ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Integral Transforms

4th June 2014, 9.45 a.m. – 11.15 a.m.

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. (a) For a function $f(z) = u(x,y)+iv(x,y)$, where $z = x+iy$, state the Cauchy-Riemann equations. Show that they are satisfied for $f(z) = z^3$. Verify that $\mathrm{d}f/\mathrm{d}z = 3z^2$.

[6 marks]

(b) Evaluate $\int_C \bar{z}\,\mathrm{d}z$, where the end-points are $a = 1$ and $b = i$, and C is the path that connects the two in a straight line.

[6 marks]

(c) Under the mapping $w = z^{1/2}$, the lines $x = c$ and $y = c$ map to curves in the uv plane. Find the equations of these curves and sketch them for $c = 2$ and $c = -2$.

[7 marks]

(d) A function $f(z)$ is analytic in some region of the complex plane. What can we say about integrals of f along different paths with the same end-point? Show that this follows from Cauchy's theorem. If the function has isolated singularities, prove that a closed contour integral can be replaced by the sum of integrals along small circles round each enclosed singularity.

[6 marks]

2. In this question you may use the polar form of the Cauchy-Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta},$$

and of the Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

(a) A function $w(z)$ has real part $u(r,\theta) = \cos(2\theta)/r^2$. Verify that this is harmonic. Given that $w(z)$ is analytic (except at the origin), find the corresponding imaginary part and express w as a function of z.

[8 marks]

(b) Consider the function

$$f(z) = \frac{1}{(z-1-i)(z-3)}.$$

If we expand $f(z)$ as a Taylor-Laurent series about $z = 0$, different series are valid in different regions of the complex plane. What are these regions? Describe the series in each region. Find the series that is valid in the region that includes the point $z = 4$.

If instead we expand about the point $z = 3$, how many regions are there?

Consider the following contours: C_1 is a circle of radius 2 centred on the origin and C_2 is a rectangle of height 1 and width 2 centred on the point $z = 3$. For each contour, traversed in an anticlockwise direction, find $\oint f(z)\,\mathrm{d}z$.

[13 marks]

(c) Find the residues of the following functions at the given points:

$$\text{(i)}\ \frac{\mathrm{e}^{2z}}{4\cosh z - 5} \quad \text{at } z = \ln 2; \qquad \text{(ii)}\ \frac{1}{z^4 \cos z} \quad \text{at } z = 0.$$

[4 marks]

3. Use Cauchy's residue theorem and a suitable choice of contour to do TWO of the following three problems. If Jordan's lemma is used, ensure that all the conditions for its validity are satisfied.

   (a) Evaluate the following integral:

$$\int_0^{2\pi} \frac{\cos\theta}{(13+12\cos\theta)^2}\,\mathrm{d}\theta.$$

   (b) Evaluate the following integral for $0 < \alpha < 1$:

$$\int_0^{\infty} \frac{1}{x^{\alpha}(x+2)}\,\mathrm{d}x.$$

   (c) Use the function $\dfrac{1}{(z-a)^2\tan\pi z}$ to prove the following result for non-integer a:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-a)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

[25 marks]

4. Show that, if $F(s)$ is the Laplace transform of $f(t)$,

$$\text{L.T.}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n F}{\mathrm{d}s^n}.$$

Hence or otherwise (but not by simply taking the L.T. of each side) show that

$$\text{I.L.T.}\left[\frac{2s^2}{(s^2+1)^2}\right] = t\cos t + \sin t.$$

[10 marks]

Use Laplace transforms to solve the following system of coupled differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} + z(t) = 2\cos t, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} - y(t) = 1,$$

subject to the boundary conditions $y(0) = -1$, $z(0) = 1$.

[15 marks]

END OF EXAMINATION PAPER