MATH 53 NOTE: 05/14/2013

SARAN AHUJA

1. Second order linear ODE with constant coefficients: homogeneous

In this note, we will discuss how to solve an equation of the form

$$ay''(t) + by'(t) + cy(t) = 0$$

Note that this can be turned into the system of ODE by letting

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

Then

$$\vec{\mathbf{x}}'(t) = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix} \vec{\mathbf{x}}(t)$$

So we can see the solution $y(t)$ should take the form $e^{\lambda t}$. By plugging that in and solve for λ, we can find solutions to the second order ODE as follows;

(1) Find the roots of characteristic equation $a\lambda^2 + b\lambda + c = 0$
(2) The roots fall into one of the three cases;
   - $\lambda_1 \neq \lambda_2$ and are both real. Then the general solution is
$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$
   - $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i$. The general solution is
$$y(t) = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$
   - $\lambda_1 = \lambda_2 = \lambda$ (must be real). The general solution is
$$y(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$$
(3) Find $y'(t)$ if the initial condition involves $y'(t_0)$.
(4) Plugging in initial conditions to find C_1, C_2

Example 1. *Solve $y'' - 5y' + 6y = 0$ with initial condition $y(0) = 1, y'(0) = 1$*

Solution. the characteristic equation is

$$\lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad (\lambda - 3)(\lambda - 2) = 0 \quad \Rightarrow \quad \lambda = 2, 3$$

The general solution is then given by

$$y(t) = C_1 e^{2t} + C_2 e^{3t}$$

We first find its derivative so we can plug in the initial condition,

$$y'(t) = 2C_1 e^{2t} + 3C_2 e^{3t}$$

Plugging in $t = 0$ to both $y(t), y'(t)$,

$$\begin{matrix} C_1 + C_2 = 1 \\ 2C_1 + 3C_2 = 1 \end{matrix} \quad \Rightarrow \quad C_1 = 2, C_2 = -1$$

So the solution to the IVP is

$$y(t) = 2e^{2t} - e^{3t}$$

Example 2. *Solve* $y'' - 4y' + 5y = 0$ *with initial condition* $y(0) = 3, y'(0) = 2$

Solution. the characteristic equation is

$$\lambda^2 - 4\lambda + 5 = 0 \quad \Rightarrow \quad (\lambda - 2)^2 = -1 \quad \Rightarrow \quad \lambda = 2 \pm i$$

The general solution is then given by

$$y(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t$$

We first find its derivative so we can plug in the initial condition,

$$y'(t) = 2C_1 e^{2t} \cos t - C_1 e^{2t} \sin t + 2C_2 e^{2t} \sin t + C_2 e^{2t} \cos t$$

Plugging in $t = 0$ to both $y(t), y'(t)$,

$$\begin{matrix} C_1 = 3 \\ 2C_1 + C_2 = 2 \end{matrix} \quad \Rightarrow \quad C_1 = 3, C_2 = -4$$

So the solution to the IVP is

$$y(t) = 3e^{2t} \cos t - 4e^{2t} \sin t$$

Example 3. *Solve* $y'' + 6y' + 9y = 0$ *with initial condition* $y(0) = 2, y'(0) = -5$

Solution. the characteristic equation is

$$\lambda^2 + 6\lambda + 9 = 0 \quad \Rightarrow \quad (\lambda + 3)^2 = 0 \quad \Rightarrow \quad \lambda = -3$$

The general solution is then given by

$$y(t) = C_1 e^{-3t} + C_2 t e^{-3t}$$

We first find its derivative so we can plug in the initial condition,

$$y'(t) = -3C_1 e^{2t} - 3C_2 t e^{-3t} + C_2 e^{-3t}$$

Plugging in $t = 0$ to both $y(t), y'(t)$,

$$\begin{matrix} C_1 = 2 \\ -3C_1 + C_2 = -5 \end{matrix} \quad \Rightarrow \quad C_1 = 2, C_2 = 1$$

So the solution to the IVP is

$$y(t) = 2e^{-3t} + te^{-3t}$$