MATH 53 NOTE: 04/04/2013

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Linear n^{th} order ODE is the ODE of the form

$$y^{(n)}(t) + a_{n-1}(t)y^{n-1}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = b(t)$$

where $a_{n-1}(t), \ldots, a_0(t), b(t)$ is given. They are terms involving constant, t, but NO $y(t)$ in there. When $b(t) = 0$, it's called homogeneous linear ODE, otherwise it is called inhomogeneous linear ODE.

Linear first order ODE is simply the ODE of the form

$$y'(t) + a(t)y(t) = b(t)$$

Note that there is no y in the terms $a(t), b(t)$.

Example 1. *Classify the following ODEs*

(1) $y' = e^{-t} + y$
(2) $y' = y(y + 2)$
(3) $y' = (\ln y)e^t$
(4) $y' = k(T - y)$
(5) $y' = 3y$

(1),(4) are inhomogeneous linear ODE, (5) is homogeneous linear ODE, (2),(3) are non-linear ODE.

In this note, we will solve the linear first order ODE with constant coefficient, i.e. $a(t) = a$. Consider the ODE

$$y'(t) + ay(t) = b(t)$$

We would like to make the left-hand side (LHS) to look like terms in the product rule. To do that we multiply both sides by $v(t)$ (to be determined), then we get

$$v(t)y'(t) + av(t)y(t) = v(t)b(t)$$

We want to make the LHS equal to

$$(v(t)y(t))' = v(t)y'(t) + v'(t)y(t)$$

Working backwards, we want to choose $v(t)$ that satisfies

$$v'(t) = av(t)$$

One such $v(t)$ is simply

$$v(t) = e^{at}$$

$v(t)$ is called an integrating factor. By letting $v(t) = e^{at}$, we now get

$$e^{at}y'(t) + ae^{at}y(t) = e^{at}b(t)$$

From the product rule, we can group the LHS and get

$$(e^{at}y(t))' = e^{at}b(t)$$

so

$$e^{at}y(t) = \int e^{at}b(t) + C$$

Thus,

$$y(t) = e^{-at}\int e^{at}b(t) + C$$

Example 2. *Solve $u'(t)+5u(t)=0$. From the discussion above, one see that the integrating factor is simply e^{5t}. Multiply both sides with that, we get*

$$e^{5t}u'(t)+5e^{5t}u(t)=0$$

The product rule tells us the LHS is just

$$(e^{5t}u(t))'=0$$

Integrating both sides, we get

$$e^{5t}u(t)=C$$

so

$$u(t)=Ce^{-5t}$$

COMMON MISTAKES (1) When multiply the whole equation with an integrating factor, don't forget to multiply the RHS as well.
(2) Don't forget the constant when integrating both side.

Example 3. $u'+2u=t, u(0)=3/4$. *The integrating factor is simply e^{2t} and the ODE becomes*

$$(e^{2t}u(t))'=te^{2t}$$

Don't forget the e^{2t} on the RHS. Integrating both side, we get

$$e^{2t}u(t)=\int te^{2t}dt+C$$

We need to use integration by part to deal with the integral on the RHS

$$\int te^{2t}dt=\int td(\frac{e^{2t}}{2})=\frac{te^{2t}}{2}-\int\frac{e^{2t}}{2}dt=\frac{te^{2t}}{2}-\frac{e^{2t}}{4}$$

so

$$e^{2t}u(t)=\frac{te^{2t}}{2}-\frac{e^{2t}}{4}+C$$

or

$$u(t)=\frac{t}{2}-\frac{1}{4}+Ce^{-2t}$$

Plug in $t=0$ and use the given assumption that $u(0)=3/4$, we get

$$3/4=u(0)=-1/4+C, \qquad \Rightarrow C=1$$

so the solution is

$$u(t)=\frac{t}{2}-\frac{1}{4}+e^{-2t}$$

1. Directional Field

Directional field is the plot to visualize the dynamic of the solution of the ODE. The horizontal axis is t and the vertical axis is u. The plot at point (t_0,u_0) is the slope represents $u'(t)$ when $u(t)=u_0$. That is, given the ODE, we plug in $t=t_0, u(t)=u_0$ and find $u'(t_0)$. We then draw the slope represents that values. For instance, when $u'(t)=0$, we draw the horizontal segment to represent the fact that u is unchanged (since $u'(t)=0$). When $u'(t)$ is positive, we draw an upward segment to represent that fact that u is moving up (since $u'(t)$ is positive).

Example 4. *Draw the directional derivative of $u'=u(u-3)$*

Note that when $u(t_0)=0$ or 3, $u'(t)=0$, so we get the flat line at those values. Try plugging $u=0.5,1,1,5,2,,$ we see that $u'(t)$ is negative and is independent of t, so we get downward sloping. See figure 1.1.13 for the complete plot.