Solutions for Week 10

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Here we constrcut a coupling (X, Y) that, with probability 1/2, we move $X_{t-1} = x$ to $X_t = (x+e^j)$ or $X_t = (x - e^j)$ for every $j \in [d]$ each with probability $\frac{1}{2d}$, and get $Y_t = Y_{t-1}$.
Or with probability 1/2, we move $Y_{t-1} = y$ to $Y_t = (y + e^j)$ or $Y_t = (y - e^j)$ for every $j \in [d]$ each with probability $\frac{1}{2d}$, and get $X_t = X_{t-1}$.
Let $d_t = \sum_{i=1}^{d} |X_t^d - Y_t^d|$, obviously, d_t will inscrease by 1 with probability 1/2 and decrease by 1 with probability 1/2. The d_t's running is like a random walk on a circle with nd vertices. What we wanna know, it's the expecation of steps to reach $d_t = 0$ and $d_t = nd$. Let A denotes the event that $d_T = 0$ or $d_T = nd$ and T denote the steps to be coupled. As what we get on random walk on a circle, the expecation

$$\mathbb{E}[A] = \Theta(n^2 d^2)$$

From Markov's Inequality, let

$$Pr[t > T] \leq \frac{\mathbb{E}[t]}{T} = \epsilon$$

Thus

$$t_{mix}(\epsilon) = T = O(\frac{n^2 d^2}{\epsilon})$$

2

We can regard the lazy random walk on the cycle as a special case of the lazy walk on $\mathbb{Z}_n^d$ when $d = 1$. From the result above, as for the lazy random walk on the $\mathbb{Z}_n^d$, the mixing time is

$$t_{mix}(c) = O(\frac{d^2 n^2}{c})$$

If we let $c < 1/2$, and let $T = \frac{kn^2 d^2}{c}$, which $k > 0$ is a big enough constant. Then using the Theorem 12.6, we get new uppor bound for mixing time

$$t_{mix}(\epsilon) \leq \lceil \frac{\ln(\epsilon)}{\ln(2c)} \rceil T = \lceil \frac{\ln(\epsilon)}{\ln(2c)} \rceil \frac{kn^2 d^2}{c} = \lfloor \frac{-\ln(\epsilon)}{-\ln(2c)} \rfloor \frac{kn^2 d^2}{c} = \lfloor \frac{\ln(1/\epsilon)}{\ln(1/(2c))} \rfloor \frac{kn^2 d^2}{c}$$

As for $c \ln(2c) \leq -\frac{1}{2} e^{-1} < 0$ when $0 < c < \frac{1}{2}$ and $\ln(\epsilon) < 0$.Thus

$$t_{mix}(\epsilon) = O(n^2 d^2 \ln(\frac{1}{\epsilon}))$$

Obviously, when $d = 1$,

$$t_{mix}(\epsilon) = O(n^2 \ln(\frac{1}{\epsilon}))$$

is the mixing time for the lazy walk on the cycle.

3

To argue that $|\Omega(b_i)| \leq (n+1)|\Omega(b_i)|$, we prove as follows. Because $b_i = \sum_{j=1}^{i} a_j$ and $b_{i-1} = \sum_{j=1}^{i-1} a_j$, obviously $b_i > b_{i-1}$. Thus,

$$\Omega(b_{i-1}) \subset \Omega(b_i)$$

Now, we focus on the $\Omega(b_i)/\Omega(b_{i-1})$. For each $x \in \Omega(b_i)/\Omega(b_{i-1})$, which means $x \in \Omega(b_i)$ and $x \notin \Omega(b_{i-1})$. From definition, we know

$$\sum_{j=1}^{n} x_j a_j > b_{i-1} = \sum_{j=1}^{i-1} a_j$$

$$\sum_{j=1}^{n} x_j a_j \leq b_i = \sum_{j=1}^{i} a_j$$

Because of the b_{i-1} is the maximun of picking all items for a_1 to a_{i-1}. Since $a_k > a_i$ when $k > i$, there must be some $x_k = 1$, where $k \geq i$. If we let $x_k = 0$ and name the new vector x', then

$$\sum_{j=1}^{n} x'_j a_j = \sum_{j=1}^{n} x_j a_j - a_k \leq b_i - a_k \leq b_i - a_i \leq b_{i-1}$$

Thus $x' \in \Omega(b_{i-1})$ now. Which means every $x \in \Omega(b_i)$ can be filp any one of the coordinate which $x_k = 1$ and $k \geq i$, then the new vector $x' \in \Omega(b_{i-1})$. And there are $(n - i + 1)$ coordinates might be chosen. Consequently,

$$\frac{\Omega(b_{i-1})}{\Omega(b_i)} \geq \frac{1}{n-i+1} \geq \frac{1}{n+1}$$

which will hold for each $i \in [k+1]$.
Now we successful prove that $|\Omega(b_i)| \leq (n+1)|\Omega(b_{i-1})|$.