CS5330: Optional problems on Martingales

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Here are solution sketches to the optional problems on martingales. If anything is unclear, please talk to me or your TA.

1. (Exercise 13.3 of MU) Let $X_0 = 0$ and for $j \geq 0$, let X_{j+1} be chosen uniformly over the real interval $[X_j, 1]$. Show that for $k \geq 0$, the sequence:

$$Y_k = 2^k(1 - X_k)$$

is a martingale.

$$\begin{aligned}
\mathbb{E}[Y_{j+1} \mid X_1, \ldots, X_j] &= \mathbb{E}[2^{j+1}(1 - X_{j+1}) \mid X_1, \ldots, X_j] \\
&= 2^{j+1}(1 - \mathbb{E}[X_{j+1} \mid X_1, \ldots, X_j]) \\
&= 2^{j+1}\left(1 - \frac{1 + X_j}{2}\right) \\
&= 2^j(1 - X_j) = Y_j
\end{aligned}$$

2. Suppose $X_0, X_1, \ldots, X_n$ are independent, identically distributed, real-valued random variables with mean 0 and variance σ^2. For any $i \geq 0$, let $S_i = \sum_{j=0}^{i} X_j$, and let $M_i = S_i^2 - i\sigma^2$. Show that $M_0, \ldots, M_n$ form a martingale with respect to $X_0, \ldots, X_n$.

$$\begin{aligned}
\mathbb{E}[M_{j+1} \mid X_0, \ldots, X_j] &= \mathbb{E}[S_{j+1}^2 - (j+1)\sigma^2 \mid X_0, \ldots, X_j] \\
&= \mathbb{E}[(S_j + X_{j+1})^2 \mid X_0, \ldots, X_j] - (j+1)\sigma^2 \\
&= S_j^2 + 2S_j\,\mathbb{E}[X_{j+1}] + \mathbb{E}[X_{j+1}^2] - (j+1)\sigma^2 \\
&= S_j^2 - j\sigma^2 = M_j
\end{aligned}$$

3. Let G be a random d-regular graph on n vertices, that is, a graph drawn uniformly from the family of all n-vertex graph in which each vertex has exactly d neighbors. Color the vertices of G red or blue independently at random.

   (a) What is the expected number of monochromatic edges in G?

   Each edge has probability 1/2 of being monochromatic. So, the expected number of monochromatic edges is $(dn/2)/2 = dn/4$.

   (b) Show that the actual number of monochromatic edges is tightly concentrated around its expectation.

   Fix any 2-coloring. Order the vertices $1, \dots, n$ arbitrarily. Let X_i be the edges between i and nodes in $\{1, \dots, i-1\}$. The number N of monochromatic edges is clearly a function f of $X_1, \dots, X_n$. We claim that f is d-Lipschitz. This is simply because there can be at most d additional monochromatic edges depending on where the edges incident to a particular vertex are placed. Hence, applying McDiarmid and part (a), we get that

$$\Pr[|N - dn/4| > \varepsilon dn] \le 2e^{-2(\varepsilon dn)^2/(d^2 n)} = 2e^{-2\varepsilon^2 n}.$$

4. Consider an n by n square grid $\{0, 1, \dots, n\}^2$, where each point is connected to each of its (at most) four neighbors. Within each inner square of the grid, we draw a diagonal from bottom left to the top right with probability 1/2.

   Let Z be the length of the shortest path from $(0,0)$ to (n,n). Show that:

$$\Pr[|Z - \mathbb{E}[Z]| \ge t] \le \exp(-\Omega(t^2/n))$$

   Hint: Consider Z as a function of $X_1, \dots, X_n$ where X_i denotes the state of the i'th column.

   As hinted, let $Z = f(X_1, \dots, X_n)$ where X_i is a vector that describes whether each cell of the i'th column has the diagonal present or not. Note that f is $(2 - \sqrt{2})$-Lipschitz because the shortest path will only cross any particular column once, and this crossing via a diagonal is of length $\sqrt{2}$ while via the sides of a cell is 2. Hence, applying McDiarmid directly:

$$\Pr[|Z - \mathbb{E}[Z]| \ge t] \le 2\exp(-2t^2/((2 - \sqrt{2})^2 n)).$$

5. Consider a random graph $G_{n,p}$ on n vertices where each edge appears with probability p independently. Let C denote the size of the largest clique in $G_{n,p}$. Show that:

$$\Pr[|C - \mathbb{E}[C]| \ge t] \le 2e^{-2t^2/n}.$$

   As in problem 3, we use the "vertex exposure martingale". Order the vertices 1 through

n arbitrarily, we let $C = C(X_1, \dots, X_n)$ where X_i is the set of edges between i and nodes $\{1, \dots, i-1\}$. The function C is clearly 1-Lipschitz. McDiarmid directly implies the result.

6. (Exercise 13.16 of MU) Let x and y be two bit strings of length n, where each character is independently 0 with probability 1/2 and 1 with probability 1/2. Show that the length of the longest common subsequence is highly concentrated around its mean.

   Write the LCS length L as a function of $x_1, \dots, x_n, y_1, \dots, y_n$. Check that this function is 1-Lipschitz. Applying McDiarmid yields: $\Pr[|L - \mathbb{E}[L]| > t] \le 2\exp(-t^2/n)$.

7. (Exercise 13.9 of MU) Consider an n-cube with $N = 2^n$ nodes. Let S be a nonempty set of vertices on the cube, and let x be a vertex chosen uniformly at random among all vertices of the cube. Let $D(x, S)$ be the minimum number of coordinates in which x and y differ over all points $y \in S$. Give a bound on:

$$\Pr[|D(x,S) - \mathbb{E}[D(x,S)]| > \lambda]$$

   Again very similar to the previous problems. S is fixed, so $D(x, S)$ is a function of the n independent coordinates of x. The function is 1-Lipschitz. So, the probability bound is $2e^{-2\lambda^2/n}$.