CS5330: Optional problems for Weeks 1 and 2

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I strongly encourage you to think about these problems and to discuss with me, your TA, or your peers if you need help. I will post solution hints in about 2 weeks.

1. (a) Formally prove linearity of expectations for a finite number of discrete random variables. That is, if $X_1, \ldots, X_n$ are discrete random variables with finite expectations, then:

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$

   Don't assume independence!

   (b) (Exercise 2.29 of MU) Show that if $X_1, X_2, \ldots$ is a sequence of discrete random variables such that $\sum_{i=1}^{\infty} \mathbb{E}[|X_i|]$ is finite, then:

$$\mathbb{E}\left[\sum_{i=1}^{\infty} X_i\right] = \sum_{i=1}^{\infty} \mathbb{E}[X_i]$$

   (c) Define the following sequence of random variables. Let X_1 be 1 with probabiity $1/2$ and -1 with probability $1/2$. For $i > 1$, let $X_i = 0$ if $X_{i-1} \geq 0$ and otherwise, let X_i be 2^{i-1} with probability $1/2$ and -2^{i-1} with probability $1/2$. Clearly, for each i, $\mathbb{E}[X_i] = 0$.
   Let $X = \sum_{i=1}^{\infty} X_i$. Show that with probability 1, $X = 1$. Why does this not contradict part (b)?

2. Let $G(n,p)$ be the *Erdős-Rényi random graph with density* p. This is simply the graph on n nodes obtained by adding an edge between nodes i and j with probability p, for every two distinct nodes $i, j \in [n]$.

   (a) What is the expected number of edges in $G(n,p)$?

   (b) What is the expected degree of the first vertex in $G(n,p)$?

   (c) What is the expected number of triangles in $G(n,p)$?

   (d) Let X be the number of subsets $\{v_1, \ldots, v_k\}$ of k vertices such that no vertex u is adjacent to all of them. What is $\mathbb{E}[X]$?

3. (a) Suppose n people go to an event and sit down at a table with chairs labeled 1 through n. The organizers of the event wanted that they should sit in order of their age, i.e., the i'th oldest person should sit in the chair labeled i. But they forgot to give this instruction to the attendees, who sit in a random order. What is the expected number of attendees who happen to sit in the correct seat?

   (b) Suppose n people go to an event and sit down randomly at a round table with n chairs. Each person has their best friend also in the group. What is the expected number of people who have their best friend seated next to them? Note: The 'best friend' relation is not symmetric (A's best friend may not have A as their best friend). Also, many people may have the same person as their best friend.

4. For each of the following, explain your answer:

   (a) Suppose that you roll a fair die that has six sides numbered $1, 2, \dots, 6$. Is the event that the number on top is a multiple of 2 independent of the event that the number of top is a multiple of 3?

   (b) Suppose that you roll a fair die that has four sides numbered $1, 2, 3, 4$. Is the event that the number on top is a multiple of 2 independent of the event that the number of top is a multiple of 3?

5. Recall that Karger's min-cut algorithm we discussed in class that errs with probability 1% runs in time $O(n^4)$ because contracting the graph to 2 vertices requires time $O(n^2)$ (since each individual edge contraction requires $O(n)$ time) while we have to repeat this $O(n^2)$ times to get the error probability down to 1%. In this problem, we see how to do better.

   (a) Suppose instead of contracting to two vertices, we contract to t vertices. In other words, suppose we repeat $n-t$ times the process of randomly choosing an edge and contracting it. If C is a cut-set of size k, argue that the probability that no edge in C is contracted is at least $t(t-1)/n(n-1)$. Set $t = n/\sqrt{2}$, so that this probability is (roughly) $1/2$.

   (b) Now consider the following FastContract algorithm: perform two independent contraction sequences to obtain graphs H_1 and H_2 each with $t = \frac{n}{\sqrt{2}}$ vertices, recursively compute the min-cut in each of H_1 and H_2, and return the smaller of the two min-cuts.

      (i) Let $T(n)$ be the running time of FastContract. Argue that it satisfies the recurrence: $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2)$. Solve the recurrence.

      (ii) Let p_i be the probability that FastContract returns a min-cut on a graph of size $n/(\sqrt{2})^i$. Note that i runs from 0 to $\Theta(\log n)$. Argue that it satisfies the recurrence:

      $$p_i \geq 1 - \left(1 - \frac{1}{2}p_{i-1}\right)^2 = p_{i-1} - \frac{1}{4}p_{i-1}^2.$$

      Show that $\frac{1}{p_i} \leq \frac{1}{p_{i-1}} + 1$ and hence, $\frac{1}{p_0} = O(\log n)$ or $p_0 = \Omega(1/\log n)$.

      (iii) Repeat FastContract an appropriate number of times to design an algorithm for min-cut that runs in time $O(n^2 \log^2 n)$ and fails to return a min-cut with probability at most 1%.

6. Assume that graph $G = (V, E)$ is a weighted graph. Give an algorithm for finding a min-cut in graph G with probability at least $\frac{1}{n(n-1)}$. (Hint: Think about choosing each edge with proportion to its weight.) Prove that your algorithm is correct.

7. Let n be an even integer and assume $n > 10$. Let G_k be the (multi)-graph on n vertices formed by taking the union of k perfect matchings which are chosen uniformly at random from the set of all perfect matchings among n points. What is the smallest k for which G_k is connected with high probability? (Recall, whp implies that this probability should be $1 - 1/n^c$ for a constant c.)

8. Recall the function $T_k : \{0,1\}^n \to \{0,1\}$ defined in class as the value of the root of a tree of depth $2k$ where the $n = 2^{2k}$ leaves are labeled with the input in $\{0,1\}^n$ and the nodes are alternately AND and OR gates at each level. Argue that any deterministic algorithm that computes $T_k(x)$ correctly for all inputs x must read all n bits of x.