CS5330: Assignment for Weeks 8 & 9

Due: Tuesday, 31st Mar 2020.

Here are solution sketches to the Week 8 & 9 problems. If anything is unclear, please talk to me or your TA.

1. Consider an irreducible Markov chain $\mathbf{X} = (X_0, X_1, \dots)$ on state space $\{1, \dots, n\}$. Let $T_i = \min\{t \geq 1 : X_t = i \mid X_0 = i\}$ be the first return time to i, and let $h_i = \mathbb{E}[T_i]$ be the hitting time from i to i. Recall that in class, we claimed that $\mathbf{X}$ has a unique stationary distribution π where $\pi_i = 1/h_i$ for all i. In this problem, you are going to prove this claim.

   First, let's show the existence of a stationary distribution. Define the random variable:

   $$V_j = |\{t \mid 0 \leq t < T_1, X_0 = 1, X_t = j\}|$$

   be the number of visits to state j between two successive visits to state 1, and let $\nu_j = \mathbb{E}[V_j]$. Note that $V_1 = 1$ with probability 1.

   (a) Show that:

   $$\nu_j = \sum_{t \geq 0} \Pr[X_t = j, t < T_1 \mid X_0 = 1]$$

   Follows from linearity of expectations.

   (b) Now, argue that the ν_j's are proportional to a stationary distribution as follows:

   $$\begin{aligned}
   \nu_j &= \sum_{t \geq 0} \Pr[X_t = j, t < T_1 \mid X_0 = 1] \\
   &= \sum_{t \geq 1} \Pr[X_t = j, t \leq T_1 \mid X_0 = 1] \\
   &= \sum_{i} P_{i,j} \sum_{t \geq 1} \Pr[X_{t-1} = i, t \leq T_1 \mid X_0 = 1] \\
   &= \sum_{i} P_{i,j} \sum_{t \geq 0} \Pr[X_t = i, t < T_1 \mid X_0 = 1] \\
   &= \sum_{i} P_{i,j} \nu_i
   \end{aligned}$$

Justify each step.

This calculation is what's called the "cycle trick" by Markov chain people. The second equality is because $X_0 = X_{T_1} = 1$ by definition. The third equality uses the fact that P is the transition matrix: $\Pr[X_t = j, t \le T_1 \mid X_0 = 1] = \Pr[X_{t-1} = i, t \le T_1 \mid X_0 = 1] \cdot P_{i,j}$. Also, the summations over t and i are interchanged. This is justified because all the summands are between 0 and 1. The fourth equality makes the transformation $t \mapsto t+1$. The final equality is from the definition of ν_i.

Next, we will argue that the stationary distribution is unique. Let π be an arbitrary stationary distribution for $\mathbf{X}$; we know one exists from above. Our goal will be to show that $\pi_i = 1/h_i$ for every i.

(c) Suppose X_0 is distributed according to π. Naturally, this means that X_t is also distributed according to π, for all $t \ge 1$.
Use the definition of h_i and the fact that $\mathbb{E}[Z] = \sum_{t\ge1} \Pr[Z \ge t]$ for any non-negative integer random variable Z to show that for any i:

$$\pi_i h_i = \sum_{t\ge1} \Pr[T_i \ge t] \Pr[X_0 = i] = \sum_{t\ge1} \Pr[X_0 = i, X_s \ne i \ \forall 1 \le s < t]$$

The first equality directly follows from the stated fact about expectations. For the second equality, note that: $\Pr[T_i \ge t] = \Pr[X_s \ne i \ \forall 1 \le s < t \mid X_0 = i]$.

(d) Manipulate the expression in (c) to get:

$$\pi_i h_i = \pi_i + \sum_{t\ge2} \Pr[X_s \ne i, \forall 0 \le s < t-1] - \Pr[X_s \ne i, \forall 0 \le s < t]$$

We use the expression from part (c). The term for $t = 1$ is $\Pr[X_0 = i] = \pi_i$. For $t \ge 2$, $\Pr[X_0 = i, X_s \ne i \ \forall 1 \le s < t] = \Pr[X_s \ne i \ \forall 1 \le s < t] - \Pr[X_s \ne i \ \forall 0 \le s < t]$. Now, we can invoke stationarity of π to get the desired result.

(e) Simplify (d) to obtain:

$$\pi_i h_i = 1 - \lim_{t\to\infty} [X_t \ne i, \forall t \ge 0]$$

Use the fact that $\mathbf{X}$ is irreducible to obtain that $\pi_i h_i = 1$.

Summing the telescoping sequence, we get $\pi_i h_i = \pi_i + \Pr[X_0 \ne i] - \lim_{t\to\infty} \Pr[X_t \ne i, \forall t \ge 0] = 1 - \lim_{t\to\infty} \Pr[X_t \ne i, \forall t \ge 0]$ where we used the fact that $\pi_i = \Pr[X_0 = i]$.
We now want to argue that the chain reaches i with probability 1. By irreducibility, from every other state j, there is a minimum T_j as well as $\varepsilon_j > 0$ such that $P_{j,i}^{T_j} > \varepsilon_j$ (where P is the transition matrix). Let $T = \max_j T_j$ and $\varepsilon = \min_j \varepsilon_j$. So, starting from anywhere, within T steps, it is guaranteed that i is not visited with probability at most ε. Hence, if the chain makes M steps, the probability that i is not visited is at most $(1-\varepsilon)^{M/T}$. As $M \to \infty$, this probability goes to 0.

2. (Exercise 7.23 of MU) One way of spreading information on a network uses a rumor-spreading paradigm. Initially one host begins with a message. Each round, every host that has the message contacts another host independently and uniformly at random from the other $n - 1$ hosts and sends that host the message.

   Implement a program to determine the number of rounds required for a message starting at the host to reach all other hosts with probability 0.9999 when $n = 128$.

   You should have found a number around 22.

3. (Exercise 7.24 of MU) The *lollipop graph* on n vertices is a clique on $n/2$ vertices connected to a path on $n/2$ vertices, as shown in Figure 7.3 of MU. The node u is a part of both the clique and the path. Let v denote the other end of the path.

   (a) Show that the expected covering time of a random walk starting at v is $\Theta(n^2)$.

   (b) Show that the expected covering time for a random walk starting at u is $\Theta(n^3)$.

   Some of you did a really excellent job with answering the problem. As an example, I am appending Tan Likai's and Kiran Gopinathan's solutions to this problem.

To solve this distribution, we use another Markov chain on $k'+1$ states, labelled 0 to k'. Except for state k', there are two possible transitions from each state i, with probability $\frac{n-k-i}{n-k}$ of moving to state $i+1$, otherwise remaining at state i. The probability of there being i non-empty bins in the distribution of k' balls and $n-k$ bins is thus the probability of being at state i after k' transitions on this Markov chain.

Using `long double` precision with C++, the minimum number of rounds for 128 machines is 22 with a probability of 0.9999610525.

Problem 3

Let $m = \frac{n}{2}$. Then the clique and the path have size m and the graph has $2m-1$ vertices.

Part (a)

First consider the expected time taken to reach u from v. From the 2-SAT analysis, the expected time is $\Theta(m^2)$ (so the lower bound is proven). Note that all vertices in the path are covered this way. It remains to cover the remaining $m-1$ vertices in the clique.

Label the clique vertices $u = c_0, c_1, c_2, ..., c_{m-1}$. We compute the expected number of steps taken to travel along the path $c_0 \to c_1 \to c_0 \to c_2 \to ... \to c_{m-1} \to c_0$. Note that for any $1 \le i, j < m$, $h_{c_0,c_i} = h_{c_0,c_j}$ and $h_{c_i,c_0} = h_{c_j,c_0}$ due to symmetry.

Fix some $c_i = w$. Compress the remaining clique vertices to a super-node x. Starting from u, label the path vertices $0, 1, ..., m-1$ (i.e. u is vertex 0). The transition probabilities between these new vertices are $P_{u,w} = \frac{1}{m}, P_{u,x} = \frac{m-2}{m}, P_{x,w} = \frac{1}{m-1}, P_{x,u} = \frac{1}{m-1}, P_{x,x} = \frac{m-3}{m-1}$.

Define t_i to be the expected number of steps to reach w from i where i is either $0, ..., n/2-1$ or u, w, x. Then $t_w = 0$.

$$\begin{aligned}
t_i &= 1 + \frac{1}{2}(t_{i-1} + t_{i+1}) \qquad \forall 1 \le i \le m-2 \\
t_{m-1} &= 1 + t_{m-2} \\
t_u &= 1 + \frac{1}{m}t_1 + \frac{m-2}{m}t_x \\
t_x &= 1 + \frac{1}{m-1}t_u + \frac{m-3}{m-1}t_x
\end{aligned}$$

Solving, we have $t_x = \frac{m-1}{2} + \frac{1}{2}t_u$, so $t_1 = -\frac{m^2+m-2}{2} + \frac{m+2}{2}t_u$. Also, $t_i - t_{i-1} = t_{i+1} - t_i + 2$ for each $1 \le i \le m-2$. So since $t_{m-1} - t_{m-2} = 1$, $t_1 - t_u = t_1 - t_0 = 2m-1$. Solving the simultaneous equation, $t_u = m - \frac{4}{m} + 5$.

So $h_{u,w} = O(m)$. To find $h_{u,w}$, note that we only have to consider the clique part of the graph. At each step, while the state is not u we have a $\frac{1}{m-1}$ probability of moving to u. So $h_{w,u}$ is the expected value of a geometric distribution with $p = \frac{1}{m-1}$, i.e. $h_{w,u} = m-1 = O(m)$.

Therefore our total time complexity is $\Theta(m^2) + (m-1)(O(m) + O(m)) = \Theta(m^2)$.

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Less rigorous but more intuitive solution

Consider the following reformulation of the problem.

Note that the following formulation is equivalent: we attempt to cover a clique of size m starting at u, except that each time we are at u we have a $1/m$ probability of entering a path with m vertices, otherwise we move within the clique. So if we move k times from u to another clique vertex, in expectation we move $\frac{k}{m-1}$ times to the path.

By the coupon collector result, the expected number of moves within the clique to cover all vertices is $O(m \log m)$. The number of steps we take from u to another clique vertex is $O(\log m)$. The expected number of times we enter the path is therefore $O(\frac{\log m}{m})$. On the path, the expected return time to u is $h_u = (\pi_u)^{-1} = (\frac{1}{2m-2})^{-1} = O(m)$. This is because on a path with k vertices, the stationary distribution is $\frac{1}{2k-2}$ on the ends and $\frac{1}{k-1}$ on all other vertices.

Therefore the total expected covering time is $\Theta(m^2) + O(n \log m) + O(m)O(\frac{\log m}{m}) = \Theta(m^2) = \Theta(n^2)$.

Part (b)

The upper bound follows from the fact that the expected covering time is at most $O(VE) = O(m^3)$. To prove the lower bound, consider the expected number of steps taken to reach v from u. We compress the $m-1$ clique vertices that are not connected to the path to a single vertex w, forming a path of length $m+1$.

Then we have $P_{w,u} = \frac{1}{m-1}$, $P_{w,w} = \frac{m-2}{m-1}$, $P_{u,w} = \frac{m-1}{m}$. The other transition probabilities remain the same.

From v, label the vertices $0, 1, ..., m$ (so u is $m-1$ and w is m). Define t_i to be the expected time taken to reach v from i. Clearly $t_0 = 0$. Then we have the following relations:

$$\begin{aligned}
t_i &= 1 + \frac{1}{2}(t_{i-1} + t_{i+1}) \qquad \forall 1 \leq i \leq m-2 \\
t_{m-1} &= 1 + \frac{1}{m}t_{m-2} + \frac{m-1}{m}t_m \\
t_m &= 1 + \frac{1}{m-1}t_{m-1} + \frac{m-2}{m-1}t_m
\end{aligned}$$

From the last two relations, we have $t_m - t_{m-1} = m-1$, and $t_{m-1} - t_{m-2} = (m-1)^2 + m$. For $1 \leq i \leq m-2$, $t_i - t_{i-1} = t_{i+1} - t_i + 2 \Rightarrow t_i - t_{i-1} = (m-1)^2 + m + 2(m-i-1) = m^2 + m - 2i - 1$. Therefore

$$\begin{aligned}
t_m &= t_0 + \sum_{k=1}^{m}(t_k - t_{k-1}) \\
&= t_0 + (m-1) + (m-1)^2 + m + \sum_{k=1}^{m-2}\left(m^2 + m - 2i - 1\right) \\
&= 1 + m^2 + (m-2)(m^2 + m - 1) - 2\sum_{k=1}^{m-2} i \\
&= 1 + m^2 + (m-2)(m^2 + m - 1) - (m-2)(m-1) \\
&= m^3 - m^2 + 1 = \Omega(m^3)
\end{aligned}$$

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So the covering time is at least $t_{u,v}$ which is $\Omega(m^3)$. Therefore the covering time is $\Theta(m^3)$.

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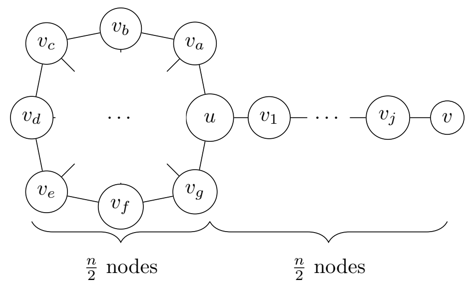

Figure 1: Lollipop graph

Question 3.a. Adopting the notation from class, we'll use $h_{i,j}$ to denote the expected hitting time from a node i to j, and $C_{i,j} = h_{i,j} + h_{j,i}$ as the commute time from i to j.

We will tackle the task of calculating the expected cover time of this graph when starting from v by subdividing it into two parts: a) the expected time to cover the nodes in the linear portion of the graph C_1, and separately b) the time to cover the nodes in the clique C_2 (after we have covered the linear subgraph, and thus are at u).

The overall cover time will then be bounded by $C_1 + C_2$.

Theorem $C_1 = \frac{n^2}{4} = O(n^2)$**.** *Proof.* This follows trivially from the fact that as this subgraph is linear, the cover time is exactly the hitting time from node v to u - i.e $h_{u,v}$. This is because reaching u is equivalent to passing through all the nodes from v to u.

Then, using the fact that the hitting time of a random walk on a linear graph is n^2 (proven during the analysis of randomised 2-SAT), we can thus conclude that the cover time of the linear segment is $\left(\frac{n}{2}\right)^2 = \frac{n^2}{4}$. $\square$

Theorem $C_2 = O(n^2)$**.** *Proof.* By the previous part, we will assume that we are now starting from node u, and will attempt to bound the time to cover the clique. In order to bound the cover time of the clique, we will create an overestimate by considering the commute time from u to all other nodes in the clique - i.e:

$$C_2 \leq \sum_{\substack{v \in V \\ v \neq u}} C_{u,v} = \sum_{\substack{v \in V \\ v \neq u}} h_{u,v} + h_{v,u}$$

The key point to notice in the subsequent analysis is that the nodes in the clique are almost all identical except for u (the node upon which we enter the clique). This is because, from any other node inside the clique, we can either move to another similar node in the clique or move to u, whereas when we are at the node u, our choices are to either move to some other node in the clique, or to move onto the linear subgraph.

We'll now consider bounding $h_{u,v}$ and $h_{v,u}$ separately.

Case 1. *Bounding* $h_{u,v} = O(n)$

When transitioning from u*, there are three possible events that could happen (setting* $m = n/2$*):*

1. *Transition directly to* v *in 1 step with probability* $\frac{1}{m}$
2. *Transition to the previous node* v_1 *on the linear subgraph with probability* $\frac{1}{m}$
3. *Otherwise, transition to some other node* v' *in the clique with probability* $\frac{m-2}{m}$

Hence, using the rules of conditional probability:

$$h_{u,v} = \underbrace{(1)\frac{1}{m}}_{\substack{\text{transition} \\ \text{directly to } v}} + \underbrace{(1 + h_{v_1,v})\frac{1}{m}}_{\substack{\text{transition} \\ \text{to linear } v_1}} + \underbrace{(1 + h_{v',v})\frac{m-2}{m}}_{\substack{\text{transition} \\ \text{to some } v'}}$$

which simplifies to:

$$h_{u,v} = \frac{1}{m} h_{v_1,v} + \frac{m-2}{m} h_{v',v} + 1$$

If we transition to some node v' *in the clique, then the next transition will fall into one of the following possibilities:*

1. *Transition directly to* v *in 1 step with probability* $\frac{1}{m-1}$
2. *Transition back to* u *with probability* $\frac{1}{m-1}$

3. Otherwise, Transition to some other node v'' in the clique with probability $\frac{m-3}{m-1}$.

Hence, using the rules of conditional probability:

$$h_{v',v} = \underbrace{(1)\frac{1}{m-1}}_{\substack{\text{transition}\\ \text{directly to } v}} + \underbrace{(1+h_{u,v})\frac{1}{m-1}}_{\substack{\text{transition}\\ \text{back to } u}} + \underbrace{(1+h_{v',v})\frac{m-3}{m-1}}_{\substack{\text{transition}\\ \text{to some } v''}}$$

which simplifies to:

$$h_{v',v} = \frac{1}{m-1}h_{u,v} + \frac{m-3}{m-1}h_{v',v} + 1$$

Rearranging in terms of $h_{u,v}$:

$$\begin{aligned} h_{v',v} &= \frac{1}{m-1}h_{u,v} + \frac{m-3}{m-1}h_{v',v} + 1 \\ \frac{2}{m-1}h_{v',v} &= \frac{1}{m-1}h_{u,v} + 1 \\ h_{v',v} &= \frac{1}{2}h_{u,v} + \frac{m-1}{2} \end{aligned}$$

Hence:

$$h_{v',v} = \frac{1}{2}h_{u,v} + \frac{m-1}{2}$$

Finally, for the case when we move onto the linear subgraph, each subsequent transition from a node v_i (until we reach u again) has two possible outcomes:

1. Transition backwards to $i-1$ with probability $\frac{1}{2}$
2. Transition forwards to $i+1$ with probability $\frac{1}{2}$

Hence:

$$h_{v_i,v} = \left(h_{v_{i-1},v} + 1\right)\frac{1}{2} + \left(h_{v_{i+1},v}\right)\frac{1}{2}$$

With an additional case when we reach v_m:

$$h_{v_m,v} = h_{v_{m-1},v} + 1$$

Substituting the expression for v_m into the case v_{m-1}:

$$\begin{aligned} h_{v_{m-1},v} &= \left(h_{v_{m-2},v} + 1\right)\frac{1}{2} + \left(h_{v_{m-1},v} + 1\right)\frac{1}{2} \\ &= \frac{1}{2}h_{v_{m-2},v} + \frac{1}{2}h_{v_{m-1},v} + 1 \\ \frac{1}{2}h_{v_{m-1},v} &= \frac{1}{2}h_{v_{m-2},v} + 1 \\ h_{v_{m-1},v} &= h_{v_{m-2},v} + 2 \end{aligned}$$

Repeating this again:

$$\begin{aligned} h_{v_{m-2},v} &= \left(h_{v_{m-3},v} + 1\right)\frac{1}{2} + \left(h_{v_{m-1},v} + 1\right)\frac{1}{2} \\ &= \frac{1}{2}h_{v_{m-3},v} + \frac{1}{2}h_{v_{m-1},v} + 1 \\ &= \frac{1}{2}h_{v_{m-3},v} + \frac{1}{2}\left(h_{v_{m-2},v} + 2\right) + 1 \\ &= \frac{1}{2}h_{v_{m-3},v} + \frac{1}{2}h_{v_{m-2},v} + 2 \\ &= h_{v_{m-3},v} + 4 \end{aligned}$$

Generalising this pattern by induction, we find that the hitting time for the $m-i$th node in the linear subgraph is related to the hitting time for the node before it by the following equation:

$$h_{v_{m-i},v} = h_{v_{m-i-1},v} + 2i$$

Hence, when at v_1, the hitting time is given by:

$$\begin{aligned} h_{v_1,v} &= h_{u,v} + 2(m-1) \\ &= h_{u,v} + 2m - 2 \end{aligned}$$

Combining all of these back into our original equation:

$$\begin{aligned}
h_{u,v} &= \frac{1}{m}h_{v_1,v} + \frac{m-2}{m}h_{v',v} + 1 \\
&= \frac{1}{m}\left(h_{u,v} + 2m - 2\right) + \frac{m-2}{m}\left(\frac{1}{2}h_{u,v} + \frac{m-1}{2}\right) + 1 \\
&= \frac{1}{m}h_{u,v} - \frac{2}{m} + \frac{m-2}{2m}h_{u,v} + \frac{(m-2)(m-1)}{2m} + 3 \\
&= \frac{1}{2}h_{u,v} + \frac{(m-2)(m-1)-4}{2m} + 3 \\
\frac{1}{2}h_{u,v} &= \frac{m^2-3m-2}{2m} + 3 \\
h_{u,v} &= \frac{m^2+3m-2}{m} = O(n)
\end{aligned}$$

Case 2. *Bounding* $h_{v,u} = O(n)$

We follow a similar analysis as before, and consider the possibilities when transitioning from v*:*

1. *Transition directly to* u *in 1 step with probability* $\frac{1}{m-1}$
2. *Transition to some other node in the clique* v' *with probability* $\frac{m-2}{m-1}$

Hence, using the rules of conditional probability:

$$\begin{aligned}
h_{v,u} &= \underbrace{(1)\frac{1}{m-1}}_{\substack{\text{transition} \\ \text{directly to } u}} + \underbrace{(1+h_{v,u})\frac{m-2}{m-1}}_{\substack{\text{transition} \\ \text{to some } v'}} \\
&= h_{v,u}\frac{m-2}{m-1} + 1
\end{aligned}$$

Bringing all terms of $h_{v,u}$ *to one side:*

$$\frac{1}{m-1}h_{v,u} = 1$$

Hence:

$$h_{v,u} = n/2 - 1 = O(n)$$

As both $h_{u,v}$ and $h_{v,u}$ are bound by $O(n)$, we can bound C_2 as follows:

$$\begin{aligned}
C_2 &\leq \sum_{\substack{v \in V \\ v \neq u}} h_{u,v} + h_{v,u} \\
&\leq \sum_{\substack{v \in V \\ v \neq u}} O(n)
\end{aligned}$$

As there are exactly $n/2$ nodes that we're iterating over:

$$\leq n \times O(n) = O(n^2)$$

$\square$

Finally, combining the two bounds on C_1 and C_2 we can thus conclude that the cover time starting from v is $O(n^2)$.

Question 3.b. We will adopt the same strategy as used for question 3.a, and upper bound the cover time by subdividing it into two parts: a) the expected time to cover the nodes in the linear portion of the graph C_1, and separately b) the time to cover the nodes in the clique C_2.

The overall cover time will then be bounded by $C_1 + C_2$.

We now note that the time to cover the clique (C_2) follows exactly the same calculation as used in question 3.a, and thus has the same upper bound of $O(n^2)$.

Theorem $C_2 = O(n^2)$**.** *Proof.* By question 3.a. $\square$

We now move to bound C_1:

Theorem $C_1 = O(n^3)$**.** *Proof.* By the previous part, we now assume that we have covered the clique and are starting from u. As such, the cover time of the remaining linear subgraph is then just the hitting time of v from u - i.e $h_{u,v}$ (following the same reasoning that was presented in question 3.a).

We will now consider the possible outcomes when transitioning from u:

1. Transition to v_1 with probability $\frac{1}{m}$
2. Transition to some other node in the clique with probability $\frac{m-1}{m}$

Applying the laws of conditional probability:

$$h_{u,v} = \underbrace{(1 + h_{v_1,v})\frac{1}{m}}_{\substack{\text{transition} \\ \text{to } v_1}} + \underbrace{(1 + h_{v',v})\frac{m-1}{m}}_{\substack{\text{transition} \\ \text{back to the clique}}}$$

which simplifies to:

$$h_{u,v} = \frac{1}{m}h_{v_1,v} + \frac{m-1}{m}h_{v',v} + 1$$

Similarly, when we are inside the clique, then the possible outcomes when transitioning are:

1. Transition to u with probability $\frac{1}{m-1}$
2. Transition to some other node in the clique with probability $\frac{m-2}{m-1}$

Applying the laws of conditional probability:

$$h_{v',v} = \underbrace{(1 + h_{u,v})\frac{1}{m-1}}_{\substack{\text{transition} \\ \text{to } u}} + \underbrace{(1 + h_{v',v})\frac{m-2}{m-1}}_{\substack{\text{transition} \\ \text{some other node } v''}}$$

Which simplifies to:

$$h_{v',v} = h_{u,v} + m - 1$$

Now, consider the possibilities when transitioning from some node v_i on the linear subgraph:

1. Transition forwards v_{i+1} with probability $\frac{1}{2}$
2. Transition backwards to v_{i-1} with probability $\frac{1}{2}$

Applying the laws of conditional probability:

$$\begin{aligned} h_{v_i,v} &= \underbrace{(1 + h_{v_{i+1},v})\frac{1}{2}}_{\substack{\text{transition} \\ \text{forwards}}} + \underbrace{(1 + h_{v_{i-1},v})\frac{1}{2}}_{\substack{\text{transition} \\ \text{backwards}}} \\ &= \frac{1}{2}h_{v_{i+1},v} + \frac{1}{2}h_{v_{i-1},v} + 1 \end{aligned}$$

With an additional special case for v_{m-1}, where with probability $\frac{1}{2}$ we hit v:

$$\begin{aligned} h_{v_{m-1},v} &= \underbrace{(1)\frac{1}{2}}_{\text{hit } v} + \underbrace{(1 + h_{v_{m-2},v})\frac{1}{2}}_{\substack{\text{transition} \\ \text{backwards}}} \\ &= \frac{1}{2}h_{v_{m-2},v} + 1 \end{aligned}$$

Plugging this into v_{m-2}:

$$\begin{aligned} h_{v_{m-2},v} &= \frac{1}{2}h_{v_{m-1},v} + \frac{1}{2}h_{v_{m-3},v} + 1 \\ &= \frac{1}{2}\left(\frac{1}{2}h_{v_{m-2},v} + 1\right) + \frac{1}{2}h_{v_{m-3},v} + 1 \\ &= \frac{1}{4}h_{v_{m-2},v} + \frac{1}{2}h_{v_{m-3},v} + \frac{3}{2} \\ \frac{3}{4}h_{v_{m-2},v} &= \frac{1}{2}h_{v_{m-3},v} + \frac{3}{2} \\ h_{v_{m-2},v} &= \frac{2}{3}h_{v_{m-3},v} + 2 \end{aligned}$$

Plugging this into v_{m-3}:

$$\begin{aligned}
h_{v_{m-3},v} &= \frac{1}{2}h_{v_{m-2},v} + \frac{1}{2}h_{v_{m-4},v} + 1 \\
&= \frac{1}{3}h_{v_{m-3},v} + \frac{1}{2}h_{v_{m-4},v} + 2 \\
\frac{2}{3}h_{v_{m-3},v} &= \frac{1}{2}h_{v_{m-4},v} + 2 \\
h_{v_{m-3},v} &= \frac{3}{4}h_{v_{m-4},v} + 3
\end{aligned}$$

Generalising this pattern by induction, we find:

$$h_{v_{m-j},v} = \frac{j}{j+1}h_{v_{m-j-1},v} + j$$

Hence, we can express $h_{v_1,v}$ as follows:

$$h_{v_1,v} = \frac{m-1}{m}h_{u,v} + m - 1$$

Substituting these all back into our expression for $h_{u,v}$:

$$\begin{aligned}
h_{u,v} &= \frac{1}{m}h_{v_1,v} + \frac{m-1}{m}h_{v',v} + 1 \\
&= \frac{1}{m}\left(\frac{m-1}{m}h_{u,v} + m - 1\right) + \frac{m-1}{m}\left(h_{u,v} + m - 1\right) + 1 \\
&= \frac{m-1}{m^2}h_{u,v} - \frac{1}{m} + \frac{m-1}{m}h_{u,v} + \frac{(m-1)^2}{m} + 2 \\
&= \frac{m^2-1}{m^2}h_{u,v} + m \\
\frac{1}{m^2}h_{u,v} &= m \\
h_{u,v} &= m^3 = O(n^3)
\end{aligned}$$

□

As such, combining these two bounds, we can now conclude that the cover time is bounded by $O(n^3)$.