CS5330: Optional problems on Markov Chains

March 30, 2020

I strongly encourage you to think about these problems and to discuss with me, your TA, or your peers if you need help. I will post solution hints in about 1 week.

1. For any transition matrix P, let $Q = (I + P)/2$. Argue that the Markov chain with transition matrix Q is aperiodic.

2. Consider the 2-state Markov chain that stays at the current state with probability p and moves to the other state with probability $1-p$. Write down the transition matrix P, and find a simple expression for P^t.

3. (a) For any matrix A, show that A and $A^\intercal$ have the same set of eigenvalues. (**Hint**: Use the fact that the eigenvalues of A are the roots of the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$, and also that the determinants of a matrix and of its transpose are the same. Show that the polynomials p_A and $p_{A^\intercal}$ are identical.)

   (b) P is a *stochastic matrix* if it has non-negative entries and each of its rows sums to 1. Show that there exists a vector π such that $\pi P = \pi$.

4. A *doubly stochastic* matrix is a stochastic matrix (see previous question) in which additionally all the columns sum to 1. Show that the uniform distribution is a stationary distribution for any Markov chain having a doubly stochastic transition matrix.

5. Let $h_{\max}$ be the maximum hitting time between any pair of vertices in an n-vertex graph G. Show that the time for a random walk to visit every vertex is $O(h_{\max} \log n)$ with high

probability. Conclude that the cover time is $O(h_{\max} \log n)$.
(**Hint**: Break a random walk of length $2k \cdot h_{\max}$ into k segments of length $2h_{\max}$. For any fixed vertex i, argue that i is visited in each segment with probability at least $1/2$. Set k so that each vertex is visited in some segment with high probability. To bound the cover time, use that $\mathbb{E}[X] = \sum_{k\geq 0} \Pr[X \geq k]$ for non-negative random variables X.)

6. Let $h_{\min}$ be the minimum hitting time between any pair of distinct vertices in an n-vertex graph G. The goal of this problem is for you to show that

$$C(G) \geq \Omega(h_{\min} \cdot \log n)$$

where $C(G)$ is the cover time of the graph.

   (a) Consider a random walk $X_0, X_1, \ldots$ where the initial state $X_0 = x$ is arbitrary. Choose a random permutation $\pi : [n] \to [n]$. For a state i, let T_i be the first time that all the states $\pi(1), \pi(2), \ldots, \pi(i)$ have been visited. Show that:

$$\mathbb{E}[T_1] \geq \left(1 - \frac{1}{n}\right) h_{\min}$$

   (b) Observe that the probability that $\pi(i)$ is visited after states $\pi(1), \ldots, \pi(i-1)$ is $\frac{1}{i}$. Using this, argue that $T_i - T_{i-1} = 0$ with probability $1 - \frac{1}{i}$.

   (c) Conditioned on $\pi(i)$ being visited after $\pi(1), \ldots, \pi(i-1)$, show that $\mathbb{E}[T_i - T_{i-1}] \geq h_{\min}$.

   (d) Conclude that:

$$\mathbb{E}[T_n] \geq \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) \cdot h_{\min}$$

7. Show that if μ and ν are two distributions on $[n]$ with probability mass functions $f : [n] \to \mathbb{R}$ and $g : [n] \to \mathbb{R}$ respectively,

$$\|\mu - \nu\| = \frac{1}{2} \sum_{i=1}^{n} |f(i) - g(i)|$$

8. Show that the Markov chain for k-coloring graphs of maximum degree Δ discussed in class is irreducible, if $k \geq \Delta + 2$. Moreover, prove that the stationary distribution of the Markov chain is the uniform distribution on k-colorings.

   Recall that each move in the Markov chain is to pick a random vertex v from the graph, a random color $c \in [k]$, and to color v with c if permitted and to otherwise leave the coloring unchanged.

9. Consider the following random walk on the hypercube $\{0,1\}^n$: with probability $1/(n+1)$, stay at current vertex; otherwise, with probability $1/(n+1)$ for each of the n neighbors, go to one of the neighbors. Note that the self-loop probability is $1/(n+1)$.

   An alternative way to view the walk is that for current state x, a random $i \in \{0,1,\dots,n\}$ is picked uniformly at random. If $i = 0$, x doesn't change; otherwise, x_i is flipped.

   Consider the following coupling (X_t, Y_t).

   - Suppose X_t and Y_t differ at only one coordinate i_0. Then, if X_t picks $i = 0$, Y_t picks i_0; if X_t picks i_0, then Y_t picks $i = 0$; else, both X_t and Y_t pick the same i.
   - Suppose X_t and Y_t differ at the subset of coordinates $S \subseteq [n]$, where $|S| > 1$. Fix a bijection $\pi : S \to S$ such that $\pi(i) \neq i$ for all $i \in S$. Then, if X_t picks $i = 0$, then Y_t also picks $i = 0$; if X_t picks $i \notin S$, then Y_t also picks i; if X_t picks $i \in S$, then Y_t picks $\pi(i)$.

   Observe that the distance between X_t and Y_t never increases. Analyze separately the expected time needed for the distance to decrease to 1 and then the expected time for the distance to go from 1 to 0. Use this to give a bound on the expected coupling time and, hence, the mixing time for this Markov chain.