CS5330: Assignment for Week 2

Due: Tuesday, 5th Feb 2020.

Please email your solutions to `arnabb@nus.edu.sg` by 5th February, 6:29 pm. I strongly encourage you to write your solutions using LaTeX.

You may discuss the problems with your classmates, though you should write up your solutions on your own. Please note the names of your collaborators in your submission.

1. Recall the definition of a treap in Lecture 1, and our observation that once the priorities are assigned to the keys, the structure of the treap is the same no matter what the insertion order is.

   Suppose n keys $x_1, \dots, x_n$ are inserted into a treap with random priorities (assumed to be distinct). By the above, without loss of generality $x_1 < x_2 < \cdots < x_n$. Let D_i be the depth of x_i in the tree. In other words, D_i is the number of ancestors x_i has in the tree. Let D_{ij} be the indicator variable that x_j is an ancestor of i in the tree.

   Suppose $j > i$ for parts (a) through (c).

   (a) What is D_{ij} if x_i has the highest priority among $\{x_i, \dots, x_j\}$?

   (b) What is D_{ij} if for some $k \in \{i+1, \dots, j-1\}$ has the highest priority among $\{x_i, \dots, x_j\}$?

   (c) What is D_{ij} if x_j has the highest priority among $\{x_i, \dots, x_j\}$?

   (d) Compute $\mathbb{E}[D_i] = \sum_{j \neq i} \mathbb{E}[D_{ij}]$, taking care to consider both $j < i$ and $j > i$.

2. n people queue up to attend a movie which has n seats. However, the first person has lost his ticket and sits in one of the empty seats uniformly at random. Subsequently, each person (and no one else has lost a ticket) sits either in his assigned seat, or if that seat is already taken sits in an empty seat uniformly at random. What is the expected number of people *not* sitting in their correct seats?

3. Given a permutation π of $\{1, 2, \dots, n\}$, let $L(\pi)$ denote the length of the longest increasing subsequence in π. Note that a subsequence may not be contiguous; for instance in the permutation $\pi = (1, 6, 4, 5, 2, 7, 3)$ for $n = 7$, the longest subsequence is $(1, 4, 5, 7)$ and so $L(\pi) = 4$. In this exercise, you need to prove $\mathbf{E}[L(\pi)] = \Theta(\sqrt{n})$ where the expectation is over a random permutation of $\{1, \dots, n\}$.

(a) Prove that $\mathbf{E}[L(\pi)] = O(\sqrt{n})$. **Hint:** Use the following fact: for any non-negative integer random variable Z, $\mathbb{E}[Z] = \sum_{z \geq 0} \Pr(Z \geq z)$. Now, for a fixed k, calculate an upper bound on the probability that $L(\pi) \geq k$.

(b) Prove that $\mathbf{E}[L(\pi)] = \Omega(\sqrt{n})$. **Hint:** Assume n is a perfect square. For $i = 1, \ldots, \sqrt{n}$, define the indicator random variable X_i which takes the value 1 if and only if some entry in $(i-1)\sqrt{n} + 1 \leq j \leq i\sqrt{n}$ satisfies $(i-1)\sqrt{n} + 1 \leq \pi(j) \leq i\sqrt{n}$. Can you relate these $\sqrt{n}$ variables with $L(\pi)$?