Solutions for Week 11

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1.1

Because this problem has "self-reducibility" property, we can solve it using FPRAS. Suppose M_0,M_1,...M_n is the satisfying assignments that when we fix the first bit equal to either 0 or 1. Without loss of generality, we assume that we fix them all to 0 (If the fraction of 1 for each bit is larger than $\frac{1}{2}$, we set it to 1 for that bit). Then we can use equations as follows

$$\#\phi = M_0 = \frac{M_0}{M_1}\frac{M_1}{M_2}...\frac{M_{n-1}}{M_n}M_n$$

Obviously, $M_n = 1$. Suppose that $q_i = \frac{M_i}{M_{i-1}} > \frac{1}{2}$, we can get

$$\#\phi = M_0 = \frac{1}{q_1}\frac{1}{q_2}...\frac{1}{q_n}$$

As we can see, the N_0 in the problem is

$$N_0 = \frac{1}{q_2}...\frac{1}{q_n}$$

and p in the problem is q_1. So we need to get N_0 recursively. And get $p = q_1$ using approximate counting.
With Chernoff Bound, to $(\epsilon/n, \delta/n)$ approximate counting, we should set sample size m

$$m = \frac{6n^2 \ln{(2n/\delta)}}{\epsilon^2}$$

Here we use the assumption that $\mu \geq \frac{1}{2}$.

The algorithm to compute each q_i is as follows.

Using each q_i and equations above, we can get approximate counting for number of $\#\phi$.

1.2

We can efficiently generate uniform samples as follows: Suppose we have $\phi_k(x) = (X_1, X_2, X_3, ..., X_k = x, ..., X_n)$, which the k bit is fixed by x. For each bit of this uniform sample, we can determine it using equation

$$X_k = \begin{cases} 1 & \text{w.p.} \frac{\#\phi_k(1)}{\#\phi} \\ 0 & \text{w,p.} \frac{\#\phi_k(0)}{\#\phi} \end{cases}$$

which $\phi_k(1)$ and $\phi_k(0)$ can be computed by black box C.

Algorithm 1 FPRAS for Computing Ratio q_i with $(\delta/n, \epsilon/n)$

Require: Satisfying assignments fixed the first $i-1$ bits to 0
Ensure: The Ratio q_i

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m = 6n^2 ln 2/δ / ε^2
C_i = 0
C_{i-1} = 0
for i = 1 to m do
    Sample x from satisfying assignments fixed the first i-1 bits to 0
    C_{i-1} = C_{i-1} + 1
    if the i-th bit of x is also 0 then
        C_i = C_i + 1
    end if
end for
q_i = C_i / C_{i-1}
return q_i
```

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2.1

Proof: When we fix $Z = z$, then $\mathbb{E}[X|Y, Z]$ is a random variable on Y.

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[X|Y, Z]|Z = z] &= \sum_y \mathbb{E}[X|Y = y, Z = z]Pr[Y = y|Z = z] \\
&= \sum_y \sum_x xPr[X = x|Y = y, Z = z]Pr[Y = y|Z = z] \\
&= \sum_y \sum_x x\frac{Pr[X = x, Y = y, Z = z]}{Pr[Y = y, Z = z]} \frac{Pr[Y = y, Z = z]}{Pr[Z = z]} \\
&= \sum_y \sum_x x\frac{Pr[X = x, Y = y, Z = z]}{Pr[Z = z]} \\
&= \sum_x x\frac{Pr[X = x, Z = z]}{Pr[Z = z]} \\
&= \sum_x xPr[X = x|Z = z] \\
&= \mathbb{E}[X|Z = z]
\end{aligned}$$

Since $\mathbb{E}[[X|Y, Z]|Z]$ is random variable on Y,

$$\begin{aligned}
\mathbb{E}[[X|Y, Z]|Z] &= \sum_z \mathbb{E}[\mathbb{E}[X|Y, Z]|Z = z] \\
&= \sum_z \mathbb{E}[X|Z = z] \\
&= \mathbb{E}[X|Z]
\end{aligned}$$

Q.E.D.

2.2

We prove these lemma in turn. Proof: When y is a constant,

$$\begin{aligned}\mathbb{E}[XY|Y=y] &= \mathbb{E}[yX|Y=y]\\ &= y\mathbb{E}[X|Y=y]\end{aligned}$$

From the definiton of condational expectation, we egt

$$\mathbb{E}[XY|Y] = Y\mathbb{E}[X|Y]$$

Take expecation on both sides,

$$\begin{aligned}\mathbb{E}[XY] &= \mathbb{E}[\mathbb{E}[XY|Y]]\\ &= \mathbb{E}[Y\mathbb{E}[X|Y]]\end{aligned}$$

Q.E.D.

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3.1

Let X_i indicate whether vectex i is a isolated vertex. Thus $X = \sum_{i=1}^{n} X_i$. For each edge, there are $\binom{n}{2}$ options. For each vertex, there are at most $(n-1)$ degree. So

$$Pr[deg(v_i) = 0] = (1 - \frac{\binom{n}{2} - (n-1)}{\binom{n}{2}})^N = (1 - \frac{n-2}{n})^{cn}$$

With linearity of expectation,

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[\sum_{i=1}^{n} X_i]\\ &= \sum_{i=1}^{n}\mathbb{E}[X_i]\\ &= n(1-\frac{n-2}{n})^{cn}\\ &= n(\frac{2}{n})^{cn}\end{aligned}$$

3.2

As we can see, X is a function of edges $E_1, E_2, ...E_{cn}$ Now we define doob edge explosure martingale, for every $i \in [0, cn]$

$$Z_i = \mathbb{E}[X|E_1, ..., E_i]$$

Obviously, $Z_0 = \mathbb{E}[X]$ and $Z_{cn} = X$. Now we prove that $Z_{i=0}^{n}$ is a martingale w.r.t (X_i).

$$\begin{aligned}
\mathbb{E}[Z_i|E_1, .., E_{i-1}] &= \mathbb{E}[\mathbb{E}[X|E_1, ..., E_i]|E_1, ..., E_{i-1}] \\
&= \mathbb{E}[X|E_1, ..., E_{i-1}] \\
&= Z_{i-1}
\end{aligned}$$

The first equality is the definition of Z_i, the second from the fact in section 2.1, and the last from the definition of Z_{i-1}. Now we can use Azuma to this doob martingale. For each E_i, it will decrease the isolated vertices at most 2 or increase isolated vertices at most 2, which means

$$|Z_i - Z_{i-1}| \leq 2$$

With Azume-Hoeffding's inequality,

$$\begin{aligned}
Pr[|Z_n - Z_0| \geq 2\lambda\sqrt{cn}] &\leq 2e^{-\frac{(2\lambda\sqrt{cn})^2}{2cn*2^2}} \\
&= 2e^{-\frac{4\lambda^2 cn}{8cn}} \\
&= 2e^{-\frac{\lambda^2}{2}}
\end{aligned}$$

Q.E.D.