MATH 53 NOTE: 04/11/2013

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Example 1. *Find the explicit solution of*

$$y' = \frac{e^{-x}}{2y+3}, \quad y(0) = -1$$

and determine the maximal interval of existence

Solution. Separate the variable,

$$(2y+3)dy = e^{-x}dx \quad \Rightarrow \quad \int (2y+3)dy = \int e^{-x}dx + C \quad \Rightarrow \quad y^2 + 3y = -e^{-x} + C$$

Plugging in $y(0) = -1$,

$$1 - 3 = -1 + C \quad \Rightarrow \quad C = -1$$

so the solution $y(x)$ satisfies

$$y^2(x) + 3y(x) = -e^{-x} - 1$$

By the quadratic formula (or completing the square), we get

$$y(x) = \frac{-3 \pm \sqrt{9 - 4(e^{-x}+1)}}{2} = \frac{-3 \pm \sqrt{5 - 4e^{-x}}}{2}$$

But $y(0) = -1$, so

$$y(x) = \frac{-3 + \sqrt{5 - 4e^{-x}}}{2}$$

The solution is well-defined only when

$$5 - 4e^{-x} \geq 0 \quad \Rightarrow \quad x \geq -\ln(5/4)$$

Thus, the maximal interval of existence is (-ln$(5/4), \infty)$.

Example 2. *Find the explicit solution of*

$$ty' + y^2 = 0, \quad y(1) = 1$$

and determine the maximal interval of existence

Solution. Separate the variable

$$-\frac{1}{y^2}dy = \frac{1}{t}dt \quad \Rightarrow \quad \int -\frac{1}{y^2}dy = \int \frac{1}{t}dt + C \quad \Rightarrow \quad \frac{1}{y(t)} = \ln|t| + C$$

Plugging in $y(1) = 1$,

$$1 = 0 + C \quad \Rightarrow \quad C = 1$$

so the solution $y(t)$ is

$$y(t) = \frac{1}{\ln|t| + 1}$$

The solution will not be defined when $\ln|t| = -1$ or $t = 0$. The first case happens when $|t| = e^{-1}$ or $t = \pm e^{-1}$. Combining all cases, the solution will not be defined when $t = -e^{-1}, 0, e^{-1}$. Since the initial is given at $t = 1$, the maximum interval of existence is (e^{-1}, ∞).

Example 3. *Find the explicit solution of*

$$y'(t) - e^{-y}\cos t = 0, \quad y(0) = y_0$$

For which y_0 is the solution $y(t)$ defined for all $-\infty < t < \infty$.

Solution Separate the variable

$$y'(t) = e^{-y}\cos t \quad \Rightarrow \quad e^y dy = \cos t dt \quad \Rightarrow \quad \int e^y dy = \int \cos t dt$$

Thus,

$$e^{y(t)} = \sin t + C$$

Plugging $y(0) = y_0$ (think of y_0 as some number given to you),

$$e^{y_0} = 0 + C \quad \Rightarrow \quad C = e^{y_0}$$

so the solution is

$$e^{y(t)} = \sin t + e^{y_0} \quad \Rightarrow \quad y(t) = \ln(\sin t + e^{y_0})$$

For solution to be defined everywhere, the term inside the ln function must be positive, that is

$$e^{y_0} > -\sin t, \text{for all } t$$

$-\sin t$ has a maximum value of 1, so we must have

$$e^{y_0} > 1 \quad \Rightarrow \quad y_0 > 0$$

Example 4. *Consider the ODE*

$$y' = y^2 - y^4$$

Determine all equilibrium points and its stability properties.

Solution The equilibrium is where $y'(t) = 0$, so we solve

$$y^2 - y^4 = 0 \quad \Rightarrow \quad y^2(1-y)(1+y) = 0 \quad \Rightarrow \quad y = -1, 0, 1$$

Drawing the directional field or just by looking at the sign of $y'(= y^2 - y^4)$ when y is in $(-\infty, -1), (-1, 0), (0, 1), (1, \infty)$. The sign is -,+,+,- respectively, so we deduce that -1 is unstable, 1 is asymptotically stable. 0 is called semistable. Note that we can solve this equation by using separation of variable and partial fraction (try it!). The solution cannot be described explicitly, but yet we can tell a lot about the solution by looking at the directional field plot.