CS5330: Assignment for Week 10

Due: Tuesday, 7th Apr 2020.

Please submit your solutions to the "Assignments/Week 10/Submissions" folder on LumiNUS by 7th April, 6:29 pm. I strongly encourage you to write your solutions using LaTeX.

You may discuss the problems with your classmates, though you should write up your solutions on your own. Please note the names of your collaborators in your submission.

1. Consider the graph on $\mathbb{Z}_n^d = \{0, 1, \ldots, n-1\}^d$ where vertices $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ are adjacent if for some $j \in [d]$, $x_j - y_j = \pm 1 \mod n$ but for all $i \neq j$, $x_i = y_i$. The lazy random walk on $\mathbb{Z}_n^d$ behaves as follows: if the walk is currently at node x, then with probability 1/2, it stays at x; otherwise, it picks $i \in [d]$ uniformly at random and conditioned on that choice of i, it moves to[1] $(x + e^i) \mod n$ with probability 1/2 and $(x - e^i) \mod n$ with probability 1/2.

   Generalize the analysis shown in class for the lazy random walk on the cycle ($\mathbb{Z}_n$) to show that for the lazy random walk on $\mathbb{Z}_n^d$, $t_{\text{mix}}(\epsilon) = O(d^2 n^2/\epsilon)$.

2. Read Theorem 12.6 (restated below in the notation used in class) and its proof in MU.

   Theorem 1. *Let $\boldsymbol{P}$ be the transition matrix for a finite, irreducible, aperiodic Markov chain M_t with $t_{\text{mix}}(c) \leq T$ for some $c < 1/2$. Then, for this Markov chain, $t_{\text{mix}}(\epsilon) \leq \lceil \ln \epsilon / \ln(2c) \rceil T$.*

   Use it to improve the dependence of the mixing time on ϵ for the lazy walk on the cycle and the lazy walk on $\mathbb{Z}_n^d$ in Problem 1 above.

3. (Exercise 11.6 in MU) The problem of counting the number of solutions to a knapsack instance can be defined as follows: Given items with sizes $a_1, a_2, \ldots, a_n > 0$ and an integer $b > 0$, find the number of vectors $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$ such that $\sum_{i=1}^n a_i x_i \leq b$. The number b can

[1] e^i is the d-dimensional vector that is 0 everywhere but 1 at the i'th coordinate

be thought of as the size of a knapsack, and the x_i denote whether or not each item is put into the knapsack. Counting solutions corresponds to counting the number of different sets of items that can be placed in the knapsack without exceeding its capacity.

Argue that if we have an FPUAS for the knapsack problem, then we can derive an FPRAS for the problem. To set the problem up properly, assume without loss of generality that $a_1 \le a_2 \le \cdots \le a_n$. Let $b_0 = 0$ and $b_i = \sum_{j=1}^{i} a_j$. Let $\Omega(b_i)$ be the set of vectors $(x_1, \ldots, x_n) \in \{0,1\}^n$ that satisfy $\sum_{j=1}^{n} a_j \le b_i$. Let k be the smallest integer such that $b_k \ge b$. Consider the equation:

$$|\Omega(b)| = \frac{|\Omega(b)|}{|\Omega(b_{k-1})|} \times \frac{|\Omega(b_{k-1})|}{|\Omega(b_{k-2})|} \times \cdots \times \frac{|\Omega(b_1)|}{|\Omega(b_0)|} \times |\Omega(b_0)|$$

You will need to argue that $|\Omega(b_{i-1})|/|\Omega(b_i)|$ is not too small. Specifically, argue that $|\Omega(b_i)| \le (n+1) \cdot |\Omega(b_{i-1})|$.