Mathematics 1 Notes

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Chapter 1

Functions and Sketching Graphs

1.1 Functions in Cartesian Co-Ordinates

1.1.1 Definitions

Range

Definition. *Values which the dependent variable can take.*

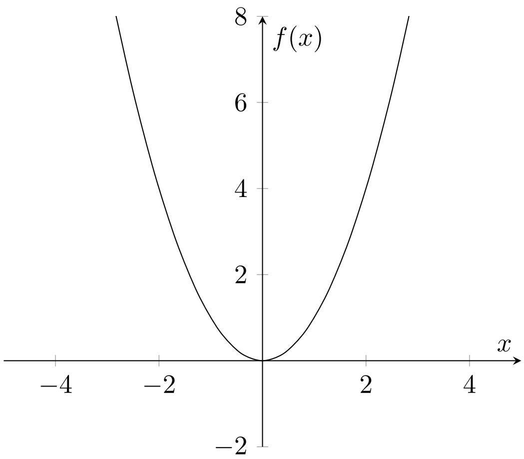


Figure 1.1: A graph plotting $f(x) = x^2$. $f(x)$ is the dependent variable and can take values greater than 0, therefore its range is $f(x) \in \mathbb{R}, f(x) \geq 0$.

Domain

Definition. *Values which the independent variable can take.*

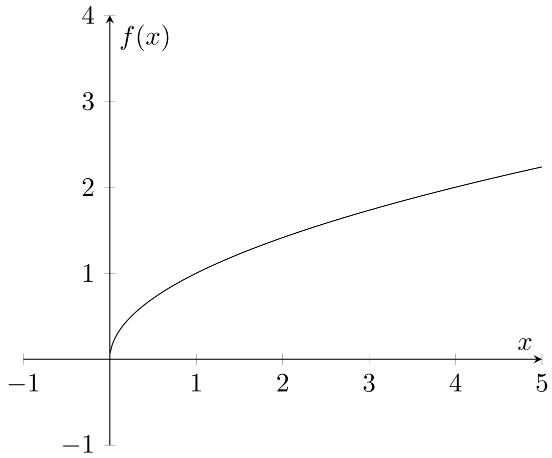


Figure 1.2: A graph plotting $f(x) = \sqrt{x}$. x is the independent variable and can only take values greater than 0, therefore its domain is $x \in \mathbb{R}, x \geq 0$.

Zeroes of a Function

Definition. *Values of the independent variable which cause the dependent variable to equal 0.*

$$f(x) = 0$$

Intercept of a Function

Definition. *Values of the dependent variable which cause the independent variable to equal 0.*

$$f(x) = f(0)$$

Even Function

Definition. *A function which, when plotted, is symmetric about the y-axis.*

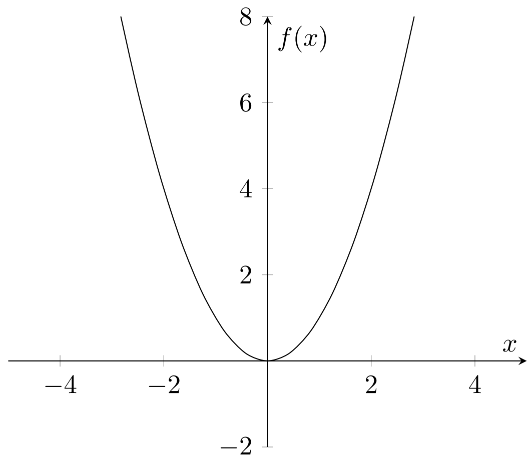


Figure 1.3: A graph plotting $f(x) = x^2$. The graph is symmetric about the y-axis.

Odd Function

Definition. *A graph which is anti-symmetric about the y-axis.*

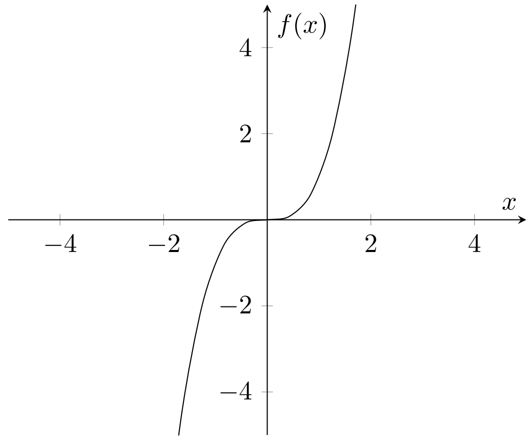


Figure 1.4: A graph plotting $f(x) = x^3$. The graph is anti-symmetric about the y-axis.

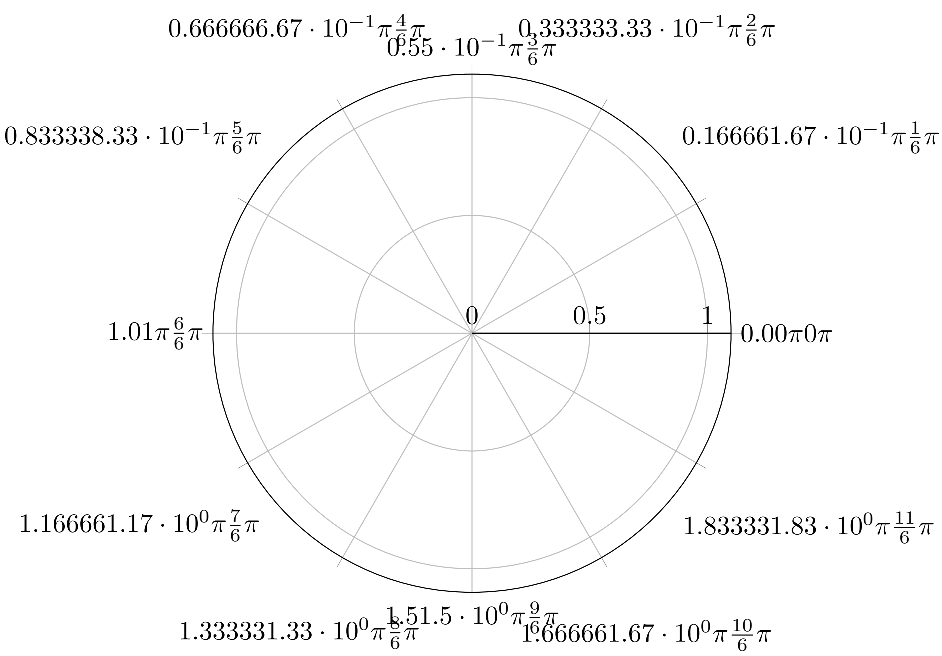
1.1.2 Oblique Asymptotes

A fraction has an oblique asymptote if the bottom of the fraction has an order n and the top of the fraction has an $n + 1$. In order to obtain the equation of the oblique asymptote we must use algebraic long division, not dividing into the negative powers. The quotient will be the equation of the oblique asymptote.

1.2 Polar Co-Ordinates

This is a co-ordinate system where a point is defined by its distance from the origin, r, and the angle the line joining the origin and the point makes with the x-axis, θ. It is then written as: (r, θ).

Often when we use polar co-ordinates, we will use the polar co-ordinate axis, as shown below:



When plotting in polar co-ordinates, one method we may wish to approach the problem may be to convert our coordinates into Cartesian co-ordinates. We may take a similar approach to a Cartesian equation, by converting it to polar coordinates. Keep in mind, only functions with circular symmetry may be represented on a polar coordinate graph.

Below are the equations needed to convert between polar and Cartesian co-ordinates and vice versa.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= tan^{-1}\left(\frac{y}{x}\right) \\ x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

$$\begin{aligned} \text{Polar} &\longleftrightarrow \text{Cartesian} \\ (r,\theta) &\longrightarrow (r\cos\theta, r\sin\theta) \\ (\sqrt{x^2+y^2}, tan^{-1}\left(\frac{y}{x}\right)) &\longleftarrow (x,y) \end{aligned}$$

Chapter 2

Calculus

2.1 Basic Differentiation

The gradient of a continuous function is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\delta y}{\delta x}$$

We want to find the equation of the instantaneous gradient. This is given as follows:

$$\begin{aligned}
\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \right) &= \frac{dy}{dx} \\
&= \lim_{\delta x \to 0} \left(\frac{f(x + \delta x) - f(x)}{\delta x} \right)
\end{aligned}$$

The last equation is how we derive equations from first principles. Below is an example.

Derivation. *The derivative of the function x^n from first principles.*

$$\begin{aligned}
\frac{d}{dy}(x^n) &= \lim_{\delta x \to 0} \left[\frac{(x + \delta x)^n - x^n}{\delta x} \right] \\
&= \lim_{\delta x \to 0} \left[\frac{(x^n + nx^{n-1}\delta x + \frac{n(n-1)}{2} x^{n-2} \delta x^2 + ... + \delta x^n) - x^n}{\delta x} \right] \\
&= \lim_{\delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} \delta x + ... + \delta x^{n-1} \right] \\
&= nx^{n-1}
\end{aligned}$$

The second derivative of a function represents the rate of change of the gradient and is known as the *curvature* of the curve.

There is further notation for time derivatives.

$$x = x(t)$$

$$\frac{dx}{dt} = \dot{x}$$

$$\frac{d^2x}{dt^2} = \ddot{x}$$

2.1.1 Common Derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\cos x$
$tanx$	sec^2x
$e^f(x)$	$f'(x)e^x$
$ln(f(x))$	$\frac{f'(x)}{f(x)}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$

2.1.2 Rules of Differentiation

Sum Rule

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

Product Rule

$$\begin{aligned} y &= u(x) \cdot v(x) \\ \frac{dy}{dx} &= u(x) \cdot v'(x) + u'(x) \cdot v(x) \end{aligned}$$

The above can generalise to multiple terms, alternating the derivative each time.

Quotient Rule

$$\begin{aligned} y &= \frac{u(x)}{v(x)} \\ \frac{dy}{dx} &= \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{(v(x))^2} \end{aligned}$$

Chain Rule

$$\begin{aligned} y &= f(u(x)) \\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \end{aligned}$$

Parametric Differentiation

$$\begin{aligned} x = f(t) \quad y &= g(t) \\ \frac{dy}{dx} &= \frac{dy}{dy} \cdot \frac{dt}{dx} \\ &= \frac{dy}{dt} \cdot \left(\frac{dx}{dt}\right)^{-1} \end{aligned}$$

Implicit Differentiation

$$\begin{aligned} y &= f(y) \\ \frac{dy}{dx} &= \frac{dy}{dx} f'(y) \end{aligned}$$

Derivatives of Logarithms

$$\frac{d}{dx}(ln(f(x))) = \frac{f'(x)}{f(x)}$$

2.1.3 Estimating Small Changes

Suppose $y = f(x_0)$. We can estimate the change in y when we make a small change in x_0.

$$\delta x = x_0 - x_1$$
$$f'(x) \approx \frac{\delta y}{\delta x}$$
$$\delta y \approx f'(x_0) \cdot \delta x$$

2.2 Partial Differentiation

Partial derivatives allow us to differentiate functions with multiple variables, by differentiating with respect to one chosen variable, and treating all other variables as constants. This allows us to find the gradient in one direction. For example, the partial derivative with respect to x for a function

$$f(x, y)$$

will represent the rate of change in the x direction, and the partial derivative with respect to y will represent the rate of change in the y direction.

2.2.1 Notation

A derivative with respect to x is always written as

$$\frac{\partial}{\partial x}$$

A function f(x,y) can have its partial derivative written as

$$f_x \equiv \frac{\partial f}{\partial x}$$

We may also write the variable we are keeping constant as the subscript of our partial derivative.

$$\left(\frac{\partial f}{\partial x}\right)_y \rightarrow \text{Partial derivative of f(x,y) where y is constant.}$$

High order partial derivatives are written as:

$$\frac{\partial^2 f}{\partial x^2} \equiv f_{xx}$$

We are also able to create mixed second order derivatives:

$$\frac{\partial^2 f}{\partial x \partial y} \equiv f_{xy}$$

2.2.2 Relationship to Total Derivative and Changing Variables

When we calculate partial derivatives, we are usually interested in the rate of change of an arbitrary variable, no matter the direction. Suppose that we have a function $z(x, y)$. We are able to get the total change in z by splitting it into the sum of its infinitesimally small gains in the individual directions.

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

In short, the above equation describes moving in the x and y direction to get the total movement in z. We are then able to take the limits as $\partial x \to 0$ and $\partial y \to 0$ and divide by dx, dy or some other infinitesimal value which will relate the total derivatives to the partial derivatives.

$$\lim_{\delta x \to 0, \delta y \to 0} (\delta z) = \lim_{\partial x \to 0, \delta y \to 0} \left(\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right)$$
$$\implies dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$
$$\implies \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

This type of equation is useful when, for example, there is a function $z(x, y)$ while $x = x(t)$ and $y = y(t)$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

2.2.3 Changing Variables

Suppose we have a function $f(x, y)$, and x and y are functions such that $x(s, t)$ and $y(s, t)$. We can then write,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial t} \cdot \frac{\partial y}{\partial t}$$

This is useful when, for example, we are converting between co-ordinate systems. For example, changing from Cartesian to polar co-ordinates. We are able to define a Cartesian function $f(x, y)$, in order to a polar $g(r, \theta)$, we must understand the relationship $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$. We then use the above equation.

2.2.4 Stationary Points

These are points where a surface is flat. Turning points of a function are defined as points where:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

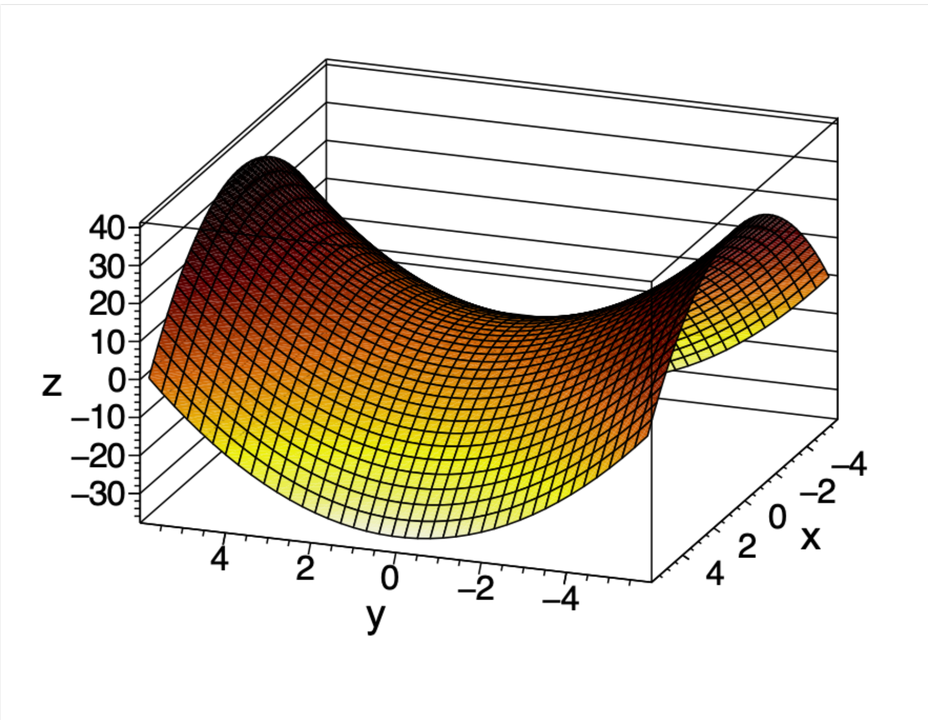
However, we are often interested in the type of turning point a function describes. There are 3 types of turning points, **minimum, maximum, and saddle points**. Additionally, functions may have multiple turning points. The way we classify a stationary point using the expression:

$$\Delta = f_{xy}^2 - f_{xx}f_{yy}$$

The condition for each type of point and a visualisation is shown below.

Saddle Point

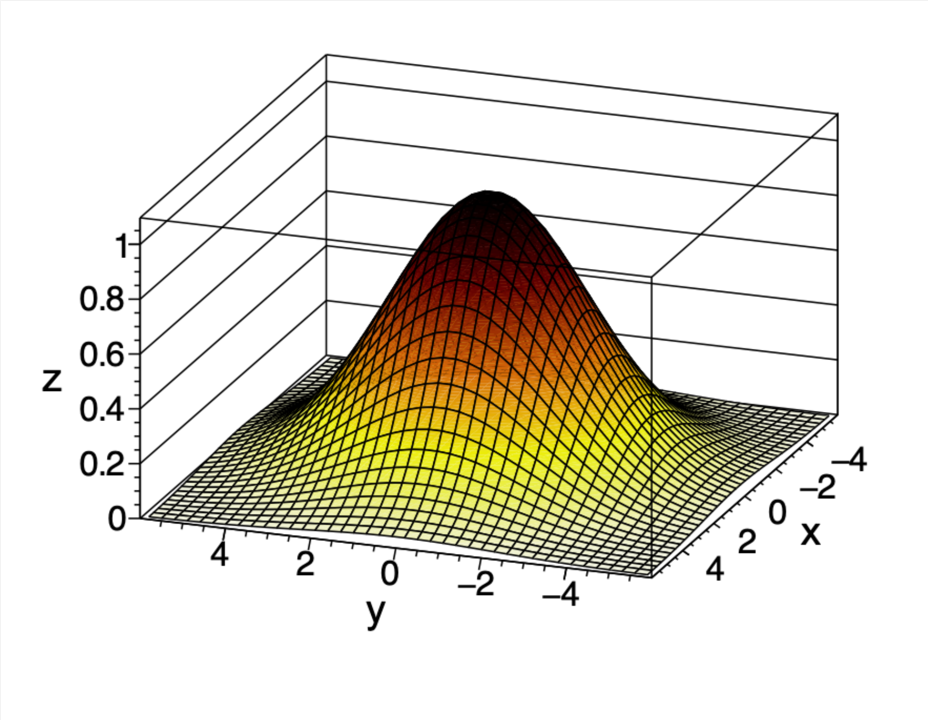
A point where one variable increases in a certain direction, while decreasing in another direction.



$$\Delta > 0 \implies \text{Saddle Point}$$

Maximum

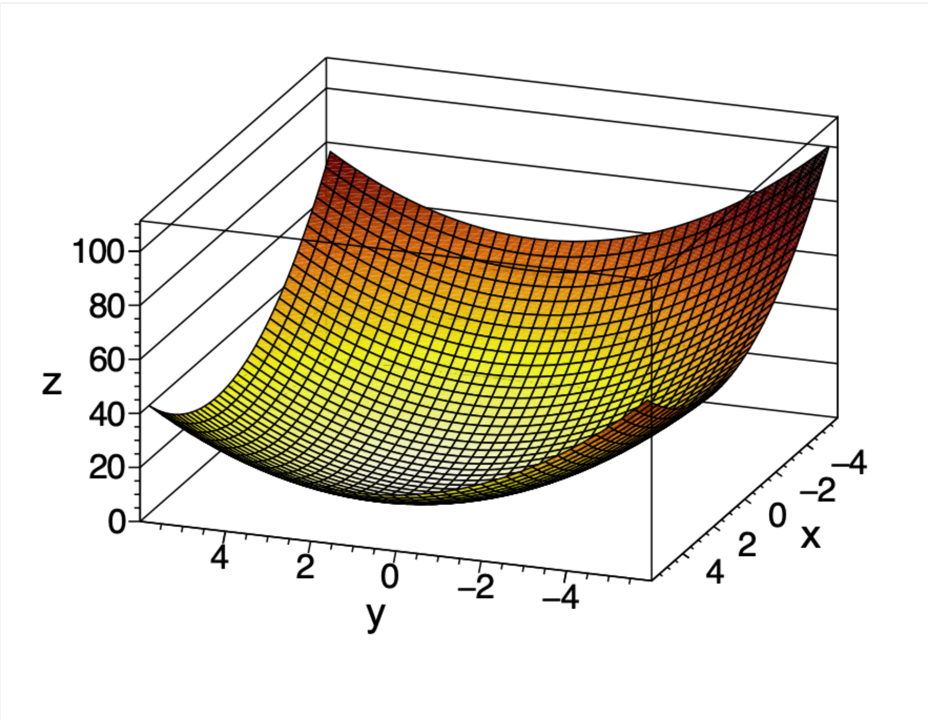
A point where the function decreases in all directions.



$$\Delta < 0 \ \& \ f_{xx} < 0 \implies \text{Maximum Stationary Point}$$

Minimum

A point where the function increases in all directions.



$$\Delta < 0 \ \& \ f_{xx} > 0 \implies \text{Minimum Stationary Point}$$

Chapter 3

Complex Numbers

Complex numbers come in the form,

$$z = x + iy$$

$$i = \sqrt{-1}$$

The **real** part of a complex number is defined

$$Re(z) = x$$

The **imaginary** part of a complex number is defined

$$Im(z) = y$$

For a quadratic equation where,

$$b^2 - 4ac < 0$$

the equation will have 2 complex roots, z and $z*$, where $z*$ is the **complex conjugate** of z, such that,

$$\begin{aligned} z &= x + iy \\ z* &= x - iy \end{aligned}$$

If we multiply a complex number by its conjugate, we get the following result.

$$\begin{aligned} z * z &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy - (iy)^2 \\ &= x^2 + y^2 \\ &= |z|^2 \\ &= Re(z)^2 + Im(z)^2 \end{aligned}$$

3.1 Complex Arithmatic

Equality:

$$\begin{aligned} z_1 &= z_2 \\ \implies x_1 &= x_2 \;\&\; y_1 = y_2 \end{aligned}$$

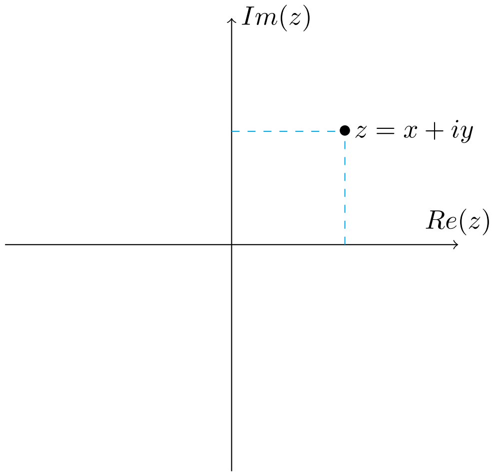
Addition and Subtraction:

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

Division When dividing complex numbers, ensure the bottom is always real by multiplying top and bottom by the complex conjugate of the top number.

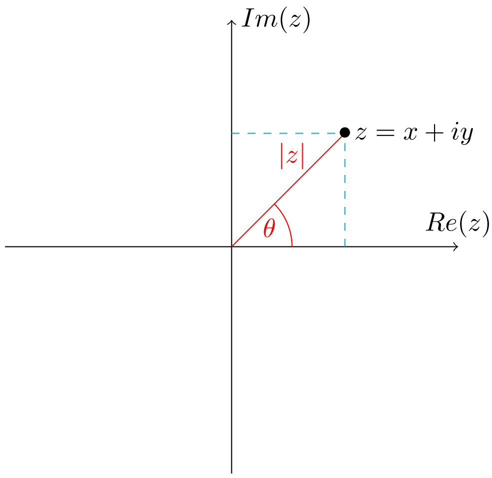
3.2 Agrand Diagram

3.2.1 Cartesian Form



3.2.2 Polar Form

This is where the complex number is defined with its **modulus** (the distance of the complex number from the origin) and **argument** (the angle the complex number makes between its line and the real axis).



We can then define

$$|z| = \sqrt{Re(z)^2 + Im(z)^2}$$
$$\theta = arg(z) = tan^{-1}\left(\frac{Im(z)}{Re(z)}\right)$$

Converting these to cartesian:

$$x = |z|\cos\theta$$
$$y = |z|\sin\theta$$

We can then write

$$z = x + iy = |z|(\cos\theta + i\sin\theta)$$
$$z* = x - iy = |z|(\cos\theta - i\sin\theta)$$

Further

$$|z_1 z_2| = |z_1||z_2| \tag{3.1}$$

$$arg(z_1 z_2) = arg(z_1) + arg(z_2) \tag{3.2}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{3.3}$$

$$arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2) \tag{3.4}$$

3.2.3 Exponential Form

We can define the polar expression for z as a function of θ.

$$z(\theta) = |z|(\cos\theta + i\sin\theta)$$

$$\begin{aligned}\frac{dz}{d\theta} &= |z|(-\sin\theta + i\cos\theta)\\ &= i|z|(\cos\theta + i\sin\theta)\\ &= iz\end{aligned}$$

since differentiating a complex number returns a complex number, we are able to map it onto the exponential function.

$$y = Ae^{ax} \implies \frac{dy}{dx} = ay$$

$$z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta} \implies \frac{dz}{d\theta} = iz$$

$$e^{i\theta} = (\cos\theta + i\sin\theta) \tag{3.5}$$

3.3 De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{3.6}$$

$$\arg(z^n) = n\arg(z) \tag{3.7}$$

3.4 Trigonometric Functions in Terms of Exponentials

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{3.8}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{3.9}$$

Chapter 4

Series and Hyperbolic Functions

4.1 Series

4.1.1 Arithmetic Progression

$$u_{n_1} = u_n + d \implies u_n = u_0 + (n+1)d \tag{4.1}$$

$$\sum_{n=0}^{N} = \frac{1}{2}N(u_0 + u_N) \tag{4.2}$$

4.1.2 Geometric Progression

$$u_{n+1} = ru_n \implies u_n = u_0 r^n \tag{4.3}$$

$$\sum_{n=0}^{N} u_0 r^n = \frac{u_0(1 - r^{N-1})}{1 - r} \tag{4.4}$$

$$S_\infty = \sum_{n=1}^{\infty} = \lim_{N\to\infty} \left[\frac{u_0(1 - r^{N-1})}{1 - r}\right]$$

For $|r| \geq 1$ the sum diverges, $\therefore S_\infty \to \infty$. For $|r| < 1, r^{N+1} \to 0 \therefore$

$$S_\infty = \frac{u_0}{1 - r} \tag{4.5}$$

4.1.3 Binomial Series

$$(a+b)^n = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} a^{n-r} b^r \tag{4.6}$$

4.1.4 Taylor Series

For a function $f(x)$ which is

1. Single valued
2. Continuous
3. N times differentiable

We can approximate the function by a polynomial of order N, if we know the value of the function at a point $x = a$.

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{4.7}$$

Further, for an infinitely differentiable function,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{4.8}$$

Maclaurin Series

This is a special Taylor series for the case $a = 0$,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0). \tag{4.9}$$

4.1.5 L'Hopital's Rule

For a function,

$$f(x) = \frac{g(x)}{h(x)}$$

it may be the case that

$$\lim_{x \to a} f(x)$$

is indeterminate. Then, in the case where

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = 0$$

we can state,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{4.10}$$

If the limit is still indeterminate, we can repeat l'Hopital's rule.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)} \tag{4.11}$$

4.2 Hyperbolic Functions

These are functions which parameterise a hyperbola (section through a cone parallel to its axis).

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \tag{4.12}$$

Chapter 5

Integration

$$\int_a^b f(x) = \lim_{\delta x \to o} \sum_{x=a}^{b} f(x)\delta x \tag{5.1}$$

5.1 Standard Integrals

$$\int \frac{f'(x)}{f(x)} dx = ln\,|f(x)| + c \tag{5.2}$$

$$\int f(x)f'(x)dx = \frac{1}{2}f^2(x) + c \tag{5.3}$$

$$\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2(f(x))^{\frac{1}{2}} + c \tag{5.4}$$

5.2 Integrals with Infinities

Sometimes integrals may diverge, for example, the case $\int_0^1 \frac{1}{x}$. In cases like these, we compute,

$$\int_a^b f(x)dx = \lim_{\epsilon \to \infty} \int_a^{b-\epsilon} f(x)dx. \tag{5.5}$$

In case of an integral which diverges between limits, for example $\int_-^1 2\frac{1}{x}$, we must split the integral,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \tag{5.6}$$

applying (5.5) where needed.

For improper integrals, we perform,

$$\int_a^\infty f(x)dx = \lim_{\epsilon \to \infty} \int_a^\epsilon f(x)dx \tag{5.7}$$

5.3 Substitution

When integrating with substitution, we need to

1. Find a suitable substitution for our integral.
2. Find an expression for dx.
3. Change the limits into our new variable.
4. Perform the integration.

Table 5.1: Common Substitutions

$f(x)$	$\int f(x)$	Substitution
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin(\frac{x}{a})+c$	$x=a\sin\theta$
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arccosh}(\frac{x}{a})+c$	$x=a\cosh u$
$\frac{1}{\sqrt{x^2+a^2}}$	$\operatorname{arcsinh}(\frac{x}{a})+c$	$x=a\sinh u$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan(\frac{x}{a})+c$	$x=a\tan\theta$

5.4 Integration by Parts

If we consider the product rule,

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

we can integrate and rearrange to get,

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx. \tag{5.8}$$

To evaluate this, we do the following,

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx. \tag{5.9}$$

5.5 Line Integrals

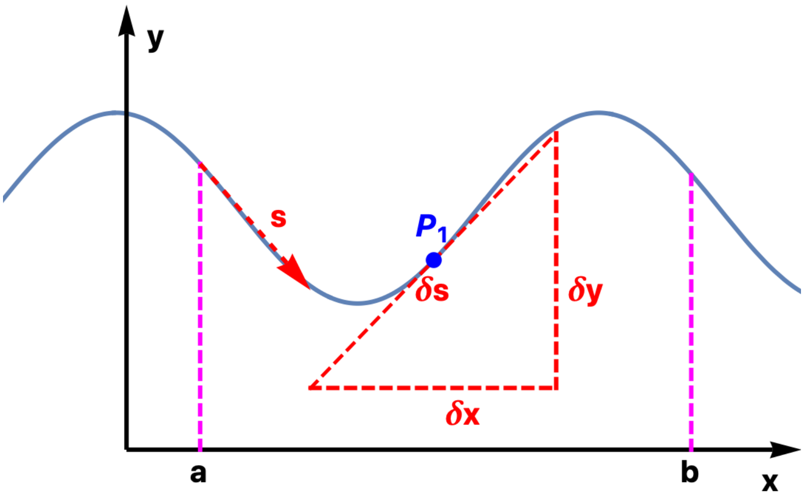


Figure 5.1: Tangent to the curve at the point P_1. Further, the length of the line will be determined between the marked points a and b.

Line integrals allow us to determine the length of lines. Let us consider the length of the tangent δs at the point P_1,

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$
$$\delta s = \sqrt{(\delta x)^2 + (\delta y)^2}.$$

We now divide both sides by δx,

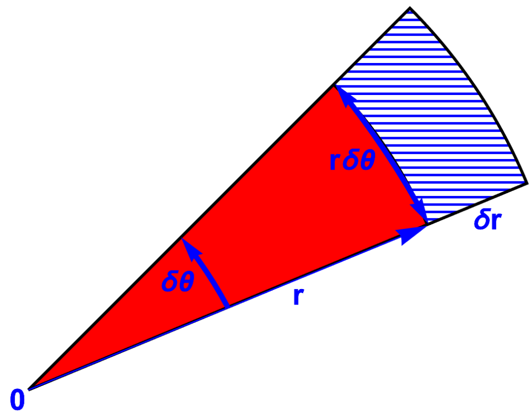
$$\frac{\delta s}{\delta x} = \frac{1}{\delta x}\sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta s}{\delta x}\right)^2}.$$

We can now take limits and integrate to get the length,

$$\lim_{\delta x \to \infty} \frac{\delta s}{\delta x} = \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{5.10}$$

5.6 Integration in Polar Co-ordinates



We wish to find the area covered by $r(\theta)$ with a width $\delta\theta$. We can define this area as,

$$A = \int_0^\theta \frac{1}{2} r^2 d\theta' \tag{5.11}$$

5.7 Volumes of Revolution

$$V = \int_a^b \pi f^2(x) dx \tag{5.12}$$

Chapter 6

Linear Algebra

Matrices are used to compactly represent systems of equations. We can define a 2×2 matrix multiplication by,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \tag{6.1}$$
$$A\mathbf{x} = \mathbf{v}$$

6.1 Common Matrices

6.1.1 2×2 Identity Matrix

$$\begin{aligned} I\mathbf{x} &= \mathbf{x} \\ I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{6.2}$$

6.1.2 Zero Matrix

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.3}$$

6.1.3 Stretch Matrix

$$\begin{aligned} M\mathbf{x} &= k\mathbf{x} \\ M &= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \end{aligned} \tag{6.4}$$

6.1.4 Reflection in x

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.5}$$

6.1.5 Rotation Through Angle θ

6.2 Matrix Arithmetic

For matrices to be equal, they must satisfy,

$$A = B \implies A_{ij} = B_{ij} \forall i, j. \tag{6.6}$$

6.2.1 Matrix Addition

$$C = A + B \implies C_{ij} = A_{ij} + B_{ij}. \tag{6.7}$$

6.2.2 Scalar Multiplication

$$B = cA \implies B_{ij} = cA_{ij}. \tag{6.8}$$

6.2.3 Matrix Multiplication

To multiply two matrices, A and B, when A is of size $n \times m$, B must be of a size $m \times p$ for the multiplication to be valid.

$$C = AB \implies C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}. \tag{6.9}$$

6.2.4 Transposing a Matrix

This is where we swap the rows and columns of the matrix,

$$A^T_{ij} = A_{ji}. \tag{6.10}$$

Transposing a column vector creates a row vector,

$$\begin{pmatrix} x \\ y \end{pmatrix}^T = \begin{pmatrix} x & y \end{pmatrix} \tag{6.11}$$

6.3 Inverse of a Matrix

The inverse of a matrix is such that,

$$AA^{-1} = I \tag{6.12}$$

or,

$$A\mathbf{x} = \mathbf{v} \implies \mathbf{x} = A^{-1}\mathbf{v}. \tag{6.13}$$

6.3.1 2×2 Matrix

We first need to consider the **determinant** of the matrix.

$$\begin{gathered} A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ \det A = A_{11}A_{22} - A_{12}A_{21} \end{gathered} \tag{6.14}$$

The inverse of A is then,

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \tag{6.15}$$

6.3.2 $3 \times 3 Matrix$

For a matrix,

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \tag{6.16}$$

its determinant is calculated by summing the determinant of its **cofactors**. These are 2×2 matrices which are generated by removing the column and row containing the selected element. For example, the co factor of A_{11} is,

$$\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33}. \end{vmatrix}$$

The sign of the determinant of the cofactor will change depending on its position in the vector, as below,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

The inverse of A can then be given as,

$$A^{-1} = \frac{1}{\det A} C^T \tag{6.17}$$

where C is the matrix of cofactors (replace each element with the determinant of its cofactor) and C^T is the transpose of the matrix of cofactors.

Orthoganol Matrix: Transpose is equal to inverse.
Symmetric Matrix: Transpose is equal to itself.
Singular (Degenerate) Matrix: Has a determinant of 0.

6.4 Homogenous Matrix Equations

For a matrix equation,

$$A\mathbf{x} = \mathbf{v}$$

where $\mathbf{v} = \mathbf{0}$, the trivial solution $\mathbf{x} = \mathbf{0}$ exists, if $|A| \neq 0$. However, in the case $|A| = 0$, the equation has an infinite number of non-trivial solutions.

6.5 Eigen Values and Eigen Vectors

Let's consider the equation,

$$A\mathbf{x} = \lambda\mathbf{x} \tag{6.18}$$

where λ is a scalar. When A is a $n \times n$ matrix, the eigenvalues λ will be satisfied by a polynomial of order n. We can rewrite (6.18) as,

$$(A - \lambda I)\mathbf{x} = \mathbf{0}. \tag{6.19}$$

Equation (6.19) is only has non-trivial solutions when,

$$|A - \lambda I| = 0.$$

The determinant is the characteristic equation. It will be a polynomial of order n, which we can solve for values of λ.

For each value of λ_i there exists an eigenvector $\mathbf{e}_i$ which we can find by solving,

$$(A - \lambda_i I)\mathbf{e}_i = \mathbf{0}. \tag{6.20}$$

We are often interested in the unit eigenvector, $\hat{\mathbf{e}}_i$ by dividing the eigenvector by its modulus.

6.6 Vector Cross Product

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{6.21}$$

6.6.1 Scalar Triple Product

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{6.22}$$

6.7 Trace of a Matrix

This is the sum of the elements along the main diagonal. For an $n \times n$ matrix,

$$tr(A) = \sum_{i=1}^{n} a_{ii}. \tag{6.23}$$

Chapter 7

Ordinary Differential Equations

7.1 Seperable ODEs

For a differential equation,

$$\frac{dy}{dx} = f(x)g(y) \tag{7.1}$$

the solution is,

$$\int \frac{1}{g(y)} dy = \int f(x) dx \tag{7.2}$$

7.2 First Order Linear ODEs

First order linear ODEs take the form,

$$\frac{dy}{dx} - P(x)y = Q(x). \tag{7.3}$$

We then use the integral,

$$I = \exp\left(\int P(x) dx\right) \tag{7.4}$$

and solve using,

$$y(x) = \frac{1}{I}\int QI dx \tag{7.5}$$

7.3 Homogeneous Differential Equations

These are equations of the form,

$$\frac{dy}{dx} = F\left(\frac{x}{y}\right) \tag{7.6}$$

or

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \tag{7.7}$$

where $g(x,y)$ and $f(x,y)$ are of the same degree. When we have an equation in this form, we use the substitution,

$$y = xv. \tag{7.8}$$

7.4 Bernoulli Equations

These are equations in the form,

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,\ \forall n \neq 0, 1 \tag{7.9}$$

where we use the substitution,

$$u = y^{-(n-1)}. \tag{7.10}$$

7.5 Second Order Differential Equations

This is an equation in the form,

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \tag{7.11}$$

usually with constant coefficients,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x). \tag{7.12}$$

For the case where $f(x) = 0$, we will use the substitution $y = Ke^{\lambda x}$ to get it in the form,

$$Ke^{\lambda x}\left(a\lambda^2 + b\lambda + c\right) = 0. \tag{7.13}$$

Because $e^{\lambda x} \neq 0$, we get the auxiliary equation,

$$a\lambda^2 + b\lambda + c = 0. \tag{7.14}$$

Which we can solve as a complex number. For two, distinct, real roots,

$$y(x) = A_1e^{\lambda_1 x} + A_2e^{\lambda_2 x}. \tag{7.15}$$

For repeated roots,

$$y(x) = (A_1x + A_2)e^{\lambda x}. \tag{7.16}$$

For complex roots,

$$y(x) = A_1e^{(\alpha+i\beta)x} + A_2e^{(\alpha-i\beta)x} \tag{7.17}$$

$$y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x). \tag{7.18}$$

$y(x)$ is known as the complementary function. We solve the equation by substituting it back into (7.12) and applying the boundary conditions.

7.5.1 Second Order ODEs With a Function on the Side

When we have equations in the form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \tag{7.19}$$

we call $f(x)$ the complementary function. We begin by solving the equation as if it were homogeneous. We denote this solution as $y_{\mathrm{CF}}(x)$. We then need to find the **particular integral** of a function, which will be added to the end of our solution to make up for dropping the $f(x)$ in the homogeneous solution.

Table 7.1: Particular Integrals

$f(x) = Ae^{ax}$	$y_{\mathrm{PI}}(x) = Be^{bx}$
$f(x) = a_1\cos(ax) + a_2\sin(ax)$	$y_{\mathrm{PI}}(x) = b_1\cos(ax) + b_2\sin(ax)$
$f(x) = a_0 + a_1x + a_2x^2 + \cdots$	$y_{\mathrm{PI}}(x) = b_0 + b_1x + b_2x^2 + \cdots$

We substitute $y_{\mathrm{PI}}(x)$ and its derivatives back into equation (7.19) to find the constants of the particular integral. We can then get our final $y(x)$ by combining the homogenous solution and the particular integral,

$$y(x) = y_{\mathrm{CF}}(x) + y_{\mathrm{PI}}(x) \tag{7.20}$$