Introduction to Astrophysics & Cosmology
Revision summary

Below you will find an overview of the most important equations and concepts in this course. This is meant as a help. It does not replace the lecture notes(!) but can be used in conjunction with those.

A summary of general astrophysics equations (not specific to this course) is available on `http://astrophysicsformulas.com/astronomy-formulas-astrophysics-formulas/astrophysics-reference/` and if there is something you don't understand in the lecture notes, it may be worth looking there. Be very aware that the same symbol can stand for different things in different contexts, for instance L can mean angular momentum or luminosity, and a is used for acceleration, for the semi-major axis, and for the scale factor. It is not sufficient to know the equation: you also need to know the meaning of the symbols. Importantly, you also need to know the units used in the equation. Some are in SI units, others are not. 'Knowing' an equation means knowing all three.

The sheet of constants will be available to you during the exam and mid-term assessment. There is no need to memorize the values of physical or astronomical constants such as astronomical unit, solar radius, etc.

All material, equations and derivations covered in the lecture notes is examinable. This is not limited to what is listed below. Simple equations you should know, and more difficult equations you should be able to derive. You may expect questions that test your knowledge, questions that test your ability to work with the equations, and questions that test your problem solving skills. Old exam papers of the last four years will give a good indication of what to expect, but be aware that the detailed syllabus may change from year to year. Tutorial questions are in many cases similar to exam-type questions and we recommend that you use them in your revision.

1. The Universe

Types of objects: planets, stars, galaxies, Universe, with masses ranging from 10^{25} to 10^{52} kg.

Numbers and units The non-SI units used in astrophysics include the Astronomical Unit (AU), parsec (pc), solar mass ($\mathrm{M}_\odot$). Powers of ten are used widely: kilo (k) 10^3, mega (M) 10^6, giga (G) 10^9. Typical accuracy is 3 significant digits but it varies per question!

Distances A basic understanding of typical distances in the Universe is required: 1 pc is roughly the distance to the nearest star, the galaxy is roughly 10 kpc in radius, and the nearest major galaxy (Andromeda) is at 1 Mpc. The edge of the (observable) Universe is at 4 Gpc.

Density: mass density ρ, units of $\mathrm{kg\,m^{-3}}$; particle (number) density n, in units of $\mathrm{m^{-3}}$. Density of water is $10^3\,\mathrm{kg\,m^{-3}}$; number density of stars in the Galaxy is roughly $1\,\mathrm{pc^{-3}}$.

Gravity

$$F = \frac{GMm}{r^2}; \quad g = \frac{GM}{r^2},$$

with F in units of Newton (N), and g the surface gravity on Earth in units of $\mathrm{m\,s^{-2}}$.

Potential energy

$$W = -\frac{GMm}{r}.$$

Over a small height h, $W = mgh$.

Escape velocity and orbital velocity

$$v_{\rm esc} = \sqrt{\frac{2GM}{R}}; \qquad v_{\rm orb} = \sqrt{\frac{GM}{R}}$$

2. Distances

parallax p is the apparent shift of position of a star when the Earth moves in its orbit by 1 AU (half the total orbit). The definition is:

$$p = \frac{1}{d},$$

where p is the parallax in arcsec, and d is the distance of the star in parsec. In SI units, it becomes

$$p = \frac{B}{d}$$

with d and B in meters and p in radians. B is the baseline given by the Earth's orbit.

Angles are either in radians, or in degrees-arcminutes-arcseconds.

Coordinates on the sky are given in right ascension (along the equator, in hours-minutes-seconds) and declination (from pole to pole in degrees. To convert from seconds to arcsecond, multiply by $15\cos\delta$ where δ is the declination.

Solid angles are measured in steradians. The full sky contains 4π sterad.

The proper motion is the speed of movement of a star on the sky, in arcseconds per year, due to its velocity v on the plane of the sky with respect to us:

$$v = \mu \times 4.74 \times d$$

where μ is the proper motion in arcsec per year, and d is the distance in pc.

Luminosity L: For stars with distance d, the observed *flux* (in $\mathrm{W\,m^{-2}}$) is

$$F = \frac{L}{4\pi d^2}$$

The magnitude m is the flux on a logarithmic scale:

$$m = -2.5 \times \log F + m_0,$$

where the constant m_0 is defined such that the star Vega has $m = 0$. Note that the log is to base 10.

The difference in magnitude between two stars is

$$m_1 - m_2 = -2.5 \times \log \frac{F_1}{F_2}$$

The absolute magnitude M is defined as the magnitude of an object (star) at a distance of 10 pc. The distance modulus equation is

$$m - M = -5 + 5 \times \log d$$

where m is the apparent magnitude (as seen on Earth), M the absolute magnitude and d the distance *in parsec.*

Types of objects for which M is known include Cepheid variables (if the pulsation period is known) and one type of supernovae.

3. Observational Astrophysics

A light wave with frequency f and wavelength λ has velocity c,

$$c = f\lambda$$

Light consists of photons, each with energy $E = hf$

Planck function describes how the emission from a blackbody, B, varies with frequency f. You do not need to memorize this equation: it will be given if you need it.

Wien's displacement law describes the wavelength of the peak of the Planck function:

$$\lambda_{\mathrm{max}} = \frac{2.897 \times 10^{-3}}{T}$$

You do not need to know the value of the constant.

The Stefan-Boltzman law is:

$$F = \sigma T^4$$

where F is the total emitted flux in $\mathrm{W\,m^{-2}}$, and T is temperature.
The total power (or luminosity) emitted by a blackbody is $P = FA$ where A is the surface area. For a spherical body (star, planet) of radius R:

$$L = \frac{F}{4\pi R^2}$$

The effective temperature T_{eff} of a star is the temperature of a blackbody that has the same luminosity L and surface area A as the star in question. It is given by

$$T_{\mathrm{eff}} = \left(\frac{L}{\sigma A}\right)^{0.25}.$$

The colour temperature is the temperature of the blackbody that gives the same colour (or shape of the spectrum) over the wavelength range of interest.

Lens formula For an object at distance u from a lens with focal length f, and an image at a distance v, the lens formula is

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

u and v are positive for a real object or image and negative for a virtual object or image. f is positive for a convex lens and negative for a conex lens. Note that different sign conventions are ion use, and the one used here may differ from those used in some text books.

The lens power is (units of dioptre)

$$P = \frac{1}{f}$$

The lens magnification is

$$M = -\frac{v}{u}$$

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A telescope uses two lenses, the objective with focal length f_0 and the eye piece f_e, with the distance between equal to $f_0 + f_e$. It turns an incoming parallel beam at angle α to the axis into an outgoing parallel beam with angle β. The magnification of a telescope is

$$M = \frac{f_o}{f_e} = \frac{\beta}{\alpha}$$

Image scale: ratio of angle on the sky and the physical size of the image in the focal plane

Parameters for a telescope

1. Image scale $p = 1/f$, in units of radians per meter (arcsec per mm is often used)
2. Collecting area $A = \pi D^2/4$ where D is the diameter.
3. Focal ratio $F = f/D$
4. Diffraction limit $\theta = 1.2\lambda/D$ (single telescope) or $\theta = \lambda/D$ (interferometer)

For optical, ground-based telescopes, the resolution may be limited by the seeing (atmosphere)

Spectroscopy For prisms: Snell's law where n is the index of refraction:

$$\frac{n_2}{n_1} = \frac{\sin\theta_1}{\sin\theta_2}$$

For gratings: constructive interference where m is an integer and d the distance between two slits:

$$d \sin\theta = m\lambda$$

Detectors The CCD photon detection efficiency is limited by the quantum efficiency (fraction of received photons liberating an electron), the transfer efficiency (fraction of electrons transfering to the next pixel during read-out) , and the dynamic range.

4. Stars and Galaxies

Stars

Parameter range of stars: mass 0.08–∼120 $M_\odot$, Radii 0.01–∼500 $R_\odot$, Temperatures 3000–200,000 K, Luminosities 10^{-4}–10^6 $L_\odot$;

HR diagram Horizontal axis: temperature or colour; vertical axis: luminosity or absolute magnitude. You should understand the axes, range of values, location of the main sequence, location of giants, supergiants, white dwarfs, and of stars of different radii and masses

Spectral classification scheme in order of temperature: OBAFGKM

Hydrogen lines Energy levels (you should understand the derivation):

$$E_n \propto -\frac{1}{n^2}$$

The lowest energy state ($n = 1$) of hydrogen is $E = -13.6\,\mathrm{eV}$.

The wavelengths of the hydrogen line transitions are given by

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

where n_1, n_2 are the quantum levels, λ is the wavelength and R is the Rydberg constant. Transitions to $n_1 = 1$ are called the Lyman series, to $n_1 = 2$ the Balmer series. Transitions with $\Delta n = 1$ are indicated with α, $\Delta n = 2$ with *beta*, etc., e.g. Lyβ. The Balmer series are in the optical part of the spectrum.

Dynamical time scale (for gravitational collapse)

$$t_{\rm dyn} \approx \sqrt{\frac{1}{G\bar{\rho}}}$$

For the Sun, this is around 1000 sec.

Newton's shell theorems see lecture notes

Hydrostatic equilibrium The balance between the pressure gradient and gravity

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

$\rho(r)$ is the density at radius r, $m(r)$ is the mass which is inside this radius (see Newton's shell theorems).

Other important equations relevant to stars are:

Ideal gas law

$$P = nkT$$

Thermal energy

$$E_{\rm th} = \frac{3}{2}NkT,$$

The potential energy for a star

$$E_{\rm p} = \int \frac{Gm(r)}{r}\rho(r)4\pi r^2\,dr \sim -\frac{GM^2}{R}$$

For a constant density star,

$$E_{\rm p} = -\frac{3}{5}\frac{GM^2}{R}.$$

Kelvin-Helmholtz timescale the time over which the internal energy is radiated away:

$$t = \frac{E_{\rm p}}{L} \approx \frac{GM^2}{RL}$$

For the Sun $t \sim 3 \times 10^7$ yr

Virial theorem $E_{\rm p} + 2E_{\rm th} = 0$

Stellar life time

$$t = fM/L$$

where f is the efficiency of the energy generation

On the main sequence $L \propto M^{3.5}$ therefore $t \propto M^{-2.5}$.

More massive stars have shorter life times.

Energy production in stars

1. Gravitation contraction: releases potential energy ΔE_p
2. Nuclear fusion: releases mass energy Δm

Stability Virial theorem stabilizes the star against fluctuations in fusion rate

Scaling relations

$$M \propto R; \qquad L \propto M^{3.5} \qquad t_{\mathrm{MS}} \propto M^{-2.5}$$

End results of stellar evolution: White dwarf (low mass stars), neutron star or black hole (high mass star). (Know approximate numbers)

Schwarzschild radius (radius of a black hole)

$$R = \frac{2GM}{c^2}$$

Orbits and planets

Kepler's laws.

Elliptical orbits, sun in focus, equal area in equal time

Third law

$$P^2 = a^3$$

with P in year and a in AU (only for orbits around the Sun or a star with the same mass); or

$$P^2 = \frac{4\pi^2 a^3}{GM}$$

where all numbers are in SI units (m, kg).

Ellipse the focal points are at distance ea from the centre

Angular momentum $L = mrv \sin\phi$, or mrv if r and v are perpendicular

Centripetal force $F = mv^2/r$.

A binary system The two masses orbit the centre of mass, given by

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Kepler's equation changes to

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)}$$

where M_1 and M_2 are the two masses, and $a = a_1 + a_2$. For circular orbits $a = r$. In a binary system, both objects have the same period.

Orbital velocity v_s of star with planet with mass M_p and orbital radius r_p

$$v_s = \frac{M_p}{M_s}\sqrt{\frac{GM_s}{r_p}}$$

Exoplanets can be found with the radial velocity method or with the transit method. They are more likely to find more massive planets closer to their star; for the radial velocity this is because the star has a higher orbital velocity, and for transit because transits are more frequent and the planet covers a larger fraction of the star.

Doppler shift

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$$

The temperature of a planet

$$T = \frac{1}{2r^{1/2}} \left(\frac{fL}{\pi\sigma}\right)^{1/4}$$

Habitable zone range of distance from the star where the planet can have liquid water on its surface

Galaxies

Types of galaxies Elliptical, spiral, barred spiral, irregular.

Properties Mass, size, amount of gas, age of stellar population and presence of star formation

Central black holes seen in larger galaxies

Rotation curves

$$M(R) = \frac{v^2 R}{G}$$

For point masses: $v \propto r^{-1/2}$; for a constant density $\rho(r) = \rho_0$, $v \propto r$.

Dark matter halo Flat rotation curves require $\rho \propto r^{-2}$, and show that galaxies have much more mass far out than is seen in stars and gas

Milky Way structure Central black hole ($10^6\,\mathrm{M}_\odot$); Bulge (old stars, $10^{10}\,\mathrm{M}_\odot$); Disk ($10^{11}\,\mathrm{M}_\odot$); halo (old stars, ($10^9\,\mathrm{M}_\odot$); dark matter halo ($10^{12}\,\mathrm{M}_\odot$)

7. Cosmology

Olbers paradox the sky is dark at night

The Hubble-Lemaitre law

$$v = H_0 r$$

where r is the distance in Mpc, v velocity in km/s, and $H_0 = 70 \pm 5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ is the Huble constant.
The Hubble flow does not apply within a galaxy or a cluster of galaxies.

The redshift of a galaxy

$$z = \frac{v}{c} = \frac{\Delta\lambda}{\lambda}.$$

More distant galaxies have higher values of z.

The Hubble time is the age of the Universe assuming constant velocities:

$$t_{\mathrm{H}} = \frac{r}{v} = \frac{1}{H_0} \approx 10^{10}\mathrm{yr}.$$

The Hubble radius

$$r_{\mathrm{H}} = ct_{\mathrm{H}} \approx 4\,\mathrm{Gpc}$$

Cosmological principle the Universe is homogeneous (looks the same from every location) and isotropic (looks the same in every direction).

The critical density of the Universe

$$\rho_{\mathrm{cr}} = \frac{3H_0^2}{8\pi G}.$$

The density parameter

$$\Omega_0 = \frac{\rho}{\rho_{\mathrm{cr}}}$$

Three universes: open, critical or closed, for $\Omega_0 < 1$, $= 1$, or > 1

Scale factor

$$a = \frac{r(t)}{r(t_0)} = \frac{1}{1+z}$$

is the factor by which distances in the Universe have changed compared to the current distance. The current value of the scale factor $a_0 = 1$.

$$\rho(t) \propto \frac{1}{a(t)^3}$$

Hubble parameter in terms of the scale factor

$$H(t) = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}$$

Density parameter $\Omega_0 = \rho_0/\rho_{\mathrm{cr}}$ where ρ_o is the actual density. Ω_0 can be < 1, 1, or > 1, for an open, critical or closed Universe respectively.
Visible and dark matter together give $\Omega_0 \sim 0.3$.

The curvature constant $k = 0$ for a critical Universe, negative for an open Universe and positive for a closed Universe

The cosmological constant Λ, representing an unknown, repulsive force. Also called *dark energy.*

The Friedmann equation with all terms becomes

$$\frac{1}{a^2}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} - \frac{1}{3}\Lambda c^2 =$$

You do not need to memorize the full equation but you should know the first term of the right hand side, understand the other terms, and be able to solve the equation under some assumptions.

Cosmic microwave background radiation: blackbody radiation that fills the Universe, with current temperature $T_0 \sim 3$K.

$$T = T_0(1+z) = \frac{T_0}{a}$$

The baryon to photon ratio

$$\eta \approx 10^{-10}$$

Nucleosynthesis in the first three minutes: role of deuterium, η, formation of helium

Dark energy and the De Sitter Universe

$$a(t) \propto e^t$$

Current Universe: 5% baryons, 25% dark matter, remainder dark energy