ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Integral Transforms

16 May 2012, 09:45 - 11:15

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. (a) For a function $f(z) = u(x,y) + iv(x,y)$ (where $z = x + iy$), state the Cauchy-Riemann equations. Show that they are satisfied for $f(z) = \exp(z^2)$.

[6 marks]

(b) Using the definition of the derivative, differentiate z^2. Show that the derivative of $|z|^2$ does not exist.

[7 marks]

(c) Evaluate $\displaystyle\int_C \frac{1}{z}\,\mathrm{d}z$, where the end-points are $a = 1$ and $b = -1$, and C is a path of your choosing which, apart from the end-points, lies entirely in the upper half plane. Without further explicit integration, write down the result for a path which lies in the lower half plane.

[6 marks]

(d) Sketch the following regions of, or paths in, the complex plane:
i) $2 < |z-3| < 4$; ii) $\pi/2 < \mathrm{Arg}[z-i] < \pi$; iii) $|z-2| = |z-4|$.

[6 marks]

2. In this question you may use the polar form of the Cauchy-Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta},$$

and of the Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

a) A function $w(z)$ has real part $u(r,\theta) = (r - R^2/r)\sin\theta$, where R is a constant. Given that $w(z)$ is analytic (except at the origin), find the corresponding imaginary part and express w as a function of z.

[9 marks]

Show that u is a harmonic function.

[3 marks]

Hence by demonstrating that it obeys the appropriate boundary conditions, show that $\phi = -E_0 u$ is the electrostatic potential outside an infinitely long earthed cylinder of radius R, placed with its axis perpendicular to the xy-plane in a constant electric field $\mathbf{E} = E_0\hat{\mathbf{y}}$. Sketch the equipotentials and field lines.

[7 marks]

b) Show that the mapping $w = \mathrm{Ln}\, z$ maps the real axis to a pair of parallel lines in the w-plane; include a sketch to show the mappings of the points $z = -2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2$. Suggest a physical problem in electrostatics which can be solved by considering this mapping (the solution is not required).

[6 marks]

3. Use Cauchy's residue theorem and a suitable choice of contour to do TWO of the following three problems. If Jordan's lemma is used, ensure that all the conditions for its validity are satisfied.

(a) Evaluate the following integral:

$$\int_0^{2\pi} \frac{1}{(5+3\cos\theta)^2}\,\mathrm{d}\theta.$$

(b) Evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(1+x^2)}\,\mathrm{d}x.$$

(c) Use an appropriate contour integral of the function $\dfrac{1}{z\cos z}$ to prove the following result:

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}.$$

[25 marks]

4. (a) Find the Taylor or Laurent series, as appropriate, of $\dfrac{z^2-1}{(z+2)(z+3)}$:

i) about $z=0$ for $|z|>3$;
ii) about $z=-2$ for $|z+2|<1$.

[13 marks]

(b) Show that the Laplace Transform of the Heaviside step function $\theta(t-t_0)$ is e^{-st_0}/s provided $t_0>0$. Hence solve the following differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+4y=\begin{cases}0 & \text{if } 0\le t<2,\\ 1 & \text{if } t>2,\end{cases}$$

subject to the initial conditions $y(0)=0$ and $y'(0)=2$.

[12 marks]

END OF EXAMINATION PAPER

PHYSICAL CONSTANTS AND CONVERSION FACTORS

SYMBOL	DESCRIPTION	NUMERICAL VALUE
c	Velocity of light in vacuum	$299\,792\,458$ m s^{-1}, exactly
μ_0	Permeability of vacuum	$4\pi \times 10^{-7}$ N A^{-2}, exactly
ϵ_0	Permittivity of vacuum where $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$	8.854×10^{-12} C^2 N^{-1} m^{-2}
h	Planck constant	6.626×10^{-34} J s
$\hbar$	$h/2\pi$	1.055×10^{-34} J s
G	Gravitational constant	6.674×10^{-11} m^3 kg^{-1} s^{-2}
e	Elementary charge	1.602×10^{-19} C
eV	Electronvolt	1.602×10^{-19} J
α	Fine-structure constant, $\frac{e^2}{4\pi\epsilon_0 \hbar c}$	$\frac{1}{137.0}$
m_e	Electron mass	9.109×10^{-31} kg
$m_e c^2$	Electron rest-mass energy	0.511 MeV
μ_B	Bohr magneton, $\frac{e\hbar}{2m_e}$	9.274×10^{-24} J T^{-1}
R_∞	Rydberg energy $\frac{\alpha^2 m_e c^2}{2}$	13.61 eV
a_0	Bohr radius $\frac{1}{\alpha}\frac{\hbar}{m_e c}$	0.5292×10^{-10} m
Å	Angstrom	10^{-10} m
m_p	Proton mass	1.673×10^{-27} kg
$m_p c^2$	Proton rest-mass energy	938.272 MeV
$m_n c^2$	Neutron rest-mass energy	939.565 MeV
μ_N	Nuclear magneton, $\frac{e\hbar}{2m_p}$	5.051×10^{-27} J T^{-1}
fm	Femtometre or fermi	10^{-15} m
b	Barn	10^{-28} m^2
u	Atomic mass unit, $\frac{1}{12} m(^{12}\mathrm{C}$ atom$)$	1.661×10^{-27} kg
N_A	Avogadro constant, atoms in gram mol	6.022×10^{23} mol^{-1}
T_t	Triple-point temperature	273.16 K, exactly
k	Boltzmann constant	1.381×10^{-23} J K^{-1}
R	Molar gas constant, $N_A k$	8.314 J mol^{-1} K^{-1}
σ	Stefan-Boltzmann constant, $\frac{\pi^2}{60}\frac{k^4}{\hbar^3 c^2}$	5.670×10^{-8} W m^{-2} K^{-4}
M_E	Mass of Earth	5.97×10^{24} kg
R_E	Mean radius of Earth	6.4×10^6 m
g	Standard acceleration of gravity	$9.806\,65$ m s^{-2}, exactly
atm	Standard atmosphere	101 325 Pa, exactly
$M_\odot$	Solar mass	1.989×10^{30} kg
$R_\odot$	Solar radius	6.96×10^8 m
$L_\odot$	Solar luminosity	3.84×10^{26} W
$T_\odot$	Solar effective temperature	5.8×10^3 K
AU	Astronomical unit, mean Earth-Sun distance	1.496×10^{11} m
pc	Parsec	3.086×10^{16} m
	Year	3.156×10^7 s