MATH 53 NOTE: 04/18/2013

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1. Eigenvalues and Eigenvectors

Definition. Given a matrix A, a real or complex number λ is called an *eigenvalue* if there exist NON-ZERO vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$, or equivalently, $(A - \lambda I)\vec{v} = 0$. This non-zero vector is called *eigenvector*.

The last condition in the definition (there exist non-zero vector such that $(A - \lambda I)\vec{v} = 0$) holds if and only if

$$\det(A - \lambda I) = 0$$

This equation is called *characteristic equation* and is what you will use to find eigenvalues. In the case of 2 by 2, this will give a quadratic equation for λ.

Once you find eigenvalues, the next thing is to find an associated eigenvector, this can be done by solving for $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ that satisfies

$$(A - \lambda I)\vec{v} = 0$$

For the case of 2 by 2 matrix, this will give you two equivalent equations, pick one equation and we will get a relationship between two components v_1, v_2. If you only want to find one eigenvector, just plug in either v_1 or v_2 by 1 (or any non-zero number you want) and solve for the other one.

Let's see some example to see how this works.

Example 1. *Find all eigenvalues and eigenvectors of*

$$\begin{bmatrix} -4 & -3 \\ 1 & 0 \end{bmatrix}$$

First we solve for eigenvalues by solving

$$\det(A - \lambda I) = 0 \quad \Rightarrow \quad \det \begin{bmatrix} -4-\lambda & -3 \\ 1 & -\lambda \end{bmatrix} = 0 \quad \Rightarrow \quad (-4 - \lambda)(-\lambda) - (1)(-3) = 0$$

$$\Rightarrow \quad 4\lambda + \lambda^2 + 3 = 0 \quad \Rightarrow \quad (\lambda + 3)(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda = -1, -3$$

Next, we will find an eigenvector associated to eigenvector $= -1$, this can be done by plugging $\lambda = -1$ and solve for $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that

$$(A - (-1) \cdot I)\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} -4 - (-1) & -3 \\ 1 & 0 - (-1) \end{bmatrix} \vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} -3 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

From this we have two equations for v_1, v_2,

$$\begin{matrix} -3(v_1 + v_2) = 0 \\ v_1 + v_2 = 0 \end{matrix} \quad \Rightarrow \quad v_1 + v_2 = 0 \quad \Rightarrow \quad v_2 = -v_1$$

Then we have

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So eigenvectors associated to eigenvalue -1 are any multiple of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If we want to find just one eigenvector, then from

$$v_2 = -v_1$$

we can just plug in $v_1 = 1$, then $v_2 = -v_1 = -1$ and we have $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as an eigenvector. Similarly for eigenvalue -3, we get

$$(A-(-3)\cdot I)\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} -4-(-3) & -3 \\ 1 & 0-(-3) \end{bmatrix}\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

From this we have two equations for v_1, v_2,

$$\begin{array}{l} -v_1 - 3v_2 = 0 \\ v_1 + 3v_2 = 0 \end{array} \quad \Rightarrow \quad v_1 + 3v_2 = 0 \quad \Rightarrow \quad v_1 = -3v_2$$

then we can just plug in $v_2 = 1$, then $v_1 = -3v_2 = -3$ and we have $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ as an eigenvector.

Example 2. *Find all eigenvalues and eigenvectors of*

$$\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$$

First we solve for eigenvalues by solving

$$\det(A-\lambda I) = 0 \quad \Rightarrow \quad \det\begin{bmatrix} 7\text{-}\lambda & 1 \\ -4 & 3-\lambda \end{bmatrix} = 0 \quad \Rightarrow \quad (7-\lambda)(3-\lambda)-(-4)(1) = 0$$

$$\Rightarrow \quad 21 - 10\lambda + \lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda^2 - 10\lambda + 25 = 0 \quad \Rightarrow \quad (\lambda-5)^2 = 0 \quad \Rightarrow \quad \lambda = 5$$

Next, we will find an eigenvector associated to eigenvector $= 5$, this can be done by plugging $\lambda = 5$ and solve for $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that

$$(A-(5)\cdot I)\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} 7-5 & 1 \\ -4 & 3-5 \end{bmatrix}\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

From this we have two equations for v_1, v_2,

$$\begin{array}{l} 2v_1 + v_2 = 0 \\ -4v_1 - 2v_2 = 0 \end{array} \quad \Rightarrow \quad 2v_1 + v_2 = 0 \quad \Rightarrow \quad v_2 = -2v_1$$

Suppose we only want to find just one eigenvector, then from

$$v_2 = -2v_1$$

we can just plug in $v_1 = 1$, then $v_2 = -2v_1 = -2$ and we have $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ as an eigenvector.

Example 3. *Find all eigenvalues and eigenvectors of*

$$\begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix}$$

First we solve for eigenvalues by solving

$$\det(A-\lambda I) = 0 \quad \Rightarrow \quad \det\begin{bmatrix} -1/2-\lambda & 1 \\ -1 & -1/2-\lambda \end{bmatrix} = 0 \quad \Rightarrow \quad (-1/2-\lambda)^2-(-1)(1) = 0$$

$$\Rightarrow \quad 1/4 + \lambda + \lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda^2 + \lambda + 5/4 = 0 \quad \Rightarrow \quad \lambda = \frac{-1 \pm \sqrt{1-4(5/4)}}{2} = -1/2 \pm i$$

Next, we will find an eigenvector associated to eigenvector $= -1/2+i$, this can be done by plugging $\lambda = -1/2+i$ and solve for $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that

$$(A-(-1/2+i)\cdot I)\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} -1/2-(-1/2+i) & 1 \\ -1 & -1/2-(-1/2+i) \end{bmatrix}\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

From this we have two equations for v_1, v_2,

$$\begin{array}{l} -iv_1 + v_2 = 0 \\ -v_1 - iv_2 = 0 \end{array} \quad \Rightarrow \quad -iv_1 + v_2 = 0 \quad \Rightarrow \quad v_2 = iv_1$$

Plug in $v_1 = 1$, then $v_2 = iv_1 = i$ and we have $\begin{bmatrix} 1 \\ i \end{bmatrix}$ as an eigenvector. Similarly for eigenvalue $-1/2 - i$, we get the eigenvector (try this yourself!) $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.