MATH 53 NOTE: 05/07/2013

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1. Case III: Repeated Eigenvalues

Suppose that A has repeated eigenvalues, that is, $\lambda_1 = \lambda_2 = \lambda$, then we can solve $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$ as follows;

(1) If A is a diagonal matrix $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, then any vector is an eigenvector, so we have two independent eigenvectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We have two solutions

$$\vec{\mathbf{x}}_1(t) = e^{\lambda t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{\mathbf{x}}_2(t) = e^{\lambda t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The general solution is simply

$$\vec{\mathbf{x}}(t) = C_1 e^{\lambda t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{\lambda t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(2) In most cases, A will not be a diagonal matrix, and it can be shown that we cannot find two linearly independent eigenvector. Let $\vec{\mathbf{v}}$ be an eigenvector, then we have only one solution

$$\vec{\mathbf{x}}_1(t) = e^{\lambda t}\vec{\mathbf{v}}$$

As you have seen in class, the second solution will be of the form

$$\vec{\mathbf{x}}_2(t) = e^{\lambda t}\left(t\vec{\mathbf{v}} + \vec{\mathbf{w}}\right)$$

where $\lambda, \vec{\mathbf{v}}$ is the eigenvalue,eigenvector that we found. To find $\vec{\mathbf{w}}$, we solve

$$(A - \lambda I)\vec{\mathbf{w}} = \vec{\mathbf{v}}$$

Note that we just need to find one $\vec{\mathbf{w}}$ that satisfies the above equation. Then the general solution is simply

$$\vec{\mathbf{x}}(t) = C_1 e^{\lambda t}\vec{\mathbf{v}} + C_2 e^{\lambda t}\left(t\vec{\mathbf{v}} + \vec{\mathbf{w}}\right)$$

Let's see some example

Example 1. *Solve*

$$\vec{\mathbf{x}}'(t) = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}\vec{\mathbf{x}}(t), \quad \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Solution. Note that

$$A - \lambda I = \begin{bmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{bmatrix} \quad \Rightarrow \quad (-3-\lambda)(1-\lambda) + 4 = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = -1$$

Next, we find a corresponding eigenvector;

$$0 = A - (-1)I\vec{\mathbf{v}} = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \Rightarrow \quad -v_1 + v_2 = 0 \quad \Rightarrow \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So we have the first solution

$$\vec{\mathbf{x}}_1(t) = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

And we know that the second solution is

$$\vec{\mathbf{x}}_2(t) = e^{-t}\left(t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \vec{\mathbf{w}}\right)$$

Now we find $\vec{\mathbf{w}}$ by solving

$$(A-\lambda I)\vec{\mathbf{w}} = \vec{\mathbf{v}} \quad \Rightarrow \quad \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad -2w_1 + 2w_2 = 1 \quad \Rightarrow \quad \vec{\mathbf{w}} = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}$$

so the second solution is

$$\vec{\mathbf{x}}_2(t) = e^{-t}\left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}\right)$$

The general solution is

$$\vec{\mathbf{x}}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t}\left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}\right)$$

Plugging in $t = 0$ yields

$$C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad C_1 = -1, C_2 = 2$$

Thus, the solution to the IVP is

$$\vec{\mathbf{x}}(t) = -e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{-t}\left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}\right)$$