CS5330: Optional problems on Martingales

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I strongly encourage you to think about these problems and to discuss with me, your TA, or your peers if you need help. I will post solution hints in a few days.

1. (Exercise 13.3 of MU) Let $X_0 = 0$ and for $j \geq 0$, let X_{j+1} be chosen uniformly over the real interval $[X_j, 1]$. Show that for $k \geq 0$, the sequence:

   $$Y_k = 2^k (1 - X_k)$$

   is a martingale.

2. Suppose $X_0, X_1, \ldots, X_n$ are independent, identically distributed, real-valued random variables with mean 0 and variance σ^2. For any $i \geq 0$, let $S_i = \sum_{j=0}^{i} X_j$, and let $M_i = S_i^2 - i\sigma^2$. Show that $M_0, \ldots, M_n$ form a martingale with respect to $X_0, \ldots, X_n$.

3. Let G be a random d-regular graph on n vertices, that is, a graph drawn uniformly from the family of all n-vertex graph in which each vertex has exactly d neighbors. Color the vertices of G red or blue independently at random.

   (a) What is the expected number of monochromatic edges in G?

   (b) Show that the actual number of monochromatic edges is tightly concentrated around its expectation.

4. Consider an n by n square grid $\{0, 1, \ldots, n\}^2$, where each point is connected to each of its (at most) four neighbors. Within each inner square of the grid, we draw a diagonal from bottom left to the top right with probability 1/2.

   Let Z be the length of the shortest path from $(0,0)$ to (n,n). Show that:

   $$\Pr[|Z - \mathbb{E}[Z]| \geq t] \leq \exp(-\Omega(t^2/n))$$

Hint: Consider Z as a function of $X_1, \ldots, X_n$ where X_i denotes the state of the i'th column.

5. Consider a random graph $G_{n,p}$ on n vertices where each edge appears with probability p independently. Let C denote the size of the largest clique in $G_{n,p}$. Show that:

$$\Pr[|C - \mathbb{E}[C]| \geq t] \leq 2e^{-2t^2/n}.$$

6. (Exercise 13.16 of MU) Let x and y be two bit strings of length n, where each character is independently 0 with probability 1/2 and 1 with probability 1/2. Show that the length of the longest common subsequence is highly concentrated around its mean.

7. (Exercise 13.9 of MU) Consider an n-cube with $N = 2^n$ nodes. Let S be a nonempty set of vertices on the cube, and let x be a vertex chosen uniformly at random among all vertices of the cube. Let $D(x, S)$ be the minimum number of coordinates in which x and y differ over all points $y \in S$. Give a bound on:

$$\Pr[|D(x,S) - \mathbb{E}[D(x,S)]| > \lambda]$$