Quantum Physics

Dominik Szablonski

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Chapter 1

Introduction

1.1 De Broglie Wavelength

By combining $E = \frac{hc}{\lambda}$ and $E = cp$, we can conjecture that, for all particles,

$$\lambda p = h. \tag{1.1}$$

NOTE: When calculating the energy of particles, remember to use the equation,

$$E^2 = c^2p^2 + m^2c^4. \tag{1.2}$$

1.2 The Wavefunction

The wavefunction, is in the form,

$$\psi(x, y, z, t) \tag{1.3}$$

and gives us the probability to find a particle in the vicinity of $x, y,$ and $z.$ It can take the complex form,

$$\psi = re^{i\theta}. \tag{1.4}$$

For this course, the wave function will likely only depend on position in 1 dimension and in a particular time. The probability of finding a particle between positions x_0 and x_1 is equal to,

$$P = \int_{x_0}^{x_1} |\psi(x)|^2 \, dx. \tag{1.5}$$

The wave function should also satisfy,

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1. \tag{1.6}$$

To find the mean value of x,

$$\langle x \rangle = \int_{-\infty}^{\infty} x \, |\psi(x)|^2 \, dx. \tag{1.7}$$

The uncertainty in x is given by its standard deviation, σ which is given by,

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle \tag{1.8}$$

The wave function for a particle of definite energy may be one of two equations,

$$\psi = \sin \frac{\pi x}{L} \tag{1.9}$$

$$\psi = \sin \frac{2\pi x}{L} \tag{1.10}$$

where L is the length of the area where the particle may exist.

Chapter 2

The Momentum Operator and the Heisenberg Uncertainty Principles

We want to find the wavefunction for a particle with definite momentum p. By de Broglie relation, it implies the function will have a constant wavelength. The wavefunction will take the form,

$$\psi(x) \propto e^{\frac{ipx}{\hbar}}. \tag{2.1}$$

If we consider probability,

$$\begin{aligned} |\psi|^2 = \psi^*\psi &\propto e^{-\frac{ipx}{\hbar}} e^{\frac{ipx}{\hbar}} \\ &\propto 1 \end{aligned}$$

This implies that all particles are equally likely to be everywhere.

2.1 The Momentum Operator

Now let us consider the differential of ψ...

$$-i\hbar\frac{d\psi}{dx} = p\psi \tag{2.2}$$

we can then define the **the momentum operator**,

$$\hat{p} = -i\hbar\frac{d}{dx} \tag{2.3}$$

which returns the momentum multiplied by the wavefunction.

Now, if the wavefunction resembles (2.1), then

$$\langle p\rangle = \int_{-\infty}^{\infty} p\psi^*\psi dx = p. \tag{2.4}$$

Then, for **all** ψ,

$$\langle p\rangle = \int_{-\infty}^{\infty} \psi^*\hat{p}\psi dx. \tag{2.5}$$

Further,

$$\langle p^n\rangle = \int_{-\infty}^{\infty} \psi^*(\hat{p})\psi dx. \tag{2.6}$$

Generally, if O is an observable, for any system,

$$\langle O\rangle = \int \psi^*\hat{O}\psi. \tag{2.7}$$

When $(\hat{p})^n$, it means to use the operator n times, such that,

$$(\hat{p})^n = (-i)^n\hbar^n\frac{d^n}{dx^n}. \tag{2.8}$$

2.2 Heisenberg Uncertainty Principle

$$\Delta x \Delta p \geq \frac{1}{2}\hbar \quad \forall \psi \tag{2.9}$$

$$\Delta x \Delta p \sim \frac{\hbar}{2} \quad \forall \psi \tag{2.10}$$

2.3 Energy Uncertainty

We know the product of the uncertainties on x and p must always be greater than or equal to $\frac{\hbar}{2}$. Let's then consider a particle trapped in an area Δx and $\langle p \rangle = 0$. A time Δt later, the particle is in a new area ΔX. (See figure 2.1). Let's consider non-relativistic momentum,

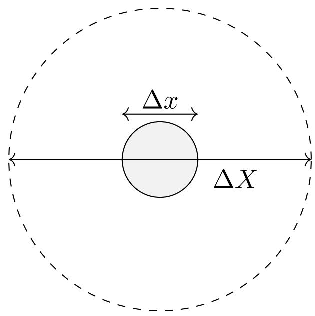

Figure 2.1:

$$p \sim m\frac{\Delta X}{\Delta t}$$

but

$$p = \Delta p = \sqrt{\langle \mathbf{p}^2 \rangle}$$

and

$$\Delta p \sim \frac{\hbar}{\Delta x}. \tag{2.11}$$

We know,

$$\frac{\hbar}{\Delta x} \sim m\frac{\Delta X}{\delta t}$$
$$\implies \Delta X \sim \frac{\hbar \Delta t}{m\Delta}$$

$$\text{Using } E = \frac{p^2}{2m}$$
$$\Delta E = \frac{p\Delta p}{m}$$
$$\text{And } \Delta X \sim \hbar.$$

We can then write,

$$\left(\frac{p}{m}\Delta t\right)\left(\frac{m\Delta E}{p}\right) \sim \hbar$$
$$\implies \Delta E \Delta t \sim \hbar$$

From this we an conclude that when we measure energy over a time Δt, there is necessarily an uncertainty ΔE.

NOTE: In questions when asked about ”width”, this means uncertainty.

2.4 Time Independent Schrodinger Equation

We can first state that non-relativistically, the kinetic energy,

$$K = \frac{p^2}{2m}$$

and the total energy,

$$E = K + V$$

where V is the potential energy. We also know that,

$$\hat{p} = -i\hbar\frac{d}{dx},$$

so we can define the Hamiltonian operator (the total energy operator) as,

$$\begin{aligned}\hat{H} &= \frac{\hat{p}^2}{2m} + \hat{V}(x) \\ &= \frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \hat{V}(x).\end{aligned}$$

We can then define a wave function $\phi(x)$ for a particle wtih definite energy E. We can then define the time independent schrodinger equation,

$$\hat{H}\phi(x) = E\phi(x) \tag{2.12}$$

who's real form is,

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + \hat{V}\phi = E\phi. \tag{2.13}$$

Setting $\hat{V}(\mathbf{x}) = -\frac{e^2}{4\pi\epsilon_0 r}$, we find that the allowed energy values of an electron in a hydrogen atom are,

$$E_n = \frac{-13.6\text{eV}}{n^2} \tag{2.14}$$

2.5 The Infinite Square Well

To determine the energy of a particle trapped in an infinite square well, we can solve the Schrodinger equation by setting $\hat{V} = 0$,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

which gives the solution,

$$\psi_n(x) = \sqrt{\frac{2}{L}}sin\frac{n\pi x}{L}. \tag{2.15}$$

We can then quantise the energies,

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2} \tag{2.16}$$