# CS5330: Assignment for Week 4

Due: Tuesday, 18th Feb 2020.

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Here are solution sketches to the Week 4 problems. If anything is unclear, please talk to me or your TA.

1. Let $f : \{0,1\}^3 \to \{0,1\}$ be defined as: $f(x_1, x_2, x_3) = 1$ iff $x_1 + x_2 + x_3 \geq 2$. Design a Las Vegas algorithm that in expectation only reads 8/3 input bits. Show that this cannot be improved using Yao's minimax principle.

   Sketch: Consider the algorithm that picks picks two of the three bits at random and reads them. It reads the third only if it needs to in order to compute $f$. For $x = 000$ or $x = 111$, it reads 2 bits with probability 1. If $x = 001$, it reads two bits (bits 1 and 2) with probability $1/3$ and all three bits with probability $2/3$, so that the expected number of bits read is $8/3$. The same analysis holds whenever $x$ has exactly two bits equal.

   For the lower bound, consider the input distribution $\mathcal{D}$ that is uniform on strings $x$ which have exactly two bits equal. Fix a deterministic algorithm $A$. The probability that the first bit queried by $A$ is one of the equal bits is $2/3$. Conditioned on the first query being one of the equal bits, the probability that the second bit is also one of the equal bits is $1/2$. So, with probability $2/3 \cdot 1/2 = 1/3$, the first two queries of $A$ are equal and hence the value of $f$ is determined. With the remaining probability, $A$ must read all three bits. Hence, the expected number of bits read by $A$ when $x \sim \mathcal{D}$ is $\geq 2/3 \cdot 2 + 1/3 \cdot 3 = 8/3$.

2. Find the minimax optimal strategies for Row and Col in the zero-sum game with two actions $\{1,2\}$ for each player, where Row pays Col $C(i,j)$ dollars if they choose actions $i$ and $j$ respectively, and $C = \begin{bmatrix} 1/2 & -3/4 \\ -1 & 3/2 \end{bmatrix}$.

   Sketch: Row's optimal strategy is to choose action 1 with probability $2/3$ and action 2 with probability $1/3$. Col's optimal strategy is to choose action 1 with probability $3/5$ and action 2 with probability $2/5$. The expected amount Row pays Col under these strategies is 0.

3. (Exercise 4.17 of MU) Suppose that we have $n$ jobs to distribute among $m$ processors. For simplicity, we assume that $m$ divides $n$. A job takes 1 step with probability $p$ and $k > 1$ steps with probability $1 - p$. Use Chernoff bounds to determine upper and lower bounds (that hold with high probability) on when all jobs will be completed if we randomly assign exactly $n/m$ jobs to each processor.

Fix the assignment of the jobs to the processors. Consider the $i$'th processor. Let $X_i$ be the number of jobs assigned to processor $i$ that are taking $k$ steps to complete. The total time for all the jobs to complete on the $i$'th processor is $\left(\frac{n}{m} - X_i\right) \cdot 1 + X_i \cdot k = \frac{n}{m} + X_i(k-1)$. The total time for all the jobs to complete on all the processors is $\frac{n}{m} + (k-1) \cdot \max_i X_i$.

Let $\mu = \mathbb{E}[X_i] = np/m$. By Chernoff bound, for $0 < \delta < 1$, $\Pr[|X_i - \mu| > \delta\mu] \leq 2e^{-\delta^2\mu/3}$. Therefore,

$$\Pr[|X_i - \mu| > 3\sqrt{\mu \log m}] \leq 2e^{-3\log m} = \frac{2}{m^3}$$

as long as $\mu > 9\log m$. Hence, by the union bound, with probability $1 - 2/m^2$, $\max_i X_i \in [\mu - 3\sqrt{\mu\log m}, \mu + 3\sqrt{\mu \log m}]$ with the upper-bound holding if $\mu > 9\log m$. Also, if $\log m > 2$, we use the large deviation version of the Chernoff bound to say that $\max_i X_i \leq \mu + 3\log m$ with high probability.