

Mathematics 2

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Chapter 1

Partial Differentiation and Multiple Integration

The differential function of a function $f = f(x, y)$ is given by,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \quad (1.1)$$

1.1 Taylor Series

The Taylor series for a function $f(x_1, x_2, \dots, x_m)$ about a point $(x_1^0, x_2^0, \dots, x_m^0)$ is,

$$f(x_1, x_2, \dots, x_m) = f(x_1^0, x_2^0, \dots, x_m^0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left((x_1 - x_1^0) \frac{\partial f}{\partial x_1} + \dots + (x_m - x_m^0) \frac{\partial f}{\partial x_m} \right)^k f(x_1, x_2, \dots, x_m), \quad (1.2)$$

where the power on the bracket applies to the amount of times each differential is applied to f . Each partial derivative is evaluated about $(x_1^0, x_2^0, \dots, x_m^0)$.

1.2 Multiple Integration

We are able to integrate over multiple variables. This allows for finding quantities such as a volume or area.

$$V = \int f(x, y, z) d\tau = \int \int \int f(x, y, z) dx dy dz = \int \left(\int \left(\int f(x, y, z) dx \right) dy \right) dz. \quad (1.3)$$

1.2.1 Areas and volumes in different coordinate systems

2D Plane Polar Coordinates

$$x = r \cos \theta \quad y = r \sin \theta \quad dA = r dr d\theta \quad (1.4)$$

3D Cylindrical Polar Coordinates

$$\text{Curved Surface: } dA = r d\theta dz \quad \text{Top Surface: } dA = r d\theta dr \quad \text{Volume: } dV = r dr d\theta dz \quad (1.5)$$

3D Spherical Polar Coordinates

$$dA = r^2 \sin \theta d\theta d\phi \quad dV = r^2 \sin \theta dr d\theta d\phi \quad (1.6)$$

There is an important identity required when working with polar co-ordinates,

$$\sin \theta d\theta = -d(\cos \theta). \quad (1.7)$$

The limits for the volume of a sphere are,

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^R r^2 dr. \quad (1.8)$$

1.2.2 Moment of Inertia

The moment of inertia is given by,

$$I = \int r_{\perp}^2 dm, \quad (1.9)$$

which can be rewritten as,

$$I = \rho \int r_{\perp}^2 dV, \quad (1.10)$$

where ρ is the density of the volume. For non-trivial axis, we use,

$$r_{\perp} = |\hat{\mathbf{a}} \times \mathbf{p}|, \quad (1.11)$$

where $\hat{\mathbf{a}}$ is the vector in the direction of the axis, and \mathbf{p} is a generic vector such that,

$$\mathbf{p} = r_i \mathbf{e}_i. \quad (1.12)$$

1.2.3 The Jacobian

When converting between co-ordinate frames, we cannot always simply replace variables. We must use,

$$dx dy dz = |J(u, v, w)| du dv dw. \quad (1.13)$$

Where, we can define,

$$x = f(u, v, w) \quad y = g(u, v, w) \quad z = h(u, v, w) \quad (1.14)$$

and then, the Jacobian is for a 2D system, the determinant of,

$$\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix} \quad (1.15)$$

and for a 3D system,

$$\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix}. \quad (1.16)$$

Chapter 2

Fields

A field is an entity whose value depends on position. A field may be either **scalar** or **vector**. The direction of vector fields may also depend on position. **Scalar fields** are often represented by **contour lines**, which connect the x and y points. **Vector fields** are often visualised using field lines. We can calculate the equations for these field lines by considering a vector field, $V = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}}$, and solving for,

$$\frac{dy}{dx} = \frac{V_y}{V_x}. \quad (2.1)$$

2.1 Gradient ∇

We can define the infinitesimal change in a scalar field as,

$$d\psi = \nabla\psi \cdot d\mathbf{r}, \quad (2.2)$$

where we define the **gradient** operator as,

$$\nabla = \frac{\partial}{\partial r_i} \mathbf{e}_i. \quad (2.3)$$

There are some properties of grad which we must be aware of. For a scalar field, $f(x, y, z)$,

- ∇f is a vector field.
- ∇f represents the maximum rate of change of f .
- ∇f is perpendicular to the contours of constant f .
- The unit vector normal to a level surface is

$$\frac{\nabla f}{|\nabla f|}. \quad (2.4)$$

There are two main types of questions where the grad operator is used,

1. Finding the unit vector at (x_0, y_0, z_0) which is normal to a level surface, $f(x, y, z)$.

$$\text{Obtain } \nabla f(x, y, z) \rightarrow \mathbf{n} = \nabla f(x_0, y_0, z_0) \rightarrow \hat{\mathbf{n}} = \frac{\nabla f(x_0, y_0, z_0)}{|\nabla f(x_0, y_0, z_0)|} \quad (2.5)$$

2. Find the rate of increase at $f(x_0, y_0, z_0)$ in the direction between, (x_1, y_1, z_1) and (x_2, y_2, z_2) .

$$\text{Eval. } \nabla f(x_0, y_0, z_0) \rightarrow d\mathbf{s} = (\Delta x, \Delta y, \Delta z) \rightarrow \hat{\mathbf{u}} = \frac{d\mathbf{s}}{|d\mathbf{s}|} \rightarrow \frac{df}{ds} = \nabla f(x_0, y_0, z_0) \cdot \hat{\mathbf{u}}. \quad (2.6)$$

where ϕ and ψ are scalar fields and K is a position independent constant scalar.

2.1.1 Grad in polar coordinates

$$\text{Cylindrical:} \quad \nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}, \quad (2.7)$$

$$\text{Spherical:} \quad \nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}. \quad (2.8)$$

2.2 Lagrangian Multipliers

These are used when wanting to find the minimum or maximum of a field when a field f is constrained by some function g . If g is a constant constraint, then we can say that,

$$dg = 0 \quad (2.9)$$

and for a minimum or maximum of f , similarly,

$$df = 0. \quad (2.10)$$

We recall that the change in a scalar field along an elemental path is,

$$d\psi = \nabla\psi \cdot d\mathbf{s}. \quad (2.11)$$

Thus, ∇f and ∇g are both perpendicular to $d\mathbf{s}$. Thus, ∇f and ∇g are parallel or anti parallel to each other. We can then state the relation,

$$\nabla f = \lambda \nabla g \quad (2.12)$$

which brings about a system of equations,

$$\begin{aligned} \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} &= 0 \end{aligned} \quad (2.13)$$

We will want to include the function of g when finding the solutions to this equation.

1. Find x, y , and z in terms of λ .
2. Substitute these into the expression for g .
3. Substitute the value found for g back into the expressions for x, y , and z to find the coordinates of the maximum and minimum rate of change.

2.3 Div

$$\text{Div}(\mathbf{F}) \equiv \nabla \cdot \mathbf{F} \quad (2.14)$$

The divergence describes the flux through an area of a vector field. A field with 0 divergence is known as a *solenoidal field*. The total flux of a vector field through a volume composed of i surfaces is given by the **divergence theorem**,

$$\begin{aligned} \sum_i \int_{S_i} \mathbf{F} \cdot d\mathbf{S}_i &= \int_V (\nabla \cdot \mathbf{F}) dV \\ \nabla \cdot \mathbf{F} &= \frac{1}{|V|} \int_S \mathbf{F} \cdot d\mathbf{S} \end{aligned} \quad (2.15)$$

Whether the flux is +ive, -ive or 0 will give us different information.

When $|\mathbf{F}| \equiv \text{const}$, we have,

$$\nabla \cdot \mathbf{F} = 0. \quad (2.16)$$

For *increasing* $|\mathbf{F}|$, the total flux is given by,

$$\int (F_R - F_L) \, ds \quad (2.17)$$

$$F_R > F_L \therefore \nabla \cdot \mathbf{F} > 0 \implies \text{Net flow out of the volume.} \quad (2.18)$$

The field is known as a *source* of flux.

For *decreasing* $|\mathbf{F}|$,

$$F_L > F_R \therefore \nabla \cdot \mathbf{F} < 0 \implies \text{Net flow into the volume.} \quad (2.19)$$

The field is then known as a sink of flux.

For non-trivial fields, the values of Div still apply.

2.3.1 Div in polar coordinates

For cylindrical polar coordinates,

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial}{\partial \theta} A_\theta + \frac{\partial}{\partial z} (A_z). \quad (2.20)$$

For spherical polar coordinates,

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (A_\phi) \quad (2.21)$$

2.3.2 Gauss' Law

$$\begin{aligned} \int \mathbf{E} \cdot d\mathbf{S} &= \int_V (\nabla \cdot \mathbf{E}) \, dV = \frac{1}{\epsilon_0} \int dQ \\ \nabla \cdot \mathbf{E} &= \frac{1}{\epsilon_0} \frac{dQ}{dV} = \frac{\rho}{\epsilon_0} \end{aligned} \quad (2.22)$$

2.4 Curl

$$\text{Curl}(\mathbf{F}) \equiv \nabla \times \mathbf{F} \quad (2.23)$$

We can interpret the physical nature of curl if we imagine dropped a ball into a field. If $(\nabla \times \mathbf{F}) \neq 0$, then the ball will experience a torque. A field with 0 curl is known as an *irrotational field*.

Curl in polar coordinates

For cylindrical polar coordinates,

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} \quad (2.24)$$

For spherical polar coordinates,

$$\nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r \sin \theta \hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r \sin \theta A_\phi \end{vmatrix} \quad (2.25)$$

2.5 The Laplacian Operator

This is defined,

$$\nabla^2 = \nabla \cdot \nabla. \quad (2.26)$$

Such that,

$$\nabla^2 = \sum_i \frac{\partial^2}{\partial r_i^2}. \quad (2.27)$$

For a scalar field ψ , the Laplacian is,

$$\nabla^2 \psi = \sum_i \frac{\partial^2 \psi}{\partial r_i^2}. \quad (2.28)$$

For a vector field, \mathbf{F} , the Laplacian is given by,

$$\nabla^2 \mathbf{F} = \nabla^2 A_i \mathbf{e}_i. \quad (2.29)$$

2.5.1 Laplacian in Polar Coordinates

In cylindrical polar coordinates,

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}. \quad (2.30)$$

In spherical polar coordinates,

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial^2 \psi}{\partial \phi^2} \quad (2.31)$$

2.6 Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{Gauss' Law} \quad (2.32)$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{No magnetic monopoles} \quad (2.33)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday's Law} \quad (2.34)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{Ampere's Law} \quad (2.35)$$

The wave equation for electromagnetic waves can then be derived,

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}. \quad (2.36)$$

2.7 Helmholtz Decomposition

If you have a smooth, rapidly varying field that vanishes faster than $\frac{1}{r}$ as $r \rightarrow \infty$,

$$\mathbf{v} = \nabla \times \mathbf{A} + \nabla \phi. \quad (2.37)$$

We know that,

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad \nabla \times \nabla \phi = 0 \quad (2.38)$$

$$\text{"B-mode"} \quad \text{"E-Modes"} \quad (2.39)$$

2.8 Surface Integrals

When we integrate over a surface, we usually integrate over a vector, given as,

$$d\mathbf{S} = \hat{\mathbf{n}} dS \quad (2.40)$$

where $\hat{\mathbf{n}}$ is a unit vector which is perpendicular and away from the surface. If our surface is described by a function f , we can compute $\hat{\mathbf{n}}$ as,

$$\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|}. \quad (2.41)$$

The elemental area dS can be rewritten in Cartesian coordinates if we consider that the elemental surface is inclined at some angle, θ to the $x - y$ plane. We then say,

$$\begin{aligned} dx dy &= dS \cos \theta = dS (\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}) \\ &= d\mathbf{S} \cdot \hat{\mathbf{k}}. \end{aligned} \quad (2.42)$$

2.8.1 Solid Angles

We often wish to integrate over a solid angle. The infinitesimal element of a solid angle is given by,

$$d\Omega = \sin \theta \, d\theta \, d\phi. \quad (2.43)$$

Surface integrals over solid angles have the infinitesimal area element,

$$d\Omega = \frac{\hat{\mathbf{r}} \cdot d\mathbf{S}}{r^2}. \quad (2.44)$$

2.8.2 Flux

The flux through of a vector field \mathbf{A} through a surface S is given by,

$$\text{FLUX} = \int_S \mathbf{A} \cdot d\mathbf{S}. \quad (2.45)$$

Often, it is a lot easier to do this using the Divergence theorem covered in the next section.

2.9 Divergence Theorem

Theorem. *The total flux of a vector \mathbf{A} through a closed surface S is related to the divergence of a vector field inside a volume V by,*

$$\int_S \mathbf{A} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{A} \, dV. \quad (2.46)$$

This theorem can also be applied to scalar fields,

$$\int_S \psi \, d\mathbf{S} = \int_V \nabla \psi \, dV. \quad (2.47)$$

2.9.1 Applications of Divergence Theorem

Maxwell's First Equation

Gauss' law can be given as follows,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_V \rho \, dV. \quad (2.48)$$

Re-writing this in terms of the divergence theorem,

$$\begin{aligned} \int_V \nabla \cdot \mathbf{E} \, dV &= \frac{1}{\epsilon_0} \int_V \rho \, dV \\ \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, \end{aligned} \quad (2.49)$$

which is Maxwell's first equation of electromagnetism.

The Continuity Equation

This is a conservation equation, given by,

$$\frac{d\rho}{dt} + \nabla \cdot \mathbf{J} = 0 \quad (2.50)$$

where ρ is an amount of some quantity, q , per unit volume, and \mathbf{J} is the flux per unit time of q .

2.10 Line Integrals

For a curve C defined by a function f and a small length element dl , the scalar line integral is defined by,

$$\int_C f \, dl. \quad (2.51)$$

For 2D line integrals, we can define,

$$(dl)^2 = (dx)^2 + (dy)^2 \quad (2.52)$$

and,

$$l = \int_C dl = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad (2.53)$$

For a curve defined by parameters, i.e., $x = g(t)$, $y = h(t)$, we write,

$$l = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \quad (2.54)$$

In polar co-ordinates,

$$dl = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta. \quad (2.55)$$

2.10.1 Vector line integrals

For a vector \mathbf{A} , we define the small length element vector as,

$$d\mathbf{l} = dx \hat{\mathbf{i}} + dy \hat{\mathbf{j}} \quad \text{Cartesian} \quad (2.56)$$

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} \quad \text{Polar.} \quad (2.57)$$

The line integral is then given by,

$$l = \int_C \mathbf{A} \cdot d\mathbf{l}. \quad (2.58)$$

2.10.2 Circulation of Vector Fields

If we wish to find the circulation of a vector \mathbf{A} along a certain path, c , we should compute $\mathbf{A} \cdot d\mathbf{l}$. If we define $c = c(x, y, z)$, then we should substitute appropriately so that we are able to integrate over $\mathbf{A} \cdot d\mathbf{l}$.

To find the circulation over polar coordinates, we want to first parametrise the curve, such that,

$$x = g(\theta) \quad y = f(\theta) \quad (2.59)$$

and obtain a vector, \mathbf{c} ,

$$\mathbf{c} = (g(\theta), f(\theta)). \quad (2.60)$$

We can then rewrite \mathbf{A} in terms of the components of \mathbf{c} and use,

$$\mathbf{A} \cdot d\mathbf{l} = \mathbf{A}(\mathbf{c}) \cdot \frac{d\mathbf{c}}{d\theta} d\theta \quad (2.61)$$

and perform the integral. For clockwise circulation this is $0 \rightarrow 2\pi$ and for anti-clockwise this is $2\pi \rightarrow 0$.

2.10.3 Stoke's Theorem

The circulation of a vector field \mathbf{B} is related to the surface integral of the curl by,

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} \quad (2.62)$$

We can use the right hand rule to determine the direction of $d\mathbf{S}$.

2.10.4 Conditions of Stoke's Theore

1. Curve is continuous,
2. The surface is simply connected,
3. There are no non-integrable singularities.

2.11 Applications in electricity and magnetism

2.12 Applications in mechanics

Chapter 3

Fourier series

Appendix A

Vector Calculus Identities

A.1 Grad Identities

$$\nabla(\psi + \phi) = \nabla\psi + \nabla\phi \quad (\text{A.1})$$

$$\nabla(K\psi) = K\nabla\psi \quad (\text{A.2})$$

$$\nabla(\psi\phi) = \psi\nabla\phi + \phi\nabla\psi \quad (\text{A.3})$$

A.2 Curl Identities

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \quad (\text{A.4})$$

$$\nabla \times (\phi\mathbf{A}) = \phi(\nabla \times \mathbf{A}) + (\nabla\phi) \times \mathbf{A} \quad (\text{A.5})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \quad (\text{A.6})$$

A.3 Div Identities

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \quad (\text{A.7})$$

$$\nabla \cdot (\phi\mathbf{A}) = (\nabla\phi) \cdot \mathbf{A} + \phi(\nabla \cdot \mathbf{A}) \quad (\text{A.8})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \quad (\text{A.9})$$

A.4 Combination Identities

$$\nabla \times \nabla\phi = 0 \quad (\text{A.10})$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (\text{A.11})$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}. \quad (\text{A.12})$$

Appendix B

Vector Calculus Proofs

B.1 $\nabla \cdot (g\mathbf{F}) = (\nabla g) \cdot \mathbf{F} + g(\nabla \cdot \mathbf{F})$

Proof.

$$\begin{aligned}\nabla \cdot (g\mathbf{F}) &= (gF_1)_x + (gF_2)_y + (gF_3)_z \\ &= g_x F_1 + g_y F_2 + g_z F_3 + g[(F_1)_x + (F_2)_y + (F_3)_z] \\ &= \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} + g \begin{pmatrix} (F_1)_x \\ (F_2)_y \\ (F_3)_z \end{pmatrix} \\ &= (\nabla g) \cdot \mathbf{F} + g(\nabla \cdot \mathbf{F})\end{aligned}$$

□

B.2 Ampere's Law