Solutions for Week 4

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1

As pigeonhole principle tells. there must be two same element in $\{x_1, x_2, x_3\}$, like two 1s, or two 0s. Thus we can design a Las Vegas algorithm like this:
1.We ramdomly choose two bits of $\{x_1, x_2, x_3\}$to read and calculate the sum of them. 2.If sum of these two bits are 2 or 0, then we got $f(x_1, x_2, x_3)$.

$$f(x_1, x_2, x_3) = \begin{cases} 1 & \text{sum of two bits is 2} \\ 2 & \text{sum of tow bits is 0} \end{cases}$$

3.When sum of these two bits is 1, we scan the last bit and calculate the sum of all. Then $f(x_1, x_2, x_3)$ is detemined as definition. We can prove it easily. Let X denotes the bits need to scan.

$$E(X) = Pr(X = 2) \cdot 2 + Pr(X = 3) \cdot 3 = \binom{3}{2} \cdot 2 + (1 - \binom{3}{2}) \cdot 3 = \frac{8}{3}$$

Yao's Minimax Principle needs to find a worst input distribution on a determistic algorithm. Because of this algorithm is for specific distribution not a randomized distribution, we can't find a more worst distribution for this problem. Thus, we can't use Yao's Minimax Principle to improve it.

2

Suppose P is mixed strategy distribution of Row and Q is mixed strategy distribution of Col. $p_i \in P = \{p_1, p_2\}$ and $q_j \in Q = \{q_1, q_2\}$. p_i denotes probability to choose action i and so do q_j. Let $V(P, Q)$ donates the expected payoff of dollars Row paying Col, we got

$$V(P, Q) = \sum_{i=1}^{2} \sum_{j=1}^{2} p_i q_j C(i, j)$$

What Row to do is

$$\max_P \min_Q V(P, Q)$$

And what Col to do is

$$\min_Q \max_P V(P, Q)$$

As Von Neumann's Minimax Theorem,

$$\max_P \min_Q V(P,Q)] \leq V^* \leq \min_Q \max_P V(P,Q)$$

$$\begin{aligned} V(P,Q) &= \frac{15}{4}p_1 q_1 - \frac{9}{4}p_1 - \frac{5}{2}q_1 + \frac{3}{2} \\ &= \frac{15}{4}(p_1 - \frac{2}{3})(q_1 - \frac{3}{5}) \end{aligned} \tag{1}$$

Thus the equilibrium point of this zero-sum game is $p_1 = \frac{2}{3}$ and $q_1 = \frac{3}{5}$. We can simplely demonstrate that when $p_1 \neq \frac{2}{3}$, q_1 could always choose some value make V less than 0. And when $q_1 \neq \frac{3}{5}$, p_1 could always choose some value make V greater than 0 Let X denotes action which Row choose and Y denotes action which Col choose So the minimax optimal strategies for Row is

$$Pr(X=1) = p_1 = \frac{2}{3}$$

$$Pr(X=2) = p_2 = \frac{1}{3}$$

$$Pr(Y=1) = q_1 = \frac{3}{5}$$

$$Pr(Y=2) = q_2 = \frac{2}{5}$$

And the expected payoff V^* is 0.

3

Each processor will get $\frac{n}{m}$ jobs. Let T_{ij} denotes j-th job on i-th processor completed in 1 step or k steps. Let T_i donotes the running steps of i-th processor.

$$T_i = \sum_{j=1}^{\frac{n}{m}} T_{ij}$$

$$T_{ij} = \begin{cases} 1 & \text{w.p. p} \\ k & \text{otherwise} \end{cases}$$

$$\mathbb{E}(T_{ij}) = p + k(1-p)$$

Suppose random variable $X_{ij} = \frac{T_{ij}-k}{1-k}$

$$X_{ij} = \begin{cases} 1 & \text{w.p. p} \\ 0 & \text{otherwise} \end{cases}$$

$$X_i = \sum_{j=1}^{\frac{n}{m}} X_{ij}$$

$$\mathbb{E}(\sum_{j=1}^{\frac{n}{m}} X_{ij}) = \sum_{j=1}^{\frac{n}{m}} \mathbb{E}(X_{ij}) = \frac{n}{m}p = \mu$$

As for

$$X_{ij} = \frac{T_{ij} - k}{1 - k}$$

We could get

$$T_i = (1-k)X_i + k\frac{n}{m}$$

Here we use Chernoff bound

$$\begin{aligned} Pr(|X_i - \mu| \geq \delta\mu) &\leq 2e^{-\frac{\mu\delta^2}{3}} \\ &\leq 2e^{-\frac{np\delta^2}{3m}} \end{aligned}$$

When we choose

$$\delta = \delta^* = \sqrt{\frac{3mc\ln{(n)}}{np}}$$

, which $c > 1$ we get

$$Pr(|X_i - \mu| \geq \delta\mu) \leq \frac{2}{n^c}$$

And with relations between T_{ij} and X_{ij}

$$\begin{gathered} |X_i - \mu| \leq \delta^*\mu \\ \left|\frac{T_i - \frac{n}{m}k}{1-k} - \mu\right| \leq \delta^*\mu \\ \frac{n}{m}(k - kp + p) + (1-k)\sqrt{\frac{3cnp\ln(n)}{m}} \leq T_i \leq \frac{n}{m}(k - kp + p) - (1-k)\sqrt{\frac{3cnp\ln(n)}{m}} \end{gathered}$$

So we got upper bound and lower bound on when all jobs will be completed as above w.h.p., which $c > 1$ and $n \geq m$.