Mathematics of Waves and Fields

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Contents

Chapter 1

Partial Differential Equations

1.1 Separation of Variables

Suppose we have a PDE whose solution is in the form, $u(r_1, r_2, \ldots, r_n)$ where there are n co-ordinates r_i, then we can solve the PDE by separation of variables by assuming a solution of the form,

$$u(r_1, r_2, \ldots, r_n) = R_1(r_1)R_2(r_2)\cdots R_n(r_n). \tag{1.1}$$

This will turn a compatible PDE into an ODE.

1.1.1 Specific solutions

We are often most interested in the specific solutions to a wave equation. In order to get a specific solution, constraints/boundary conditions must be provided. The general method is as follows,

1. Use separation of variables;
2. Build superpositions of solutions;
3. Apply boundary conditions and find appropriate constants.

1.2 Series Solutions

The general steps to solving an ODE using this method are,

1. Assume a series solution of the form,
$$y = \sum_{n=0}^{\infty} a_k x^k \tag{1.2}$$
2. Obtain the recurrence relation.

Chapter 2

Fourier Series

Given a periodic function $f(x)$ with period $2L$ in the range $-L \leq x \leq L$, the Fourier expansion is given by,

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)} \tag{2.1}$$

for,

$$\boxed{a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, \mathrm{d}x} \qquad \boxed{b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, \mathrm{d}x} \tag{2.2}$$

In order to expand a function it must meet *Dirichlet's Conditions*, so the function must,

1. be single valued,
2. have a finite number of discontinuities,
3. $\int_{-L}^{L} |f(x)| \, \mathrm{d}x$ must be finite.

We say that $\sin \frac{n\pi x}{L}$ and $\cos \frac{n\pi x}{L}$ form a *complete, orthogonal basis.* And furthermore, the Fourier series allows *an expansion of a function on a set of orthogonal basis functions.*

2.1 Exponential Fourier Series

We can further write the Fourier expansion in terms of complex exponentials,

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(i \frac{n\pi x}{L} \right)} \tag{2.3}$$

for,

$$\boxed{c_n = \frac{1}{2L} \int_{-L}^{L} f(x) \exp\left(-i \frac{n\pi x}{L} \right) \mathrm{d}x} \tag{2.4}$$

From our complex definition of the Fourier series, we can say that 2 complex functions $u(z)$ and $v(z)$ are orthogonal on the interval $a \leq z \leq b$ if,

$$\int_a^b u(z)v(z) = 0. \tag{2.5}$$

Chapter 3

Fourier Transform

If we wish to analyse non-periodic functions, we can take the limit of our range, $\lim_{L\to\infty}(-L, L)$. Let us write,

$$k_n = \frac{n\pi}{L} \qquad\qquad \Delta k = \frac{\pi}{L} \tag{3.1}$$

so,

$$\tag{3.2}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\exp\left(-ik_n x\right)\mathrm{d}x\,. \tag{3.3}$$

Let us write $F(k) = 2Lac_n$,

$$F(k_n) = a\int_{-L}^{L} f(x)\exp\left(-ik_n x\right)\mathrm{d}x \tag{3.4}$$

$$\implies f(x) = \frac{1}{2\pi a}\sum_{n=-\infty}^{\infty} F(k_n)\exp\left(ik_n x\right)\Delta k. \tag{3.5}$$

In the limit of $L \to \infty$, we obtain,

$$\boxed{F(k) = a\int_{-\infty}^{\infty} f(x)\exp(-ikx)\,\mathrm{d}x} \qquad \text{Fourier transform of } f(x). \tag{3.6}$$

$$\boxed{f(x) = \frac{1}{2\pi a}\int_{-\infty}^{\infty} F(k)\exp(ikx)\,\mathrm{d}k} \qquad \text{Inverse Fourier transform of } F(k). \tag{3.7}$$

We define constant,

$$a = \begin{cases} \text{unity} & \text{Physics} \\ \dfrac{1}{\sqrt{2\pi}} & \text{Maths} \end{cases}. \tag{3.8}$$

3.1 Manipulation of the Fourier transform

Most generally,

$$\boxed{\mathscr{F}\left\{f(ax-b)\right\} = e^{-ikb}\frac{1}{a}F\left(\frac{k}{a}\right)}. \tag{3.9}$$

3.2 Dirac-Delta Function

Dirac's original approximation of the δ function used a function $\Pi(x)$ which was defined,

$$\Pi(x) = \begin{cases} 1 & -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0 & \text{Elsewhere} \end{cases}. \tag{3.10}$$

Using this definition we can write,

$$\delta(x) = \lim_{k\to\infty} \left\{ k\Pi(kx) \right\}. \tag{3.11}$$

It can also be defined using sinc,

$$\delta(x) = \lim_{k\to\infty} \left\{ \frac{k}{\pi} \frac{\sin kx}{kx} \right\}. \tag{3.12}$$

However, the most commonly used, and most applicable form is,

$$\boxed{\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \, \mathrm{d}x} \tag{3.13}$$

We note 4 important properties of the Dirac-Delta,

1. $\lim_{x\to 0} \delta(x) = \infty$
2. $\int_{-\infty}^{\infty} \delta(x) \, \mathrm{d}x = 1$
3. $\delta(ax+b) = \frac{1}{|a|} \delta\left(x + \frac{a}{b}\right)$
4. $\delta(x) = \delta(-x)$ i.e., Dirac-Delta is even.

3.2.1 Fourier Transform of the Dirac Delta

The Fourier transform of a constant is the Dirac-Delta, i.e.,

$$\mathscr{F}\left\{1\right\} = \int_{-\infty}^{\infty} e^{ikx} \, \mathrm{d}x = 2\pi\delta(x), \tag{3.14}$$

and similarly for the inverse Fourier Transform,

$$\mathscr{F}^{-1}\left\{1\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, \mathrm{d}x = \delta(x). \tag{3.15}$$

Taking the Fourier transform of the Dirac-Delta,

$$\mathscr{F}\left\{\delta(x)\right\} = \int_{-\infty}^{\infty} \delta(x) e^{-ikx} \, \mathrm{d}x = 1 \tag{3.16}$$

and thus clearly,

$$\mathscr{F}\left\{\delta(x - x_0)\right\} = \int_{-\infty}^{\infty} \delta(x - x_0) e^{-ikx} \, \mathrm{d}x = e^{-ikx_0}. \tag{3.17}$$

The derivative of the Dirac-Delta is found by,

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ike^{-ikx} \, \mathrm{d}x = \mathscr{F}\left\{ik\right\} \tag{3.18}$$

and most generally,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\delta(x) = \mathscr{F}\left\{(ik)^n\right\}. \tag{3.19}$$

3.3 Parseval's Theorem

Theorem.

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x = a \int_{-\infty}^{\infty} |F(k)|^2 \,\mathrm{d}k} \tag{3.20}$$

where $a = 1$ for mathematical symmetry, and $a = \frac{1}{2\pi}$ for physical symmetry.

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k) e^{ikx} \,\mathrm{d}k \int_{-\infty}^{\infty} F^*(k) e^{ik'x} \,\mathrm{d}k' \,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} F(k) \,\mathrm{d}k \int_{-\infty}^{\infty} F^*(k') \,\mathrm{d}k' \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} \,\mathrm{d}x}_{\delta(k-k')} \\ &= \int_{-\infty}^{\infty} |F(k)|^2 \,\mathrm{d}k\end{aligned} \tag{3.21}$$

□

3.4 Convolution

We define the convolution $h(x)$ of two functions $f(x)$ and $g(x)$ as,

$$\begin{aligned}h(x) = f(x) * g(x) &= \int_{-\infty}^{\infty} f(x - x') g(x') \,\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} f(x') g(x - x') \,\mathrm{d}x' \,.\end{aligned} \tag{3.22}$$

If we define the Fourier transforms of $f(x)$ and $g(x)$,

$$F(k) = \mathscr{F}\{f(x)\} \qquad\qquad G(k) = \mathscr{F}\{g(x)\} \tag{3.23}$$

then the Fourier transform of the convolution is given by,

$$\mathscr{F}\{h(x)\} = \mathscr{F}\{f(x) * g(x)\} = F(k)G(k). \tag{3.24}$$

Proof. Let us define $\zeta = x - x'$, then,

$$\begin{aligned}H(k) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x \,\mathrm{d}x' \, f(\zeta) g(x') e^{-kx} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x' \,\mathrm{d}\zeta \, f(\zeta) g(x') e^{-ik(\zeta + x')} \\ &= \int_{-\infty}^{\infty} g(x') e^{ikx'} \,\mathrm{d}x' \int_{-\infty}^{\infty} f(\zeta) e^{ik\zeta} \,\mathrm{d}\zeta\end{aligned} \tag{3.25}$$

which clearly corresponds to the product of the two transforms. □

3.5 Wave Packets

In 1 dimension, a forward travelling wave is defined by,

$$\phi(x, t) = e^{-i(kx - \omega t)} \tag{3.26}$$

which satisfies the 1 dimensional wave equation,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}. \tag{3.27}$$

By substituting in the travelling wave, we find,

$$k^2 = \frac{1}{c^2}\omega^2 \implies \omega = ck. \tag{3.28}$$

A plane wave $\phi(\mathbf{x}, t) = e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ satisfies the 3 dimensional wave equation,

$$\boldsymbol{\nabla}^2\phi = \frac{1}{c^2}\frac{\partial\phi}{\partial t}. \tag{3.29}$$

Where we have,

$$\omega = c|k| \tag{3.30}$$

for a plane travelling along k.

Returning to the 1 dimensional wave, we can sum these travelling waves along the $+x$ direction,

$$\phi(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(k)e^{ik(x-ct)}\,\mathrm{d}k \tag{3.31}$$

where $G(k)$ is the Fourier transform os $\phi(x, 0)$. This wave satisfies the wave equation as all components of the wave travel at the same velocity c. We are also able to use the wave equation to describe waves in *non-dispersive media*, i.e., those where the velocity of the waves depends on wavelength,

$$v_p(k) = \frac{\omega(k)}{k}. \tag{3.32}$$

The most general wave can be written as,

$$\boxed{\phi(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(k)e^{i(kx-\omega(t)t)}\,\mathrm{d}k}. \tag{3.33}$$

3.5.1 Dispersion

A dispersive wave packet will have the following properties,

- The envelope wave of the wave packet will move with group velocity,

$$\boxed{v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = v_p + k\frac{\mathrm{d}v_p}{\mathrm{d}k}}. \tag{3.34}$$

- The dispersive effects of the wave are a second order effect. i.e., we must expand any approximations to the second order. We will always assume $\omega \equiv \omega(k)$.

Chapter 4

Special Functions

4.1 Taylor Expansion

The Taylor expansion about x_0 is,

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0)f'(x) + \frac{(x - x_0)^2}{2!} + \cdots \\ &= \sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x - x_0)^n}{n!} \end{aligned} \tag{4.1}$$

Let us then redefine this as a simple series,

$$f(x) = \sum_{n=0}^{\infty} u_n \tag{4.2}$$

we can then define the convergence criteria,

$$\lim_{n\to\infty} |r_n| = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1. \tag{4.3}$$

4.2 Hermit's Equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2ny = 0, \forall y, n \in \mathbb{Z} \tag{4.4}$$

We can obtain solutions to Hermit's equation by assuming a series solution,

$$y = \sum_{k=0}^{\infty} x^k. \tag{4.5}$$

Substituting this into Hermit's equation,

$$\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - 2x\sum_{k=0}^{\infty} ka_k x^{k-1} + 2n\sum_{k=0}^{\infty} a_k x^k = 0 \tag{4.6}$$

Let us shift k such that, $k \to k + 2$,

$$\begin{aligned} \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+1}x^k - 2\sum_{k=0}^{\infty} ka_k x^k + 2n\sum_{k=0}^{\infty} a_k x^k = 0 \\ \sum_{k=0}^{\infty} \left\{(k+2)(k+1)a_{k+2} - (2k-2n)a_k\right\} x^k = 0 \end{aligned} \tag{4.7}$$

from which we obtain the recurrence relation,

$$a_{k+2} = \frac{2(k-n)}{(k+2)(k+1)}a_n. \tag{4.8}$$

Using this recurrence relation, we are able to form a solution for y, starting at $k = 0$ and $k = 1$ to obtain the even and odd solutions respectively,

$$\begin{aligned} y =& a_0 \left[1 - \frac{2n}{2!}x^2 - \frac{2n(4-2n)}{4!}x^4 + \cdots \right] + \\ & a_1 \left[x + \frac{(2-2n)}{3!}x^3 + \frac{(2-2n)(6-2n)}{5!}x^5 + \cdots \right]. \end{aligned} \tag{4.9}$$

Let us note that at $k = n$ the series will terminate.

Given the solution to Hermit's equation, by considering different values of n, we are able to obtain *Hermit's Polynomials* which are discussed in the section below.

4.2.1 Hermit's Polynomials

We denote Hermit's polynomials by $y \equiv H_n(x)$. By simply looking at eq. (4.9), we see that the first three even Hermit polynomials are,

$$H_0(x) = 1 \qquad H_2(x) = 1 - 2x^2 \qquad H_4(x) = 1 - 4x^2 + \frac{4}{3}x^4, \tag{4.10}$$

and the first 3 odd ones are,

$$H_1(x) = x \qquad H_3(x) = x - \frac{2}{3}x^3 \qquad H_5(x) = x - \frac{4}{3}x^3 + \frac{4}{5}x^5 \tag{4.11}$$

In physics, we often normalise the Hermit polynomials such that the highest order term is positive and has a coefficient 2^n.

Orthogonality of Hermit Polynomials

Hermit Polynomials satisfy the orthogonality relation,

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{x^2}\,\mathrm{d}x = \sqrt{\pi}n!2^m\delta_{nm}. \tag{4.12}$$

This means that Hermit's polynomials can be used as a basis for series expansion of a function. We can further define a normalised Hermit function,

$$\psi_m(x) = \left(\frac{1}{\sqrt{\pi}m!2^m}\right)^{1/2} H_m e^{\frac{x^2}{2}} \tag{4.13}$$

which satisfies,

$$\int_{-\infty}^{\infty} \psi_m(x)\psi_n(x)\,\mathrm{d}x = \delta_{mn} \tag{4.14}$$

4.3 Legendre's Equation

$$\boxed{(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \ell(\ell+1)y = 0 \quad l \geq 0, l \in \mathbb{Z}} \tag{4.15}$$

We solve Legendre's equation by series expansion, from which we obtain,

$$(1-x^2)\sum_{n=2} n(n-1)a_n x^{n-2} - 2x\sum_{n=1} na_n x^{n-1}\ell(\ell+1)\sum_{n=0} a_n x^n. \tag{4.16}$$

Let us shift the sums so we only have terms in powers of n,

$$\sum_{n=0} \left\{(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n x^n - 2na_n x^n + \ell(\ell+1)a_n\right\} x^n = 0. \tag{4.17}$$

From which we can easily obtain a general recurrence relation,

$$\boxed{a_{n+2} = \frac{n(n+1) - \ell(\ell+1)}{(n+2)(n+1)} a_n} \tag{4.18}$$

thus the solution for Legendre's equation has even and odd parts given below,

$$\begin{aligned} y =& a_0 \left[(1 - \ell(\ell+1))\frac{x^2}{2!} + (\ell-2)(\ell(\ell+1)(\ell+3))\frac{x^4}{4!} + \cdots \right] && \text{Even} \\ &+ a_1 \left[x - (\ell-1)(\ell+2)\frac{x^3}{3!} + (\ell-3)(\ell-1)(\ell+2)\frac{x^5}{5!} + \cdots \right] && \text{Odd} \end{aligned} \tag{4.19}$$

which we can clearly see terminates at $\ell = n$, which allows the series to converge.

4.3.1 Legendre Polynomials

The steps to finding Legendre polynomials $P_n(x)$ are as follows,

- Decide whether the polynomial is odd or even, and choose which part of y you will use.
- Find the coefficients of the polynomial $y(x)$ in terms of a_0 for even polynomials and a_1 for odd polynomials.
- Set $y(0) = 1$ to find a value for a_1 or a_0.
- Evaluate the final polynomial.

Orthogonality of Legendre Polynomials

Legendre polynomials are orthogonal over the interval $|x| \leq 1$, i.e.,

$$\int_{-1}^{1} P_l(x)P_m(x)\,\mathrm{d}x = 0 \quad m \neq l. \tag{4.20}$$

Let us recall eq. (4.15), and rewrite,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\partial P_l}{\partial x}\right] = -l(l+1)P_l(x) \tag{4.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\partial P_m}{\partial x}\right] = -m(m+1)P_m(x). \tag{4.22}$$

Multiply eq. (4.21) by P_m, and eq. (4.22) by P_l, and take them away from each other. We have,

$$\begin{aligned} \text{LHS} =& \int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\partial P_l}{\partial x}\right] P_m\,\mathrm{d}x \\ &- \int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\partial P_m}{\partial x}\right] P_l\,\mathrm{d}x \end{aligned} \tag{4.23}$$

Evaluating eq. (4.23) by parts, we have,

$$u = P_m \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\partial P_l}{\partial x}\right] \tag{4.24}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial P_m}{\partial x} \qquad v = (1-x^2)\frac{\partial P_l}{\partial x} \tag{4.25}$$

and similarly for the latter half of the equation. we can then write,

$$\begin{aligned} \text{LHS} =& \underbrace{\left[(1-x^2)\frac{\mathrm{d}P_l}{\mathrm{d}x}P_m\right]_{-1}^{1}}_{0} - \int_{-1}^{1}(1-x^2)\frac{\mathrm{d}P_m}{\mathrm{d}x}\frac{\mathrm{d}P_l}{\mathrm{d}x}\,\mathrm{d}x \\ &- \underbrace{\left[(1-x^2)\frac{\mathrm{d}P_m}{\mathrm{d}x}P_l\right]_{-1}^{1}}_{0} - \int_{-1}^{1}(1-x^2)\frac{\mathrm{d}P_l}{\mathrm{d}x}\frac{\mathrm{d}P_m}{\mathrm{d}x}\,\mathrm{d}x \\ =& 0 \end{aligned} \tag{4.26}$$

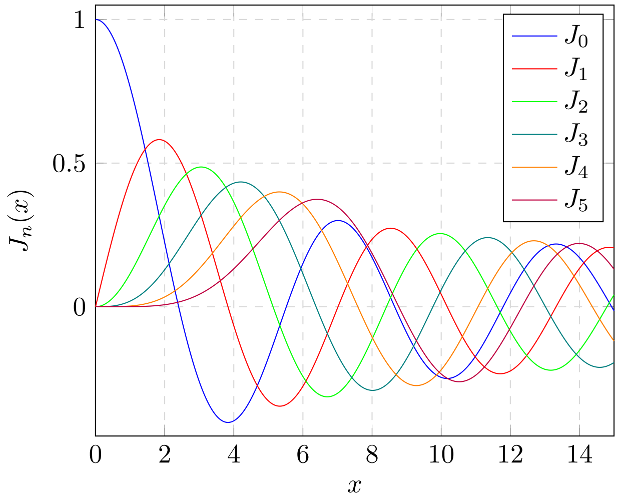

Figure 4.1: The first six Bessel functions.

We then have that, for $n \neq m$, the LHS is,

$$[m(m+1) - l(l+1)]\, P_l(x)P_m(x) = 0 \tag{4.27}$$

Furthermore, we can show,

$$\boxed{\int_{-1}^{1} P_l(x)P_m(x)\,\mathrm{d}x = \frac{2}{2l+1}\delta_{lm}}. \tag{4.28}$$

4.3.2 Legendre Polynomial Expansion

We can use the Legendre polynomials to perform a Legendre series expansion,

$$f(x) = \sum_{l=0}^{\infty} c_l P_l(x), \quad -1 \leq x \leq 1 \tag{4.29}$$

where the coefficients are given by,

$$c_l = \frac{1}{2}(2l+1) \int_{-1}^{1} \int_{-1}^{1} f(x)P_l(x)\,\mathrm{d}x\,. \tag{4.30}$$

4.4 Bessel Functions

The most general Bessel equation is given by,

$$\boxed{x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - m^2)y = 0} \tag{4.31}$$

We can find a general solution by,

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n \tag{4.32}$$

where we impose a boundary condition that $y(0)$ must be finite. Substituting eq (4.32) into eq. (4.31),

$$x\frac{\mathrm{d}x}{\mathrm{d}y} = \sum_{n=0}^{\infty} a_n(n+s)x^{n+s} \tag{4.33}$$

$$x^2\frac{\mathrm{d}^2x}{\mathrm{d}y^2} = \sum_{n=0}^{\infty} a_n(n+s)(n+s-1)x^{n+s} \tag{4.34}$$

$$\tag{4.35}$$

$$\begin{aligned}(x^2-m^2)y &= \sum_{n=0}^{\infty} a_n x^{n+s+2} - m^2 \sum a_n x^{n+s} \\ &= \sum_{n=2}^{\infty} a_{n-2} x^{n+s} - m^2 \sum a_n x^{n+s}\end{aligned} \tag{4.36}$$

Putting all these terms together, we can get rid of the x^{n+s} as we require the equation to be true $\forall x$.

$$\sum_{n=0}^{\infty} a_n \left[(n+s)(n+s-1)+(n+s)-m^2\right] + \sum_{n=2}^{\infty} a_{n-2} = 0. \tag{4.37}$$

We must first consider $n=0$ and $n=1$ before we can find the recurrence relation. For $n=0$,

$$\begin{aligned} a_0 s(s-1) + a_0 s - m^2 a_0 &= 0 \\ a_0 s^2 - m^2 a_0 & \\ \implies s = \pm m. & \end{aligned} \tag{4.38}$$

For $n=1$,

$$\begin{aligned} a_1 s(s+1) + a_1(s+1) - m^2 a_1 &= 0 \\ a_1(s+1)(s+1) - m^2 a_1 &= 0 \\ \implies a_1\left[(s+1)^2 - m^2\right] & \end{aligned} \tag{4.39}$$

for which we require $a_1 = 0$ unless $s = \pm m = -\frac{1}{2}$. Otherwise, there will be no odd terms in our solution.

We may now move onto the general recurrence relation,

$$\begin{aligned} \left[(n+s)(n+s-1)+(n+s)-m^2\right] a_n - a_{n-2} &= 0 \\ \left[(n+s)^2 - m^2\right] a_n + a_{n-2} &= 0 \end{aligned} \tag{4.40}$$

we then have the recurrence relation,

$$a_n = -\frac{a_{n-2}}{(n+m)^2 - m^2} = -\frac{a_{n-2}}{(2m+n)n} \tag{4.41}$$

which can be further generalised to,

$$a_{2j} = (-1)^j \frac{m!}{2^{2j} j!(m+j)!} a_0 \tag{4.42}$$

where $j \in \mathbb{Z}^+$. We can then write the general solution to Bessel's equation,

$$y(x) = \sum_{n=0}^{\infty} a_n x^{m+n} = \sum_{j=0}^{\infty} a_{2j} x^{m+2j}. \tag{4.43}$$

We often rewrite $y(x)$ as,

$$y(x) = a_0 m! 2^m \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j} 2^m j!(m+j)!} x^{m+2j} \tag{4.44}$$

where we obtain *Bessel's function of the first kind*,

$$\boxed{J_m(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{m+2j}} \tag{4.45}$$

which obeys the orthogonality condition,

$$\int_0^L x J_p\left(\frac{\chi_n x}{l}\right) J_p\left(\frac{\chi_m x}{l}\right) \mathrm{d}x = \frac{l^2}{2}\left[J_{p+1}\left(\chi_m\right)\right]^2 \delta_{mn} \tag{4.46}$$

where,

$$J_p\left(\chi_n\right) = 0 \quad n \in \mathbb{Z}^+ \tag{4.47}$$

i.e., χ_n is the nth zero point of the pth Bessel function.

For non-integer values of m, we must use the *gamma function*,

$$J_m(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(m+j+1)} \left(\frac{x}{2}\right)^{x+2j} \tag{4.48}$$

where Γ is defined,

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} \, \mathrm{d}t \,. \tag{4.49}$$

We find that,

$$J_m(-x) = (-1)^m J_m(x) \tag{4.50}$$

so the mth Bessel function is even for even m, and odd for odd m.

Convergence of the solution

By considering,

$$|r_n| = \left| \frac{a_{n+2} x^{n+2}}{a_n x^n} \right| = \left| \frac{x^2}{(n+2+m)} \right| \tag{4.51}$$

which clearly converges, as in the limit $n \to 0$, $|r_n| \to 0 \; \forall x$

4.4.1 Modes in a Circular Membrane

Consider a circular membrane clamped at its edges. Using the typical 3D wave equation, we use separation of variables,

$$\phi = F(r,\theta)T(t) \tag{4.52}$$

where $F(r,\theta) = R(r)\Theta(\theta)$. We have,

$$\frac{1}{F}\nabla^2 F = \frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2 \tag{4.53}$$

and applying the Laplacian in polar coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + k^2 F = 0. \tag{4.54}$$

Let us multiply eq. (4.54) by r^2/F,

$$\underbrace{\frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) k^2 r^2}_{n^2} + \underbrace{\frac{1}{\Theta}\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}}_{-n^2} = 0. \tag{4.55}$$

We then have,

$$\frac{1}{\Theta}\frac{\mathrm{d}\Theta}{\mathrm{d}\theta} + n^2 = 0 \tag{4.56}$$

$$\frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + k^2 r^2 - n^2 = 0 \tag{4.57}$$

We can clearly see that eq. (4.56) produces a cyclic solution. However, rearranging eq. (4.57),

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + (k^2 r^2 - n^2)R = 0 \tag{4.58}$$

which is Bessel's equation, so, $R(r) \approx J_n(kr)$. Given the boundary condition,

$$R(a) = 0 \tag{4.59}$$

we have,

$$J_n(ka) = 0 \tag{4.60}$$

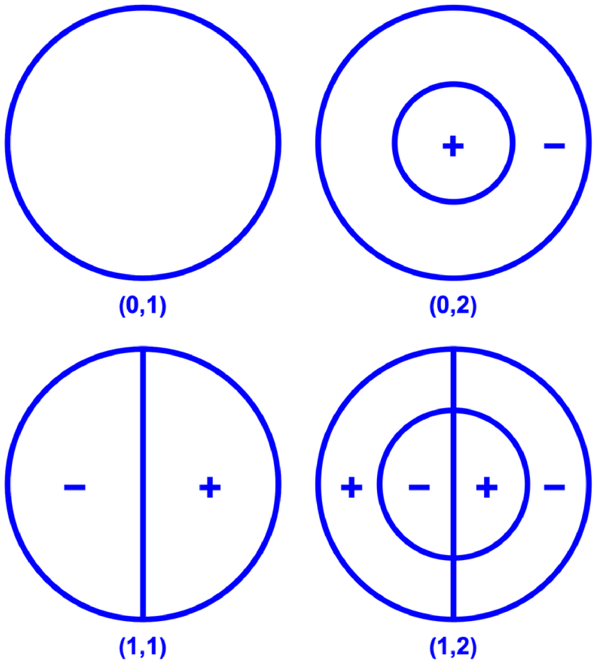

Figure 4.2

so we require,

$$k_{nm}a = \chi_{nm} \tag{4.61}$$

where χ_{nm} is the mth root of the nth Bessel function of the first kind. We then require the angular frequency of the modes of oscillation of the membrane to be,

$$\omega_{nm} = k_{nm}c = \frac{\chi_{nm}c}{a}. \tag{4.62}$$

Some normal modes of the membrane are shown in figure 4.2.

NOTE: Frequencies of vibration are not integer multiples of the fundamental mode, as it might be with a regular wave.

The final solution for the vibrations of the membrane is given by,

$$\phi = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} J_n(k_{nm}r)\left[A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)\right]\cos(\omega_{nm}t) \tag{4.63}$$

4.5 Spherical Waves

The spherical wave equation is governed by the Laplacian in spherical polar coordinates. The spherical wave equation is given by,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}. \tag{4.64}$$

Let us for now consider the solution on the surface, $r = a$. Let us use a separation of variables,

$$F(\theta, \phi, t) = Y(\theta, \phi)T(t) \tag{4.65}$$

We have,

$$\frac{1}{Ya^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Ya^2\sin^2\theta}\frac{\partial^2 Y}{\partial\theta^2} = \frac{1}{c^2T}\frac{\mathrm{d}^2T}{\mathrm{d}t^2} = -\frac{l(l+1)}{a^2}. \tag{4.66}$$

Analysing the time equation,

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2} = -\frac{l(l+1)}{a^2c^2}T \tag{4.67}$$

we have a trivial SHM solution,

$$T = A\sin(\omega t) + B\cos(\omega t) \tag{4.68}$$

where

$$\omega^2 = \frac{c^2}{a^2}l(l+1). \tag{4.69}$$

For the angular solution, we use separation of variables,

$$Y = \Theta(\theta)\Phi(\phi) \tag{4.70}$$

to obtain,

$$\begin{aligned} \frac{1}{sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \frac{1}{\sin^2\theta}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} &= -l(l+1)\Theta\Phi \\ \underbrace{\frac{\sin\theta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta) + l(l+1)\sin^2\theta}_{m^2} + \underbrace{\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}}_{-m^2} &= 0 \end{aligned} \tag{4.71}$$

Let us consider a solution of constant ϕ, and set $m = 0$. Our equation is then purely in θ,

$$\frac{\sin\theta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + l(l+1)\sin^2\theta = 0. \tag{4.72}$$

We can solve this by a substitution $x = \cos\theta$,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta \qquad \Longrightarrow \qquad \mathrm{d}\theta = -\frac{\mathrm{d}x}{\sin\theta} \tag{4.73}$$

Our equation then becomes,

$$-\frac{\cancelto{1}{\sin^2\theta}}{\Theta}\frac{\mathrm{d}}{\mathrm{d}x}\left(-\sin^2\theta\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right) + l(l+1)\cancelto{1}{\sin^2\theta} = 0 \tag{4.74}$$

If we note that $\sin^2\theta = 1 - x^2$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right) + l(l+1)\Theta = 0 \tag{4.75}$$

which is Legendre's polynomials. Our solution is then,

$$\Theta = P_l(x). \tag{4.76}$$

The general solution, considering only θ and t dependence is then,

$$f(\theta, t) = \sum_{l=0}^{\infty} P_l(\cos\theta)\left(A_l\cos\omega t + B_l\sin\omega t\right) \tag{4.77}$$

4.5.1 ϕ Dependent Solution

Let us now consider solutions where $m \neq 0$. Our angular equations become,

$$\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = -m^2 \tag{4.78}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^2\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left[l(L+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \tag{4.79}$$

Eq. (4.78) is trivial,

$$\Phi = A\sin m\phi + B\cos m\phi, \quad m \in \mathbb{Z}^+. \tag{4.80}$$

Eq. (4.79) can be solved if we consider,

$$x = \cos\theta \tag{4.81}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta \therefore \mathrm{d}\theta = \frac{\mathrm{d}x}{\sin\theta} \tag{4.82}$$

we can rewrite eq. (4.79) as,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(1 - x^2\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left[l(l+1) - \frac{m^2}{1-x^2}\right]\Theta = 0. \tag{4.83}$$

For $m = 0$, we obtain Legendre polynomial solutions. However, for $m \neq 0$, we obtain associated Legendre polynomials,

$$\Theta = P_l^m(x), \quad |m| \leq l. \tag{4.84}$$

We then find the total angular solution is,

$$Y(\theta,\phi) = P_l^m(\cos\theta)\left[A\sin m\phi + B\cos m\phi\right] \tag{4.85}$$

4.5.2 Waves Inside the Sphere (r dependence)

If we take into account r dependence, we retain the angular solutions from before. Focusing on the radial and time solutions,

$$\frac{1}{Rr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{l(l+1)}{r^2} = \frac{1}{c^2T}\frac{\mathrm{d}^2T}{\mathrm{d}t^2} = -k^2. \tag{4.86}$$

We see the time solutions are SHM. Focussing on the radial equation,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left[k^2r^2 - l(l+1)\right] = 0. \tag{4.87}$$

Let us recall the boundary condition $R(a) = 0$, where a is the surface of the sphere. Let $x = kr$ to rewrite eq. (4.86),

$$\boxed{x^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}R}{\mathrm{d}x} + \left[x^2 - l(l+1)\right]R = 0} \tag{4.88}$$

which is the form of the *spherical Bessel equation.* The solution to the spherical Bessel equation are the spherical Bessel functions,

$$R = j_l(kr). \tag{4.89}$$

It can be shown that,

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+\frac{1}{2}}(x). \tag{4.90}$$

Applying the boundary condition,

$$j_l(k_{lm}a) = 0. \tag{4.91}$$

The final solution to the spherical wave equation is given by,

$$f = \sum_{l,m,n}^{\infty} j_l(k_{ln}r)P_l^m(\cos\theta)\left[\sin(m\phi) + \cos(m\phi)\right]\left[A_{lmn}\cos(\omega_{lmn}t) + B_{lmn}\cos(\omega_{lmn}t)\right] \tag{4.92}$$

Appendix A

Examples: Differential Equations

A.1 1D Wave Equation

The 1 dimensional wave equation for a wavefunction ϕ is given by,

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}}. \tag{A.1}$$

A.1.1 Euler/d'Alembert solution

We can find a general solution to eq. (A.1) by using the substitution,

$$v = x - ct \Leftarrow \text{ Backward component} \tag{A.2}$$

$$u = x + ct \Leftarrow \text{ Forward component.} \tag{A.3}$$

Computing the derivative with respect to x by the chain rule,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial u} = \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial u}\right)\phi \tag{A.4}$$

from which the second derivative follows,

$$\frac{\partial^2 \phi}{\partial x^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2 \phi. \tag{A.5}$$

Similarly for the time component,

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)^2 \phi. \tag{A.6}$$

Applying eqs. (A.5) and (A.6) to eq. (A.1), we find,

$$\begin{aligned} &\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + 2\frac{\partial}{\partial u}\frac{\partial}{\partial v}\right) = \frac{c^2}{c^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} - 2\frac{\partial}{\partial u}\frac{\partial}{\partial v}\right) \\ \implies &\left(\frac{\partial}{\partial u}\frac{\partial}{\partial v}\right)\phi = 0 \implies \text{The solution is a sum of backward and forward components.} \end{aligned} \tag{A.7}$$

Thus the general solution to eq. (A.1) is,

$$\boxed{\phi = \phi(x + ct) - \phi(x - ct)}. \tag{A.8}$$

A.2 Laplace's Equation

Laplace's equation is given by,

$$\boxed{\nabla^2 \phi = 0} \tag{A.9}$$

and can be readily solved using separation of variables.

A.3 Diffusion Equation

The diffusion equation is given by,

$$\boxed{\boldsymbol{\nabla}^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \tag{A.10}$$

which reduces to Laplace's equation for a steady-state system. We will use it to analyse the temperature distribution on a circular plate.

Given we have a circular plate, we wish to use the polar form of the Laplacian. We have,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^r T}{\partial \theta^r} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}. \tag{A.11}$$

The circular plate has an insulating boundary, and has a cyclic boundary condition,

$$\left.\frac{\partial T}{\partial r}\right|_{r=a} = 0 \tag{A.12}$$

$$T(r,\theta,t) = T(r,\theta+2n\pi,t). \tag{A.13}$$

We then solve the diffusion equation by separation of variables, $T = R(r)\Theta(\theta)\tau(t)$,

$$\frac{1}{Rr}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{\Theta r^2}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = \frac{1}{\alpha^2\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = -k^2. \tag{A.14}$$

The *time equation* is,

$$\frac{\mathrm{d}\tau}{\tau} = -k^2\alpha^2\,\mathrm{d}t \tag{A.15}$$

which has an exponential solution,

$$\tau = Ae^{-k^2\alpha^2 t}. \tag{A.16}$$

Let us analyse the θ and r dependence in eq. (A.14),

$$\frac{r}{R}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + k^2r^2 = -\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = m^2 \tag{A.17}$$

The cyclic equation is given by,

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\theta} = -m^2\Theta \tag{A.18}$$

which has a trigonometric solution,

$$\Theta = A\cos(m\theta) + B\sin(m\theta) \tag{A.19}$$

where we require $m \in \mathbb{N}^+$ to satisfy the cyclic boundary condition.

The *radial equation* is given by,

$$\boxed{r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + r\frac{\mathrm{d}r}{\mathrm{d}R} + (k^2r^2 - m^2)R = 0} \tag{A.20}$$

which is Bessel's equation. The radial solution is then given by,

$$R = J_m(kr) \tag{A.21}$$

where $J_m(kr)$ is an m^{th} order Bessel function.