# CS5330: Assignment for Week 11

Due: Tuesday, 14th Apr 2020.

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Please submit your solutions to the "Assignments/Week 11/Submissions" folder on LumiNUS by 14th April, 6:29 pm. I strongly encourage you to write your solutions using LaTeX.

You may discuss the problems with your classmates, though you should write up your solutions on your own. Please note the names of your collaborators in your submission.

1. Let $\phi$ be a Boolean formula on $n$ variables $x_1, \ldots, x_n$, and let $\#\phi$ be the number of satisfying assignments to $\phi$.

   (a) Suppose we have an algorithm $S$ that uniformly samples from the satisfying assignments of $\phi$. The following is an informal description of an FPRAS for $\#\phi$. State the algorithm formally and analyze it.

   *Draw some samples from $S$. Define $p$ to be the fraction of samples that have 0 in their first bit. Suppose $p \geq 1/2$. Then, estimate $p$ using Chernoff bounds. Produce the formula $\phi|_{x_1=0}$ obtained from $\phi$ by fixing $x_1$ to be 0. Recursively estimate $N_0 = \#\phi|_{x_1=0}$. Return $N_0/p$.*

   (b) Suppose we have an algorithm $C$ that computes $\#\phi$. Using $C$ as a black box, show how to efficiently generate uniform samples from the satisfying assignments of $\phi$. (**Hint**: Apply $C$ on sub-formulas, e.g., $\phi|_{x_1=0}$.)

2. Prove the following:

   (a) $\mathbb{E}[\mathbb{E}[X \mid Y, Z] \mid Z] = \mathbb{E}[X \mid Z]$

   (b) $\mathbb{E}[Y \cdot \mathbb{E}[X \mid Y]] = \mathbb{E}[XY]$

3. Let $G_{n,N}$ be the uniform distribution on $n$-vertex graphs with exactly $N$ edges. Suppose $N = cn$ for some constant $c > 0$. Let $X$ be the expected number of isolated vertices (i.e., vertices of degree 0) for a random graph from $G_{n,N}$.

   (a) Determine $\mathbb{E}[X]$.

   (b) Using Azuma, show that $\Pr[|X - \mathbb{E}[X]| \geq 2\lambda\sqrt{cn}] \leq 2e^{-\lambda^2/2}$.