ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Integral Transforms

8th June 2016, 2.00 p.m. – 3.30 p.m.

Answer **ALL** parts of question 1 and **TWO** other questions

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. (a) For a function $f(z) = u(x, y) + iv(x, y)$ (where $z = x + iy$), state the Cauchy-Riemann equations.

Show that they are satisfied for $f(z) = \mathrm{e}^z$, and hence find $\dfrac{\mathrm{d}f}{\mathrm{d}z}$.

[6 marks]

(b) Evaluate the following integral, $\displaystyle\int_1^i \frac{1}{z}\,\mathrm{d}z$, along a straight line path. Comment on your result.

[You may use $\int(2x^2 - 2x + 1)^{-1}\mathrm{d}x = -\arctan(1 - 2x) + c$.]

[9 marks]

(c) Find the Taylor-Laurent series about $z = 0$ of the function

$$\frac{z}{(z+1)(z-2)} = \frac{1}{3}\left(\frac{1}{(z+1)} + \frac{2}{(z-2)}\right),$$

valid for the region $1 < |z| < 2$.

[5 marks]

(d) A function $w = f(z)$ has three roots and no singularities in some region of the complex plane. C_0, C_1 and C_3 are circular paths in the z plane, which encircle zero, one and three of the roots respectively. Under the mapping $z \to f(z)$, the circle C_i is mapped to the curve C_i'. Sketch possible paths C_0', C_1' and C_3' in the w plane.

[5 marks]

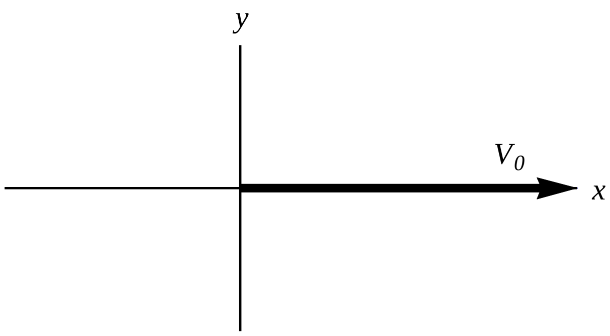
2. (a) An analytic function $f(z)$ has imaginary part $v(x, y) = x^2 + y - y^2$. Show that $v(x, y)$ is harmonic, and find the corresponding real part $u(x, y)$ of $f(z)$ given that $u(0, 0) = 1$. Express $f(z)$ in terms of z.

[8 marks]

(b) Consider the mapping $w = \sqrt{z}$. Show that lines of constant $\mathrm{Re}[w] = a$ are given by the equation $y = \pm 2a\sqrt{a^2 - x}$, and find the equation of lines of constant $\mathrm{Im}[w] = b$. (It will help to start with $w^2 = z$.) Sketch the lines for $a = 0, 1, 2$ and $b = 0, 1, 2$. Why are no new curves generated for negative a, b?

[10 marks]

(c) Use the mapping and your sketch above to describe the potential and field lines around a charged semi-infinite plate held at potential V_0. Take the plate to lie in the xz plane with its left-hand edge along the z-axis, so that a cross-sectional view is given below. (In this paragraph only, z refers to the third spatial dimension.)



Show that the potential is given by $\phi(r, \theta) = -Ar^{1/2}\sin(\theta/2) + V_0$, where A is a constant and r and θ are polar coordinates in the xy plane.

[7 marks]

3. (a) Find the Fourier transform $F(k)$ of

$$f(x) = \frac{1}{(x+a+ib)(x-a+ib)}$$

where a and b are real and positive; ensure your answer is valid for all real k.
If Jordan's lemma is used, ensure that all the conditions for its validity are satisfied.

[13 marks]

(b) Use Cauchy's residue theorem and a suitable choice of contour to do **ONE** of the following integrals.

(i)

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 3\cos\theta}\, \mathrm{d}\theta$$

(ii)

$$\int_0^{\infty} \frac{\sqrt{x}}{1+x^2}\, \mathrm{d}x$$

[12 marks]

4. (a) Find the position of, and residues at, the poles of $f(z) = \dfrac{1}{z^3 \cos(\pi z)}$. Hence show that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots = \frac{\pi^3}{32}.$$

[13 marks]

(b) (i) If $f(t)$ and $g(t)$ both vanish for $t < 0$, their convolution can be written

$$h(t) = \int_0^\infty \theta(t - t')\, f(t - t')\, g(t')\, \mathrm{d}t'.$$

Show that the Laplace transform of $h(t)$ is $F(s)G(s)$, where $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$ respectively.

[6 marks]

(ii) Use the result above to solve the following differential equation, subject to the initial conditions $y(0) = 0$ and $y'(0) = 0$:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - y = \frac{1}{\cosh t}.$$

[Hint: do not attempt to find the Laplace transform of the right-hand side.]

[6 marks]

END OF EXAMINATION PAPER