Dynamics Notes

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December 10, 2023

Contents

Chapter 1

Kinematics

1.1 Kinematics of a Particle in 1 Dimension

A particles velocity v and acceleration a can be defined in terms of derivatives:

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Where x is the displacement of the particle. If these properties can be described as functions of time, such that $x = x(t)$, $v = v(t)$, or $a = a(t)$, they are then known as equations of motion.

1.1.1 Determining Motion and Position of a Particle

Given we know the function of a certain particles motion, we are able to find equations of its position, velocity, acceleration. This may be done through differentiating an equation of motion, or through the use of integrals and splitting variables. The latter is shown below.

Uniform Acceleration

First we will look at the case where acceleration is a constant, such that:

$$\frac{dv}{dt} = a(t) = a$$

In order to obtain the acceleration, we must integrate both sides of the equation and split the variables.

$$\int_{v_0}^{v} dv = \int_{t}^{0} a \ dt'$$

The bounds on either side of the equation correspond to each other. This is so that the velocity v_0 corresponds to time 0. We use t' rather than t in order to distinguish it from the upper limit.

Completing this integral we get:

$$v = v_0 + at$$

We may recognise this as one of our SUVAT equations.

We can go further with this and derive the equation for displacement using a similar method.

$$\frac{dx}{dt} = v_0 + at$$

$$\int_{x_0}^{x} dx = \int_{0}^{t} (v_0 + at)dt'$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

Non-Uniform Acceleration

The process is similar for non-uniform acceleration, except that we cannot consider acceleration to be a constant and we must instead perform an integral on equation of acceleration.

1.1.2 Determining Distance Travelled

When doing problems like these, it is important to realise that the equation describing the particles motion may not be linear, and that the particle may move back, which will decrease displacement but increase distance. So, we must find the 0s of the equation of velocity and then solve for distance by taking times between the 0s of the velocity function.
The process for finding distance travelled goes like this:

1. Find equations of velocity and of displacement (if not already given).

2. Find values of time when velocity equals 0.

   - Suppose these times are $t_1, t_2, t_3 ... t_n$, where n is the order of the equation describing velocity and $t_n > 0$.

3. If you are told to find the distance between a time t_α and t_β, and the times t_n fall somewhere between the values t_α and t_β, calculate the modulus value of displacement from $t_\alpha \to t_n$, then from $t_n \to t_\beta$. Add these values together to gain total displacement.

1.2 Kinematics in 2D and 3D

Position vectors are written as:

$$\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$

Where $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ are all unit vectors which are perpendicular to each other. The magnitude of the position is then:

$$|r| = \sqrt{x^2 + y^2 + z^2}$$

Velocity is then written as:

$$v = \lim_{\Delta t \to \infty} \left(\frac{\Delta \boldsymbol{x}}{\Delta t} \right) = \frac{d\boldsymbol{x}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dx}\hat{\boldsymbol{\jmath}} + \frac{dz}{dt}\hat{\boldsymbol{k}} = v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}}$$

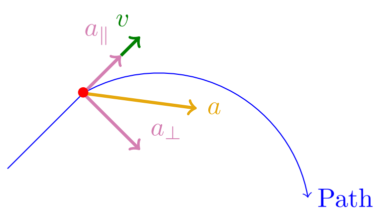
$$|v| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Acceleration is written as:

$$a = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dx}\hat{\boldsymbol{\jmath}} + \frac{dv_z}{dt}\hat{\boldsymbol{k}} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}$$

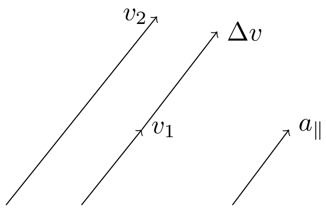
$$|v| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

For particles travelling on curved paths, a is not in the same direction as v. In these cases, we can model particle's motion by splitting the acceleration into component parts.



Below we are able to visualise the different effects the components of acceleration have on velocity.

Parallel



As we can see, the acceleration parallel to velocity only increases the speed but does not change direction.

Perpendicular

Intuitively, we may say that acceleration perpendicular to velocity only changes its directions. We can prove this by first taking the dot product of velocity:

$$\boldsymbol{v} \cdot \boldsymbol{v} = v^2$$

We will then take the time derivative of this and use the product rule:

$$\frac{d}{dt}(v^2) = \frac{dv}{dt} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \frac{dv}{dt} = 2\boldsymbol{a} \cdot \boldsymbol{v}$$

From the equation of the dot product:

$$\boldsymbol{A} \cdot \boldsymbol{B} = |A|\,|B|\,cos\theta$$

Since $\boldsymbol{a}$ and $\boldsymbol{v}$ are perpendicular, we know $cos\theta = 0$, therefore $2\boldsymbol{a} \cdot \boldsymbol{v} = 0 \implies$ acceleration perpendicular to velocity does not affect its direction.

1.2.1 Projectile Motion

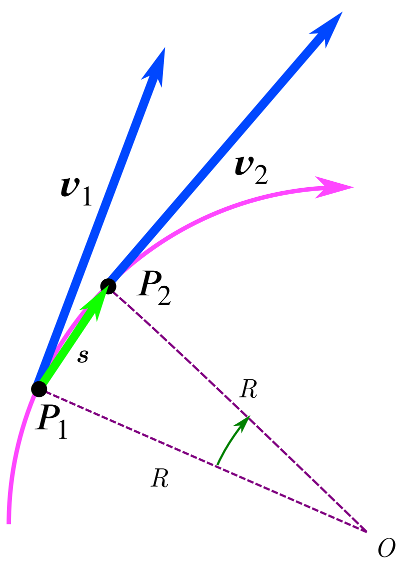
This is 2 dimensional motion in which a particle is projected obliquely with an angle to the horizontal. In order to properly model this motion we must state:

$$a_x = 0 \qquad a_y = -g$$

Further equations of motion follow from these.

1.2.2 Uniform Circular Motion

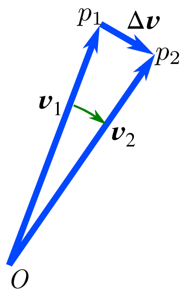
This is motion who's acceleration is always perpendicular to velocity and towards the centre of the circle the particle is following. We will model this motion below.



The angle between OP_1 and OP_2 is $\Delta\phi$ The particle moves from P_1 to P_2 in a time Δt. This change is given by:

$$\Delta \boldsymbol{v} = \boldsymbol{v_1} - \boldsymbol{v_2}$$

We simplify the vectors, as shown in the diagram below:



The triangle Op_1p_2 is similar to the triangle OP_1P_2. We can therefore say the angle between the two lines is the same. Further, the ratio of the sides is similar for similar triangles.

$$\implies \frac{|\Delta \boldsymbol{v}|}{v_1} = \frac{\Delta s}{R} = \Delta\phi$$

This only works for small $\Delta\phi$ because Δs is not measuring arc length, so Δs is a worse approximation for arc length as $\Delta\phi$ approaches 0.

We can then say that magnitude of the average acceleration is:

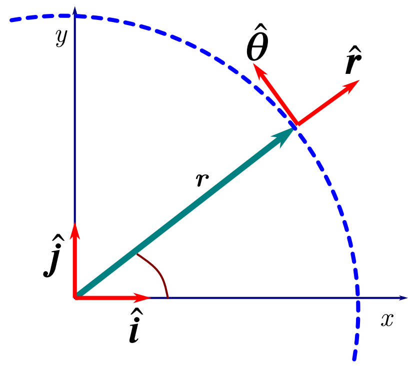
$$a_{\text{avg}} = \frac{|\Delta \boldsymbol{v}|}{\Delta t} = \frac{v_1}{R} \cdot \frac{\Delta S}{\Delta t}$$

We can then take the limit as P_2 approaches P_1 and $\Delta t \to 0$ to get the instantaneous velocity. We will drop any subscripts because our function will now describe a generic point P.

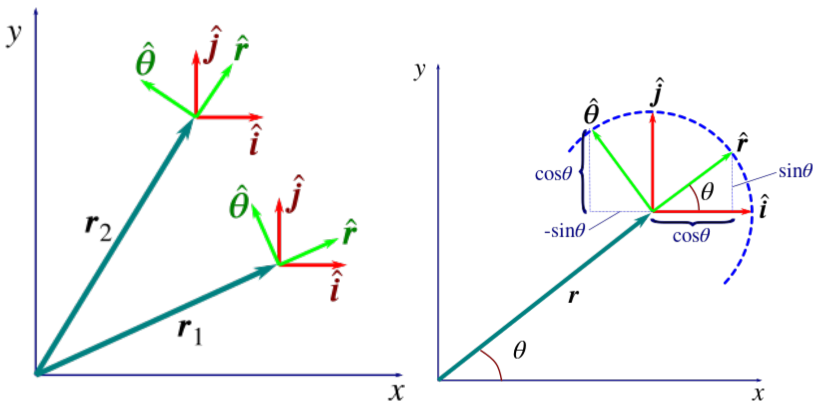
$$a = \lim_{\Delta t \to 0} \left(\frac{v_1}{R} \cdot \frac{\Delta s}{\Delta t} \right) = \frac{v}{R} \cdot \frac{ds}{dt} \implies a = \frac{v^2}{R}$$

1.3 Kinematics in Planar Polar Coordinates

Notation and interpretation of polar co-ordinates is covered in the maths notes. In dynamics, we must be able to use polar co-ordinates to model motion of particles. We can begin by defining the unit vectors used in polar co-ordinates, $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$.



The main difference between the Cartesian unit vectors, $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$, and polar unit vectors is that the direction of polar unit vectors, $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$, changes as you move through space, whereas Cartesian unit vectors are the same at every point in space. As shown below, the direction of the unit vectors depends on the angle θ between the polar unit vectors and the Cartesian unit vectors. The diagrams below show how the unit vectors vary with position:



We can see that both unit vectors $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{r}}$ both depend on θ. Therefore, they are both vector functions of θ. Since $\hat{\boldsymbol{r}}$ is in the direction of $\boldsymbol{r}$, it is a vector from the centre to a point on the circumference of a unit circle. We can then define $\hat{\boldsymbol{r}}$ in terms of its x and y components.

$$\hat{\boldsymbol{r}} = (|\hat{\boldsymbol{r}}|cos\theta, |\hat{\boldsymbol{r}}|sin\theta) = (cos\theta, sin\theta)$$

Therefore,

$$\hat{\boldsymbol{r}} = cos\theta\hat{\boldsymbol{i}} + sin\theta\hat{\boldsymbol{j}} = \hat{\boldsymbol{r}}(\theta) \tag{1.1}$$

We can define this more straightforwardly by taking the direction of $\hat{\boldsymbol{r}}$ as the derivative of $\boldsymbol{r}$ with respect to its magnitude r without change in direction.

$$\frac{d\boldsymbol{r}}{dr} = \hat{\boldsymbol{r}}$$

We can then go about defining unit vector $\hat{\boldsymbol{\theta}}$. We know that $\hat{\boldsymbol{\theta}}$ is $\perp$ to $\hat{\boldsymbol{r}}$. Therefore, we can say...

$$\hat{\boldsymbol{\theta}} = -sin\theta\hat{\boldsymbol{i}} + cos\theta\hat{\boldsymbol{j}} \tag{1.2}$$

We should also note that,

$$\hat{\boldsymbol{\theta}} = \frac{d\hat{\boldsymbol{r}}}{d\theta}$$

However, this being equal to the derivative of the $\hat{\boldsymbol{r}}$ is not a proof of the fact that $\hat{\boldsymbol{\theta}}$ is tangential to $\hat{\boldsymbol{r}}$.

1.3.1 Position in Polar Coordinates

Now that we have defined the unit vectors, we are able to define an equation of position, we can define the position vector $\boldsymbol{r}$ in terms of polar co-ordinates.

$$\begin{aligned} \boldsymbol{r} &= r\hat{\boldsymbol{r}} \\ &= r(cos\theta\hat{\boldsymbol{i}} + sin\theta\hat{\boldsymbol{j}}) \end{aligned} \tag{1.3}$$

Dot Product in Polar Co-Ordinates

Let's say we have two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, such that

$$\boldsymbol{a} = a_r\hat{\boldsymbol{r}} + a_\theta\hat{\boldsymbol{\theta}} \tag{1.4}$$

$$\boldsymbol{b} = b_r\hat{\boldsymbol{r}} + b_\theta\hat{\boldsymbol{\theta}} \tag{1.5}$$

We can calculate their dot product as normal because their unit vectors are perpendicular to each other. So,

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_r b_r + a_\theta b_\theta$$

1.3.2 Velocity in Polar Co-Ordinates

We understand that

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}$$

So in polar coordinates this becomes

$$\boldsymbol{v} = \frac{d}{dt}(r\hat{\boldsymbol{r}})$$

We can then use the product rule,

$$\boldsymbol{v} = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\hat{\boldsymbol{r}}}{dt}$$

The second term of the above equation is the tangential component of velocity in the direction of $\hat{\theta}$. We can simplify this by the following.

$$\begin{aligned}\frac{d\hat{\boldsymbol{r}}}{dt} &= \frac{d}{dt}(cos\theta\hat{\boldsymbol{\imath}} + sin\theta\hat{\boldsymbol{\jmath}}) \\ &= -\frac{d\theta}{dt}sin\theta\hat{\boldsymbol{\imath}} + \frac{d\theta}{dt}cos\theta\hat{\boldsymbol{\jmath}} \\ &= \dot{\theta}(-sin\theta\hat{\boldsymbol{\imath}} + cos\theta\hat{\boldsymbol{\jmath}}) \\ &= \dot{\theta}\hat{\boldsymbol{\theta}} \Longleftarrow 1.2\end{aligned} \tag{1.6}$$

It follows that,

$$\begin{aligned}\boldsymbol{v} &= \frac{dr}{dt}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \\ &= \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\end{aligned} \tag{1.7}$$

We have then split the velocity into two components. Its *radial component*, $\dot{r}$, and its *tangential velocity*, $\dot{\theta}$. **NOTE:** $\dot{\theta}$ is angular velocity in circular motion, often written as ω.

1.3.3 Acceleration in Polar Coordinates

Evaluating the equation for acceleration in polar co-ordinates is rather complex. The derivation is shown below.

$$\begin{aligned}\boldsymbol{a} &= \frac{d\boldsymbol{v}}{dt} \\ &= \frac{d}{dt}(\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) \\ &= \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\frac{d\hat{\boldsymbol{r}}}{dt} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt}\end{aligned}$$

$$1.6 \Longrightarrow \frac{d\hat{\boldsymbol{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}}$$

$$\begin{aligned}\therefore \boldsymbol{a} &= \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt} \\ &= \ddot{r}\hat{\boldsymbol{r}} + 2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt}\end{aligned}$$

We can now attempt to find an expression for $\frac{d\hat{\boldsymbol{\theta}}}{dt}$.

$$\begin{aligned}\frac{d\hat{\boldsymbol{\theta}}}{dt} &= \frac{d}{dt}(-sin\theta\hat{\boldsymbol{\imath}} + cos\theta\hat{\boldsymbol{\jmath}}) \\ &= -\dot{\theta}cos\theta\hat{\boldsymbol{\imath}} - \dot{\theta}sin\theta\hat{\boldsymbol{\jmath}} \\ &= -\dot{\theta}(cos\theta\hat{\boldsymbol{\imath}} + sin\theta\hat{\boldsymbol{\jmath}}) \\ &= -\dot{\theta}\hat{\boldsymbol{r}} \Longleftarrow 1.1\end{aligned}$$

We can finally conclude,

$$\begin{aligned} \boldsymbol{a} &= \ddot{r}\hat{\boldsymbol{r}} + 2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\boldsymbol{r}} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}} \end{aligned} \tag{1.8}$$

We can attempt to understand this equation a bit more.

- $\ddot{r}$ **represents the linear radial acceleration.** This represents the particle's acceleration along the radial line. It does not change θ.
- $r\dot{\theta}^2$ **represents the centripetal acceleration.** This represents the constantly changing direction of the velocity with respect to x due to acceleration directed towards the centre of motion. It has units of ms^{-2}.
- $r\ddot{\theta}$ **represents the pure angular acceleration.** This has nothing to do with the radial velocity of the particle and only describes how the angle changes with time. Its units are $rads^{-2}$
- $2\dot{r}\dot{\theta}$ **is often known as the Coriolis acceleration.**

1.3.4 Cylindrical Coordinates

These are co-ordinates which describe a sphere. They consist of 2 length unit vectors and an angle unit vector.

1.4 Relativity and Reference Frames

For a reference frame S' moving with respect to another stationary reference frame S at a velocity u, we can convert between position r and velocity v in the reference frame S to the position r' and velocity v' using galilean transformation equations.

$$r = r' + ut \tag{1.9}$$

$$v = v' + u \tag{1.10}$$

Chapter 2

Newton's Laws

2.1 Newton's First Law

Definition. *Every object continues in its state of rest or uniform motion in a straight line unless acted upon by a resultant force.*

Force not required to maintain motion, however force is required to change motion.

2.1.1 Inertia

Definition. *The resistance of an object to a change in its state of motion is called inertia.*

We quantify inertia by mass.

This law is sometimes referred as Newton's law of inertia. We use the concept of "inertial frames of reference" as those which obey Newton's laws.

2.1.2 Non-Inertial Frames of Reference

We can test whether or not we are in a non-inertial frame by observing any fictitious forces. For example, the "centrifugal force" experienced by an object within a rotating frame.

The reason behind this force is that, in the rotating reference frame, the object doesn't experience a force. However, relative to a stationary reference frame, the object is experiencing a force. Therefore, the object will attempt to "catch up" with the stationary frame by experiencing a fictitious force.

2.2 Newton's Second Law

Definition. *The acceleration of an object is directly proportional to the resultant force acting on it and inversely proportional to its mass. The direction of the acceleration is in the direction of the resultant force.*

$$\boldsymbol{a} = \boldsymbol{F}_{\text{constant}} m \tag{2.1}$$

2.2.1 Applying Newton's Second Law to Circular Motion

We are able to define the magnitude of the centripetal acceleration of a body undergoing circular motion as

$$a = \frac{v^2}{r}$$

We know this force is directed towards the centre of the circle, so in vector notation is

$$\boldsymbol{a} = -\frac{v^2}{r}\hat{\boldsymbol{r}}$$

The centripetal force then follows.

$$\boldsymbol{F} = -\frac{mv^2}{r}\hat{\boldsymbol{r}} \tag{2.2}$$

2.3 Newton's Third Law

Definition. *The acceleration of an object is directly proportional to the resultant force acting on it and inversely proportional to its mass. The direction of the acceleration is in the direction of the resultant force.*

This law may also be stated as,

Definition. *Action and reaction are equal and opposite, and act on different bodies.*

We must keep this last statement in mind when determining force diagrams and resolving forces.

Chapter 3

Friction and Non-Uniform Circular Motion

3.1 Friction

Definition. *The force which opposes relative motion of 2 surfaces in contact with each other.*

3.1.1 Dynamic Friction

Definition. *Friction experienced by a moving body at constant speed.*

$$F_k = \mu_k N \tag{3.1}$$

where

$$N \rightarrow \text{ Normal Force} \quad \mu_k \rightarrow \text{Co-efficient of friction}$$

3.1.2 Static Friction

For a non-moving body at an angle θ between its axis and the vertical, the co-efficient of friction is given by,

$$\mu_s = \tan(\theta) \tag{3.2}$$

Chapter 4

Energy

4.1 Centre of Mass

Definition. *This is the point on a body, or on a system of bodies, which acts as though all mass was concentrated at that point and all external forces were applied there.*

The centre of mass of a system of particles is given by the weighted average of each particle's position,

$$x_{\text{cm}} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}. \tag{4.1}$$

The total mass of a system is often noted as M, so we can write,

$$M x_{\text{cm}} = \sum_{i=1}^{n} m_i x_i. \tag{4.2}$$

We can also write this in vector notation,

$$M \mathbf{r}_{\text{cm}} = \sum_{i=1}^{n} m_i \mathbf{r}_i. \tag{4.3}$$

For continuous bodies, we take the entire body as being made up of infinitesimal particles,

$$r_{\text{cm}} = \frac{1}{M} \int \mathbf{r} dm. \tag{4.4}$$

4.2 Work

For motion under a straight line under a constant force, the word done is given by,

$$W = \mathbf{F} \cdot \mathbf{x}. \tag{4.5}$$

For curved paths, we must use the **line integral**.

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}. \tag{4.6}$$

4.3 Work-Energy Theorem

Definition. *The work done by a resultant force on a particle is equal to the change of kinetic energy on the particle.*

This theorem is proved for various cases below.

4.3.1 Constant Unbalance Force

From the equations of linear motion, we know,

$$v^2 = v_0^2 + 2ax.$$

We can rearrange this in terms of acceleration, to then obtain the force.

$$a = \frac{v^2 - v_0^2}{2x}$$
$$F = m\frac{v^2 - v_0^2}{2x}.$$

From (4.5),

$$W = m\frac{v^2 - v_0^2}{2x}x$$
$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \quad \square.$$

4.3.2 Force in a Straight Line

$$W = \int \mathbf{F}d\mathbf{x} = \int_{x_0}^{x_1} F(x)dx.$$

From (2.1),

$$F = ma = m\frac{dv}{dt}.$$

From the chain rule,

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

so,

$$\begin{aligned} W &= \int_{x_0}^{x_1} Fdx = m\int_{x_0}^{x_1} v\frac{dv}{dx}dx = m\int_{v_0}^{v_1} vdv \\ &= m\int_{v_0}^{v_1} vdv = m\left[\frac{v^2}{2}\right]_{v_0}^{v_1} \\ &= \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2. \quad \square \end{aligned}$$

4.3.3 General Case

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}.$$

We can find work done from position a to position b,

$$W = \int_a^b \mathbf{F}\cdot d\mathbf{r} = \int_a^b m\frac{d\mathbf{v}}{dt}d\mathbf{r}$$
$$\to d\mathbf{r} = \mathbf{v}dt$$
$$W = \int_{t_a}^{t_b} m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}dt,$$

now applying the reverse product rule,

$$\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \frac{1}{2}\frac{d\left(v^2\right)}{dt}.$$

Therefore,

$$W = \int_{t_a}^{t_b}\frac{1}{2}m\frac{d\left(v^2\right)}{dt}dt = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2. \quad \square$$

4.3.4 Kinetic Energy of a System of Bodies

$$K = \frac{1}{2}Mv_{\text{cm}}^2 + \sum \frac{1}{2}m_1 u_1^2. \tag{4.7}$$

We can then make this simpler, by using the reduced mass, μ.

$$K = \frac{1}{2}Mv_{\text{cm}}^2 + \mu v_{\text{rel}}^2, \tag{4.8}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

4.4 Power

Definition. *The rate of doing work.*

$$P = \mathbf{F} \cdot \mathbf{v} \tag{4.9}$$

$$\langle P \rangle = \frac{\Delta W}{\Delta t} \tag{4.10}$$

4.5 Potential Energy

In its most general form, the change in potential energy takes the form of the form of the path integral,

$$\Delta U = -\int_{\mathbf{r_0}}^{\mathbf{r_1}} \mathbf{F} \cdot d\mathbf{r} \tag{4.11}$$

4.5.1 Gravitational Potential Energy

For distances where we don't need to account for the change in gravitational field strength, the potential energy of an object at a distance y above the ground is is,

$$U(y) = mgy. \tag{4.12}$$

However, for large distances this becomes,

$$U(r) = -\frac{GmM}{r}. \tag{4.13}$$

The work done by gravity between y_1 and y_2 is equal to,

$$W_{\text{g}} = mgy_1 - mgy_2 = U_1 - U_2 = -\Delta U. \tag{4.14}$$

The work done may also be expressed in the form of kinetic energy,

$$W_{\text{g}} = K_2 - K_1$$
$$U_1 - U_2 = K_2 - K_1$$
$$U_1 + K_1 = U_2 + K_2$$

and we have derived a conservation law.

4.5.2 Elastic Potential Energy

$$W_{\text{el}} = \int_{x_1}^{x_2} -kx dx = \frac{kx_1}{2} - \frac{kx_2}{2} \tag{4.15}$$

4.5.3 Conservative and Non-Conservative Forces

Conservative forces are those where the total work done on a system is 0. This means that the work is reversible and the total *mechanical* (potential and kinetic) energy of the system is conserved.

Definition. *The total initial mechanical energy plus the work done by non-conservative forces equals the total final mechanical energy; this is also equal to the change in kinetic energy.*

4.5.4 Properties of Conservative Forces

1. The work done by a conservative force is reversible.
    - $W_{A \to B} = -W_{B \to A}$
2. The work done by a conservative force while moving in a closed path is 0.
3. The work done by a conservative force only depends on the starting and finishing points. (The path the particle takes from A to get to B does not matter.)
4. The total work done can always be expressed as the difference between the initial and final values of the potential energy.

It is important to understand that the total energy when there are no external forces acting is **constant**.

$$E = K + U \tag{4.16}$$

4.5.5 Potential Energy Graphs

Potential energy graphs can be useful to analyse in order to learn about the operation of a mechanical system. **Stable equilibria** are minimum turning points on a potential energy graph. **Unstable equilibria** are maximum turning points on a potential energy graph.

Chapter 5

Momentum and Colissions

Linear momentum is defined as,

$$\mathbf{p} = m\mathbf{v}. \tag{5.1}$$

Newton's first law is defined as,

$$\mathbf{F} = \frac{d}{dt}\mathbf{p}. \tag{5.2}$$

For a system of particles, the momentum is defined as,

$$\begin{aligned} \mathbf{P} &= M\mathbf{v}_{\text{cm}} \\ \frac{d}{dt}\mathbf{P} &= \mathbf{F}_{\text{ext}}. \end{aligned} \tag{5.3}$$

A conservation law applies to momentum, if the sum of external forces acting on a system is 0. In this case,

$$\frac{d}{dt}\mathbf{P} = 0 \implies \mathbf{P} \text{ is constant.} \implies \mathbf{P}_{\text{i}} = \mathbf{P}_{\text{f}} \tag{5.4}$$

which is known as the **principle of conservation of momentum.**

Definition. *If no external forces act on a system of interacting bodies, then the total momentum of the system is conserved.*

Impulse is the difference in momentum. It can be seen as the area under the curve of a force-time graph,

$$\mathbf{J} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F} dt. \tag{5.5}$$

We can further define the average force on a system as,

$$\mathbf{F}_{\text{avg}} = \frac{\mathbf{J}}{t_f - t_i}. \tag{5.6}$$

5.1 Types of Collision

5.1.1 Elastic Collisions

These are collisions where all kinetic energy is conserved.
For two bodies colliding elastically, their initial relative velocities are equal to the negative of their relative velocities after the collision,

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}). \tag{5.7}$$

5.1.2 Inelastic Collisions

This is where kinetic energy is not conserved, and the objects stick together.

5.1.3 Partially Inelastic Collisions

Where some kinetic energy is conserved.

5.1.4 Coefficient of Restitution

$$e = \left| \frac{v_{1f} - v_{2f}}{v_{1i} - v_{2i}} \right| \tag{5.8}$$

Chapter 6

Rotation of Rigid Bodies

Consider a rigid body composed of i particles with positions p_i.

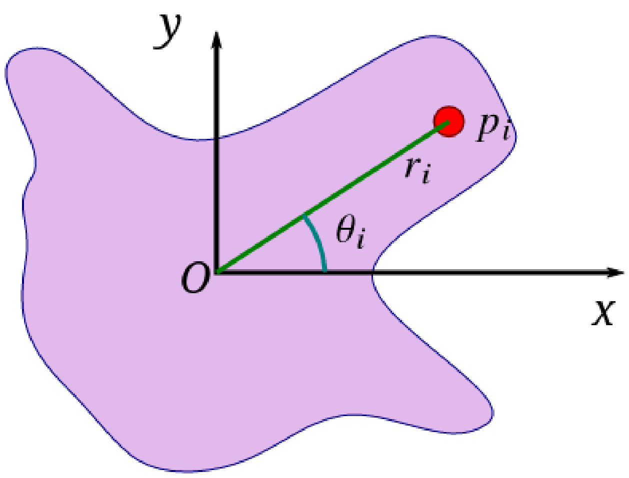


Figure 6.1: Rigid body with its axis of rotation at O, with the axis of rotation pointing out of the page. The particle p_i is at a perpendicular distance r_i from the axis of rotation.

As the body rotates, the change in the angle $d\theta$ is called the *angular displacement*.

The angular velocity is the same for all particles,

$$\omega_z = \frac{d\theta}{dt}. \tag{6.1}$$

ω is a vector property. Its direction, indicated by its subscript, indicates the direction of the axis of rotation.

The rate of change of angular velocity is,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{6.2}$$

Suppose a particle p_i moves through a path ds_i as the rigid body rotates in a time dt around the axis of rotation. We define the *perpendicular distance* from the axis of rotation as $r_{\perp i}$. The linear velocity of the particle is then,

$$v_t = \frac{ds_i}{dt} = r_{\perp i}\frac{d\theta}{dt} = r_{\perp i}\omega. \tag{6.3}$$

The tangential acceleration is then,

$$a_t = \frac{dv_i}{dt} = r_{\perp i}\frac{d\omega}{dt} = r_{\perp i}\alpha. \tag{6.4}$$

Each particle will also have a centripetal acceleration,

$$a_r = \frac{v_t^2}{r_{\perp i}} = \frac{(r_{\perp i}\omega)^2}{r_{\perp i}} = \omega^2 r_{\perp i} \tag{6.5}$$

6.1 Rotational Kinetic Energy

For a particle p_i,

$$K_i = \frac{1}{2} m_i v_i^2$$

$$K = \frac{1}{2}\left(\sum_i m_i r_{\perp i}^2\right)\omega^2$$

6.1.1 Moment of Inertia

Definition. *The moment of inertia, I, about a given axis is defined as the sum $\sum_i m_i r_{\perp i}^2$ over all particles that makeup the body.*

So,

$$I = \sum_i m_i r_{\perp i}^2 \tag{6.6}$$

$$I = \int r_\perp^2 \, dm \tag{6.7}$$

The kinetic energy can then be defined,

$$K = \frac{1}{2} I \omega^2 \tag{6.8}$$

Generally, it is true that the further the mass fro the axis of rotation, the greater the moment of inertia about the axis.

6.2 Torque

Torque is the analogue to force in rotational motion. It is defined as the vector product of the distance between where the force acts and the axis of rotation, and the force,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \tag{6.9}$$

Consider the situation presented in figure 6.2.

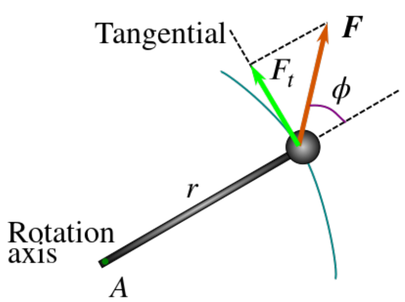


Figure 6.2: A mass at the end of a pole of negligible mass of length r rotating about the centre of rotation A experiences a force $\mathbf{F}$.

The tangential force is,

$$F_t = ma_t$$

we can then say torque is,

$$\tau = F_t r = ma_t r = mr^2\alpha.$$

Now, let's consider this as the i^{th} particle in a rigid body,

$$\tau_{\text{net}i} = m_i r_{\perp i}^2 \alpha.$$

For all particles in the rigid body,

$$\sum_i \tau_{\text{net}i} = \sum_i m_i r_{\perp i}^2 \alpha$$

$$\tau_{\text{total}} = I\alpha \tag{6.10}$$

6.3 Parallel Axis Theorem

Theorem. *The moment of inertia about an axis through the centre of mass of an object is related to the moment of inertia about another parallel axis given by the equation,*

$$I = I_{cm} + Md^2 \tag{6.11}$$

where d is the distance between the two axis.

6.4 Kinetic Energy, Work Done, and Power

Consider a force F acting on a rotating object. It rotates through an angle $d\theta$ and travels a path $\mathbf{s} = \mathbf{r}d\theta$, doing work in the process. So,

$$dW = \mathbf{F} \cdot d\mathbf{s} = F_t r d\theta = \tau d\theta.$$

The work done between two angles is then,

$$W = \int_{\theta_1}^{\theta_2} \tau d\theta. \tag{6.12}$$

We can then define power,

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

$$P = \tau\omega \tag{6.13}$$

6.5 Non-Slip Motion

$$\frac{ds}{dt} = R\frac{d\theta}{dt} = \omega \tag{6.14}$$

$$v_{\text{cm}} = \omega R \tag{6.15}$$

6.5.1 Kinetic Energy of a Rolling Wheel

$$K = \frac{1}{2} I_{\text{cm}} \omega^2 + \frac{1}{2} M v_{\text{cm}}^2 \tag{6.16}$$

6.6 Angular Momentum

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \tag{6.17}$$

6.6.1 Newton's Second Law of Rotation

The rate of change of angular momentum equals the net torque acting on that particle.

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} \\ \frac{d\mathbf{L}}{dt} &= m\left(\frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r}\frac{d\mathbf{v}}{dt}\right) = m(\mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a}) \\ \implies \frac{d\mathbf{L}}{dr} &= m\mathbf{r} \times \mathbf{a} = \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau} \end{aligned} \tag{6.18}$$

For a system of particles,

$$\frac{d\mathbf{L}_{\text{sys}}}{dt} = \boldsymbol{\tau}_{\text{ext}}. \tag{6.19}$$

For rigid bodies rotating about an axis,

$$\mathbf{L} = I\boldsymbol{\omega}. \tag{6.20}$$

6.7 Instantaneous Rotation

$$v_{\text{cm}} = 2v_{\text{top}} \tag{6.21}$$

6.8 Rolling With Slipping

For an object to be rolling without slipping, it must meet the condition,

$$v = \omega R. \tag{6.22}$$

To solve equations with rolling while slipping, follow the steps:

1. Model the system as if it was sliding.
2. Model the system as if it was rotating.
3. You can find the time/other properties up until the motion becomes completely rotational by substituting the condition, $v = \omega R$.

Chapter 7

Gravitation

Newton's law of gravity, in vector form, is given by,

$$\mathbf{F}_{21} = -\frac{Gm_1m_2}{r_{12}^3}\mathbf{r}_{12} \tag{7.1}$$

where r_{12} is the magnitude of $\mathbf{r}_{12}$.

7.1 Gravitational Potential Energy

GPE can be given by solving the integral,

$$W_{\text{grav}} = \int_R^\infty \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}.$$

GPE is then,

$$U(r) = W_{\text{grav}} = -\frac{Gm_1m_2}{r_{12}}. \tag{7.2}$$

For a system of bodies, the total gravitational potential will be the sum of the GPE between each pair of bodies.

7.2 Escape Speed

$$v = \sqrt{\frac{2GM_E}{R}} \tag{7.3}$$

7.3 Kepler's Laws

1. All planets move in an elliptical orbit, with the sun at one focus of the ellipse.
2. A line that connects a planet to the sun sweeps out equal areas in equal times. Such that
$$\frac{d\theta}{dt} = \frac{\pi ab}{P}\frac{2}{r^2} \tag{7.4}$$
3. The square of the period of any planet about the sun is proportional to the cube of that planet's mean distance from the sun,
$$T^2 = a^3 \tag{7.5}$$

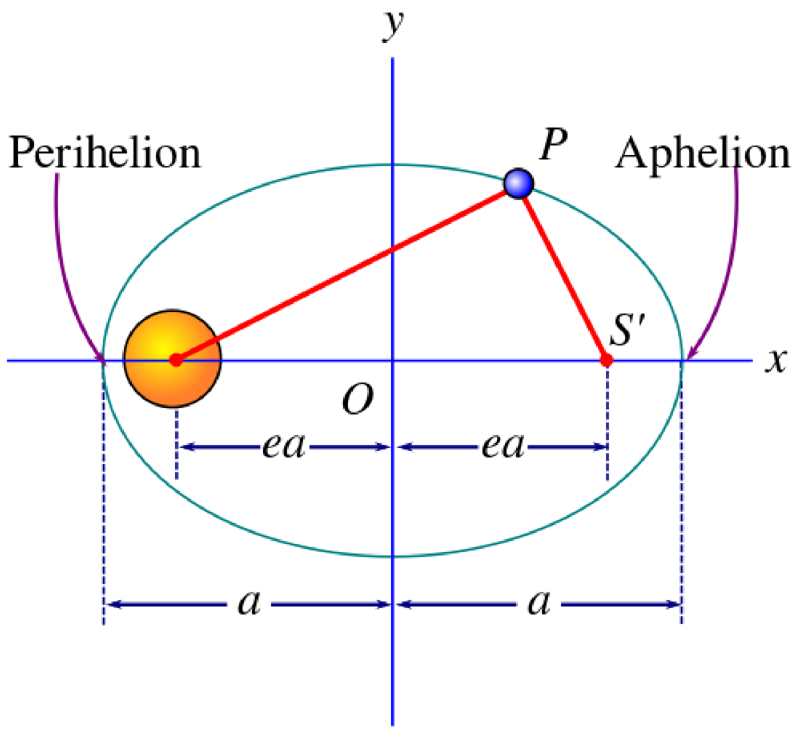


Figure 7.1: An elliptical orbit.