ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Statistical Mechanics

7th June 2023, 2.00 p.m. - 3.30 p.m.

Answer **ALL** parts of question 1 and **TWO** other questions

---

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

Information which may be used in this paper:

The density of states for a spinless particle in two dimensions is $g(k) = \dfrac{Ak}{2\pi}$.

The symbol β is defined as $\beta = 1/(k_B T)$.

You may use $k_B = 8.6173 \times 10^{-5}$ eV K^{-1} and $\hbar c = 1973$ eV Å.

The symbol n_Q is defined as $n_Q = \left(\dfrac{mk_B T}{2\pi\hbar^2}\right)^{3/2}$.

The translational partition function for a single non-relativistic particle in a box is $Z_1 = V g_s n_Q$, where g_s is the spin degeneracy.

In the classical limit, the chemical potential for an ideal gas is given by $\mu = -k_B T \ln\left(\dfrac{g_s n_Q}{n}\right)$.

The Planck formula for the energy density of black-body radiation is

$$u(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega\beta} - 1}.$$

Stirling's approximation is $\ln N! \approx N \ln N - N$ for $N \gg 1$.

$\displaystyle\sum_{n=0}^{\infty} (n+1)\, x^n = (1-x)^{-2} =$ for $|x| < 1$.

The following integrals may be useful (note $n!! \equiv n(n-2)(n-4)\ldots 1$ for odd n.)

$$\int_0^\infty \frac{x^{1/2}}{e^x + 1} dx = 0.678094 \qquad \int_0^\infty \frac{x^{1/2}}{e^x - 1} dx = 2.31516$$

$$\int_0^\infty \frac{x}{e^x + 1} dx = \frac{\pi^2}{12} \qquad \int_0^\infty \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}$$

$$\int_0^\infty \frac{x^{3/2}}{e^x + 1} dx = 1.15280 \qquad \int_0^\infty \frac{x^{3/2}}{e^x - 1} dx = 1.78329$$

$$\int_0^\infty \frac{x^2}{e^x + 1} dx = 1.80309 \qquad \int_0^\infty \frac{x^2}{e^x - 1} dx = 2.40411$$

$$\int_0^\infty \frac{x^{5/2}}{e^x + 1} dx = 3.08259 \qquad \int_0^\infty \frac{x^{5/2}}{e^x - 1} dx = 3.74453$$

$$\int_0^\infty \frac{x^3}{e^x + 1} dx = \frac{7\pi^4}{120} \qquad \int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$$

$$\int_0^\infty x^n e^{-x/a} dx = n!\, a^{n+1} \qquad \int_0^\infty x^n e^{-x^2/a^2} dx = \frac{(n-1)!!\sqrt{\pi} a^{n+1}}{2^{(n/2+1)}} \quad \text{for even } n.$$

1. a) Distinguish between the microstates and macrostates of a many-particle system, giving examples.

[5 marks]

b) One microjoule of energy is added to a kilogram of water at temperature 27°C. By what factor does the number of available microstates increase? (Express your answer as 10^x.)

[5 marks]

c) The energy E of a photon gas at temperature T is proportional to VT^4, and its entropy is given by $S = \frac{4}{3}E/T$. Derive an expression for the pressure P in terms of E and the volume V.

[4 marks]

d) Two otherwise identical black bodies have different temperatures. The temperature of the hotter body is such that $k_B T = 0.4$ eV. At one particular frequency, such that $\hbar\omega = 0.2$ eV, the intensity of the radiation from the hotter body is twice that from the cooler body. What is the temperature of the cooler body?

[5 marks]

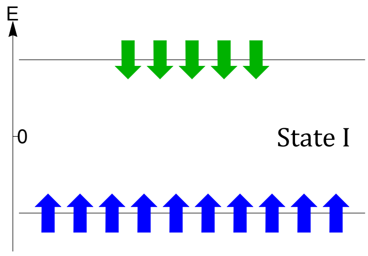
e) Electrons are confined with a density of 3×10^{28} m^{-3}. Calculate the Fermi temperature. Estimate the chemical potential at $T = 1000$ K and at $T = 10^6$ K.

[6 marks]

2. i) Write down the Boltzmann distribution, clearly explaining the meaning of the symbols you use and stating in what circumstances it holds.

[5 marks]

ii) Consider a system of N distinguishable non-interacting spin-$\frac{1}{2}$ particles, each of magnetic moment μ. The following diagram represents the energy levels of each particle in an external field B_1, and the arrows indicate the occupancy of the two levels at some temperature T_1. In the diagram, one third of particles are in the higher-energy state. (Only a fraction of the particles are shown.)



Give an expression for T_1 in terms of μ and B_1.

Denoting the state shown by (I), draw two more diagrams indicating energy levels and occupancies for the following pairs of temperature and magnetic field:

(II) $B = 2B_1$, $T = T_1$

(III) $B = B_1$, $T = \frac{1}{2}T_1$.

For the transitions from (I) to (II) and from (II) to (III), is work done on or by the system? Is heat transferred to or from the system?

[8 marks]

iii) Briefly describe how such a system may be used to reach low temperatures.

[4 marks]

iv) For the system above, find the partition function and hence the magnetisation and entropy of the system as a function of temperature. Sketch the magnetisation as a function of temperature for two values of the magnetic field, B_1 and $2B_1$. Hence comment on whether the process described above can be used to reach zero temperature.

[8 marks]

3. i) By considering the wave function in two dimensions of a spinless particle free to move within a square of side L, derive the density of states $g(k)$.

[7 marks]

ii) Consider a two-dimensional relativistic spin-zero gas of indistinguishable particles, whose energy and momentum are related by $E = \sqrt{p^2c^2 + m^2c^4}$. The particles are free to move within an area A, and the temperature and density are such that the gas is in the classical regime.

Show that the single-particle partition function of the system is given by

$$Z_1 = \frac{A}{2\pi}\left(\frac{\mathrm{e}^{-mc^2\beta}(\beta mc^2 + 1)}{\beta^2\hbar^2c^2}\right).$$

What is the partition function of the N-particle system?

[8 marks]

[You may use the following result: $\displaystyle\int_{x_0}^{\infty} x\mathrm{e}^{-x}\mathrm{d}x = (x_0 + 1)\,\mathrm{e}^{-x_0}$.]

iii) Show that the average energy of the system is given by $N\langle E_1\rangle$, where

$$\langle E_1\rangle = mc^2 + 2k_BT - \frac{mc^2\,k_BT}{mc^2 + k_BT}.$$

Find the limits both for $mc^2 \gg k_BT$ and for $mc^2 \ll k_BT$, and comment on your results in the context of equipartition.

[10 marks]

4. i) Explain the conditions for a system to be described by the grand-canonical distribution. State the associated thermodynamic potential.

Define the grand partition function, defining any symbols you use, and state how it is related to the relevant thermodynamic potential.

[7 marks]

ii) A system has a set of single-particle states of energy $\{\varepsilon_i\}$, and is in contact with a reservoir of spin-$\frac{1}{2}$ particles at chemical potential μ and temperature T. Find an expression for the average number of particles in a state of energy ε_i.

[3 marks]

iii) At very high temperatures, electrons and positrons can be regarded as a dense gas of ultra-relativistic particles. The number of particles is not conserved. Explain why that means that the chemical potential is equal to zero.

Show that the number of electrons is equal to

$$N_{\mathrm{e}} = CV \left(\frac{k_B T}{\hbar c} \right)^3 ,$$

and determine the constant C.

[8 marks]

iv) Find an expression for the total energy of the electrons, and comment on its temperature dependence. Qualitatively, how would the result change for spinless particles?

[7 marks]

END OF EXAMINATION PAPER