MATH 53 NOTE: 04/16/2013

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1. Exact equation

Recall that the first order ODE
$$M(x,y) + N(x,y)y' = 0$$
is called *exact* if
$$M_y(x,y) = N_x(x,y)$$
In this case, there exist $H(x,y)$ such that
$$H_x(x,y) = M(x,y), \quad H_y(x,y) = N(x,y)$$
and the solution is given by
$$H(x, y(x)) = C$$
Note that *separable* equation is a special case of exact equation. When the ODE is separable, we get
$$M_y = 0 = N_x$$
To find H, we start with either $H_x = M$ or $H_y = N$, then we take the integral. Since we are dealing with partial derivative, the constant is a function of another variable. Then we need the other condition to determine the constant. This might be a bit confusing for now, but let's look at some examples to see how this works.

Example 1. *Determine if the following ODE is exact or not.*

(1) $2y + xy' = 0$
(2) $2xy + x^2y' = 0$
(3) $xy' = -1 - 2x^2$
(4) $e^{x^2+y}(1+2x^2) + xe^{x^2+y}y' = 0$
(5) $3xy + y^2 + (x^2+xy)y' = 0$
(6) $3x^2y + xy^2 + (x^3+x^2y)y' = 0$

Answer (1),(3),(5) are not exact, but after multiplying by integrating factors, we get (2),(4),(6) [respectively] which are exact.

Example 2. *Solve*
$$3x(xy-2) + (x^3+2y)y' = 0$$
with initial condition $y(0) = 1$

Solution Note that
$$M(x,y) = 3x(xy-2), \quad N(x,y) = x^3 + 2y$$
and
$$M_y = 3x^2 = N_x$$
so the ODE is exact. We want to find $H(x,y)$ such that
$$H_x = 3x(xy-2) = 3x^2y - 6x, \quad H_y = x^3 + 2y$$
We start with the second condition. Taking the integral with respect to y gives
$$H(x,y) = x^3y + y^2 + C(x)$$
Taking derivative of this gives
$$H_x(x,y) = 3x^2y + y^2 + C'(x)$$

Comparing with the first condition gives

$$C'(x) = -6x, \quad C(x) = -3x^2$$

So we have that

$$H(x,y) = x^3y + y^2 - 3x^2$$

and the general solution is given by

$$x^3y(x) + y(x)^2 - 3x^2 = C$$

Given the initial condition $y(0) = 1$, we plug in $x = 0, y(x) = y(0) = 1$ to find that $C = 1$. The solution is

$$x^3y(x) + y(x)^2 - 3x^2 = 1$$

We can write this out explicitly using quadratic formula to get

$$y(x) = \frac{-x^3 \pm \sqrt{x^6 - 4(-3x^2 - 1)}}{2}$$

Checking with initial $y(0) = 1$, we know it's the solution with the plus sign, so the explicit solution is

$$y(x) = \frac{-x^3 + \sqrt{x^6 + 12x^2 + 4}}{2}$$

Example 3. *Solve*

$$(\cos y + y\cos x) + (\sin x - x\sin y)y' = 0$$

with initial condition $y(0) = -1$

Solution Note that

$$M(x,y) = \cos y + y\cos x, \quad N(x,y) = \sin x - x\sin y$$

and

$$M_y = -\sin y + \cos x = N_x$$

so the ODE is exact. We want to find $H(x,y)$ such that

$$H_x = \cos y + y\cos x, \quad H_y = \sin x - x\sin y$$

This time, we take the first condition and take the integral with respect to x and get

$$H(x,y) = x\cos y + y\sin x + C(y)$$

Taking derivative with respect to y,

$$H_y(x,y) = -x\sin y + \sin x + C'(y)$$

Comparing with the second condition, we get

$$C'(y) = 0 \quad \Rightarrow \quad C(y) = C$$

So the general solution is

$$H(x,y) = C \quad \Rightarrow \quad x\cos y + y\sin x = C$$

Plugging in initial condition $y(0) = -1$, we get

$$C = H(0,-1) = 0 + 0 = 0$$

So the solution is

$$x\cos y + y\sin x = 0$$

In this case, we cannot find an explicit formula.