CS5330: Assignment for Weeks 8 & 9

Due: Tuesday, 31st Mar 2020.

Please submit your solutions to the "Assignments/Week 89/Submissions" folder on LumiNUS by 31st March, 6:29 pm. I strongly encourage you to write your solutions using LaTeX.

You may discuss the problems with your classmates, though you should write up your solutions on your own. Please note the names of your collaborators in your submission.

1. Consider an irreducible Markov chain $\mathbf{X} = (X_0, X_1, \dots)$ on state space $\{1, \dots, n\}$. Let $T_i = \min\{t \geq 1 : X_t = i \mid X_0 = i\}$ be the first return time to i, and let $h_i = \mathbb{E}[T_i]$ be the hitting time from i to i. Recall that in class, we claimed that $\mathbf{X}$ has a unique stationary distribution π where $\pi_i = 1/h_i$ for all i. In this problem, you are going to prove this claim.

   First, let's show the existence of a stationary distribution. Define the random variable:

   $$V_j = |\{t \mid 0 \leq t < T_1, X_0 = 1, X_t = j\}|$$

   be the number of visits to state j between two successive visits to state 1, and let $\nu_j = \mathbb{E}[V_j]$. Note that $V_1 = 1$ with probability 1.

   (a) Show that:
   $$\nu_j = \sum_{t \geq 0} \Pr\left[X_t = j, t < T_1 \mid X_0 = 1\right]$$

   (b) Now, argue that the ν_j's are proportional to a stationary distribution as follows:
   $$\begin{aligned} \nu_j &= \sum_{t \geq 0} \Pr\left[X_t = j, t < T_1 \mid X_0 = 1\right] \\ &= \sum_{t \geq 1} \Pr\left[X_t = j, t \leq T_1 \mid X_0 = 1\right] \\ &= \sum_i P_{i,j} \sum_{t \geq 1} \Pr\left[X_{t-1} = i, t \leq T_1 \mid X_0 = 1\right] \\ &= \sum_i P_{i,j} \sum_{t \geq 0} \Pr\left[X_t = i, t < T_1 \mid X_0 = 1\right] \\ &= \sum_i P_{i,j} \nu_i \end{aligned}$$

   Justify each step.

Next, we will argue that the stationary distribution is unique. Let π be an arbitrary stationary distribution for $\mathbf{X}$; we know one exists from above. Our goal will be to show that $\pi_i = 1/h_i$ for every i.

(c) Suppose X_0 is distributed according to π. Naturally, this means that X_t is also distributed according to π, for all $t \geq 1$.

Use the definition of h_i and the fact that $\mathbb{E}[Z] = \sum_{t\geq 1} \Pr[Z \geq t]$ for any non-negative integer random variable Z to show that for any i:

$$\pi_i h_i = \sum_{t\geq 1} \Pr[T_i \geq t] \Pr[X_0 = i] = \sum_{t\geq 1} \Pr[X_0 = i, X_s \neq i \;\; \forall 1 \leq s < t]$$

(d) Manipulate the expression in (c) to get:

$$\pi_i h_i = \pi_i + \sum_{t\geq 2} \Pr[X_s \neq i, \forall 0 \leq s < t-1] - \Pr[X_s \neq i, \forall 0 \leq s < t]$$

(e) Simplify (d) to obtain:

$$\pi_i h_i = 1 - \lim_{t\to\infty} [X_t \neq i, \forall t \geq 0]$$

Use the fact that $\mathbf{X}$ is irreducible to obtain that $\pi_i h_i = 1$.

2. (Exercise 7.23 of MU) One way of spreading information on a network uses a rumor-spreading paradigm. Initially one host begins with a message. Each round, every host that has the message contacts another host independently and uniformly at random from the other $n-1$ hosts and sends that host the message.

   Implement a program to determine the number of rounds required for a message starting at the host to reach all other hosts with probability 0.9999 when $n = 128$.

3. (Exercise 7.24 of MU) The *lollipop graph* on n vertices is a clique on $n/2$ vertices connected to a path on $n/2$ vertices, as shown in Figure 7.3 of MU. The node u is a part of both the clique and the path. Let v denote the other end of the path.

   (a) Show that the expected covering time of a random walk starting at v is $\Theta(n^2)$.

   (b) Show that the expected covering time for a random walk starting at u is $\Theta(n^3)$.