Solutions for Week 2

Bao Jinge

1

(a) As the definition of a treap in Lecture 1, which treap perseves max-heap, $D_{ij} = 0$ under conditions.
(b) As the definition of a treap in Lecture 1, which treap perseves max-heap, $D_{ij} = 0$ under conditions.
(c) As the definition of a treap in Lecture 1, which treap perseves max-heap, $D_{ij} = 1$ under conditions.
(d) As we can see, the depth of node i is equal to the number of ancestor of node i. If we define a matrix to descripe the D_{ij}, we can find that the number of ancestors of node i is the sum of i-th row except for i-th element of row i(beacause a node can't be his own ancestor). So

$$E(D_i) = \sum_{j \neq i} E(D_{ij})$$

As what we found in above questions, iff x_j has the highest priority among $x_i, ..., x_j$(when $j > i$) or $x_i, ..., x_j$(when $j < i$) that $D_{ij} = 1$. So

$$\begin{aligned} E(D_i) &= \sum_{j \neq i} E(D_{ij}) \\ &= \sum_{1 \leq j < i} E(D_{ij}) + \sum_{i < j \leq n} E(D_{ij}) \\ &= \sum_{1 \leq j < i} E(D_{ij}) \frac{1}{i - j + 1} + \sum_{i < j \leq n} \frac{1}{j - i + 1} \\ &= \sum_{1 \leq j \leq i} E(D_{ij}) \frac{1}{i - j + 1} + \sum_{i \leq j \leq n} \frac{1}{j - i + 1} - 2 \\ &= H(i) + H(n - i + 1) - 2 \\ &= O(\ln n) \end{aligned} \tag{1}$$

where H(i) is harmonic number that $H(n) = \sum_{k=1}^{n} \frac{1}{k}$. So $E(D_i) = H(i) + H(n - i + 1) - 2$

2

Suppose $Pr(k)$ denotes that person k finds that his seat has been occupied on his turn. Obviously, we can find that when $k = 2$,

$$Pr(k) = \frac{1}{n-1}$$

. When $k > 2$, it might be one of last $k-1$ person occupies his seat. Thus

$$Pr(k) = \frac{1}{n} + \sum_{i=2}^{k-1} Pr(i)\frac{1}{n-i+1} = \frac{1}{n+2-k}$$

Suppose X_i denotes whether i-th person's seat was occupied by others or not. When $X_i = 1$, his seat was occupied by others, otherwise $X_i = 0$. So the expecation of people not sitting on their own seat is

$$E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$$

As above, we know that when $i \geq 2$, $Pr(k) = \frac{1}{n+2-k}$. Here we get

$$E(X_i) = 1 \cdot Pr(i) + 0 \cdot (1 - Pr(i))$$

Thus,

$$E(x) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n-1} 1/i = H(n-1)$$

where H(n) denotes Harmonic number.

3

(a) Let $X_{n,k}$ dontes the number of increasing subsequences of the π that have length of k. Noting that this is equal to the sum, over all $\binom{n}{k}$ subsequences of the length k, of the probability for the subsequence to be increasing, where $1 \leq k \leq n$. We can get

$$E(X_{n,k}) = \frac{1}{k!}\binom{n}{k}$$

Thus

$$Pr(L(\pi) \geq k) = Pr(X_{n,k} \geq 1) \leq E(X_{n,k}) = \frac{1}{k!}\binom{n}{k} \leq \frac{n^k}{(\frac{k}{e})^{2k}}$$

As hint gives, we got

$$E(L(\pi)) = \sum_{k \geq 0} Pr(L(\pi) \geq k) = \sum_{k \geq 0}^{n} Pr(L(\pi) \geq k) \leq \sum_{k \geq 0}^{n} \frac{n^k}{(\frac{k}{e})^{2k}}$$

here we fixing some $\delta > 1$ and taking $k = \lceil \delta e \sqrt{n} \rceil$ we have

$$Pr(L(\pi) \geq k) \leq (\frac{1}{\delta})^{2k} \leq (\frac{1}{\delta})^{2\delta e \sqrt{n}}$$

And then

$$E(L(\pi)) \leq \sum_{k \geq 0}^{n} \frac{n^k}{(\frac{k}{e})^{2k}} \leq \sum_{k \geq 0}^{n} (\frac{1}{\delta})^{2\delta e \sqrt{n}} \leq \delta e \sqrt{n}$$

So $E(L(\pi)) = O(\sqrt{n})$
(b) As hint gives, when n is a perfect square we can find that $[1, 2, 3, ..., n]$ can be divided as $\sqrt{n}$ intervals of length $\sqrt{n}$. By the same way, we can divided a pertutation π into $\sqrt{n}$ continuous parts. Here we have

$$L(\pi) \geq \sum_{i=1}^{\sqrt{n}} X_i$$

$$\begin{aligned} E(L(\pi)) &\geq E(\sum_{i=1}^{\sqrt{n}} X_i) \\ &= \sum_{i=1}^{\sqrt{n}} E(X_i) \\ &= \sum_{i=1}^{\sqrt{n}} \sqrt{n} * \frac{1}{\sqrt{n}} \\ &= \sqrt{n} \end{aligned} \tag{2}$$

When n is not a perfect square, there must be a $n_0 < n$ which is a perfect square and $\lfloor \sqrt{n} \rfloor_0 = \sqrt{n_0}$. And we got

$$\begin{aligned} E(L(\pi)) &\geq E(\sum_{i=1}^{\sqrt{n_0}} X_i) \\ &= \sum_{i=1}^{\sqrt{n_0}} E(X_i) \\ &= \sum_{i=1}^{\sqrt{n_0}} \sqrt{n_0} \cdot \frac{1}{\sqrt{n_0}} \\ &= \sqrt{n_0} \\ &= \lfloor \sqrt{n} \rfloor \end{aligned} \tag{3}$$

So we got $E(L(\pi)) = \Omega(\sqrt{n})$.

To sum up results from (a) and (b), we got $E(L(\pi)) = \Theta(\sqrt{n})$