RIP suffices for LP decoding

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Overview

Suppose $Ax^* = Ax$ and $\|x^*\|_1$ is minimal. Let $h = x^* - x$.

- Reorder the coordinates so that (i) $x_1 \dots x_k$ are the k heaviest elements of x, and (ii) $|h_{k+1}| \geq \dots \geq |h_n|$
- $T_0 = \{1 \dots k\}$, $T_j = \{(j-1)M + k + 1 \dots jM + k\}$, $T_{01} = T_0 \cup T_1$, for $M = 5k$
- $\eta = k^{-1/2} \|x_{k+1\dots n}\|_1$.

Three ingredients:

1. $\left\|h_{T_0^c}\right\|_1 \leq \|h_{T_0}\|_1 + O(\sqrt{k}\eta)$
2. $\left\|h_{T_{01}^c}\right\|_2 \leq \left\|h_{T_0^c}\right\|_1 / \sqrt{M} = O\left(\|h_{T_0}\|_2 + \eta\right)$
3. $\|h_{T_{01}}\|_2 \leq O(\eta)$, and therefore $\|h\|_2 \leq O(\eta)$

ℓ^1 Concentration

Lemma 1: x feasible and $\|x^*\|_1$ is minimal implies $\left\|h_{T_0^c}\right\|_1 \leq \|h_{T_0}\|_1 + O(\sqrt{k}\eta)$.

Proof: Observe that $\|x\|_1 = \|x_{T_0}\|_1 + \left\|x_{T_0^c}\right\|_1$ and

$$
\begin{aligned}
\|x\|_1 &\geq \|x^*\|_1 \\
&= \|x + h\|_1 \\
&= \|x_{T_0} + h_{T_0}\|_1 + \left\|x_{T_0^c} + h_{T_0^c}\right\|_1 \\
&\geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 - \left\|x_{T_0^c}\right\|_1 + \left\|h_{T_0^c}\right\|_1
\end{aligned}
$$

so $\left\|h_{T_0^c}\right\|_1 \leq \|h_{T_0}\|_1 + 2\left\|x_{T_0^c}\right\|_1 \leq 2\sqrt{k}\eta$.

Bounding the Tail

Lemma 2: $\left\|h^{c}_{T_{01}}\right\|_2 \leq O\left(\left\|h_{T_{01}}\right\|_2 + \eta\right)$.

Proof: We have $|(h_{T_0^c})_j| \leq \left\|h_{T_0^c}\right\|_1 / j$. Thus

$$\left\|h_{T_{01}^c}\right\|_2^2 \leq \left\|h_{T_0^c}\right\|_1^2 \sum_{j=M+1}^{d} \frac{1}{j^2} \leq \left\|h_{T_0^c}\right\|_1^2 / M.$$

Bounding the Head

Lemma 3: $\|h_{T_{01}}\|_2 \leq O(\eta)$.

Proof:

$$\begin{aligned} 0 &= \|A(x^* - x)\|_2 = \|Ah\|_2 \\ &\geq \|Ah_{T_{01}}\|_2 - \left\| \sum_{j \geq 2} Ah_{T_j} \right\|_2 \\ &\geq \|Ah_{T_{01}}\|_2 - \sum_{j \geq 2} \|Ah_{T_j}\|_2 \\ &\approx \|h_{T_{01}}\|_2 - \sum_{j \geq 2} \|h_{T_j}\|_2 . \end{aligned}$$

Need to bound $\sum_{j \geq 2} \|h_{T_j}\|_2$ above.

Bounding the Head

Each term in T_{j+1} is smaller than the average term in T_j, i.e., $|h_{T_{j+1}}| \leq \|h_{T_j}\|_1 / M$, so $\|h_{T_{j+1}}\|_2^2 \leq M \|h_{T_j}\|_1^2 / M^2 = \|h_{T_j}\|_1^2 / M$. Thus

$$
\begin{aligned}
\sum_{j \geq 2} \|h_{T_j}\|_2 &\leq \sum_{j \geq 1} \|h_{T_j}\|_1 / \sqrt{M} \\
&= \|h_{T_0^c}\|_1 / \sqrt{M} \\
&\leq \|h_{T_0}\|_1 / \sqrt{M} + O(\eta) \\
&\leq \sqrt{k/M}(\|h_{T_0}\|_2 + O(\eta)) \\
&\leq \sqrt{k/M}(\|h_{T_{01}}\|_2 + O(\eta)).
\end{aligned}
$$

Thus $\|h_{T_{01}}\|_2 \leq \sqrt{k/M}(\|h_{T_{01}}\|_2 + \eta) \leq (1/2)(\|h_{T_{01}}\|_2 + \eta)$, so

$$\|h_{T_{01}}\|_2 \leq O(\eta).$$