Relativity Notes

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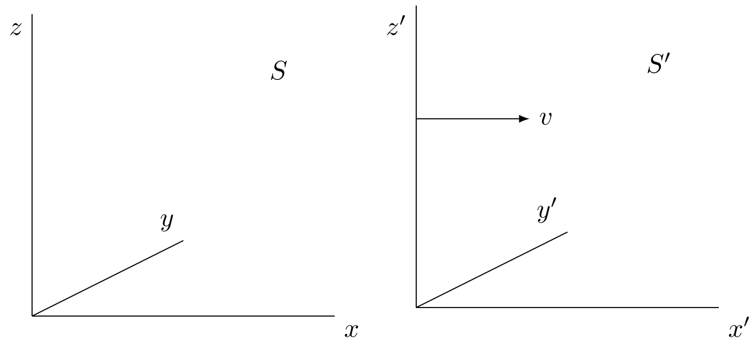
Chapter 1

Introduction to Relativity

1.1 Galilean Transformation Equations

These equations allow us to transform a set of co-ordinates in one frame of reference to another frame of reference.

Suppose a stationary reference frame S with co-ordinates (x, y, z, t). Further suppose an inertial reference frame S' with constant velocity v parallel to the x-axis and co-ordinates (x', y', z', t'). (See below)



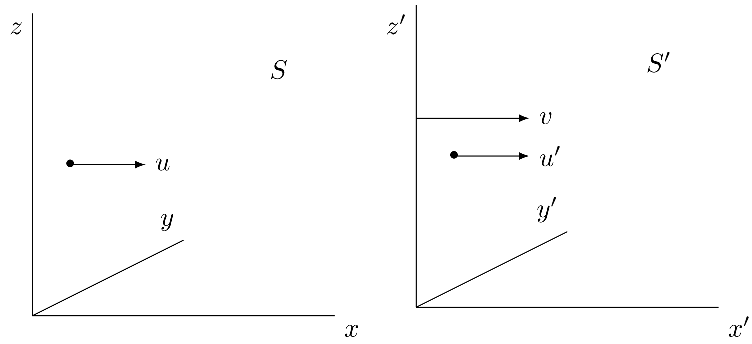
We may wish to be able to change between these two co-ordinate systems. Using the Galilean transformations and supposing that time is absolute such that $t = t'$, we can say:

$$x = x' - vt$$

$$x' = x + vt$$

We can further extend this to relativistic velocity.

In this case, we have a particle travelling at a velocity u relative to a stationary observer in a reference frame S. However, u is affected by the velocity v of the reference frame S', and relative to this reference frame the particle is moving at a speed u'.



Knowing this, we can then write:

$$u = u' + v$$

Keeping in mind that u is the velocity of the particle as seen by a stationary observer.

Chapter 2

Einstein's Special Relativity

2.1 Einstein's Postulates

1. *Identical isolated experiments in different inertial frames deliver identical results.*

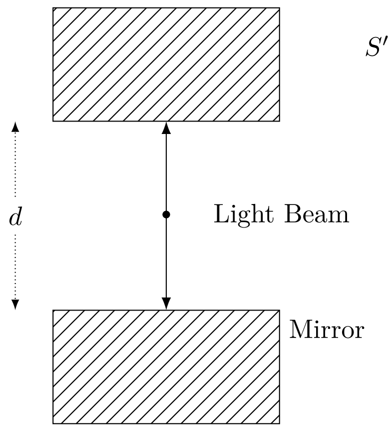
   This means that, no matter in what inertial frame we do our experiments, we will always get consistent results. Therefore, if experiments in electricity and magnetism are correct **and** Maxwell's equations are correct:

2. *The speed of light in a vacuum is the same in all inertial frames.*

$$c = \frac{1}{\mu_0 \epsilon_0}$$

These postulates lead to very strange effects, such as the one showcased below:

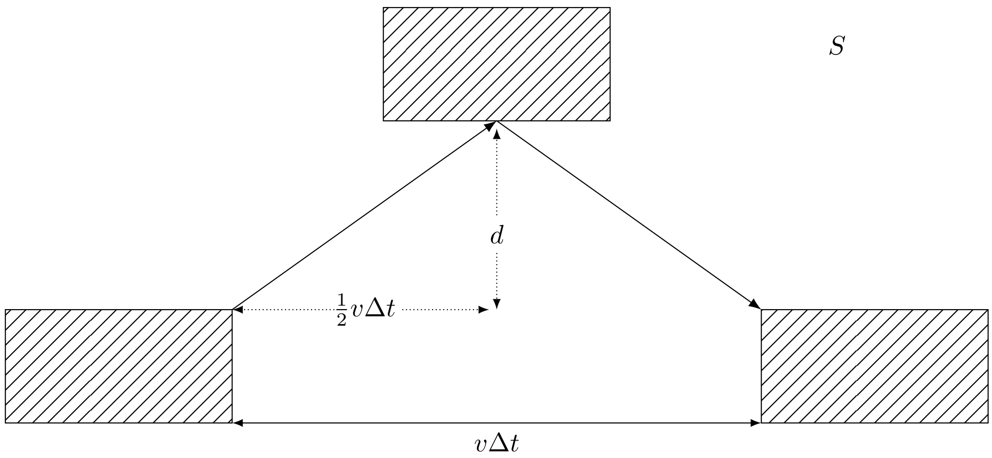
2.1.1 Time Dilation



The above showcases a diagram of a light clock relative to a moving train on which it is positioned in reference frame S'. The light clock works so that 1 tick is counted when light travels from one mirror to the next and back. We can then say that the time for a single tick is:

$$\Delta t_0 = \frac{2d}{c}$$

Lets say the light clock ticks away on a train with a velocity v. What would the light clock look like relative to a stationary observer? We can visualise this below. Note that we must keep in mind that the speed of light is always the constant value c.



The above diagram shows that the light travels in a different path relative to the observer. We can find the relative distance the light travelled rather trivially using Pythagoras:

$$d_{light} = \sqrt{(\frac{1}{2}v\Delta t)^2 + d^2}$$

Further, finding the time the light took to travel between the mirrors is trivial. However, we must remember that the speed of light is constant. So it would be wrong to perform a Galilean addition of velocities. Therefore, the equation for the time it took for light to travel would be:

$$\Delta t = \frac{\sqrt{(\frac{1}{2}v\Delta t)^2 + d^2}}{c}$$

$$\Delta t = \frac{2d}{c} \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

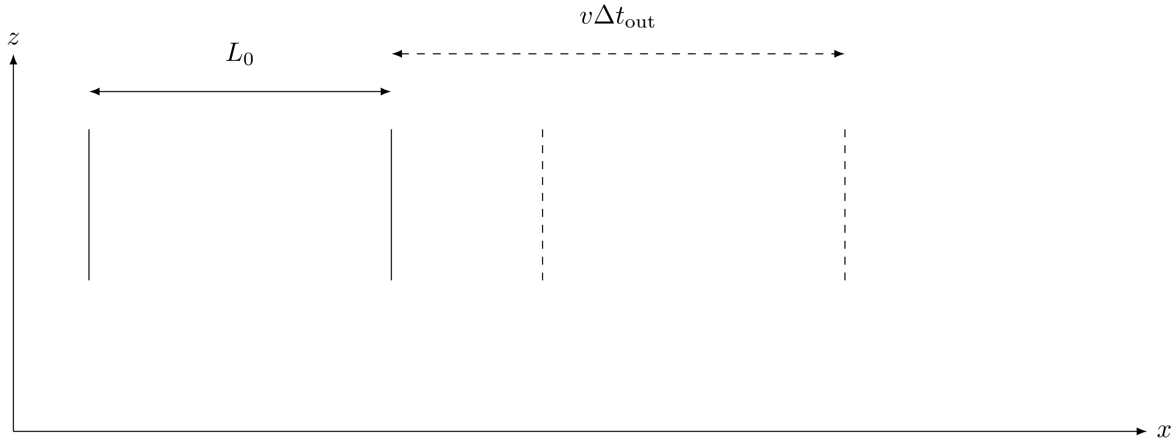
We will recall that $\Delta t_0 = \frac{2d}{c}$, which is the time it takes for a single tick of the clock. We can conclude then that as velocities approach the speed of light, time seems to slow down, relative to stationary observers. To simplify the equation further, we can write:

$$\Delta t = \Delta t_0 \cdot \gamma$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

2.1.2 Length Contraction

Consider our light clock, this time being placed horizontally, with the mirrors a length L_0 apart measured in their rest frame and moving parallel to the x-axis with a velocity v relative to S.



We can say that the light travels a distance:

$$L + v\Delta t_{\text{out}} = c\Delta t_{\text{out}} \tag{2.1}$$

For the return leg, the light will travel a shorter distance.

$$c\Delta t_{\text{in}} = L - v\Delta t_{\text{in}} \tag{2.2}$$

We are then able to derive the total time for the light to travel.

$$\begin{aligned} 2.1 \implies \Delta t_{\text{out}} &= \frac{L}{c-v} \\ 2.2 \implies \Delta t_{\text{in}} &= \frac{L}{c+v} \\ \therefore \text{ total time } \Delta t &= \frac{L}{c-v} + \frac{L}{c+v} \\ \text{In the clock's frame: } \Delta t_0 &= \frac{2L_0}{c} \end{aligned}$$

We are able to equate the total time Δt to the time dilation equation.

$$\Delta t = \gamma \Delta t_0 = \gamma \frac{2L_0}{c} = \frac{L}{c-v} + \frac{L}{c+v}$$

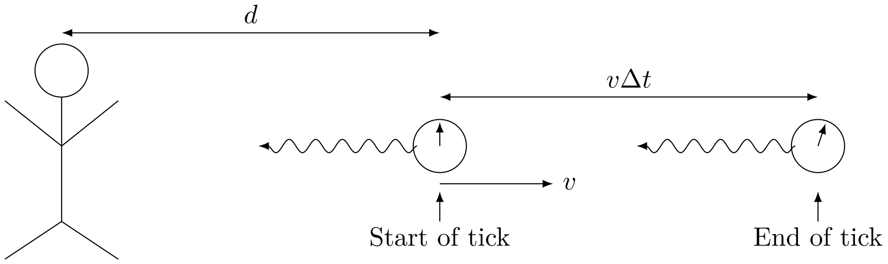
We can then rearrange these terms in terms of the observed length and rest length to get:

$$L = \frac{L_0}{\gamma} \tag{2.3}$$

We can see the reason this phenomenon is called length contraction is because γ is always greater than 1, therefore the observed length will shrink.

2.2 Doppler Effect For Light

We can think about how relativity effects the wavelength of a moving light source by considering a clock being observed by an observer. Lets say that every full oscillation of the wave is represented by a tick of the clock, ΔT, which is the same length of time as the period of the wave.



The clock is a distance d from the observer at the start of the tick and the light from the clock will arrive at a time $\frac{d}{c}$ later. At the end of the tick, the clock is further away by a distance $v\Delta t$, and the light emitted at the end will arrive at a time $\frac{v\Delta t}{c} + \Delta t$ after the first tick. We can then define the full time of the tick of the clock, or rather the period of the wave.

$$\begin{aligned} \Delta T &= \Delta t(1 + \frac{v}{c}) \\ &= \gamma \Delta t_0 (1 + \frac{v}{c}) \\ &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta t_0 \left(1 + \frac{v}{c}\right) \\ &= \Delta t_0 \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \end{aligned}$$

We can rename some of these variables so that they're better for waves.

$$\Delta T = \Delta T_0 \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \tag{2.4}$$

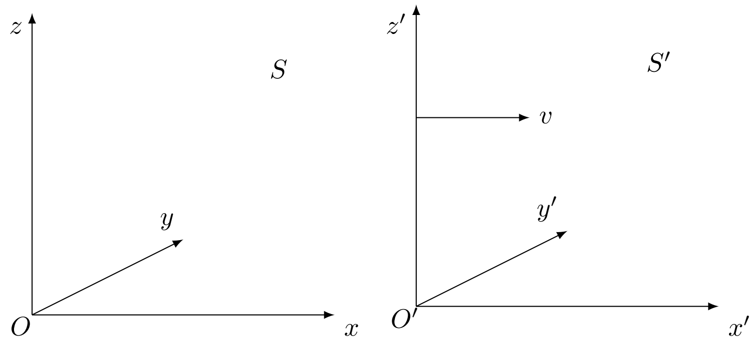
$$\Delta f = \Delta f_0 \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \tag{2.5}$$

Where ΔT is the period of the wave and Δf is its frequency.

Chapter 3

Lorentz Transformations

3.1 Derivation of Lorentz Transformations



By assuming the most general, linear form for transformations between inertial frames such that

$$x' = ax + bt \tag{3.1}$$
$$t' = dx + et \tag{3.2}$$

we can apply Einstein's postulates to derive the Lorentz transformation equations.
We can begin by stating that the origin O' (where $x' = 0$) moves along the x-axis at speed v according to $x = vt$. We can then substitute this into (3.1).

$$\begin{gathered} 0 = avt + bt \\ \therefore -\frac{b}{a} = v \end{gathered} \tag{3.3}$$

Similarly, the origin O moves along the x' axis at speed $-v$ according got $x' = -vt'$. Using (3.1), we can say

$$x' = bt$$
$$t' = et$$

$$-\frac{b}{e} = v \tag{3.4}$$

Combining (3.3) and (3.4), we can conclude,

$$e = a$$
$$b = -av$$

Substituting this into (3.1) and (3.2), we can simplify our general equations to 2 unknowns.

$$\begin{aligned} x' &= at - avt \\ v' &= dx + at \end{aligned} \tag{3.5}$$

To reduce this equation further, let's implement Einstein's second postulate by considering a photon emitted at O and O' when $O' = O$. It will travel in S along the x and x' axis at speed c. Its position will be such that $x = ct$ and $x' = ct'$. Using (3.5)

$$\begin{aligned} ct' &= act - avt \\ t' &= dct + at \\ \therefore d &= -\frac{av}{c^2} \end{aligned} \tag{3.6}$$

Applying (3.6) to (3.5),

$$\begin{aligned} x' &= a(x - vt) \\ t' &= a\left(t - \frac{vx}{c^2}\right) \end{aligned} \tag{3.7}$$

We can now eliminate a by applying Einstein's second postulate. Applying this postulate, (3.7) should take the same form in all inertial frames. We can then change change the frame which our equations describe.

$$\begin{aligned} x =& a(x' + vt') \\ t =& a\left(t' + \frac{vx'}{c^2}\right) \end{aligned} \tag{3.8}$$

Solving for x,

$$\begin{aligned} x &= a\left(ax - avt + avt - \frac{av^2x}{c^2}\right) \\ &= a^2x\left(1 - \frac{v^2}{x^2}\right) \end{aligned}$$

$$a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \tag{3.9}$$

We can fully derive the Lorentz transformations by applying (3.9) to (3.8).

$$x' = \gamma(x - vt) \tag{3.10}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{3.11}$$

$$x = \gamma(x' + vt') \tag{3.12}$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{3.13}$$

The Lorentz velocity transformation equations for an object travelling at a velocity v in a reference frame travelling at a velocity u are given by,

$$v_x = \frac{v'_x + u}{1 + \frac{uv_x}{c^2}} \tag{3.14}$$

$$v_y = \frac{v'_y}{\gamma\left(1 + \frac{uv'_x}{c^2}\right)} \tag{3.15}$$

Chapter 4

Spacetime

4.1 Invariance

Spacetime is a relativistic co-ordinate system under which spacetime vectors and distances are invariant when switching between inertial frames.

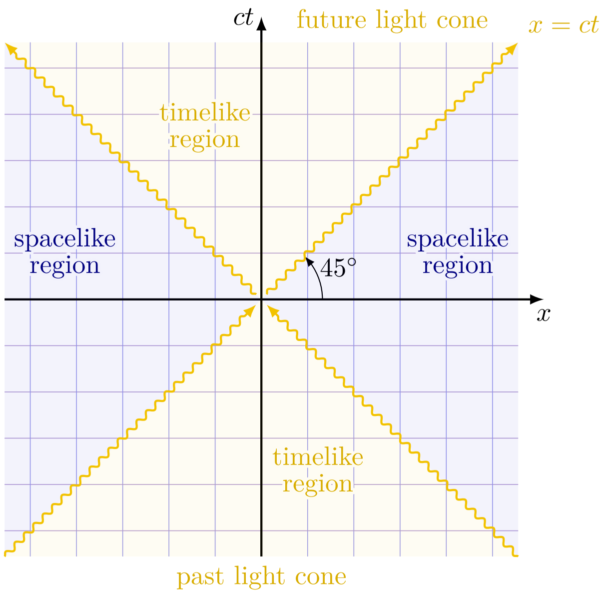


Figure 4.1: This is a space-time diagram. We plot ct on the vertical axis to represent time. The gradient of a line represents its speed. Therefore, a photon will always make a 45-degree angle with the vertical. Further, this means that any object starting at the origin can only move through the timelike region.

Distances in space are **not invariant** between inertial frames using relativistic geometry. They are invariant, however, in spacetime. Such that,

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2. \tag{4.1}$$

4.1.1 Vectors

For a vector in Euclidean space, its length is given by,

$$\text{Length} = \sqrt{\Delta \mathbf{x} \cdot \Delta \mathbf{x}} = |\Delta \mathbf{x}|.$$

In space time, we must define a vector $\underset{\sim}{x}$, such that,

$$\underset{\sim}{x} = (ct, \mathbf{x}),$$

where $\mathbf{x}$ is a regular spacial vector. We can then say,

$$\Delta \underset{\sim}{x} \cdot \Delta \underset{\sim}{x} = (c\Delta t)^2 - \Delta \mathbf{x} \cdot \Delta \mathbf{x} \tag{4.2}$$

4.2 Path Integral

The path integral of a space time diagram gives us the amount of time experienced by the object travelling along the path. For the absolute time, $\Delta\tau$, the time an object has aged travelling along its world line is given by,

$$\Delta t = \int_0^{\Delta\tau} = \sqrt{1 - \frac{v^2}{c^2}} dt. \tag{4.3}$$

4.3 Energy and Momentum

Classical moment is variant between inertial frames in relativity. We can then define,

$$m\frac{d\underset{\sim}{x}}{d\tau} = \underset{\sim}{P} \tag{4.4}$$

where $d\tau$ is the proper time, $\frac{ds}{dt}$, which we use as it is invariant in spacetime. This then derives to,

$$\underset{\sim}{P} = \gamma(\frac{mc^2}{c}, m\mathbf{v}), \tag{4.5}$$

the law then follows,

$$\sum_i \underset{\sim}{P_i} = \sum_j \underset{\sim}{P_j}. \tag{4.6}$$

The $\frac{\gamma mc^2}{c}$ component represents the total energy of a particle, where for a moving particle, the total energy E is given by,

$$E = \gamma mc^2,$$

and the kinetic energy is given by,

$$KE = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2 \tag{4.7}$$

where mc^2 is the energy of the particle rest. We ca further define the total energy of a system using,

$$E^2 = c^2p^2 + m^2c^4. \tag{4.8}$$

We can further derive a quantity known as the **invariant mass** from equation (4.8).

$$M^2 = \frac{E_{\text{tot}}^2}{c^2} - P_{\text{tot}}^2. \tag{4.9}$$