Explicit Bound on Cover Time and Cover Time of Random Walk on Complete Bipartite Graph

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1 Introduction

Let (X_t) be a finite Markov chain with state space Ω. The cover time τ_{cov} of a Markov chain is the first time at which all the states have been visited. The deterministic version of cover time, which is called *maximal cover time*, is defined as

$$t_{\text{cov}} = \max_x \mathbb{E}_x \tau_{\text{cov}}$$

We also define the *maximal hitting time*, which is

$$t_{\text{hit}} = \max_x \mathbb{E}_x \tau_y$$

for any $x, y \in \Omega$, where τ_y is the first hitting time of state y. Intuitively, one can easily see that there should be a monotonic relationship between maximal cover time and maximal hitting time. The classic bound of cover time given by Matthew is one example of such relationship.

Theorem 1.1. *Let (X_t) be an irreducible finite Markov chain on n states, then*

$$t_{cov} \leq t_{hit} \left(1 + \frac{1}{2} + ... + \frac{1}{n}\right)$$

Matthew gave a proof using simple randomization method. A slight modification of Matthew method can be used to prove the lower bound.

Theorem 1.2. *Let $A \subset X$. Set $t^A_{\min} = \min_{a,b\in A, a\neq b} \mathbb{E}_a \tau_b$. Then*

$$t_{cov} \geq \max_{A\subseteq\Omega} t^A_{\min}\left(1 + \frac{1}{2} + ... + \frac{1}{|A|-1}\right)$$

For the proof of Theorem (1.1) and (1.2), please see [1]. The recent development in computing cover time is done by Ding, Lee, Peres [2]. They exhibit a strong connection between cover times of graphs, Gaussian free field, and Talagrand's theory of majorizing measures. The discrete Gaussian free field of a graph G is defined as follows

Definition 1.3. *The Gaussian free field (GFF) on a graph $G = (V, E)$ is a centered Gaussian process $\{\eta_v\}_{v\in V}$ with $\eta_{v_0} = 0$ for some fixed $v_0 \in V$ with the following property*

$$\mathbb{E}(\eta_u - \eta_v)^2 = \frac{\kappa(x, y)}{2|E|}$$

where $\kappa(x, y) = \mathbb{E}_x\tau_y + \mathbb{E}_y\tau_x$ is a commuting time between x and y.

Their main result then states that

Theorem 1.4. *For any graph $G = (V, E)$, we have*

$$t_{cov}(G) \asymp |E|\left(\mathbb{E}\max_{v\in V}\eta_v\right)^2$$

where $\{\eta_v\}$ is a Gaussian free field on G. In other words, there exist a universal constant $A > 0$ such that

$$\frac{1}{A}|E|\left(\mathbb{E}\max_{v\in V}\eta_v\right)^2 \leq t_{cov}(G) \leq A|E|\left(\mathbb{E}\max_{v\in V}\eta_v\right)^2$$

To prove Theorem (1.4), they introduced a δ-*blanket time*, which is denoted by $\tau_{\mathrm{bl}}(\delta)$. It is defined as the first time $t \geq 1$ such that every $u, v \in V$, we have

$$\frac{N_u}{\pi(u)} \geq \delta\frac{N_v}{\pi(v)}$$

where N_w denotes the number of visit to state w and $\pi(w)$ denotes the stationary probability at w. Similar to the hitting time and cover time, a deterministic version of a blanket time is given by

$$t_{\mathrm{bl}} = \max_x \mathbb{E}_x[\tau_{\mathrm{bl}}]$$

We will refer to it as a *maximal δ-blanket time*. They also define a maximal blanket time for the associated continuous-time random walk (the same random walk with independent holding time exponentially distributed with mean 1). We will denote this quantity by $t_{\mathrm{bl}}^C(\delta)$. It is easy to see that

$$t_{\mathrm{cov}} = t_{\mathrm{cov}}^C \leq t_{\mathrm{bl}}^C(\delta) \tag{1}$$

The following theorem provides a bound for $t_{\mathrm{bl}}^C(\delta)$.

Theorem 1.5. *Consider a graph $G = (V, E)$. For any fixed $0 < \delta < 1$, there exist a constant $C = C(\delta)$ such that*

$$t_{bl}^C(G, \delta) \leq C(\delta) \cdot |E| \cdot \left(\mathbb{E} \max_{v \in V} \eta_v\right)^2$$

where $\{\eta_v\}$ is the associated Gaussian Free Field

Combining this result with (1), we have

$$t_{\mathrm{cov}}(G) \leq C(\delta) \cdot |E| \cdot \left(\mathbb{E} \max_{v \in V} \eta_v\right)^2$$

which proves one side of Theorem (1.4). They also exhibit a strong asymptotic upper bound

Theorem 1.6. *For every graph $G = (V, E)$, if $t_{hit}(G)$ denotes the maximal hitting time in G, and $\{\eta_v\}_{v \in V}$ is the Gaussian free field on G, then*

$$t_{cov}(G) \leq \left(1 + C\sqrt{\frac{t_{hit}(G)}{t_{cov}(G)}}\right) \cdot |E| \cdot \left(\mathbb{E} \max_{v \in V} \eta_v\right)^2$$

At the end of the paper, they consider the extent to which Theorem (1.6) is sharp. That is, for a family of graphs $\{G_n\}$, they are curious to know when the asymptotic property,

$$t_{\mathrm{cov}}(G) \sim |E| \cdot \left(\mathbb{E} \max_{v \in V} \eta_v\right)^2 \tag{2}$$

holds. It has been shown that property (2) holds for both the family of complete graphs and the family of regular trees. Theorem (1.5), Theorem (1.6) and property (2) will be the focus of the final project.

2 Outline

In the first part of the paper, I will find an explicit constant in Theorem (1.4) and Theorem (1.6). This can be done easily by directly following all the bounds in the proves of both theorems and their associated lemmas.

In the second part, I will show that a complete bipartite graph satisfies property (2). First, I apply the Matthews upper and lower bound to get an asymptotic of t_{cov}. Then by using the asymptotic property of maximum of iid standard normal random variables and Slepian's lemma, the asymptotic of maximum of Gaussian Free Field can be found.

3 Explicit Bound on Cover Time

In this section, I will find an explicit universal constant in Theorem (1.5) and (1.6).

Theorem 3.1. *Consider a graph $G = (V, E)$. For any fixed $0 < \delta < 1$, we have*

$$t_{bl}^C(G, \delta) \leq C(\delta) \cdot |E| \cdot \left(\mathbb{E} \max_{v \in V} \eta_v\right)^2$$

where

$$C(\delta) = 4(2\pi + 1)\left(2(1 + 2\pi) + e^{-2(1+2\pi)}\right)\left(\frac{1}{a_\delta^2} + \frac{2}{b_\delta^2} + \frac{1}{a_\delta'^2}\right)$$

and

$$a_\delta = \sqrt{2} - \sqrt{1 + \sqrt{\delta}}, \quad b_\delta = \sqrt{1 - \sqrt{\delta}}, \quad a'_\delta = \sqrt{1/\delta} - 1$$

Proof. In the proof of Theorem (1.5) (Theorem 2.3 in [2]), it has been shown that

$$\left\{\tau_{\text{bl}}^C \geq 2|E|t\sqrt{\delta}\right\} \subset \left\{\min_x L_{\tau(t)}^x \leq \sqrt{\delta}t\right\} \bigcup \left\{\max_x L_{\tau(t)}^x \geq t/\sqrt{\delta}\right\} \tag{3}$$

$$\mathbb{P}\left(\min_x L_{\tau(t)}^x \leq \sqrt{\delta}t\right) \leq \mathbb{P}\left(\inf_x \eta_x \leq -a_\delta\sqrt{t}\right) + \mathbb{P}\left(\sup_x |\eta_x| \geq b_\delta\sqrt{t}\right) \tag{4}$$

and

$$\mathbb{P}\left(\max_x L_{\tau(t)}^x \geq t/\sqrt{\delta}\right) \leq \mathbb{P}\left(\max_x \eta_x \geq a'_\delta\sqrt{t}\right) \tag{5}$$

where η_x is a Gaussian Free Field on G, L_t^v is a local time at v, $\tau(t) = \inf\{s : L_s^{v_0} > t\}$ for a fixed vertex v_0 and

$$a_\delta = \sqrt{2} - \sqrt{1+\sqrt{\delta}}, \quad b_\delta = \sqrt{1-\sqrt{\delta}}, \quad a'_\delta = \sqrt{1/\delta} - 1 \tag{6}$$

To use all the bounds above, we need to apply the following lemma (see, for example, [5]).

Lemma 3.2. *Consider a Gaussian process $\{\eta_x : x \in V\}$ and define $\sigma = \sup_{x\in V} \sqrt{\mathbb{E}\eta_x^2}$. Then for $\alpha > 0$*

$$\mathbb{P}\left(|\sup \eta_x - \mathbb{E}\sup \eta_x| > \alpha\right) \le 2e^{-\alpha^2/2\sigma^2}$$

We will use the Lemma (3.2) above to prove the following bound

Lemma 3.3. *Consider a Gaussian free field $\{\eta_x : x \in V\}$ on a finite graph $G = (V, E)$, and define $\sigma = \sup_{x\in V} \sqrt{\mathbb{E}\eta_x^2}, \Lambda = \mathbb{E}\sup \eta_x$. Then for any $\beta \ge 4(1 + 2\pi)/a^2$, the following holds*

$$\mathbb{P}(\sup \eta_x \ge a\sqrt{t}) \le 2e^{-a^2\beta/2}$$

where $t = t(\beta) = \beta(\Lambda^2 + \sigma^2)$.

Proof. First, let x^* be such that $\mathbb{E}\eta_{x^*}^2 = \sigma^2$, and recall that v_0 denotes the fixed vertex such that $\eta_{v_0} = 0$. Then we have

$$\Lambda \ge \mathbb{E}\max(\eta_{v_0}, \eta_{x^*}) = \mathbb{E}\max(0, \eta_{x^*}) = \frac{\sigma}{\sqrt{2\pi}} \tag{7}$$

Thus, we have

$$\frac{\frac{\Lambda}{\sigma}}{\sqrt{(\frac{\Lambda}{\sigma})^2 + 1} - 1} = \frac{\sigma}{\Lambda}\left(\sqrt{(\frac{\Lambda}{\sigma})^2 + 1} + 1\right) = \sqrt{1 + (\frac{\sigma}{\Lambda})^2} + \frac{\sigma}{\Lambda}$$

$$\le \sqrt{1+2\pi} + \sqrt{2\pi} \le 2\sqrt{1+2\pi}$$

From the assumption that $\beta \ge 4(1 + 2\pi)/a^2$, it then follows that

$$\sqrt{\beta} \ge \frac{2(1+2\pi)}{a} \ge \frac{1}{a}\frac{\frac{\Lambda}{\sigma}}{\sqrt{(\frac{\Lambda}{\sigma})^2 + 1} - 1}$$

which is equivalent to

$$\frac{a\sqrt{\beta}}{\sqrt{2}}\sqrt{(\frac{\Lambda}{\sigma})^2+1}-\frac{\Lambda}{\sqrt{2}\sigma}\geq\frac{a\sqrt{\beta}}{\sqrt{2}}$$

That is,

$$\frac{a\sqrt{t}-\Lambda}{\sqrt{2}\sigma}\geq\frac{a\sqrt{\beta}}{\sqrt{2}}\geq 0$$

Therefore,

$$\frac{(a\sqrt{t}-\Lambda)^2}{2\sigma^2}\geq\frac{a^2\beta}{2} \tag{8}$$

Now using (8) and Lemma (3.2), it follows that

$$\mathbb{P}(\sup\eta_x\geq a\sqrt{t})=\mathbb{P}(\sup\eta_x-\Lambda\geq a\sqrt{t}-\Lambda)\leq$$

$$\mathbb{P}(|\sup\eta_x-\Lambda|\geq a\sqrt{t}-\Lambda)\leq e^{-\frac{(a\sqrt{t}-\Lambda)^2}{2\sigma^2}}\leq e^{-a^2\beta/2}$$

and we have proved the lemma $\square$

Back to the proof of the theorem, from (3),(4),(5) and Lemma (3.3), we have

$$\begin{aligned}
\mathbb{P}\left(\tau_{\mathrm{bl}}^C\geq 2|E|t\right) &\leq \mathbb{P}\left(\min_x L^x_{\tau(t)}\leq\sqrt{\delta}t\right)+\mathbb{P}\left(\max_x L^x_{\tau(t)}\geq t/\sqrt{\delta}\right)\\
&\leq \mathbb{P}\left(\inf_x\eta_x\leq -a_\delta\sqrt{t}\right)+\mathbb{P}\left(\sup_x|\eta_x|\geq b_\delta\sqrt{t}\right)+\mathbb{P}\left(\sup_x\eta_x\geq a'_\delta\sqrt{t}\right)\\
&\leq \mathbb{P}\left(\sup_x\eta_x\geq a_\delta\sqrt{t}\right)+\mathbb{P}\left(\sup_x\eta_x\geq b_\delta\sqrt{t}\right)\\
&\quad+\mathbb{P}\left(\inf_x\eta_x\leq -b_\delta\sqrt{t}\right)+\mathbb{P}\left(\sup_x\eta_x\geq a'_\delta\sqrt{t}\right)\\
&= \mathbb{P}\left(\sup_x\eta_x\geq a_\delta\sqrt{t}\right)+\mathbb{P}\left(\sup_x\eta_x\geq b_\delta\sqrt{t}\right)\\
&\quad+\mathbb{P}\left(\sup_x\eta_x\geq b_\delta\sqrt{t}\right)+\mathbb{P}\left(\sup_x\eta_x\geq a'_\delta\sqrt{t}\right)\\
&= \mathbb{P}\left(\sup_x\eta_x\geq a_\delta\sqrt{t}\right)+2\mathbb{P}\left(\sup_x\eta_x\geq b_\delta\sqrt{t}\right)+\mathbb{P}\left(\sup_x\eta_x\geq a'_\delta\sqrt{t}\right)
\end{aligned} \tag{*}$$

For any c, let $\beta_0 = 4(1+2\pi)/c^2$, and thus, by Lemma (3.2),

$$\begin{aligned}\int_0^\infty \mathbb{P}(\sup\eta_x \geq c\sqrt{t})d\beta &= \int_0^{\beta_0} \mathbb{P}(\sup\eta_x \geq c\sqrt{t})d\beta + \int_{\beta_0}^\infty \mathbb{P}(\sup\eta_x \geq c\sqrt{t})d\beta \\ &\leq \beta_0 + \int_{\beta_0}^\infty e^{-c^2\beta/2}d\beta \\ &= \beta_0 + \frac{2}{c^2}e^{-c^2\beta_0/2} \\ &= \frac{1}{c^2}\left(4(1+2\pi) + 2e^{-2(1+2\pi)}\right) \qquad (**)\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\tau_{\mathrm{bl}}^C &= \int_0^\infty \mathbb{P}(\tau_{\mathrm{bl}}^C \geq y)dy \\ &= 2|E|(\Lambda^2+\sigma^2)\int_0^\infty \mathbb{P}(\tau_{\mathrm{bl}}^C \geq 2|E|t(\beta))d\beta \\ &\leq 2|E|\Lambda^2(2\pi+1)\int_0^\infty \mathbb{P}(\tau_{\mathrm{bl}}^C \geq 2|E|t(\beta))d\beta \qquad \text{[from (7)]} \\ &\leq 2|E|\Lambda^2(2\pi+1)\left(4(1+2\pi)+2e^{-2(1+2\pi)}\right)\left(\frac{1}{a_\delta^2}+\frac{2}{b_\delta^2}+\frac{1}{a_\delta'^2}\right) \qquad \text{[from (*) ,(**)]} \\ &= C(\delta)\cdot|E|\cdot\left(\mathbb{E}\sup_x \eta_x\right)^2\end{aligned}$$

where

$$C(\delta) = 4(2\pi+1)\left(2(1+2\pi)+e^{-2(1+2\pi)}\right)\left(\frac{1}{a_\delta^2}+\frac{2}{b_\delta^2}+\frac{1}{a_\delta'^2}\right)$$

and we have proved the theorem □

Choosing $\delta = 10^{-10}$ for instance, we get $C(\delta) \approx 3322.1 \leq 3323$. Thus, we have an explicit bound on cover time

$$t_{\mathrm{cov}} \leq C'\cdot|E|\cdot\left(\mathbb{E}\sup_x \eta_x\right)^2, \quad C' = 3323 \qquad (9)$$

Next, we find an explicit universal constant in Theorem (1.6)

Theorem 3.4. *For any graph $G = (V, E)$. Let $t_{hit}(G)$ be the maximal hitting time in G. Then,*

$$t_{cov}(G) \leq \left(1 + C\sqrt{\frac{t_{hit}(G)}{t_{cov}(G)}}\right) \cdot |E| \cdot \left(\mathbb{E}\sup_{v\in V} \eta_v\right)^2$$

where

$$C = 1.5 \times 10^7$$

Proof. [1] In the proof of Theorem (1.6) (Theorem 2.8 in [2]), it has been shown that for any $\beta > 0$ and

$$\sigma^2 = \max_v \mathbb{E}\eta_v^2, \quad S = \sum_v d_v \eta_v^2, \quad Q = \sum_v d_v \eta_v, \quad t(\beta) = \frac{1}{2}(\mathbb{E}\max_v \eta_v + \beta\sigma)^2$$

we have

$$\mathbb{P}(\tau(t) \geq 2|E|t + \sqrt{2t}\beta|E|\sigma + \beta|E|\sigma^2) \leq \mathbb{P}(S \geq \beta|E|\sigma^2) + \mathbb{P}(|Q| \geq \beta|E|\sigma) \tag{10}$$

$$\mathbb{P}(|Q| \geq \beta|E|\sigma) \leq 2e^{-\beta^2/8} \tag{11}$$

and

$$\mathbb{P}(\tau_{\text{cov}} > \tau(t)) \leq 4e^{-\beta/8} \tag{12}$$

Note that

$$\mathbb{E}[S] = \mathbb{E}[\sum_v d_v \eta_v^2] \leq \sigma^2 \sum_v d_v = 2|E|\sigma^2$$

Also, let $\sigma_u^2 = \mathbb{E}[\eta_u^2] \leq \sigma^2$, we get

$$\mathbb{E}[S^2] = \sum d_u d_v \mathbb{E}[\eta_u^2 \eta_v^2] \leq \sum d_u d_v (\mathbb{E}[\eta_u^4])^{1/2} (\mathbb{E}[\eta_v^4])^{1/2}$$

$$= \sum d_u d_v (3\sigma_u^4)^{1/2} (3\sigma_v^4)^{1/2} \leq 3\sigma^4 (\sum d_u)(\sum d_v) = 12|E|^2\sigma^4$$

and

$$\mathbb{E}[S^3] = \sum d_u d_v d_w \mathbb{E}[\eta_u^2 \eta_v^2 \eta_w^2] \leq \sum d_u d_v d_w (\mathbb{E}[\eta_u^8])^{1/4} (\mathbb{E}[\eta_v^8])^{1/4} (\mathbb{E}[\eta_u^4])^{1/2}$$

$$= \sum d_u d_v d_w (105\sigma_u^8)^{1/4} (105\sigma_v^8)^{1/4} (3\sigma_w^4)^{1/2} \leq 18\sigma^6 (\sum d_u)(\sum d_v)(\sum d_w) = 144|E|^3\sigma^6$$

[1] The proof presented in this paper is slightly different than the original proof provided in [2] due to a minor mistake in the original proof.

That is, let $U = S/|E|\sigma^2$, we have the bound

$$\mathbb{E}U \leq 2, \quad \mathbb{E}U^2 \leq 12, \quad \mathbb{E}U^3 \leq 144$$

Since $U \geq 0$, by Markov inequality, it follows that

$$\begin{aligned}\mathbb{P}(S \geq \beta|E|\sigma^2) &= \mathbb{P}(U \geq \beta) = \mathbb{P}((U+1)^3 \geq (\beta+1)^3) \\ &\leq \frac{\mathbb{E}[(U+1)^3]}{(\beta+1)^3} \leq \frac{187}{(\beta+1)^3} \end{aligned} \tag{13}$$

From (10), (11), (12), and (13), we get

$$\mathbb{P}(\tau_{\text{cov}} \geq 2|E|t + \sqrt{2t}|E|\sigma + \beta|E|\sigma^2) \leq \frac{187}{(\beta+1)^3} + 2e^{-\beta^2/8} + 4e^{-\beta/8}$$

Let

$$u(\beta) = 2|E|t + \sqrt{2t}|E|\sigma + \beta|E|\sigma^2, \quad f(\beta) = \frac{187}{(\beta+1)^3} + 2e^{-\beta^2/8} + 4e^{-\beta/8}$$

Then

$$u(0) = |E|(\mathbb{E}\sup_v \eta_v)^2, \quad u'(\beta) = A + B\beta$$

where

$$A = 3|E|\sigma\mathbb{E}\sup \eta_v + |E|\sigma^2, \quad B = 4|E|\sigma^2$$

Thus,

$$\begin{aligned}\mathbb{E}[\tau_{\text{cov}}] &= \int_0^\infty \mathbb{P}(\tau_{\text{cov}} \geq y)dy \\ &= \int_0^{u(0)} \mathbb{P}(\tau_{\text{cov}} \geq y)dy + \int_{u(0)}^\infty \mathbb{P}(\tau_{\text{cov}} \geq y)dy \\ &\leq u(0) + \int_0^\infty \mathbb{P}(\tau_{\text{cov}} \geq u(\beta))u'(\beta)d\beta \\ &\leq u(0) + \int_0^\infty f(\beta)(A + B\beta)d\beta \\ &= u(0) + A\int_0^\infty f(\beta)d\beta + B\int_0^\infty \beta f(\beta)d\beta \end{aligned}$$

It is straightforward to compute the integral of $f(\beta)$ and $\beta f(\beta)$ and get

$$\int_0^\infty f(\beta)d\beta \le 131, \quad \int_0^\infty \beta f(\beta)d\beta \le 358$$

Therefore,

$$\begin{aligned}
\mathbb{E}[\tau_{\text{cov}}] &\le u(0) + 131A + 358B \\
&= |E|(\mathbb{E}\sup\eta_v)^2 + 393|E|\sigma\mathbb{E}\sup\eta_v + 1563|E|\sigma^2 \\
&= |E|(\mathbb{E}\sup\eta_v)^2\left[1 + 393\frac{\sigma}{\mathbb{E}\sup\eta_v} + 1563\frac{\sigma^2}{(\mathbb{E}\sup\eta_v)^2}\right] \\
&\le |E|(\mathbb{E}\sup\eta_v)^2\left[1 + (393 + 1563\sqrt{2\pi})\frac{\sigma}{\mathbb{E}\sup\eta_v}\right] \qquad \text{[from(7)]}
\end{aligned}$$

Note that

$$\sigma^2 = \max_v \mathbb{E}\eta_v^2 = \max_v \mathbb{E}(\eta_v - \eta_{v_0})^2 = \max_v \frac{\mathbb{E}_v\tau_{v_0} + \mathbb{E}_{v_0}\tau_v}{2|E|} \le \frac{t_{\text{hit}}}{|E|}$$

and from (9),

$$\frac{1}{(\mathbb{E}\sup_x\eta_x)^2} \le \frac{C'|E|}{t_{\text{cov}}}, \quad C' = 3323$$

Therefore,

$$\frac{\sigma}{\mathbb{E}\sup_x\eta_x} \le \sqrt{C'}\sqrt{\frac{t_{\text{hit}}}{t_{\text{cov}}}}$$

Hence,

$$t_{\text{cov}}(G) \le |E|(\mathbb{E}\sup\eta_v)^2\left[1 + C\sqrt{\frac{t_{\text{hit}}}{t_{\text{cov}}}}\right]$$

where

$$C = (393 + 1563\sqrt{2\pi})\sqrt{3323} \le 1.5 \times 10^7$$

as desired. □

4 Cover Time of Random Walk on Complete Bipartite Graph

In this section, we will show that the random walk on complete bipartite graph satisfies property (2). Let us first formally define a random walk on complete bipartite graph

Definition 4.1. *Let $G_{k,m}$ be a graph with $n = k+m$ vertices. We will always assume that $k \geq m$. Denote the vertices by $u_i, v_j, 1 \leq k, 1 \leq j \leq m$, then the random walk on complete bipartite graph $G_{k,m}$ is a markov chain with following transition probability*

$$K(u_i, u_j) = K(v_i, v_j) = 0, K(u_i, v_j) = \frac{1}{m}, K(v_j, u_i) = \frac{1}{k} \tag{14}$$

Since we are interested in asymptotic property of cover time, we need to consider a sequence of $G_{k,m}$. We will do so by fixing a ratio k/n. Thus, it is more convenient to denote the same graph described above by $G_{n,p}$ where $k = \lfloor pn \rfloor$. We will fix $p \geq 1/2$ as we take n to infinity. Which notation is refered to will be clear in the context.

We first exhibit a tight bound on $t_{\text{cov}}(G_{n,p})$

Theorem 4.2. *Consider a random walk on $G_{n,p} = G_{k,m}$. Then*

$$2k(1 + \frac{1}{2} + ... + \frac{1}{k-1}) \leq t_{cov}(G_{k,m}) \leq 2k(1 + \frac{1}{2} + ... + \frac{1}{n})$$

In particular,

$$t_{cov}(G_{n,p}) \sim 2k \log n \sim 2pn \log n$$

Proof. Let V, W denote the set of v_i, w_j in Definition (4.1). Using first step analysis, it is easy to compute

$$\mathbb{E}_v[\tau_{v'}] = 2k, \quad \mathbb{E}_w[\tau_{w'}] = 2m, \quad \mathbb{E}_v[\tau_w] = 2m-1, \quad \mathbb{E}_w[\tau_v] = 2k-1 \tag{15}$$

for all $v, v' \in V$ and $w, w' \in W$. Recall that we let k denotes the larger partite, that is, $k \geq m$, thus

$$t_{\text{hit}} = 2k$$

Applying Theorem(1.1), we have

$$t_{\text{cov}}(G_{k,m}) \leq 2k(1 + \frac{1}{2} + ... + \frac{1}{n})$$

Now choose $A = V$ in Theorem(1.2), we have $t^A_{\min} = 2k, |A| = k$, therefore

$$t_{\text{cov}}(G_{k,m}) \geq 2k(1 + \frac{1}{2} + ... + \frac{1}{k-1})$$

Since p is fixed, as $n \to \infty$, we have

$$k \sim pn, \quad 1 + \frac{1}{2} + ... + \frac{1}{k-1} \sim \log k \sim \log p + \log n \sim \log n$$

Therefore,

$$t_{\text{cov}}(G_{n,p}) \sim 2k \log n \sim 2pn \log n$$

and we complete the proof. □

Next, we find an upper bound on the maximum of Gaussian Free Field on $G_{n,p}$.

Theorem 4.3. *Let $\{\eta_{n,x}, x \in G_{n,p}\}$ denotes the Gaussian Free Field on $G_{n,p}$, then*

$$\mathbb{E} \sup_x \eta_{n,x} \sim \sqrt{\frac{2 \log n}{m}} \sim \sqrt{\frac{2 \log n}{(1-p)n}}$$

Proof. For convenience's sake, I will suppress n in $\eta_{n,x}$. Let V, W denote the set of v_i, w_j in Definition (4.1). Recall the definition of Gaussian Free Field, $\{\eta_x\}$ satisfies

$$\mathbb{E}(\eta_x - \eta_y)^2 = \frac{\mathbb{E}_x \tau_y + \mathbb{E}_y \tau_x}{2|E|}$$

Thus, fix $v_0 \in V$ to be the vertex such that $\eta_{v_0} = 0$ and from (15), it follows that

$$\mathbb{E}\eta_v^2 = \frac{2}{m}, \quad \mathbb{E}\eta_w^2 = \frac{1}{k} + \frac{1}{m} - \frac{1}{km}, \quad \mathbb{E}\eta_v\eta_{v'} = \mathbb{E}\eta_v\eta_w = \frac{1}{m}, \quad \mathbb{E}\eta_w\eta_{w'} = \frac{1}{m} - \frac{1}{km} \tag{16}$$

for all $v, v' \in V$ and $w, w' \in W$. By Slepian's Lemma (see [3], for example), we have

$$\mathbb{E} \sup_x \eta_{n,x} \leq \mathbb{E} \sup_x \zeta_{n,x}$$

where $\zeta_{n,x}$ satisfies

$$\mathbb{E}\zeta_v^2 = \frac{2}{m}, \quad \mathbb{E}\zeta_w^2 = \frac{1}{k} + \frac{1}{m} - \frac{1}{km}, \quad \mathbb{E}\zeta_v\zeta_{v'} = \mathbb{E}\zeta_v\zeta_w = \mathbb{E}\zeta_w\zeta_{w'} = \frac{1}{m}$$

Thus, $\zeta_{n,x}$ can be written as

$$\zeta_v = \sqrt{\frac{1}{m}}\xi + \sqrt{\frac{1}{m}}\xi_v, \quad \zeta_w = \sqrt{\frac{1}{m}}\xi + \sqrt{\frac{1}{k} - \frac{1}{km}}\xi_w$$

where ξ, ξ_v, ξ_w are iid $N(0,1)$. Thus,

$$\mathbb{E}\sup_x \zeta_{n,x} = \mathbb{E}\sup_{v,w}\{\sqrt{\frac{1}{m}}\xi_v, \sqrt{\frac{1}{k} - \frac{1}{km}}\xi_w\} = \sqrt{\frac{1}{m}}\mathbb{E}\sup_{v,w}\{\xi_v, a\xi_w\} \tag{17}$$

where $a = \sqrt{\frac{1}{k} - \frac{1}{km}}/\sqrt{\frac{1}{m}} \leq 1$. Before we proceed, we need to prove a following lemma

Lemma 4.4. *Consider an i.i.d. standard normal random variables $X_1, ..., X_n$. If a_i is a constant in [0,1], then there exist a constant M independent of n such that*

$$\mathbb{E}\sup a_i X_i \leq \mathbb{E}\sup X_i + \frac{C}{2^n}$$

Proof. Note that

$$\mathbb{E}[\sup a_i X_i - \sup X_i] = \mathbb{E}[\sup a_i X_i - \sup X_i | \sup X_i \geq 0]\mathbb{P}(\sup X_i \geq 0)$$

$$+\mathbb{E}[\sup a_i X_i - \sup X_i | \sup X_i < 0]\mathbb{P}(\sup X_i < 0)$$

$$\leq 0 + \mathbb{E}[-X_1 | X_1 < 0]\frac{1}{2^n} = \frac{M}{2^n}$$

and we are done. □

Applying the above lemma to the last term in (17) with $a_v = 1, a_w = a < 1$, we have

$$\mathbb{E}\sup_x \eta_x \leq \sqrt{\frac{1}{m}}\left(\mathbb{E}\sup_x \xi_x + \frac{C}{2^n}\right)$$

Recall the well known asymptotic property of maximum of iid standard normal random variables (see [4] for example), we have

$$\mathbb{E}\sup_x \xi_x \sim \sqrt{2\log n}$$

Therefore,

$$\lim_{n\to\infty} \frac{\mathbb{E}\sup_x \eta_x}{\sqrt{\frac{2\log n}{m}}} \leq 1$$

For the other side, we have shown that

$$t_{\text{hit}} = 2k, \quad t_{\text{cov}} \geq 2k(1 + \frac{1}{2} + ... + \frac{1}{k-1})$$

Thus,

$$\lim_{n\to\infty} \frac{t_{\text{hit}}(G_{n,p})}{t_{\text{cov}}(G_{n,p})} = 0$$

By Theorem (1.6), it follows that

$$\lim_{n\to\infty} \sqrt{\frac{|E|}{t_{\text{cov}}(G_{n,p})}} \mathbb{E} \sup_x \eta_x \geq 1$$

But $|E| = mk$ and from Theorem(4.2), we know that $t_{\text{cov}}(G_{n,p}) \sim 2pn \log n \sim 2k \log n$, therefore,

$$\lim_{n\to\infty} \frac{\mathbb{E} \sup_x \eta_x}{\sqrt{\frac{2 \log n}{m}}} \geq 1$$

Hence,

$$\mathbb{E} \sup_x \eta_x \sim \sqrt{\frac{2 \log n}{m}}$$

as desired. □

As a corollary of Theorem (4.2) and Theorem (4.3), we have eventually shown that a complete bipartite graph $G_{p,n}$ with a fix ratio p satisfies

$$t_{\text{cov}}(G_{n,p}) \sim |E| \left(\mathbb{E} \sup_x \eta_x \right)^2$$

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