ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Integral Transforms

24th May 2011, 2.00 p.m. - 3.30 p.m.

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. (a) For a function $f(z) = u(x,y) + iv(x,y)$ (where $z = x + iy$), state the Cauchy-Riemann equations. Show that they are satisfied for $f(z) = 1/z$ for $z \neq 0$.

[6 marks]

(b) Evaluate $\int_C \bar{z}\, dz$, where the end-points are $a = 1$ and $b = i$, and C is the path which follows the axes and passes through the origin. Should you expect the same answer for a straight-line path between the same end-points?

[6 marks]

(c) Show that under the mapping $w = e^z$, the line $x = a$ maps to the curve $u^2 + v^2 = c^2$ and the line $y = b$ maps to a portion of the line $v = mu$, and express c and m in terms of a and b respectively. (u and v are the real and imaginary parts of w.)

[7 marks]

(d) Find the Laplace transform of $t \sin \omega t$ (for real ω).

[6 marks]

2. (a) An analytic function $w(z)$ has imaginary part $v(x,y) = x^2 + y - y^2$. Find the corresponding real part $u(x,y)$ of $w(z)$ given that $u(0,0) = 1$. Express $w(z)$ in terms of z.

[8 marks]

(b) Find the Taylor or Laurent series, as appropriate, of the function $\dfrac{z}{(z+1)(z-2)}$:

i) about $z = 0$, for the region $|z| < 1$;
ii) about $z = 2$, for the region $|z - 2| > 3$.

[8 marks]

(c) Find the first three terms of the Taylor-Laurent series about $z = 0$ of $z^{-2} \tan z$ (you may use standard Taylor series for functions like $\sin z$ if required). What is the nature of the singularity and the value of the residue? What is the radius of convergence of the series?

[9 marks]

3. Use Cauchy's residue theorem and a suitable choice of contour to calculate the following integrals. If Jordan's lemma is used, ensure that all the conditions for its validity are satisfied.

(a)

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 3\cos\theta}\, \mathrm{d}\theta$$

[13 marks]

(b)

$$\int_{-\infty}^{\infty} \frac{x}{(x+1)(x^2+4)}\, \mathrm{d}x$$

[12 marks]

4. (a) Use contour integration to find the Fourier transform of $\dfrac{1}{1+x^2}$ for both positive and negative k. (Where Jordan's lemma is used, ensure that all the conditions for its validity are satisfied.)

[15 marks]

(b) We define a "top-hat function" $R(x)$ which is zero for $|x| \geq 1/2$ and 1 for $|x| < 1/2$; also a "triangle function" $T(x)$ which is zero for $|x| \geq 1$ and $1 - |x|$ for $|x| < 1$. Sketch these functions. Find their Fourier transforms. Explain the relation between the two.

(You may use the following integral: $\displaystyle\int_0^1 (1-x)\cos(kx)\, \mathrm{d}x = \frac{2}{k^2}\sin^2\!\left(\frac{k}{2}\right)$.)

[10 marks]

END OF EXAMINATION PAPER

PHYSICAL CONSTANTS AND CONVERSION FACTORS

SYMBOL	DESCRIPTION	NUMERICAL VALUE
c	Velocity of light in vacuum	299 792 458 m s^{-1}, exactly
μ_0	Permeability of vacuum	$4\pi \times 10^{-7}$ N A^{-2}, exactly
ϵ_0	Permittivity of vacuum where $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$	8.854×10^{-12} C^2 N^{-1} m^{-2}
h	Planck constant	6.626×10^{-34} J s
$\hbar$	$h/2\pi$	1.055×10^{-34} J s
G	Gravitational constant	6.674×10^{-11} m^3 kg^{-1} s^{-2}
e	Elementary charge	1.602×10^{-19} C
eV	Electronvolt	1.602×10^{-19} J
α	Fine-structure constant, $\frac{e^2}{4\pi\epsilon_0 \hbar c}$	$\frac{1}{137.0}$
m_e	Electron mass	9.109×10^{-31} kg
$m_e c^2$	Electron rest-mass energy	0.511 MeV
μ_B	Bohr magneton, $\frac{e\hbar}{2m_e}$	9.274×10^{-24} J T^{-1}
R_∞	Rydberg energy $\frac{\alpha^2 m_e c^2}{2}$	13.61 eV
a_0	Bohr radius $\frac{1}{\alpha}\frac{\hbar}{m_e c}$	0.5292×10^{-10} m
Å	Angstrom	10^{-10} m
m_p	Proton mass	1.673×10^{-27} kg
$m_p c^2$	Proton rest-mass energy	938.272 MeV
$m_n c^2$	Neutron rest-mass energy	939.565 MeV
μ_N	Nuclear magneton, $\frac{e\hbar}{2m_p}$	5.051×10^{-27} J T^{-1}
fm	Femtometre or fermi	10^{-15} m
b	Barn	10^{-28} m^2
u	Atomic mass unit, $\frac{1}{12} m(^{12}C$ atom)	1.661×10^{-27} kg
N_A	Avogadro constant, atoms in gram mol	6.022×10^{23} mol^{-1}
T_t	Triple-point temperature	273.16 K, exactly
k	Boltzmann constant	1.381×10^{-23} J K^{-1}
R	Molar gas constant, $N_A k$	8.314 J mol^{-1} K^{-1}
σ	Stefan-Boltzmann constant, $\frac{\pi^2}{60}\frac{k^4}{\hbar^3 c^2}$	5.670×10^{-8} W m^{-2} K^{-4}
M_E	Mass of Earth	5.97×10^{24} kg
R_E	Mean radius of Earth	6.4×10^6 m
g	Standard acceleration of gravity	9.806 65 m s^{-2}, exactly
atm	Standard atmosphere	101 325 Pa, exactly
$M_\odot$	Solar mass	1.989×10^{30} kg
$R_\odot$	*Solar radius*	6.96×10^8 *m*
$L_\odot$	Solar luminosity	3.84×10^{26} W
$T_\odot$	Solar effective temperature	5.8×10^3 K
AU	Astronomical unit, mean Earth-Sun distance	1.496×10^{11} m
pc	Parsec	3.086×10^{16} m
	Year	3.156×10^7 s