Solutions for Week 89

Bao Jinge A0214306U e0522065@u.nus.edu

1

1.1 a

According to the definition, led V_{jt} indicate whether $X_t = j$ or not with $X_0 = 1$. Thus,

$$V_j = \sum_{0 \le t < T_1} V_{jt}$$

Using the linearity of expectation, we get

$$\begin{aligned}
v_j = \mathbb{E}[V_j] &= \mathbb{E}[\sum_{0 \le t < T_1} V_{jt}] \\
&= \sum_{0 \le t < T_1} \mathbb{E}[V_{jt}] \\
&= \sum_{0 \le t < T_1} Pr[X_t = j, X_0 = 1] \\
&= \sum_{0 \le t < T_1} Pr[X_t = j | X_0 = 1] Pr[X_0 = 1] \\
&= \sum_{t \ge 0} Pr[X_t = j, t < T_1 | X_0 = 1]
\end{aligned}$$

1.2 b

$$\begin{aligned}
v_j &= \sum_{t \ge 0} Pr[X_t = j, t < T_1 | X_0 = 1] \\
&= \sum_{t \ge 1} Pr[X_t = j, t \le T_1 | X_0 = 1] \\
&= \sum_i Pr_{i,j} \sum_{t \ge 1} Pr[X_t - 1 = i, t \le T_1 | X_0 = 1] \\
&= \sum_{t \ge 1} Pr[X_t = j, t \le T_1 | X_0 = 1] \\
&= \sum_i Pr_{i,j} \sum_{t \ge 1} Pr[X_t - 1 = i, t \le T_1 | X_0 = 1] \\
&= \sum_i Pr_{i,j} v_i
\end{aligned}$$

The first row comes from result from section (a).
The second row and the forth row use the property of summation notation and memoryless property of Markov Chain, which replace t using $t-1$ and replace $t-1$ using t separately.
The third row comes from transition of Markov Chain' definition.
The last row comes from the definition of v_i.
From derivation above, we know that the v_j's are proportional to a stationary distribution according to definition of stationary distributiion.

1.3 c

Because of X_0 is distributed accoring to π,

$$\pi_i = Pr[X_0 = i]$$

According to the definition of h_i and the fact of expecation, we get

$$\begin{aligned}\pi_i h_i &= Pr[X_0 = i] \sum_{t\geq 1} Pr[T_i \geq t] \\ &= \sum_{t\geq 1} Pr[X_0 = i] Pr[T_i \geq t] \\ &= \sum_{t\geq 1} Pr[X_0 = i, X_s \neq i, \forall 1 \leq s < t]\end{aligned}$$

The last row comes from the definition of T_i, which means if $X_0 = i$ and $X_{T_i} = i$, then no element between X_0 and X_{T_i} wii be equal to i. Otherwise, it will contradict to definition of T_i.

1.4 d

From result derived from section (c), we get

$$\begin{aligned}\pi_i h_i &= \sum_{t\geq 1} Pr[X_0 = i, X_s \neq i, \forall 1 \leq s < t] \\ &= Pr[X_0 = i, X_s \neq i, \forall 1 \leq s < 1] + \sum_{t\geq 2} Pr[X_0 = i, X_s \neq i, \forall 1 \leq s < t] \\ &= Pr[X_0 = i] + \sum_{t\geq 2} Pr[X_0 = i, X_s \neq i, \forall 1 \leq s < t] \\ &= \pi_i + \sum_{t\geq 2} Pr[X_0 = i, X_s \neq i, \forall 1 \leq s < t] \\ &= \pi_i + \sum_{t\geq 2} Pr[X_s \neq i, \forall 1 \leq s < t] - Pr[X_s \neq i, \forall 0 \leq s < t] \\ &= \pi_i + \sum_{t\geq 2} Pr[X_s \neq i, \forall 0 \leq s < t-1] - Pr[X_s \neq i, \forall 0 \leq s < t]\end{aligned}$$

The last row use the memoryless property of Markov Chain.

1.5 e

Because X is irreducible and its state space is finite, so this markov chain is ergodic, whicn means every state will be traversed in the future, namely $p^n_{i,j} > 0$. Thus,

$$\lim_{t\to\infty} [X_t \neq i, \forall t \geq 0] = 0$$

Thus,

$$\pi_i h_i = 1$$

Q.E.D.

2

The number of rounds required is about 14.
Code is written in C++, as follows

```
    long long total_round = 0;
    int times = 1000000;
    int Ntimes= 1000000;
    while (times--) {
        srand(time(NULL));
        int round = 0;
        memset(q, false , sizeof(q));
        q[0] = true;
        int cnt = 1;
        while (cnt < NNodes) {
            for (int i = 0; i < NNodes; i++) {
                p[i] = rand() % NNodes;
                while (p[i] == i) p[i] = rand() % NNodes;
            }
            int tmp_q[200];
            for (int i = 0; i < NNodes; i++) {
                tmp_q[i] = q[i];
            }

            for (int i = 0; i < NNodes; i++) {
                if (tmp_q[i]) {
                    if (!q[p[i]]) cnt++;
                    q[p[i]] = q[p[i]] + true;
                }
            }
            round++;
//     cout << "round = " << round << ", cnt = " << cnt <<endl;
        }
        cout << round << endl;
        total_round = total_round + round;
    }

    cout << total_round << endl;
    cout << "# of avg_round = " << total_round * 1.0 / Ntimes << endl;
```

3

3.1 i

We partition the lolipop graph into two parts. One for clique $G_{loli} = \{V_{loli}, E_{loli}\}$ and a line graph G_{line} with $(n/2)$ vertices. Let C_v denote the expected covering time of a random walk starting at v and C_{loli} denote the expected covering time of G_{loli}, we can build cover time like this

$$C_v = h_{v,u} + C_{loli}$$

For the line graph, caculating $h_{v,u}$ can be ragarded as the analysis in 2-SAT Problem. So we can safely get

$$h_{v,u} = (\frac{n}{2})^2$$

For G_{loli}, we have upper bound that

$$C_{loli} \leq \sum_{w \in V, w \neq u} h_{u,w} + h_{w,u}$$

Without loss of generality, suppose h_{u,w^*} is the larget hitting time from u to other vertice in the G_{loli}, where $w^* \in V$ and $w^* \neq u$. Let other $(n/2 - 2)$ nodes are contracted into a composition vertice w'. Let $x_1, x_2, ...x_{n/2}$ denote the vetices on the line starting from the closet vetice to u. We get system of equations below,

$$\begin{cases}
h_{u,w^*} & = \frac{1}{n/2} h_{x1,w^*} + \frac{n/2-1}{n/2} h_{w',w^*} + \frac{1}{n/2} \cdot 0 + 1 \\
h_{w',w^*} & = \frac{n/2-1-2}{n/2-1} h_{w',w'} + \frac{1}{n/2-1} h_{u,w^*} + \frac{1}{n/2-1} \cdot 0 + 1 \\
h_{x_1,w^*} & = \frac{1}{2} h_{u,w^*} + \frac{1}{2} h_{x_2,w^*} + 1 \\
h_{x_2,w^*} & = \frac{1}{2} h_{x1,w^*} + \frac{1}{2} h_{x_3,w^*} + 1 \\
... & \\
h_{x_{n/2-1},w^*} & = \frac{1}{2} h_{x_{n/2-2},w^*} + \frac{1}{2} h_{x_{n/2},w^*} + 1 \\
h_{x_{n/2},w^*} & = h_{x_{n/2-1},w^*} + 1
\end{cases}$$

As we can see, there are $(\frac{n}{2} + 2)$ variables and $(\frac{n}{2} + 2)$ equations. We have an unique solution that

$$h_{u,w^*} = \frac{n^2 + 18n - 8}{4n}$$

As for $h_{u,u}$,

$$h_{u,u} = \frac{2|E|}{deg(u)} = \frac{1}{deg(u)} \sum_{w \in V_{loli}, w \neq u} (1 + h_{w,u})$$

Thus, we get

$$\sum_{w \in V_{loli}, w \neq u} h_{w,u} = 2|E| - \frac{n}{2} = \frac{n^2}{2} - \frac{n}{2}$$

Consequently, we have upper bound below

$$C_{loli} \leq \sum_{w \in V_{loli}, w \neq u} h_{u,w} + h_{w,u} \leq \frac{5n^3 + 12n^2 - 44n + 16}{8n} = O(n^2)$$

Now, we bound the C_u,

$$C_v \geq h_{u,v} = (\frac{n}{2})^2 = \Omega(n^2)$$
$$C_v \leq O(n^2)$$

So,

$$C_v = \Theta(n^2)$$

Q.E.D.

3.2 ii

Let C_u denote the expected covering time of a random walk starting at u, we can know

$$C_u = \max\{C_{loli}, h_{u,v}\}$$

Suppose W denote the set of vertices in lolipop except u.

$$\begin{cases} h_{u,v} & = \frac{n/2-1}{n/2} h_{W,v} + \frac{1}{n/2} h_{x_1,v} + 1 \\ h_{W,v} & = \frac{n/2-2}{n/2-1} h_{W,v} + \frac{1}{n/2-1} h_{u,v} + 1 \\ h_{x_1,v} & = \frac{1}{2} h_{u,v} + \frac{1}{2} h_{x_2,v} + 1 \\ h_{x_2,v} & = \frac{1}{2} h_{x_1,v} + \frac{1}{2} h_{x_3,v} + 1 \\ ... & \\ h_{x_{n/2-1},v} & = \frac{1}{2} h_{u,v} + \frac{1}{2} \cdot 0 + 1 \end{cases}$$

Here are $n+1$ equations and $n+1$ variables, so we have an unique solution. We can get

$$h_{u,v} = \frac{n^3}{8} = \Theta(n^3).$$

We can get bound that

$$C_u = \max\{\Theta(n^2), \Theta(n^3)\} = \Theta(n^3)$$

Q.E.D.