Solutions for Week 5

Bao Jinge (E0522065)

1

Supposing we have m vectors in set of ϵ-orthogonal unit vectors. When $\epsilon \geq 1$, all vectors in V will be included in set of ϵ-orthogonal unit vectors and $m = 2^d$.
When $\epsilon = 0$, we could get $m = d$ or $m = 0$ when d is even or odd.
When $\epsilon < 0$, obviously, $m = 0$.
Now we discuss the situation when $0 < \epsilon < 1$. Suppose we fix a vector u and then select a vector v randomly in set V, whose coordinates are u_k and v_k, which $k \in [d]$.
We have

$$\mathbb{E}[u_k v_k] = u_k \mathbb{E}[v_k] = 0$$

Let X_{ij} denotes the inner product of v^i and v^j, when we fix v^i. So

$$-\frac{v_k^i}{\sqrt{d}} \leq v_k^j \leq \frac{v_k^i}{\sqrt{d}}$$

Using Hoeffding Bound, we get

$$Pr(|X_{ij}| > \epsilon) = Pr(|\sum_{k=1}^{d} v_k^i v_k^j - 0| > \epsilon) \leq 2e^{-\frac{2\epsilon^2}{\sum_{k=1}^{d}(2\frac{v_i^k}{\sqrt{d}})^2}} = 2e^{-\frac{d\epsilon^2}{2}}$$

Then we use Union bound for all pairs of vector in set of pairwise ϵ-orthogonal unit vectors. Then we got

$$Pr(\sum_{i=1}^{m}\sum_{j=i+1}^{m}(|X_{ij}| > \epsilon)) <= \sum_{i=1}^{m}\sum_{j=i+1}^{m} Pr(|X_{ij}| > \epsilon) <= \binom{m}{2} 2e^{-\frac{d\epsilon^2}{2}}$$

Let

$$\binom{m}{2} 2e^{-\frac{d\epsilon^2}{2}} \leq m^2 e^{-\frac{d\epsilon^2}{2}} \leq \frac{1}{m^c}$$

which $c > 0$. We got

$$m \leq e^{\frac{d\epsilon^2}{2(c+2)}}$$

So the cardinality of ϵ-orthogonal set i can construct is

$$m = \begin{cases} 0 & \epsilon < 0 \\ 0 & \epsilon = 0, \text{d is odd} \\ d & \epsilon = 0, \text{d is even} \\ e^{\frac{d\epsilon^2}{2(c+2)}} & 0 < \epsilon < 1 \\ 2^d & \epsilon \geq 1 \end{cases}$$

2

3

Suppose the running time of randomized quicksort is T. Without loss of generality, we focus on an element z.
With randomized quicksort can be viewed as a recursion tree, we could know that $T \le hn$ if the depth of recursion tree is h.
Focus on one element z. Let S denotes each node in recursion tree. As for randomization, we define balanced partition that partition step divides S such that

$$\frac{|S|}{4} \le |S_l|$$

and

$$|S_r| \le \frac{3|S|}{4}$$

At root of the tree, $|S_{root}| = n$.
Here we will prove that among the $c \ln n$ partitioning steps, at least $\frac{c}{4} \ln n$ results in balanced paritions for any element z. For any element z, after $\frac{c}{4} \ln n$ balanced partitions, z will end into a partition with size of $n(\frac{3}{4})^{\frac{c}{4} \ln n}$.
Let size of partition $n(\frac{3}{4})^{\frac{c}{4} \ln n} \le 1$. We could get $c \ge 14$, which means that any element z will end up in its leaf of recursion tree after $\frac{c}{4} \ln n$ balanced partitions when $c \ge 14$.
Let X_i indecates z is in balanced partition in level i, we could easily get

$$Pr(X_i = 1) = \frac{\frac{3k}{4} - \frac{k}{4}}{k} = \frac{1}{2}$$

Because X_i are mutually independent, let $X = \sum_{i=1}^{c \ln n} X_i$ and $\mu = \mathbb{E}[X] = \frac{c \ln n}{2}$. Using Chernoff Bound, we get

$$Pr(X \le \frac{c}{4} \ln n) \le Pr(X - \frac{c}{2} \ln n < -\frac{c}{4} \ln n) \quad \le e^{-\frac{c}{16} lnn} = \frac{1}{n^{\frac{c}{16}}}$$

Let $c = 32$, we have

$$Pr(X \le 8 \ln n) \le \frac{1}{n^2}$$

which means that element z can't reach its final leaf of recursion tree after $32 \ln n$ levels with probability $\le \frac{1}{n^2}$.
Using Union Bound, at least one of n input elements can't reach its final leaf of recursion tree after $32 \ln n$ levels with probability $\le \frac{1}{n}$.
Thus, after $32 \ln n$ levels of recursions, all element will reach its final leaf with probability $\ge 1 - \frac{1}{n}$, namely with high probability
With there are $O(n)$ comparisons for each level. So Randomized Quicksort will terminate after $32n \ln n$ with high probability.
In other way, RQS runs in time $O(n \log n)$ with high probability.