Introduction to Astrophysics and Cosmology Lecture Notes

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December 15, 2023

Contents

Chapter 1

The Universe and Its Contents

These notes go over the various bodies found in space as well as some values and units used throughout astrophysics and cosmology.

1.1 Classification of Objects

Planets

Definition. *Large objects which don't generate energy through nuclear fusion.*

This includes moons, asteroids, comets, i.e. most dense rocky or gaseous objects. Up to 13x Jupiter's mass ($M_{Jupiter}$).

Brown Dwarfs

Between a planet and a star. They are usually not heavy enough to produce their own energy via fusion of ${}^{2}_{1}$H. They are 13x - 80x $M_{Jupiter}$.

Stars

Definition. *Large objects which generate energy via the fusion of* ${}^{1}_{1}H$.

They are over 80x $M_{Jupiter}$ and include star remnants (objects formed after the collapse of stars):

- White dwarfs
- Neutron stars
- Black holes (However not all black holes are formed from the collapse of stars).

Nebulae

Definition. *Clouds of gas composed mostly of* ${}^{1}_{1}H$*, dust, and other material.*

Those clouds may collapse to form new stars.

Galaxies

Definition. *Structures of stars and clouds which are bound by their mutual gravitation.*

Gravitational Waves

Definition. *Ripples in space-time caused by gravitational events such as the merging of two black holes.*

1.2 Common Values and Units

Common Astrophysical and Cosmological Units		
Name	Symbol	Si Unit
Solar Mass	$M_{\odot}$	$1.989 \times 10^{30} kg$
Astronomical Unit	AU	$1.5 \times 10^{11} m$
Parsec	pc	$3.09 \times 10^{16} m$

Typical Values of Astrophysical and Cosmological Objects	
Name	Value
Mass of Planet	$10 \times 10^{25} kg$
Mass of Star	$10 \times 10^{30} kg$
Mass of Galaxy	$10 \times 10^{40} kg$ / $10 \times 10^{10} M_{\odot}$
Mass of Universe	$10 \times 10^{50} kg$ / $10 \times 10^{22} M_{\odot}$
Distance of Earth $\rightarrow$ Proxima-Centauri	10×10^{25} / $1.3pc$
Distance of Earth $\rightarrow$ Edge of the Observable Universe	10×10^{26} / $3Gpc$

1.3 Density

Mass Density

The common equation for density is:

$$\rho = \frac{M}{V}$$

But, as most objects we deal with in astrophysics and cosmology are spherical, we can write:

$$\rho = \frac{M}{\frac{4\pi}{3} R^3}$$

Number Density

We often refer to the total number of objects as N, and the number density of those objects in a given space as n. The equations for n are shown below.

$$n = \frac{\rho}{m}$$

Where ρ is the mass density of the space and m is the mass of the object.

Density for Different Forms of Hydrogen

State	Equation
Molecular Gas	$n = \frac{\rho}{m_{H_2}}$
Atomic Gas	$n = \frac{\rho}{m_H}$
Ionised Gas	$n = 2\frac{\rho}{m_H}$

1.3.1 Gravity

$$F = G\frac{mM}{r^2}$$

$$G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$$

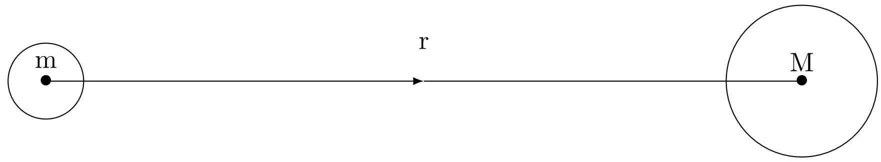
We can only use G when working with standard, SI units. The constant is small which means that to have a noticeable effect of the gravitational force we need large masses. In addition, this force is only attractive, therefore the force can only get stronger with more mass.

Let us now derive the equation for the acceleration due to gravity.

$$F = ma = m \cdot g = G\frac{m \cdot M_{\text{Earth}}}{R_{\text{Earth}}^2}$$
$$g = G\frac{M_{\text{Earth}}}{R_{\text{Earth}}^2} = 9.81ms^{-1}$$

From the above equation, we can tell that acceleration due to gravity does not depend on the object's mass.

A further property of the gravitational force is that it acts radially. The below diagram showcases this:



1.3.2 Gravitational Energy

We are able to derive a new, more accurate equation for gravitational potential energy using the equation for the gravitational force. We can do this by understanding the relation between work done and force.

$$W = F\delta s$$

Where δs is the change in distance due to the force. We will get a function of instantaneous work done when $\delta s \to 0$. Therefore, we are able to take the integral of the work done with respect to the distance, which we will call r. We can then equate this to the gravitational force equation. Below we also take r_1 as the radius of the planet and s as the distance above the surface of the planet from which an object is falling.

$$\begin{aligned} W &= \int F dr = \int G\frac{mMG}{r^2}dr \\ &= -\left[G\frac{mM}{r}\right]_{r_1}^{r_1+s} \\ &= \left[G\frac{mM}{r}\right]_{r_1+s}^{r_1} \\ &= GMm\left(\frac{1}{r_1} - \frac{1}{r_1+s}\right) \end{aligned}$$

Escape Velocity

We can use the above equation to find the escape velocity from a planet. We can do this by taking the limit:

$$\lim_{s\to\infty}\left(\frac{1}{r_1+s}\right) = 0$$

We can then simplify our equation of gravitational potential energy and equate it to the kinetic energy on impact with the planet.

$$W_{\text{esc}} = \frac{GMm}{r_1} = \frac{1}{2}mv_{\text{esc}}^2$$
$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}$$

Orbital Velocity

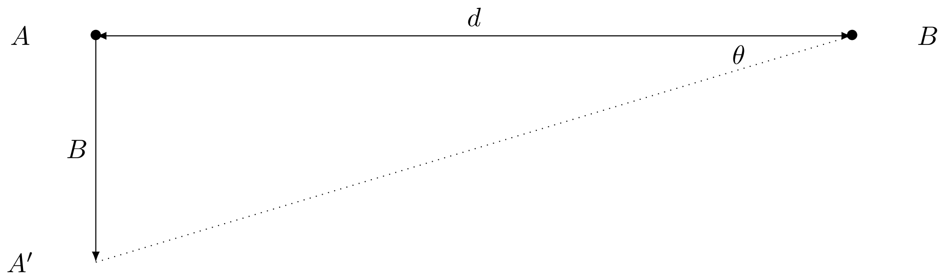
This will be the velocity of an object orbiting around a body producing a gravitational force. Through this, we will also be able to derive a simpler equation for escape velcoity.

$$F_c = F_g$$
$$\frac{mv^2}{r} = G\frac{Mm}{r^2}$$
$$v_{\text{orb}} = \sqrt{\frac{GM}{r}}$$
$$v_{\text{esc}} = \sqrt{2}v_{\text{orb}}$$

Chapter 2

Distances

2.1 Parallax



In this example, we are measuring the distance from A to B. We can do this by moving down a distance B, known as the baseline, and measuring the change in angle, θ. We can then use simple trigonometry to find d.

$$tan\theta = \frac{B}{d}$$

Because we deal with small angles, we are able to use the small angle approximation:

$$tan\theta \approx \theta$$

$$\theta = \frac{B}{d}$$

Where θ is in radians. You can see how this equation becomes difficult for small angles and large distances. We can then use this formula to define the parsec.

$$d = \frac{1}{p}$$

Where the baseline is equal to 1 AU, d is measured in parsec, and p is the parallax, an angle measured in arcseconds. We can convert between radians and arcseconds using:

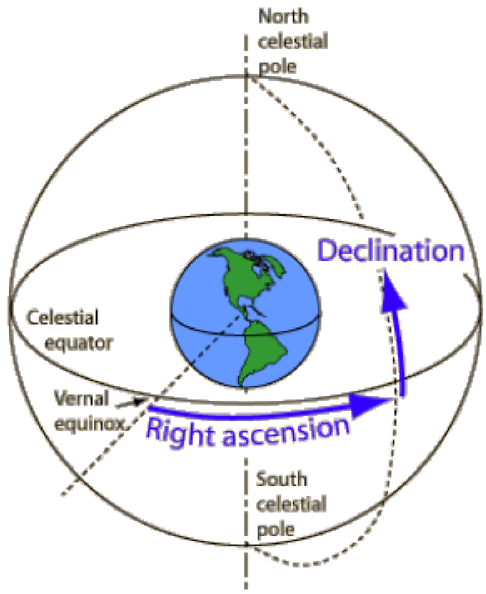
$$1_{\text{Rad}} = \frac{180 \cdot 3600}{\pi}"_{\text{Arcsecond}}$$

$$1_{\text{Degree}} \rightarrow 60_{\text{Arcmin}} \rightarrow 3600_{\text{Arcsec}}$$

2.2 Solid Angle

This is a method of measuring area of an object on the sky. The units of solid angles are called **Steradians** and is equal to a square radian. The entire sky has a solid angle of 4π steradians. This follows the equation for the surface area of a sphere, $4\pi r^2$.

2.3 Celestial Co-Ordinate System



This is a system fixed to the vernal equinox. It is defined by the right ascension of a body and its declination. The declination is measured form the south to the north pole and is denoted by δ, and can take on values $-90 \leq \delta \leq 90$. The right ascension is split into 24 hours, each hour is split into 60 seconds, etc. A right ascension of 1 second is equal to 15 arcseconds at the equator. We are able to convert between these units with the equation below:

$$\Delta\alpha_{\text{arcsec}} = 15 \cdot \Delta\alpha_{\text{s}} \cdot cos\delta$$

2.4 Parallax and Proper Motion

Proper Motion

Definition. *Change in position due to the velocity of the star in the plane of the sky.*

However, proper motion does not account for the line of sight velocity of the star. The equation for proper motion with corresponding units is show below:

$$\mu_{\text{arcsec year}^{-1}} = \frac{v_{kms^{-1}}}{4.74 \cdot d_{pc}}$$

2.5 Distance Due To Brightness

We are able to measure a distance to a star using its luminosity. We understand that luminosity decreases as an inverse square of distance. We can write this relationship as:

$$F = \frac{L}{4\pi d^2} \tag{2.1}$$

Where L is measured in Watts (W), and F is the Flux, which is the energy per second per square meter received from the star.

2.5.1 Magnitude

Magnitude is a logarithmic way of measuring flux through the flux ratio between two bodies. It is a logarithmic measurement. Below is the equation for the difference magnitude between 2 stars related to their flux ratio.

$$m_1 - m_2 = -2.5log\left(\frac{F_1}{F_2}\right)$$

2.5.2 Absolute Magnitude and Distance Modulus

Absolute magnitude M is the magnitude of a star as it would be seen from $10pc$. This is a standard measurement of magnitude so that stars at the same distance but at different luminosity will have different magnitudes.

$$\frac{F_{10pc}}{F_{\text{obs}}} = \frac{L}{4\pi d_{10pc}^2}\frac{4\pi d^2}{L} = \left(\frac{d}{d_{10pc}}\right)^2$$

Further, we can say:

$$\frac{F_{10pc}}{F_{\text{obs}}} = 10^{-0.4\times(M-m)}$$

Combining the two equations, and putting d in units of parsec, we can say:

$$\begin{aligned} M - m &= -2.5\log\left(\frac{d}{10}\right)^2 \\ M &= m - 5\log\frac{d}{10} \\ &= m - 5\log d + 5 \end{aligned}$$

We can then define the distance modulus,

$$m - M = -5 + 5\log d \tag{2.2}$$

Where d is in parsec. Often, distances in astronomy are often given as the distance modulus. The actual distance may then be found using 2.2.

Chapter 3

Observational Astronomy

3.1 Black Body Radiation

Definition. *An ideal object with perfect efficiency, i.e. it absorbs and emits radiation and photos with . Its emission depends entirely on temperature and surface area.*

3.1.1 Planck Function

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\left(\frac{1}{e^{\frac{hc}{kT\lambda}}-1}\right)Wm^{-2}m^{-1} \tag{3.1}$$

$$\begin{aligned} \lambda &\rightarrow \text{Wavelength} \\ T &\rightarrow \text{Temperature} \\ h &\rightarrow \text{Planck Constant} \\ c &\rightarrow \text{Speed of Light} \\ k &\rightarrow \text{Boltzman Constant} \end{aligned}$$

This function describes the spectrum emitted by a blackbody and describes flux in terms of temperature.

Blackbody Spectrum

As we can see from figure **??**, for lower temperatures, the function has a lower peak values of flux, but greater peak wavelength. We can find this wavelength using **Wien's Displacement Law**.

$$\lambda_{\text{peak}} = \frac{2.897 \times 10^{-3}}{T}m \tag{3.2}$$

We are also able to obtain the flux from the Planck function using the **Stefan Boltzmann Law**.

$$\begin{gathered} F = \sigma T^2 Wm^{-2} \\ \sigma \rightarrow \text{ Stefan Boltzmann Constant} \end{gathered} \tag{3.3}$$

3.1.2 Real Emission Spectra

The blackbody spectrum does not cover a large area of the visible colour spectrum, nor does it fit perfectly onto the Planck function as most stars are not perfect blackbodies. In addition, real emissions do not fit perfectly onto the Planck function due to layers of stars having different immensities and colours. Therefore, we can determine two different kinds of temperature in stars. **Colour Temperature**, which most closely fits the perceived colour of the star and **Effective Temperature** which most effectively fits the measured luminosity.

$$T_{\text{eff}} = \left[\frac{L}{\sigma 4\pi R^2}\right]^{\frac{1}{4}} \tag{3.4}$$

3.2 Telescopes and Optics

The focal point, F, of a lens is the position at which all rays converge. The focal length, f, of a lens is defines as the distance from the origin to the focal point.

$$\begin{aligned} \text{Convex } &\rightarrow f > 0 \\ \text{Conxave } &\rightarrow f < 0 \end{aligned}$$

The power of a lens is defined as,

$$P = \frac{1}{f} D \tag{3.5}$$

$$D \implies \text{dioptres} \implies m^{-1}$$

When locating an image, draw a ray parallel to the x-axis from the top of the image until it reaches the y-axis, and then draw it through the focal point. Then, draw a ray going from the top of the image through the optical origin. The point at which these two lines intersect is where the image is located.

For an object horizontal distance u away from the origin, which produces an image at a distance v from the origin, we can define the focal length f as

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} \tag{3.6}$$

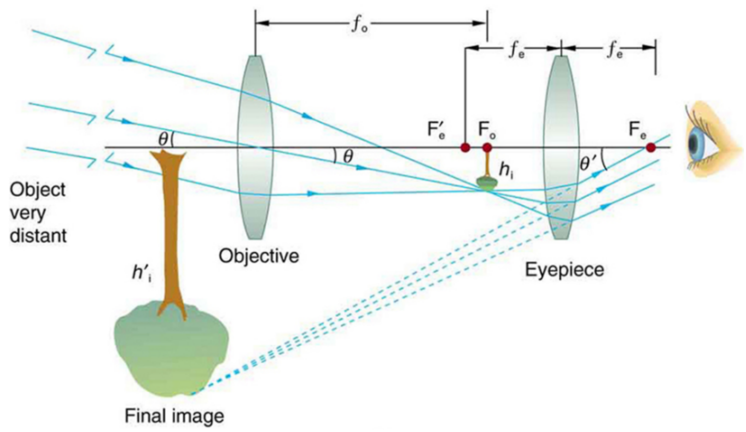
The magnification of the object in the image is then defined as

$$M = -\frac{v}{u} \tag{3.7}$$

3.2.1 Properties of Images at Different Distances in Concave Lenses

Distance	Orientation	Type of Image	Magnification
$u < f$	Upright	Virtual	Increase
$u = f$	Upright	Virtual	Increase, at infinite Distance
$f < u < 2f$	Inverted	Real	Increase
$u = 2f$	Inverted	Real	Same Size
$u > 2f$	Inverted	Real	Diminished
$u \rightarrow \infty$	Image forms on the focal plane	Real	Diminished

3.2.2 Refracting Telescopes



These telescopes use 2 lenses, one to focus the image and the other to make the rays of light parallel so that they are observable. telescopes magnify the **angle** of what we are observing. This magnification is given by

$$M = \frac{f_o}{f_e} \tag{3.8}$$

However, these telescopes are susceptible to chromatic aberrations. An alternative are **reflective telescopes**.

3.2.3 Reflective Telescopes

These telescopes are more simple to build, produce cleaner images, but are small and are susceptible to spherical aberrations.

Telescopic Parameters

Magnification

$$M = \frac{f_o}{f_e} = \frac{\theta'}{\theta} \tag{3.9}$$

Image Scale

$$P = \frac{\theta}{s} = \frac{1}{f} \tag{3.10}$$

$$s \rightarrow \text{ Scale/focal plane size}$$
$$\theta \rightarrow \text{ Incoming angle}$$

Collecting Area

$$A = \frac{\pi D^2}{4} \tag{3.11}$$

Focal Ratio

$$F = \frac{f}{D} \tag{3.12}$$

Resolution
It is set by the **diffraction limit** of the telescope, which is the smallest angle observable under perfect conditions.

$$\theta = 1.2 \times \frac{\lambda}{D} \tag{3.13}$$

Atmospheric Turbulence For an air pocket of size r_o, the resulting resolution is,

$$\theta = 1.2 \times \frac{\lambda}{r_o} \tag{3.14}$$

3.2.4 Interferometry

This is a way of combining multiple telescopes. If we have N telescopes, the total collection area is,

$$A = \frac{N\pi D^2}{4}, \tag{3.15}$$

and if they are a distance d apart, the resolution becomes,

$$\theta = \frac{\lambda}{d}. \tag{3.16}$$

3.3 Spectroscopy

3.3.1 Refraction

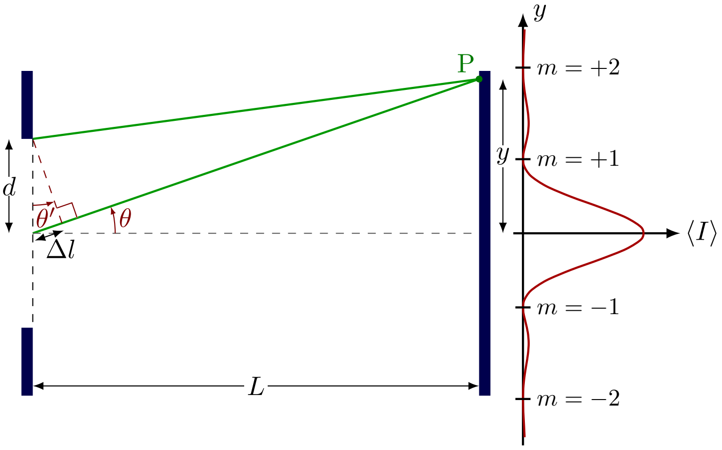
This is a method of being able to seperate wavelengths of light. For light travelling through different materials, we say,

$$\frac{n_1}{n_2} = \frac{\sin(\theta_2)}{\sin(\theta_1)} \tag{3.17}$$

Where θ_1 is the angle the light makes incident on the material, and θ_2 is the angle the light makes inside the material. n is defined as the refractive index of a material. It is defined by the speed v of light travelling through a material,

$$n = \frac{v_{\text{material}}}{c} \tag{3.18}$$

3.3.2 Diffraction



We are able to categorise wavelengths by observing maxima of light through a double slit. These will be points at which the light goes under constructive interference. We are able to relate the wavelength of light observed to its angular position as shown,

$$d \sin\theta = m\lambda \tag{3.19}$$

where m is an integer.

3.4 Detectors

Detectors work using the photoelectric effect; photons hit the surface of a detector, an electron is then freed up, then returns to the detector which creates a charge.

When thinking about detectors, we must consider the following:

- **Quantum Efficiency -** Fraction of photons actually liberating an electron.
- **CCD Charge Transfer Efficiency** Fraction of electrons transferred to the next pixel during the read-out.
- **Dynamic Range** Highest number of electrons that can be counted.

Chapter 4

Stars and Galaxies

4.1 Hertzsprung-Russell Diagram

This is a plot of wavelength of light vs. absolute magnitude of a star. It helps us in categorising stars.

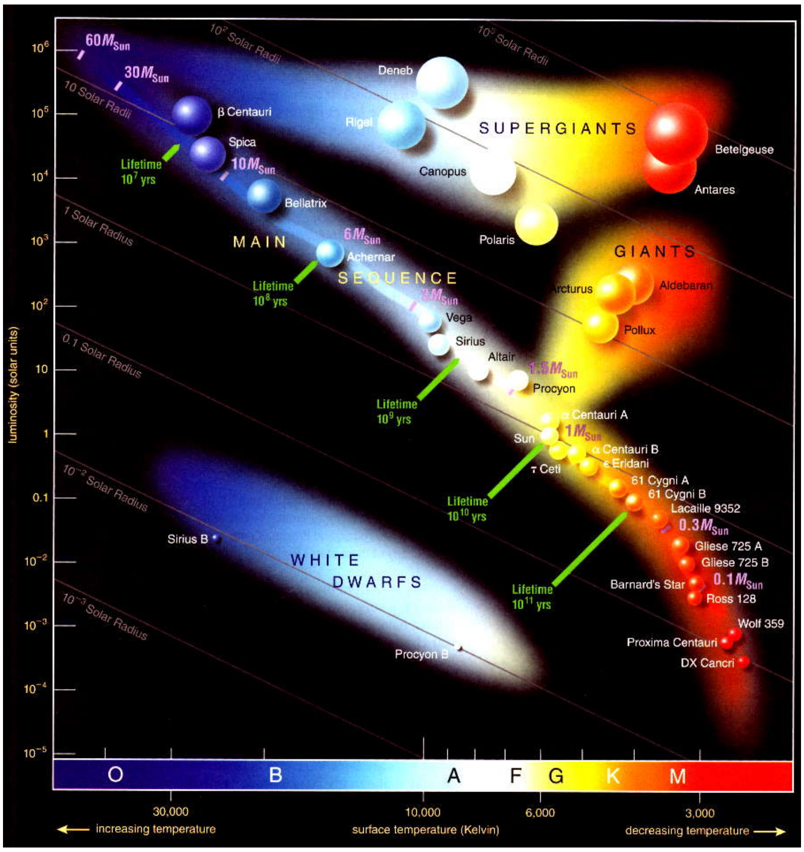


Figure 4.1: Hertzsprung-Russell Diagram

Most stars will lie on the main sequence. If a star lies on the main sequence. We can think about the equation,

$$L = 4\pi R^2 \sigma T^4$$

and then by taking logs on both sides,

$$\log L = 4 \log T + 2 \log R + \log 4\pi\sigma \tag{4.1}$$

whose slope follows the general slope of the main sequence line, given that radius is constant (see figure 4.2).

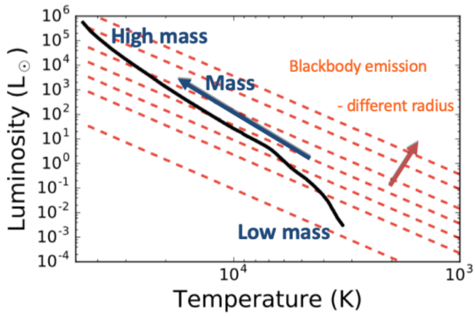


Figure 4.2: Main sequence line compared with equation (4.1) for various radii.

Figure 4.1 shows us that most stars exist on the main sequence line, and, that the further a star evolves the further to the right of the main sequence it exists. As stars approach the final steps of their evolution, they go into a "turn-off" to become supergiants. Low mass stars then move onto becoming white dwarfs (see table 4.1). From this turn-off, we can analyse the luminosity of a star cluster to estimate its age.

Table 4.1: HR Ages

Group	Masses	Evolution
Main Sequence	All Masses	90% of the evolution of a star
Giants	Low Masses	After exhaustion of hydrogen in the core
Supergiants	High Masses	After exhaustion of hydrogen in the core
White Dwarfs	Low Masses	Final remnant

4.2 Hydrogen Emission Lines

The reason stars emit specific spectra of light is because of electrons within atoms. Below is a semi-classical derivation of the emitted wavelength of a hydrogen atom.

Electrons are both particles and waves, and can only exist when their orbits are described as standing waves. Therefore, the circumference of its orbit must be equal to an integer number of wavelengths,

$$2\pi r = n\lambda.$$

The wavelength of an electron depends on its momentum,

$$\lambda = \frac{h}{p} = \frac{h}{m_e v},$$

therefore,

$$\begin{aligned} 2\pi r &= n\frac{h}{m_e v} \\ \implies v &= \frac{hn}{m_e 2\pi r}. \end{aligned} \tag{4.2}$$

The electron is kept in orbit by the electric force, acting as the centripetal force. This is given as,

$$\frac{e^2}{4\pi\epsilon_0 r^2} = \frac{m_e v^2}{r}. \tag{4.3}$$

Substituting v from (4.2) into (4.3), we get the expression,

$$r = n^2 \left(\frac{h}{2\pi}\right)^2 \frac{4\pi\epsilon_0}{m_e e^2}, \tag{4.4}$$

known as the **Bohr radius**.

Further, the energy of an electron consists of kinetic and potential energy, given by the expression,

$$E_n = \frac{1}{2}m_e v^2 - \frac{e^2}{4\pi\epsilon_0 r}. \tag{4.5}$$

Substituting (4.2) and (4.4) into (4.5), we find that the energy only depends on n,

$$E_n = -\frac{1}{n^2}\left(\frac{\pi}{2h}\right)^2 \frac{m_e e^4}{(4\pi\epsilon_0)^2}. \tag{4.6}$$

All the terms in (4.6) are constants apart from n, therefore,

$$E_n = -\frac{13.6}{n^2}\text{eV}. \tag{4.7}$$

When changing energy levels, so when transitioning from n_1 to n_2, we apply the formula,

$$E = 13.6\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)\text{eV}. \tag{4.8}$$

We can use the relation between energy of a photon and its wavelength to get,

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \tag{4.9}$$

where R is the Rydberg constant.

Then, for larger nuclei with a proton number Z,

$$\frac{1}{\lambda} = RZ^2\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right).$$

4.3 Hydrostatic Equilibrium and Virial Theorem

A star is compressed by its gravitational force. We can calculate the dynamical time, which is the time taken for a star to collapse under its own gravitational force.

$$\begin{aligned}
v_{\text{esc}} &= \sqrt{\frac{2GM}{R}} \\
t_{\text{dyn}} &= \frac{R}{v_{\text{esc}}} \\
M &= \frac{4}{3}\pi R^3 \boldsymbol{\rho} \\
\therefore t_{\text{dyn}} &= \sqrt{\frac{3}{8\pi}\frac{1}{G\boldsymbol{\rho}}} \\
&= \sqrt{\frac{1}{G\rho}}
\end{aligned}$$

Pressure then acts by expanding the star, so that it does not collapse in on itself within hours. Knowing this, we can derive an expression which all stars must satisfy in order to be stable. To do this, we must suppose a spherical star, and within that star is a shell of thickness dr and radius r. It has an area dA and mass dM. We can derive an expression using **Newton's Shell Theorems**.

Theorem 4.3.1. *The gravitational force from a spherically symmetric mass distribution can be calculated as if the entire mass is located at the centre of the sphere.*

Theorem 4.3.2. *The gravitational force of a spherically symmetric shell on any object inside, regardless of the object's location within the shell, is 0.*

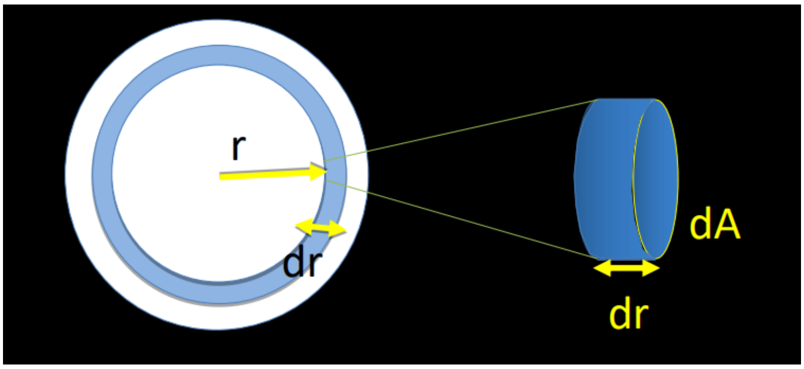


Figure 4.3: A star with a radius r and a shell within it with a thickness dr, mass dM, and area dA.

Applying this,

$$F_g = \frac{Gm\left(r\right)dm}{r^2}, \tag{4.10}$$

where $m(r)$ is the mass located inside the radius r. (See figure 4.3.)

Because the star is stable, the forces should cancel out.

$$\begin{aligned}
F_g &= P(r)dA - P(r+dr)dA \\
P(r+dr) &= P(r) + \frac{dP}{dr}dr \\
F_g + \frac{dP}{dr}drdA &= 0 \\
dm &= drdA\rho(r) \\
\frac{dP}{dr} &= -\frac{Gm(r)\rho(r)}{r^2}
\end{aligned}$$

The final line above is a condition which all stars must satisfy. It is the pressure gradient which opposes the gravitational force. However, stars do eventually die as they radiate energy. We must then find the total energy of the star. This will be its **gravitational potential energy.**

First we can begin by saying that mass is given by,

$$m(r)\int_0^R 4\pi r^2\rho(r)dr$$

$$m(r) = \frac{4\pi}{3}r^3\rho.$$

We can then use the gravitational energy equation,

$$E(r) = -\frac{Gm(r)m_{\text{shell}}}{r} = -\frac{Gm(r)4\pi r^2\rho dr}{r}.$$

We can now integrate over r,

$$E_p = -\int_0^R \frac{Gm(r)4\pi r^2\rho}{r}dr = -\int_0^R G\frac{16\pi^2}{3}\rho^2 r^4 dr$$

where R is the final outer radius of the star. This integral becomes,

$$E_p = -\frac{3}{5}\frac{GM^2}{R}. \tag{4.11}$$

4.3.1 Kelvin-Helmholtz Timescale

$$t_{KH} = \frac{E_p}{L} \approx \frac{GM^2}{RL} \tag{4.12}$$

This describes how long it will take a star to radiate out all of its energy. However, this time scale is still too short. We must then think about the thermal energy of a star.

The ideal gas laws tell us,

$$PV = NkT \implies P = \frac{N}{V}kT = nkT$$

where N is the number of particles and n is the particle density. We must then define the thermal energy density,

$$\epsilon = \frac{3}{2}nkT = \frac{3}{2}P.$$

The total thermal energy is then given by,

$$\begin{aligned}
E_{\text{th}} &= \int_0^R \epsilon 4\pi r^2 dr \\
E_p &= -\int_0^R \frac{Gm(r)\rho(r)}{r} 4\pi r^2 dr \\
&= \int_0^R \frac{dP}{dr} 4\pi r^2 dr \\
&= -3\int_0^E P(r) 4\pi r^2 dr \\
\epsilon(r) &= \frac{3}{2}P(r) \\
\therefore E_p = -2\int_0^R \epsilon(r) 4\pi r^2 dr &= -3\int_0^R P(r) 4\pi r^2 dr
\end{aligned}$$

The above then gives way to the virial theorem, which states,

$$E_{\text{th}} = -\frac{1}{2}E_p \tag{4.13}$$

$$E_{\text{kinetic}} = -\frac{1}{2}E_p \tag{4.14}$$

4.3.2 Where does the sun get its energy?

The total density and temperature of the sun is just enough to be bale to sustain itself. It produces energy through the following process:

PP Cycle

This is known as the Proton-Proton cycle, which is shown in figure 4.4.

The reaction follows as so:

1. $\mathrm{H} + \mathrm{H} \longrightarrow 2\,\mathrm{H} + \mathrm{e}^+ + v$
   - Releases -1.44MeV of energy, takes 14×10^9 years.
2. $2\,\mathrm{H} + \mathrm{H} \longrightarrow 3\,\mathrm{He} + \gamma$
   - Releases -5.49MeV, takes 6 seconds.
3. $3\,\mathrm{He} + 3\,\mathrm{He} \longrightarrow 4\,\mathrm{He} + 2\,\mathrm{H}$
   - Releases -12.85MeV, takes 10^6 years.

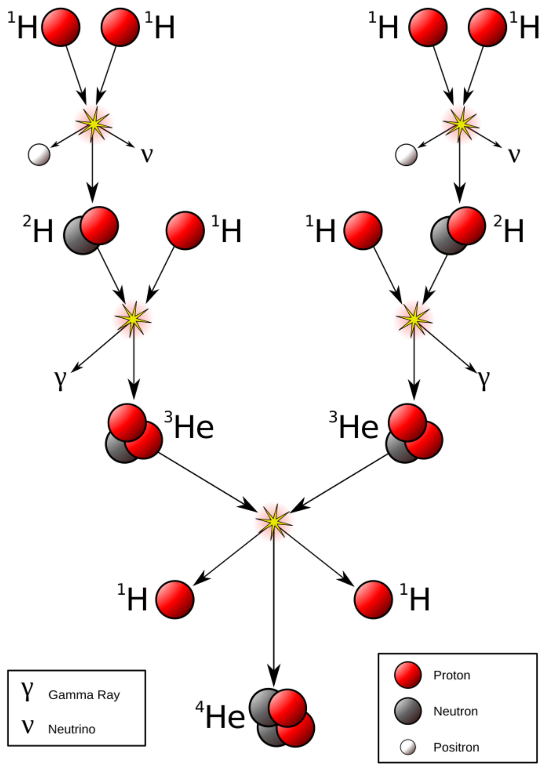


Figure 4.4: Scheme of the proton–proton branch I reaction.

4.3.3 Nuclear Time-Scale

This timescale will tell us how long a star should stay on the main sequence. The equation for this timescale is,

$$t = \frac{fM}{L}, \tag{4.15}$$

which takes into account how long it will take for a star's mass to be converted into energy. f is the efficiency factor. We can make this equation simpler for **main sequence** stars.

$$\begin{aligned} L_{\text{MS}} &\propto M^{3.5} \\ t_{\text{MS}} &\propto M^{-2.5} \\ t_{\text{MS}} &= 10^{10}\left(M^{-2.5}\right). \end{aligned}$$

Further, we can approximate the stellar radius for main sequence stars;

$$\begin{aligned} \frac{dP}{dr} &\propto -\frac{M\rho}{R^2} \\ \text{Assume } \frac{dP}{dr} &= -\frac{P}{R} \\ \frac{M\rho}{R^2} &\propto \frac{P}{R} \\ P = nkT &\implies \rho T \\ &\text{Assuming temperature is} \\ &\text{the same/similar for all stars} \\ &\implies M \propto R \end{aligned} \tag{4.16}$$

4.3.4 Stellar Remnants

When a star dies, it stops fusing hydrogen, or whatever other element it is composed of. Hydrogen fusion occurs for stars with a mass $> 0.008 M_{\odot}$.

Brown Dwarfs under deuterium fusion.

Red Giants fuse helium through the heating of their core, however their surface cools.

Massive stars simply continue fusing. They are usually composed of an iron core with layers of elements which keep fusing and creating energy. (See figure 4.5). Their iron core is unable to fuse, and simply burns. (See figure 4.6).

Figure 4.5: Massive star layer placeholder

Figure 4.6: Fusion fission graph placeholder

End products of star remnants

	Low to Average Mass	High Mass Stars	Very High Mass Stars
Product	White Dwarf	Neutron Stars	Black Holes
Mass	$M < 1.4M_\odot$	$2M_\odot < M < 3M_\odot$	$M > M_\odot$
Radius	$0.01R_\odot$	10km	R_{Sch}
Characteristics	Not hot enough to ignite carbon, so fusion ends.	Composed entirely of neutrons	Form when a star collapses into a singularity. Crete material around themselves.

The radius of a black hole is given by the **Schwarschild Radius**, which uses the fact that light cannot escape a black hole to state,

$$v_{\text{esc}} = c = \sqrt{\frac{2GM}{R}}$$

so,

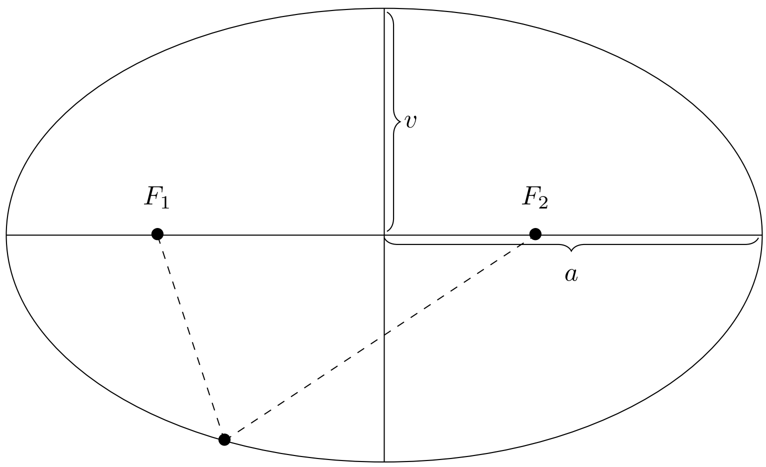
$$R_{\text{Sch}} = \frac{2GM}{c^2} \tag{4.17}$$

Chapter 5

Orbits and Exoplanets

5.1 Kepler's Laws

Law 1. *Planets orbit the sun in ellipses with the at one focus.*



- $a \rightarrow$ Semi-Major Axis
- $b \rightarrow$ Semi-Minor Axis
- $r_{xF_1} + r_{xF_2} = 2a$
- Eccentricity $\rightarrow e = \sqrt{1 - \frac{b^2}{a^2}}$
- $\left(\frac{x}{a}\right)^2 + \left(\frac{x}{b}\right)^2 = 1$
- Distance between focus and centre $\rightarrow FO = ae$

Law 2. *Planets sweep out equal areas in equal times.*

This is a consequence of the conservation of angular momentum, which states,

$$\begin{aligned} L &= mvr\sin\theta \\ L &= m\mathbf{r}\times\mathbf{v} \end{aligned} \tag{5.1}$$

Generalisation

Let us assume a perfectly circular orbit of a mass M.

$$
\begin{aligned}
L &= mrv \\
v &= r\frac{d\theta}{dt} \\
L &= mr^2\frac{d\theta}{dt} \\
\frac{dL}{dt} = 0 \quad \frac{dL}{dt} &= m\left(\frac{dr^2}{dt}\frac{d\theta}{dt} + \frac{d^2\theta}{dt^2}r^2\right) \\
\frac{dr^2}{dt} = \frac{d^2\theta}{dt^2} = 0 & \\
&\implies \frac{d\theta}{dt}\text{is conserved} \\
&\therefore \text{No angular acceleration} \\
&\therefore \text{Gravity must act be radially.}
\end{aligned} \tag{5.2}
$$

Law 3. *The period P and semi-major axis of the orbit are related through $P^2 = a^2$.*

Where P is in years and a is in AU. However, this only applies in our own solar system.

Generalisation

Assuming a perfectly circular orbit, assume the force acting on a planet is $F_g = Ar^\beta$ and the centripetal force is $F_g = \frac{mv^2}{r}$. Assuming Kepler's law is correct, we can state,

$$
\begin{aligned}
P^2 &= Cr^2 \\
v = \frac{2\pi r}{P} & \\
Ar^\beta &= \frac{4\pi^2 m}{Cr^2}
\end{aligned}
$$

where A, C, and β are some constants. We can conclude from the above equation,

$$\beta = -2 \qquad A = \frac{4\pi^2}{C}$$

which implies that gravity scales by a factor of r^{-2} and is proportional to the mass of the orbiting object. If we factor in another mass, and replace all constants with a single G, we get Newton's law of gravity,

$$F_g = -\frac{GMm}{r^2}.$$

We are able to rearrange our equation for Ar^β to get the general form of Kepler's third law,

$$P^2 = \frac{4\pi^2}{GM}a^3. \tag{5.3}$$

5.2 Binary Orbits

We will now consider how two planets of similar mass orbiting each other and how they effect each other's motion. The two planets will orbit their shared, stationary centre of mass, calculated using,

$$x_{\text{cm}} = \frac{\sum m_i x_i}{\sum m_i}.$$

5.2.1 Kepler's Third Law for a Binary System in Circular Orbit

$$F_g = \frac{Gm_1m_2}{(r_1+r_2)^2}$$

$$F_{cp_1} = \frac{m_1v_1^2}{r_1^2}$$

$$F_{cp_2} = \frac{m_2v_2^2}{r_2^2}$$

$$v_i = \frac{2\pi r_i}{P}$$

$$\frac{Gm_2}{(r_1+r_2)^2} = \frac{4\pi^2 r_1}{P^2} \tag{5.4}$$

$$\frac{Gm_1}{(r_1+r_2)^2} = \frac{4\pi^2 r_2}{P^2} \tag{5.5}$$

(5.4) + (5.5):

$$\frac{G(m_1+m_2)}{(r_1+r_2)^2} = \frac{4\pi^2(r_1+r_2)}{P^2}$$

$$P^2 = \frac{4\pi^2}{G}\left(\frac{r^3}{m_1+m_2}\right)$$

where

$$r = r_1 + r_2.$$

For elliptical orbits,

$$r = a = a_1 + a_2.$$

5.2.2 Velocities of Star-Planet Systems

$$M_* >> m$$

$$r_* + r \approx r$$

$$mvm_*v_*$$

$$F_{cp} = F_g$$

$$\implies \frac{mv^2}{r} = \frac{GmM_*}{(r+r_*)}$$

$$\implies \frac{v^2}{r} = \frac{GM_*}{r^2}$$

$$\implies v = \sqrt{\frac{GM_*}{r^2}}$$

$$v_* = \frac{m}{M_*}\sqrt{\frac{GM_*}{r}} \tag{5.6}$$

5.3 Exoplanet Detection

Planets are objects with a mass below $13_\odot$, and which are spherical and dominate their orbit. Below outlines the various methods of detecting exoplanets.

5.3.1 Radial Velocity

This is a method where we measure the star's wobble (in the case the we view the planet edge-on). We measure v via the Doppler shift

$$v_{\text{Doppler}} = c\frac{\Delta\lambda}{\lambda_0}. \tag{5.7}$$

If the orbit is not perfectly edge on, we must resolve its angle of inclination with the plane of the sky,

$$v_{\text{Doppler}} = v_{\text{orbit}} \sin i. \tag{5.8}$$

We can the combine this with equation (5.6) to get the mass of the star as,

$$m \sin i = M_* v_{*\text{ Doppler}} \sqrt{\frac{r}{GM_*}}. \tag{5.9}$$

5.3.2 Transit Method

This method is used for edge-on orbit. As the exoplanet orbits its host star, it will eventually eclipse it and cause a dip in brightness. The brightness will give a period which can be measured. However, the inclination must be very close to 90°. The calculation we can use is related to the change in flux we observe as the planet passes the star,

$$\frac{\Delta F}{F} = \left(\frac{R}{R_*}\right)^2. \tag{5.10}$$

5.3.3 Habitability

This describes planets which are rocky and of a surface temperature that allows for liquid water to exist on it. To calculate the temperature on the surface of the exoplanet, assume it to be blackbody. The received luminosity is,

$$L_{\text{rec}} = \pi R^2 F.$$

We can then consider the luminosity absorbed. This will be determined by a factor called albedo, ω,

$$L_{\text{absorbed}} = (1-\omega)\pi R^2 F.$$

Since we are considering it as a blackbody, the luminosity it emits is then,

$$L_{\text{Emit}} = 4\pi R^2 \sigma T^4.$$

The absorbed and emitted luminosities should be equal, so

$$T = \left(\frac{(1-\omega)L_*}{16\pi\sigma}\right)^{\frac{1}{4}} \sqrt{\frac{1}{d}}. \tag{5.11}$$

The temperature required for the habitable zone should satisfy,

$$270\text{k} < T < 320\text{k}.$$

Chapter 6

Galaxies

Galaxies are gravitationally bound systems of stars, planets nebulae, black holes, and gas. They are bound by their own mutual gravity. Stars can only be formed within them. Their mass is between, $10^7 M_\odot < M < 10^{13} M_\odot$.

Galaxies can take different forms. Table 6.1 breaks these down.

Table 6.1: Types of Galaxies

Spiral	**Elliptical**	**Irregular**
Have a central bulge/bar.	Smooth ellipsoid shape	Don't have a clear structure
Have spiral arms where star formation occurs	Composed of mostly old stars and dust, star formation does not occur here.	Gas rich with very active star formation
Have a stellar halo	Products of merger galaxies	These may be dwarf galaxies or galaxies undergoing mergers.
A subsection of spiral galaxies are **barred spirals**. They are the most common type in the universe, mostly appearing in older galaxies.		

The milky way is a type of *spiral galaxy.* Its centre is red, which implies it is old. It is around 30kpc wide and 0.5kpc thick.

6.1 Central Black Hole

Most large galaxies contain black holes. Black holes have active acreting disk of gas which fuel star formation. This disk drives a massive jet which can drive the remaining gas out of the galaxy.

6.2 Rotation Curves

We can estimate the mass of a galaxy if we know the period of orbit and rotational velocity of a star, we are able to find the velocity of the galaxy up to the radius that the star lies on in the galaxy,

$$M(R) = \frac{v^2 R}{G}. \tag{6.1}$$

Kepler's laws tell us that,

$$v \propto R^{-\frac{1}{2}} \tag{6.2}$$

however, for constant density,

$$M(R) \propto \rho \tag{6.3}$$
$$v \propto R \tag{6.4}$$

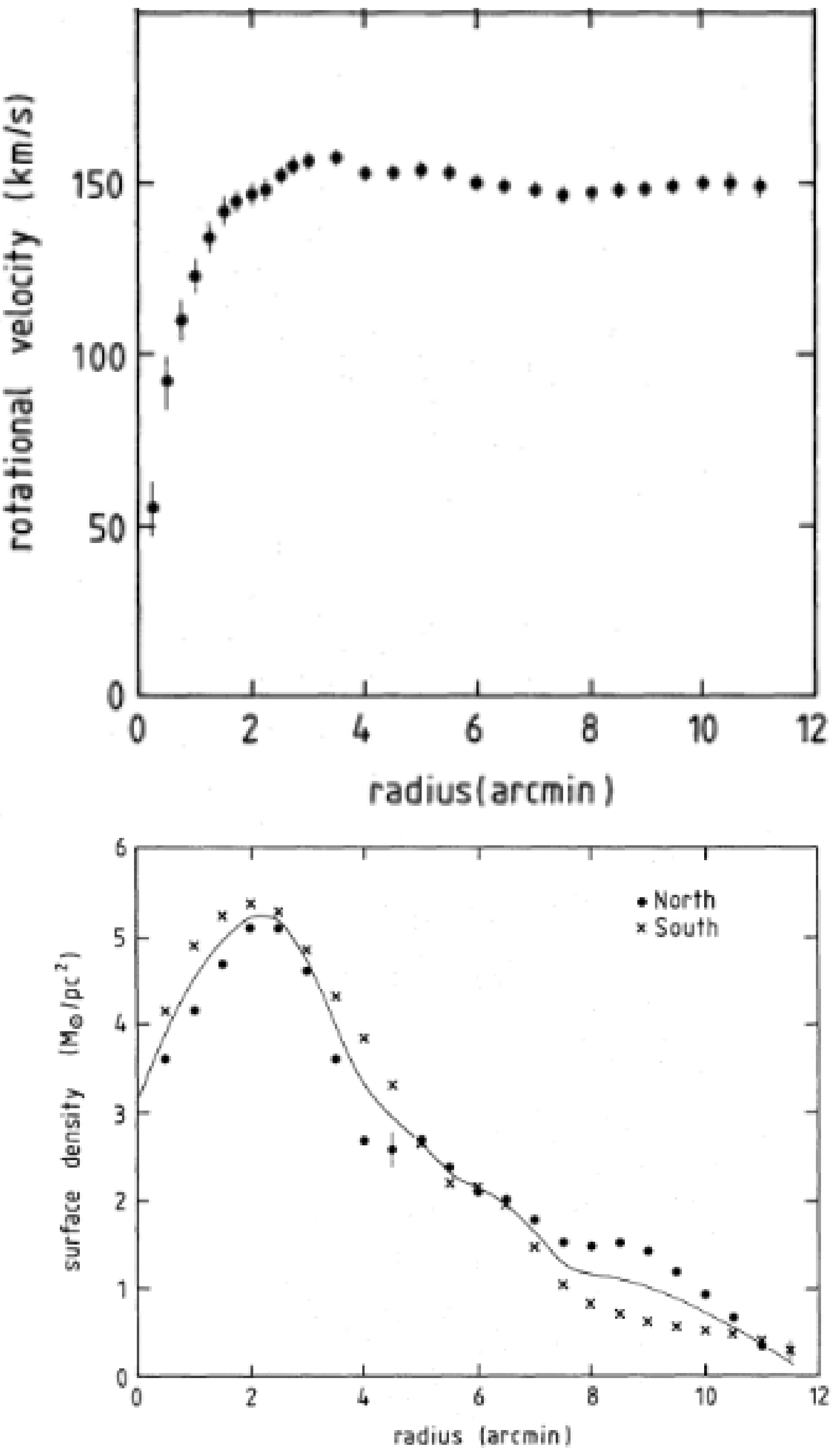


Figure 6.1: Rotation curve and mass distribution of the galaxy NGC 3198

However, as can be seen in figure 6.1, the rotational velocity and observed mass of the galaxy does not match. From the rotational curve, the mass of the system should be modelled by,

$$\rho \propto e^{-R} \tag{6.5}$$

so an additional mass component is necessary, filled in by dark matter.

6.2.1 Dark Matter

- Composes 85% of matter in the universe.
- It is not baryonic.
- Distributing in large spherical halos around galaxies.

Chapter 7

Cosmology

7.1 Hubble-Lemaitre Law

$$v = H_0 r$$

7.2 Redshift

$$z = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{v}{c} \tag{7.1}$$

z is the redshift. It is detected from wavelengths of hydrogen lines. Distances to galaxies are often quoted in redshift rather than distance.

At larger distances, velocities become relativistic. We need to use the relativistic form,

$$1 + z = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \tag{7.2}$$

Galaxies aren't ripped apart by hubble flow because the force of gravity is stronger than the force causing the expansion of the universe. This leads to observing negative redshift.

7.2.1 Time Of Distant Galaxies

We view distant objects as if they were in the past. Galaxies with high redshifts are seen as they were close to the time of their formation.

7.3 Star Formation

Young stars are luminous and hot. This means that they are more visible in the ultraviolet spectrum. Massive stars burn very bright and die fast. They burn through their fuel much faster and they burn in the ultraviolet. We can use UV images to measure star formation rate in galaxies.

Peak star formation occurred in the early universe ($z \approx 2$).

7.4 The Cosmological Principle

The universe is...

1. **Homogeneous** - Same at any location.
2. **Isotropic** - Same in all directions.

7.4.1 The Galilean Principle

Definition. *Laws of physics must be the same in all inertial frames.*

7.5 The Critical Density

Hubble flow will causes galaxies to flow apart, while gravity will attempt to keep them together. We want to know if gravity is strong enough to halt the Hubble flow.

$$\begin{aligned}
F &= m\frac{d^2r}{dt2} \\
F_G &= -G\frac{m(r)m_G}{r^2} \\
m(r) &= \frac{4\pi}{3}\rho r^3 \\
F_G &= -\frac{4\pi}{3}\rho M_g r \\
U &= -G\frac{m(r)m_G}{r} \\
&= -G\frac{4\pi}{3}\rho m_G r^2 \\
K &= \frac{1}{2}m_G v^2 = \frac{1}{2}m_G H_0^2 r^2 \\
E_{\text{tot}} &= m_G\left(\frac{1}{2}H_0^2 - G\frac{4\pi}{3}\rho\right)
\end{aligned} \tag{7.3}$$

From this we can conclude that if...

- $E > 0$, the universe moves with more force than gravity. Gravity will slow the Hubble flow.
- $E < 0$, gravity will stop and reverse the Hubble flow.
- $E = 0$, Gravity will stop the Hubble flow, but take an infinitely long time to do so.

The critical density is then,

$$\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G}. \tag{7.4}$$

We can also then define the **true density**,

$$\Omega_0 = \frac{\rho_0}{\rho_{crit}}. \tag{7.5}$$

An $\Omega_0 = 1$ implies we live in a redshift of 0. The universe is currently at a true density of 0.3.

7.6 Scale Factor

If we consider a galaxy receding from us, we can define the scale factor as,

$$a(t) = \frac{r(t)}{r(t_0)}. \tag{7.6}$$

We can define a functional Hubble parameter in terms of the scale factor,

$$H(t) = \frac{1}{a(t)}\frac{da(t)}{dt}. \tag{7.7}$$

We can also define it in terms of the redshift,

$$1 + z = \frac{1}{a(t)}. \tag{7.8}$$

7.7 General Relativity

Einstein suggested that laws of physics should apply in non-inertial frames too. The consequence of this is that gravity is not a force. If you experience no force, you move in a straight line. So, non-inertial frames curve spacetime.

7.7.1 Friedman Equation

$$H(t)^2 = \frac{8\pi G\rho(t)}{3} - \frac{kc^2}{a^2} \tag{7.9}$$

where a is the expansion rate and k is the curvature of the universe.

Proof.

$$\frac{d^2r}{dt^2} = -\frac{4\pi}{3}G\rho(t)r$$
$$r(t) = a(t)r_0$$
$$\frac{d^2a(t)}{dt^2} = -\frac{4\pi}{3}G\rho(t)a(t0$$
$$\rho(t) = \rho_0\frac{r}{a(t)^3}$$
$$\frac{d^2a(t)}{dt^2} = -\frac{4\pi}{3}G\frac{G\rho_0}{a(t)^2}$$
$$\dot{a}\frac{d\dot{a}}{dt} = -\frac{4\pi}{3}G\frac{\rho_0}{a(t)^2}\dot{a}$$
$$\dot{a}d\dot{a}(t) = -\frac{4\pi}{3}G\frac{\rho_0}{a(t)^2}da$$
$$\frac{1}{2}\dot{a}(t)^2 = \frac{4\pi}{3}G\frac{\rho_0}{a(t)} + C$$
$$\left(\frac{da}{dt}\right)^2 = \frac{8\pi}{3}G\frac{\rho_0}{a(t)} + C$$
$$C = kc^2$$
$$\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = H(t)^2 = \frac{8\pi}{3}G\rho(t) - \frac{kc^2}{a^2}$$

□

7.7.2 Curvature

$$\begin{aligned}\rho(t) &= \Omega(t)\rho_{\text{crit}}(t)\\ \Omega(t) - 1 &= \frac{kc^2}{H(t)^2a(t)^2}\end{aligned} \tag{7.10}$$

7.8 The Big Bang

Order in which the universe formed:

1. Inflation occurs. The universe is expands at super-luminous speed.

$$T(t) = \frac{T_0}{a(t)} = T_0(1+z) \tag{7.11}$$

2. After inflation, the universe is made of hot, dense, opaque plasma. No light escapes.

3. < 1**s** $\rightarrow$ Nucleosynthesis begins. There is initially an equilibrium between protons and neutrons.

$$e^- + p \leftrightarrow v_e + n \tag{7.12}$$

4. At 1s, the reaction stops and the ratio of neutrons to protons is $1:5$. The neutrons are not stable, with a half life of 10 minutes.

$$n \rightarrow p + e^- + \overline{v}_e \tag{7.13}$$

5. **Deutirium fusion begins.** At 1 minute, photon density is not high enough to dissociate deutirium, so it is stable. This causes neutron decay to stop.

$$n + p \rightarrow D + 2.2\text{MeV} \tag{7.14}$$

^{4}He is then formed from deutirium. However, temperatures are too low for fusion of carbon to occur.

6. At 3 minutes, nucleosynthesis ends.

7.8.1 Helium Mass Ratio

$$Y = \frac{2N_n}{N_n + N_p} \tag{7.15}$$

7.8.2 Cosmic Microwave Background

Formed during the **epoch of recombination,**

$$e^- + p \rightarrow H + \gamma. \tag{7.16}$$

Formed at temperatures of around 3000K. Free electron density was too low for scattering of photons, so the Universe becomes transparent.

7.8.3 Baryon Density

$$\eta = \frac{N_b}{N_\gamma} = \frac{\rho_{\text{crit}}\Omega}{m_b}\frac{1}{N_\gamma} \tag{7.17}$$

7.9 Dark Energy

This is a concept used to describe the acceleration of the expansion of the universe. We can use the **cosmological constant** to describe it. We can add it to the Friedmann equation,

$$\frac{1}{a(t)^2}\left(\frac{da(t)}{dt}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{kc^2}{a(t)^2} + \frac{\Lambda}{3}. \tag{7.18}$$

Λ is very small quantity.

From this chapter, we can conclude that our universe progressed from being dominated by:

$$\text{Radiation} \rightarrow \text{Matter, } a(t) \propto t^{2/3} \rightarrow \text{Dark Energy, } a \propto e^t \tag{7.19}$$