MATH 53 NOTE: 05/16/2013

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1. Second order linear ODE with constant coefficients: inhomogeneous

In this note, we will discuss how to solve an equation of the form

$$ay''(t) + by'(t) + cy(t) = g(t)$$

when $g(t) \neq 0$. To solve this equation, we do the following

(1) Solve the homogeneous equation to get two fundamental solutions $y_1(t), y_2(t)$
(2) Find a particular solution $y_p(t)$ of the original inhomogeneous equation.
(3) The general solution is simply $y_p(t) + C_1 y_1(t) + C_2 y_2(t)$.
(4) Find $y'(t)$ if need to, then plugging in the initial condition to find C_1, C_2

The first step is discussed in the previous note. To find a particular solution, we will use what is called the method of undetermined coefficients. It just means that we will guess the form of $y_p(t)$ (particular solution). The idea is to get the form of $y_p(t)$ to be the same as $g(t)$. For example

$g(t)$	$y_p(t)$
$3t^2 + 1$	$At^2 + Bt + C$
t	$At + B$
$\cos 2t$	$A\cos t + B\sin t$
e^{-3t}	Ae^{-3t}

See page 272 for more detail. The problem will arise if one of the term in our guess is a solution to the homogeneous one since it will be zero when plugging in to the ODE. It turns out that we can get around that by multiplying the guessed form by t. If that form still contains the solution (could happen in the case of repeated root), then we will need to multiply by t^2 instead. After guessing the form, then we need to plug it in to determine all the coefficients.

Example 1. *Find general solution of $y'' + 5y' + 6y = 3t^2 + 1$*

Solution. the characteristic equation is

$$\lambda^2 + 5\lambda + 6 = 0 \quad \Rightarrow \quad (\lambda + 3)(\lambda + 2) = 0 \quad \Rightarrow \quad \lambda = -2, -3$$

The general solution to the homogeneous is then given by

$$y(t) = C_1 e^{-2t} + C_2 e^{-3t}$$

Now we find a particular solution of the form (because $g(t) = 3t^2 + 1$)

$$y_p(t) = At^2 + Bt + C$$

We plug it back to our inhomogeneous equation to find A, B, C,

$$\begin{array}{r} y'(t) = 2At + B \\ y''(t) = 2A \end{array} \quad \Rightarrow \quad 3t^2+1 = y''+5y'+6y = 2A+5(2tA+B)+6(At^2+Bt+C) = 6At^2+(10A+6B)t+2A+5B+6C$$

So

$$\begin{array}{r} 6A = 3 \\ 10A + 6B = 0 \\ 2A + 5B + 6C = 1 \end{array} \quad \Rightarrow \quad \begin{array}{r} A = 1/2 \\ B = -5/6 \\ C = 25/36 \end{array}$$

So

$$y_p(t) = \frac{t^2}{2} - \frac{5t}{6} + \frac{25}{36}$$

The general solution is then given by

$$y(t) = \frac{t^2}{2} - \frac{5t}{6} + \frac{25}{36} + C_1 e^{-2t} + C_2 e^{-3t}$$

Example 2. *Find general solution of* $y'' + 4y = \cos 3t$

Solution. the characteristic equation is

$$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda = \pm 2i$$

The general solution to the homogeneous is then given by

$$y(t) = C_1 \cos 2t + C_2 \sin 2t$$

Now we find a particular solution of the form (because $g(t) = \cos 3t$)

$$y_p(t) = A\cos 3t + B\sin 3t$$

We plug it back to our inhomogeneous equation to find A, B, C,

$$\begin{aligned} y'(t) &= -3A\sin 3t + 3B\cos 3t \\ y''(t) &= -9A\cos 3t - 9B\sin 3t \end{aligned}$$

So

$$\cos 3t = y'' + 4y = -9A\cos 3t - 9B\sin 3t + 4(A\cos 3t + B\sin 3t) = -5A\cos 3t - 5B\sin 3t$$

Then we get

$$\begin{matrix} -5A = 1 \\ -5B = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} A = -1/5 \\ B = 0 \end{matrix}$$

So

$$y_p(t) = -\frac{\cos 3t}{5}$$

The general solution is then given by

$$y(t) = -\frac{\cos 3t}{5} + C_1 \cos 2t + C_2 \sin 2t$$

Example 3. *Find general solution of* $y'' + 3y' + 2y = 2e^{-t}$

Solution. the characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad (\lambda + 2)(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda = -1, -2$$

The general solution to the homogeneous is then given by

$$y_{hom}(t) = C_1 e^{-t} + C_2 e^{-2t}$$

Now we find a particular solution of the form (because $g(t) = 2e^{-t}$)

$$y_p(t) = Ae^{-t}$$

But this won't work since e^{-t} is a solution. If we plug this in, we will simply get zero and we cannot find A that will make it equal to $g(t)$. So we multiply by t and use

$$y_p(t) = Ate^{-t}$$

We plug it back to our inhomogeneous equation to find A,

$$\begin{aligned} y'(t) &= A(e^{-t} - te^{-t}) \\ y''(t) &= A(te^{-t} - 2e^{-t}) \end{aligned} \quad \Rightarrow \quad 2e^{-t} = y'' + 3y' + 2y = A(te^{-t} - 2e^{-t}) + 3A(e^{-t} - te^{-t}) + 2Ate^{-t} = Ae^{-t}$$

So $A = 2$ and we get

$$y_p(t) = 2te^{-t}$$

The general solution is then given by

$$y(t) = 2te^{-t} + C_1 e^{-t} + C_2 e^{-2t}$$