MATH 53 NOTE: 04/30/2013

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1. Solving $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$

Recall that our strategy for solving $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$ is as follows;

- First, we look for two "independent" solutions, called them $\vec{\mathbf{x}}_1(t), \vec{\mathbf{x}}_2(t)$.
- Form a general solutions by taking a linear combination of those two solutions, that is, the general solution is
$$C_1\vec{\mathbf{x}}_1(t) + C_2\vec{\mathbf{x}}_2(t)$$
- Plug in the initial condition (if given) to find C_1, C_2.

Let's look at the one dimensional problem to get some idea of how to find a solution for 2D problem. The analogue of this equation in 1D is simply
$$u'(t) = cu(t)$$
and a solution is given by
$$u(t) = e^{ct}$$
From that, we will first look for a solution of the form
$$\vec{\mathbf{x}}(t) = e^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix}$$
We want to find out which λ, a, b will make $\vec{\mathbf{x}}(t)$ a solution. By plugging this back in $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$, we get that
$$\lambda e^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix} = Ae^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix} \quad \Rightarrow \quad A\begin{bmatrix} a \\ b \end{bmatrix} = \lambda\begin{bmatrix} a \\ b \end{bmatrix}$$
That is, we have just shown that **if λ is an eigenvalue and $\begin{bmatrix} a \\ b \end{bmatrix}$ is a corresponding eigenvector of matrix A, then**
$$\vec{\mathbf{x}}(t) = e^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix}$$
is a solution to $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$.

The recipe for finding two solutions will depend on eigenvalues of matrix A, which can be divided into three cases;

- A has two different real eigenvalues $\lambda_1 \neq \lambda_2$
- A has complex eigenvalues $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i$
- A has repeated eigenvalues $\lambda_1 = \lambda_2$

2. Case I: two different real eigenvalues $\lambda_1 \neq \lambda_2$

Assume A has two different real-valued eigenvalues $\lambda_1 \neq \lambda_2$. Let $\vec{v}_1, \vec{v}_2$ denote the corresponding eigenvectors. In this case, we have two solution right away
$$\vec{\mathbf{x}}_1(t) = e^{\lambda_1 t}\vec{v}_1, \quad \vec{\mathbf{x}}_2(t) = e^{\lambda_2 t}\vec{v}_2$$
so the general solution is given by
$$\vec{\mathbf{x}}(t) = C_1e^{\lambda_1 t}\vec{v}_1 + C_2e^{\lambda_2 t}\vec{v}_2$$

Example 1. *Solve system of ODE*
$$x'(t) = 3x(t) + 2y(t), y'(t) = x(t) + 2y(t), x(0) = 0, y(0) = -3$$

Solution. We first write an equation in a matrix form

$$\vec{\mathbf{x}}'(t) = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \vec{\mathbf{x}}(t), \qquad \vec{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

We then find an eigenvalue of $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$,

$$\lambda I - A = \begin{bmatrix} \lambda - 3 & -2 \\ -1 & \lambda - 2 \end{bmatrix} \quad \Rightarrow \quad (\lambda - 3)(\lambda - 2) - 2 = 0 \quad \Rightarrow \quad \lambda^2 - 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda = 4, 1$$

Start with $\lambda_1 = 4$,

$$4I - A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \quad \Rightarrow \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have found one solution to the system of ODE,

$$\vec{\mathbf{x}}_1(t) = e^{4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Next, we work with $\lambda_2 = 1$,

$$1I - A = \begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix} \quad \Rightarrow \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We have found a second solution to the system of ODE,

$$\vec{\mathbf{x}}_2(t) = e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Now that we have found two solutions, the general solution is simply

$$\vec{\mathbf{x}}(t) = C_1 e^{4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Plugging in initial condition $t = 0$ yields

$$\begin{bmatrix} 0 \\ -3 \end{bmatrix} = \vec{\mathbf{x}}(0) = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2C_1 + C_2 \\ C_1 - C_2 \end{bmatrix}$$

Solving this, we get $C_1 = -1, C_2 = 2$, thus the solution to the IVP is

$$\vec{\mathbf{x}}(t) = -e^{4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2e^{-4t} + 2e^{t} \\ -e^{4t} - 2e^{t} \end{bmatrix}$$