ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Vector Spaces

32nd January 2023, 2.00 p.m. - 3.30 p.m.

Answer **ALL** parts of question 1 and **TWO** other questions

---

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. a) An operator $\hat{A}$ is termed anti-hermitian if it satisfies $\hat{A}^\dagger = -\hat{A}$. Show that the eigenvalues of an anti-hermitian operator $\hat{A}$ are pure imaginary.

[6 marks]

b) Consider the real functions $\phi_n(x) = p_n(x)e^{-x^2/2}$ where $p_n(x)$ are polynomials of degree n in the real variable x. The inner product is given by,

$$\langle \phi_n | \phi_m \rangle = \int_{-\infty}^{\infty} \phi_n(x)\phi_m(x)\, \mathrm{d}x.$$

Find $p_2(x)$, given that the functions ϕ_0, ϕ_1 and ϕ_2 are mutually orthogonal with respect to the inner product. Your answer for p_2 does not need to be normalised.

You may use without proof the standard integrals,

$$\int_{-\infty}^{\infty} e^{-x^2}\, \mathrm{d}x = \sqrt{\pi} \quad \text{and} \quad \int_{-\infty}^{\infty} x^2 e^{-x^2}\, \mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

[8 marks]

c) Integrate the function $f(z) = \overline{z}$ along a straight-line path between $z = 0$ and $z = 1 + i$.

State without proof whether the value of this integral could change if an alternative path is used. Justify your answer.

[5 marks]

d) For a function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, state the Cauchy-Riemann conditions for $f(z)$ to be differentiable. Show that they are satisfied for $f(z) = e^z$.

[6 marks]

2. A two-dimensional complex vector space has an orthonormal-basis $|0\rangle$ and $|1\rangle$. The Pauli operators on this space are defined as

$$\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \quad \hat{\sigma}_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|, \quad \text{and} \quad \hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|.$$

a) Write down the matrix representation of all three Pauli operators with respect to the $\{|0\rangle, |1\rangle\}$ basis.

[3 marks]

b) Find the eigenvalues and corresponding normalised eigenvectors for $\hat{\sigma}_y$.

[5 marks]

c) A rotation operator in this two-dimensional vector space can be defined as

$$\hat{R}(\theta) = \exp(-i\theta\hat{\sigma}_y),$$

where θ is a real constant. Find the spectral representation of $\hat{R}(\theta)$, and show that it is unitary for all values of θ.

[4 marks]

d) Show that the rotation operator in part (c) has the matrix representation,

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

with respect to the $\{|0\rangle, |1\rangle\}$ basis.

[7 marks]

e) Consider the operator,

$$\hat{\rho} = \beta\hat{\sigma}_x + \gamma\hat{\sigma}_z,$$

where β and γ are real constants. Show that the rotation operator $\hat{R}(\theta)$ diagonalises $\hat{\rho}$ when

$$\theta = -\frac{1}{2}\arctan\left(\frac{\beta}{\gamma}\right).$$

[6 marks]

3. For this question, the following identities may be used without proof:

$$\cosh(A \pm B) = \cosh(A)\cosh(B) \pm \sinh(A)\sinh(B) \quad \text{and} \quad \cosh^2 A - \sinh^2 A = 1.$$

a) A function $f(z) = u + iv$, where $z = x + iy$, has real part $u(x, y) = \cosh x \cos y$. Show that u satisfies $\nabla^2 u = 0$. Given that f is analytic and that $f(0) = 1$, find the corresponding imaginary part $v(x, y)$ and hence show that $f(z) = \cosh z$.

[9 marks]

b) Consider the mapping $Z = X + iY = \operatorname{arccosh} z$. Show that under the inverse mapping, $z = \cosh Z$, lines of constant $X = a \neq 0$ map to the ellipses

$$\frac{x^2}{\cosh^2 a} + \frac{y^2}{\sinh^2 a} = 1\,.$$

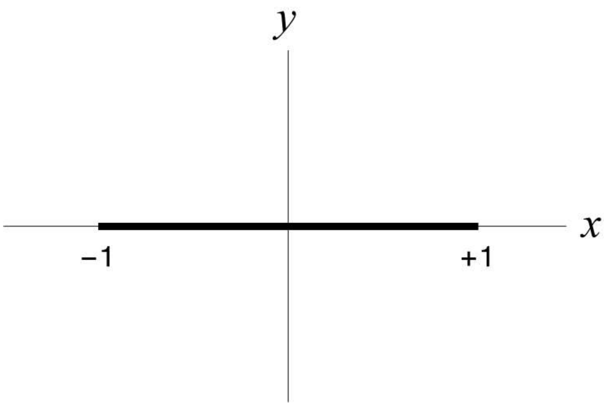
Also show that the line $X = 0$ maps to the segment of the x-axis with $-1 \leq x \leq 1$.

[5 marks]

c) For the mapping considered in part (b), find the equations of the curves to which lines of constant $Y = b$ are mapped for $-\pi < b \leq \pi$. Sketch curves of constant X and curves of constant Y on the xy-plane.

[5 marks]

d) The function $Z(x, y) = \operatorname{arccosh}(x + iy)$ can be regarded as the complex potential around an infinitely long charged strip of metal. The cross-section of this strip is indicated by the heavy line in the figure below.



State how the field lines and equipotentials are related to the curves sketched in part (c). Use the Cauchy–Riemann conditions to show that the magnitude of the electric field $\mathbf{E}(x, y)$ is given by

$$E = \left|\frac{\mathrm{d}Z}{\mathrm{d}z}\right|.$$

Hence show that $E \approx 1/(x^2 + y^2)^{1/2}$ at large distances from the metal strip.

[6 marks]

4. In the following problems involving contour integration, the contour should be sketched and the positions of any poles indicated on your sketch. If Jordan's lemma is used, show that the conditions for its validity are satisfied.

a) Evaluate the following real, principal-value integral

$$⨍_{-\infty}^{\infty} \frac{\cos x}{x-1}\,\mathrm{d}x.$$

[11 marks]

b) Find the poles and residues of the function $f(z) = \dfrac{\cot(\pi z)}{(a^2+z^2)}$, where a is a non-zero real number.

By considering the integral of $f(z)$ around a suitable contour, show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2} = \frac{\pi}{a\tanh(\pi a)}.$$

[14 marks]

END OF EXAMINATION PAPER