$$

These constraints can be organized in matrix form, as $A_{eq}x = b_{eq}$ with $A_{eq} \in \mathbb{R}^{N\times 2N}$, $b_{eq} \in \mathbb{R}^{N\times 1}$.

$$A_{eq} = \begin{bmatrix}
1 & 0 & (\dots) & 0 & 0 & a_2\Delta t & 0 & (\dots) & 0 \\
(a_3\Delta t - 1) & 1 & (\dots) & 0 & 0 & 0 & a_2\Delta t & (\dots) & 0 \\
(\dots) & (\dots) & (\dots) & (\dots) & (\dots) & (\dots) & (\dots) & (\dots) & (\dots) \\
0 & 0 & (\dots) & (a_3\Delta t - 1) & 1 & 0 & 0 & (\dots) & a_2\Delta t
\end{bmatrix}$$

$$b_{eq} = \begin{bmatrix}
\Delta t(a_1\dot{q}_{solar}(1) + a_2(\dot{q}_{occ}(1) - \dot{q}_{vent}(1)) + a_3T_{amb}(1)) + (1 - a_3\Delta t)T_b(1) \\
\Delta t(a_1\dot{q}_{solar}(2) + a_2(\dot{q}_{occ}(2) - \dot{q}_{vent}(2)) + a_3T_{amb}(2)) \\
\Delta t(a_1\dot{q}_{solar}(3) + a_2(\dot{q}_{occ}(3) - \dot{q}_{vent}(3)) + a_3T_{amb}(3)) \\
(\dots) \\
\Delta t(a_1\dot{q}_{solar}(N) + a_2(\dot{q}_{occ}(N) - \dot{q}_{vent}(N)) + a_3T_{amb}(N))
\end{bmatrix}$$

3.2 Inequality Constraints

At every point, we're subject to:

$$0 \leq \dot{q}_{ac}(k) \leq \dot{q}_{ac,max} \Leftrightarrow \begin{cases}\dot{q}_{ac}(k) \leq \dot{q}_{ac,max} \\ \dot{q}_{ac}(k) \geq 0\end{cases} \tag{23}$$

$$T_{min} \leq T_b(k) \leq T_{max} \Leftrightarrow \begin{cases}T_b(k) \leq T_{max} \\ -T_b(k) \leq -T_{min}\end{cases} \tag{24}$$

Similarly to our equality constraints, we can write these equations in matrix form, as $Ax = b$ with $A \in \mathbb{R}^{3N\times 2N}$, $b \in \mathbb{R}^{3N\times 1}$.

$$A = \begin{bmatrix} I_N & 0_N \\ -I_N & 0_N \\ 0_N & I_N \end{bmatrix} \qquad b = \begin{bmatrix} T_{max} \\ -T_{min} \\ \dot{q}_{ac,max} \end{bmatrix}$$

The constrained Quadratic Programming Problem is now fully defined can now be solved using *Matlab*.