CS5330: Sample Endterm

April 10, 2020

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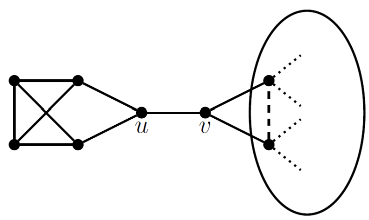
1. Give an example of a family $\mathcal{H}$ of hash functions $h : [n] \to \{0, 1\}$ such that $\mathcal{H}$ is strongly $(n-1)$-uniform but not strongly n-uniform.

2. 

   In the graph above, the ellipse on the right side is a 3-regular graph with $n-6$ vertices where the dashed edge is absent. Show that $h_{u,v} = O(1)$ but $h_{v,u} = \Omega(n)$. (**Hint**: What can you say about $h_{u,u}$?)

3. Consider a Markov chain on n points $\{0, 1, \ldots, n-1\}$ lying in order on a circle. At each step, the chain stays at the current point with probability $1/2$ or moves to the next point in the clockwise direction with probability $1/2$. Find the stationary distribution and show that, for any $\varepsilon > 0$, the mixing time $t_{\text{mix}}(\varepsilon)$ is $O(n^2 \log(1/\varepsilon))$.

4. Design a Markov chain on the independent sets of a graph where in the stationary distribution, $\pi_x \propto \lambda^{|I_x|}$ where I_x is the independent set corresponding to the state x and λ is a parameter.

5. Consider an n-cube with $N = 2^n$ nodes. Let S be a nonempty set of vertices on the cube, and let x be a vertex chosen uniformly at random among all vertices of the cube. Let $D(x, S)$ be the minimum number of coordinates in which x and y differ over all points $y \in S$.

   (a) Show that
   $$\Pr[|D(x,S) - \mathbb{E}[D(x,S)]| > \lambda] \le 2\exp\left(-\frac{2\lambda^2}{n}\right)$$

(b) By plugging in $\lambda = \mathbb{E}[D(x, S)]$ into part (a), show that if $|S| \geq \alpha \cdot 2^n$, then show that:

$$\mathbb{E}[D(x, S)] \leq c_\alpha \sqrt{n}$$

where c_α is a constant depending on α.