

Fig. 3.2. Parallel plate capacitor with the plates separated by a distance d which have been charged by a power supply with potential difference ΔV . The electric field is in the z -direction and will be directed from the positively charged plate to the negative.

WEEK 7 SUPPORTING TEXTS:

Grant & Phillips; Sections 2.1, 2.2 (2.2.2) and start of 2.3

Griffiths; Section 4.1, focus on 4.1.2 and 4.1.3 then 4.2 and start of 4.3

3.3 Capacitance, relative permittivity & dielectrics

Consider a parallel plate capacitor of two plates separated by a distance d . The plates are connected to a power supply with potential difference ΔV , until the plates carry charges $+Q$ and $-Q$ respectively. The power supply is disconnected leaving the potential difference between the plates. If we draw a Gaussian surface around the plates as illustrated in Figure 3.2, then we can determine the electric field under the assumption that there is a vacuum between the plates. Gauss' law gives

$$\int \mathbf{E} \cdot d\mathbf{S} = E_z A = \frac{Q}{\epsilon_0}, \quad (3.12)$$

where A is the surface area of the plates, and we have aligned the system along the z -axis (that is the plates lie in the $x - y$ -plane). Hence we deduce a constant electric field

$$E_z = \frac{Q}{A\epsilon_0}. \quad (3.13)$$

The potential difference between the top and bottom plates is

$$\Delta V = E_z d = \frac{Qd}{A\epsilon_0}. \quad (3.14)$$

We **define capacitance** to be the ratio between the charge and potential difference,

$$C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d}, \quad (3.15)$$

whose unit is the Farad, [F]. This is a property of the system which is only dependent on the geometry of the system, in this case, only on A and d . We can also compute the

Substance	ϵ_r
Air (1atm)	1.00059
Air (100atm)	1.0548
Teflon	2.1
Polyethylene	2.25
Mylar	3.1
Glass	5 – 10
Water	80.4
Strontium Titanate	310

Table 3.1. *Typical values of the relative permittivity (dielectric constant) for some materials. Note that these values are dependent on enviromental properties such as density, pressure and temperature.*

energy of a parallel plate capacitor,

$$U = \frac{1}{2}\epsilon_0 \int |\mathbf{E}|^2 dV = \frac{1}{2}\epsilon_0 \left(\frac{\Delta V}{d}\right)^2 Ad = \frac{1}{2}C(\Delta V)^2. \quad (3.16)$$

If we now add a material (for example, glass) between the plates, the measured voltage is observed to drop, meaning that the capacitance increases. We define the *relative permittivity* to be

$$\epsilon_r = \frac{C}{C_{\text{vacuum}}}, \quad (3.17)$$

which is also referred to as the *dielectric constant*. The quantity C is the measured value of the capacitance with some material separating the plates and C_{vacuum} is the capacitance for the same geometry, that is the same physical separation and shape of the parallel plates, but separated by a vacuum. The values of ϵ_r are listed in Table 3.1 for some typical materials

The class of materials which exhibit this kind of behaviour are known as *dielectrics*. They are electrical insulators, meaning that the materials have low conductivity, and the electrons are bound into atoms and molecules. Hence, there are few free electrons. When an electric field is applied to a dielectric the intrinsic dipoles within the material can become aligned, while atoms/molecules with no intrinsic dipole moment become polarized creating dipole. We will discuss the possible mechanisms for these phenomena in the following sections.

3.4 Polarization

Polarization of a material object occurs when the constituents of the substance align in some preferred direction associated with an electric field. To understand how the atoms

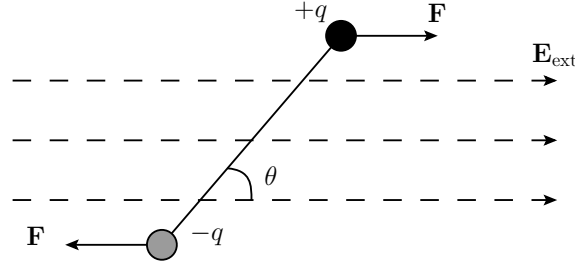


Fig. 3.3. Dipole in an external electric field.

of a material object respond to an applied electromagnetic field we will consider the action of a single electric dipole in an external electric field \mathbf{E}_{ext} as shown in Figure 3.3. Recall that the dipole moment is $\mathbf{p} = q\mathbf{d}$.

The work done by the dipole is the energy change due to the dipole

$$U_{\text{ext}} = -qE_{\text{ext}}.d \cos \theta, \quad (3.18)$$

where θ is the angle between the dipole and the external field. The torque on the dipole is

$$\tau_{\text{ext}} = F.d \sin \theta = dqE_{\text{ext}} \sin \theta. \quad (3.19)$$

If $\theta = 0$ the energy due to the external field is minimized and the torque is zero. Therefore, dipoles will attempt to rotate to align with the applied field to minimize the energy and eliminate torque.

For a general dipole, the energy due to and torque on a dipole in an applied electric field \mathbf{E}_{ext} are calculated via

$$U_{\text{ext}} = -\mathbf{p} \cdot \mathbf{E}_{\text{ext}}, \quad \boldsymbol{\tau}_{\text{ext}} = \mathbf{p} \times \mathbf{E}_{\text{ext}}. \quad (3.20)$$

The lowest energy state is achieved by setting $\mathbf{p} \propto \mathbf{E}_{\text{ext}}$ which corresponds to a state with zero torque.

Therefore, if an electric field is applied to a material, two things happen:

- (i) intrinsic dipoles will align to minimize energy and eliminate torques;
- (ii) atoms and molecules can be polarized, inducing a dipole moment.

We define a macroscopic vector field, the polarization, \mathbf{P} , whose units are $[\text{Cm}^{-2}]$ to represent the combined effect. The polarization vector can be computed from

$$\mathbf{P} = n\mathbf{p}, \quad (3.21)$$

where n is the number density of atoms or molecules and \mathbf{p} is the average dipole moment

due to the alignment of intrinsic dipoles and induced dipoles in the material. The polarization of a material will in general be a function of the applied electric field, $\mathbf{P} \equiv \mathbf{P}(\mathbf{E})$ which is called a constitutive relation.

For a linear isotropic material we will take the constitutive relation to be

$$\mathbf{P} = \chi_E \varepsilon_0 \mathbf{E}. \quad (3.22)$$

In index notation this reads

$$P_i = \chi_E \varepsilon_0 E_i. \quad (3.23)$$

The coefficient χ_E is the *electric susceptibility*, and determines the strength of the response of the polarization properties of the material due to an applied electric field \mathbf{E} . This will be a property of the material and will be a function of environmental properties of the material. We will see later that it is related to the dielectric constant, ε_r .

We can generalize (3.23) and introduce (a) anisotropic direction dependancies and (b) non-linear responses. These complicated anisotropic non-linear responses can be encoded in an expression of the form

$$P_i = \varepsilon_0 \sum_{j=1}^3 \chi_{ij}^{(1)} E_j + \varepsilon_0 \sum_{j,k=1}^3 \chi_{ijk}^{(2)} E_j E_k. \quad (3.24)$$

Taking the simple case where $\chi_{ijk}^{(2)} = 0$, if the matrix $\chi_{ij}^{(1)}$ is diagonal and all entries are equal then the material is isotropic then (3.23) is recovered. The off-diagonal terms in $\chi_{ij}^{(1)}$ reflect anisotropic responses. The final term, $\chi_{ijk}^{(2)}$, is a tensor and contains information about quadratic responses of the polarization vector to the applied electric field. Including the anisotropic and non-linear responses will describe systems closer to reality (for instance, if a substance is constructed from a regular lattice of atoms there will be a natural set of preferred directions), but they will make the problem incredibly complicated. For this reason we will focus on the linear isotropic response (3.23).

3.4.1 Mechanisms for polarization

We shall now return to linear isotropic materials, for which the polarization in terms of the applied electric field is given by

$$\mathbf{P} = n\mathbf{p} = \chi_E \varepsilon_0 \mathbf{E}, \quad (3.25)$$

and we will consider mechanisms for the generation of the internal polarization of the material. There many different mechanisms but we will focus on two: one due to the alignment of intrinsic dipoles and one where dipoles are created within a dielectric material.

As we have already mentioned, a realistic dielectric material will have an enormously

large number of atoms ($\sim 6 \times 10^{23}$ atoms per mole). These atoms will be arranged in highly complicated structures within the substance. It may therefore seem a fruitless task to attempt to construct a model for how a dielectric material will respond to an applied electric field. However, we can use our intuition for the behaviour of a single dipole to build simple theoretical pictures that will enable us to understand the macroscopic behavior of a dielectric material in the presence of an applied electric field.

3.4.1.1 Alignment

When the atoms or molecules of a dielectric material have an intrinsic dipole moment, \mathbf{p}_{int} , then in the absence of an externally applied electric field there is no energy penalty in having the dipoles have random alignments. This is shown in Figure 3.4(a). However, when an external electric field, \mathbf{E}_{ext} , is applied to the dielectric material there are large energies associated with dipoles which have large angles between the dipole moment and the electric field, which is what we calculated in (3.18). To reduce this energy the dipoles collectively align with the applied electric field; this is shown in Figure 3.4(b). The polarization that this induces can be calculated[†], and is given by

$$\mathbf{P}_{\text{align}} = \frac{np_{\text{int}}^2}{3k_{\text{B}}T} \mathbf{E}_{\text{ext}}. \quad (3.26)$$

This formula is of the form (3.23), where the electric susceptibility χ_{E} is inversely proportional to the temperature.

Note:

- The largest possible field in air is $\approx 10^6 \text{ Vm}^{-1}$, beyond which the molecules break down. This threshold can be a little larger in dielectrics.
- The energy at this threshold electric field is $\Delta U_{\text{ext}} = 2pE_{\text{ext}} \approx 2 \times 10^{-4} \text{ eV} \ll U_{\text{thermal}} = \frac{3}{2}k_{\text{B}}T \approx 4 \times 10^{-2} \text{ eV}$, at room temperature. This means that complete alignment will not take place even at the very largest electric fields possible. However, the fact that there are so many atoms means that there will be an observable polarization even at relatively low values of the electric field.

3.4.1.2 Induced

We will now consider a material that does not have any intrinsic dipoles. When an electric field is applied to this material, the electron “cloud” around an atom will distort relative to the “cloud” without an applied field. This will create a dipole from a given atom because the negative electrons are shifted relative to the positive nucleus as shown in Figure 3.5. These induced dipoles will then align with the applied electric field, in a manner similar to that discussed in §3.4.1.1. This mechanism is sometimes called atomic

[†] Grant & Phillips p 62-64

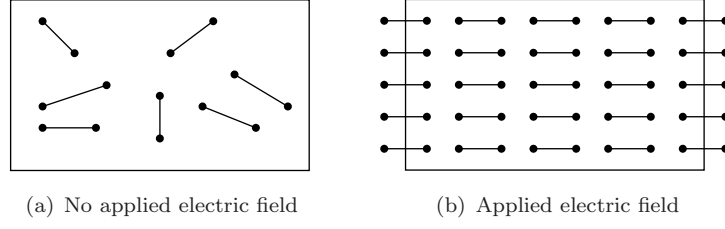


Fig. 3.4. Schematic of the distribution of intrinsic dipoles within a substance. In (a) there is no applied external electric field and the dipoles have a “random” alignment. In (b) there is an external electric field, which has the effect of aligning the intrinsic dipoles to reduce the energy of the dipoles.

polarization. In Figure 3.6 we give a schematic showing how the induced polarization generates a surface charge density due to an applied electric field.

We can calculate the induced dipole moment around a given atom due to some externally applied field \mathbf{E}_{ext} . If the external field causes an offset d for a distribution of radius R_0 , then

$$qE_{\text{ext}} = \frac{q^2 f}{4\pi\epsilon_0 d^2} \quad (3.27)$$

is the force due to the offset, with $f \approx d^3/R_0^3$ is the fractional charge offset. Hence,

$$qd \approx 4\pi\epsilon_0 R_0^3 E_{\text{ext}} \quad (3.28)$$

is the induced dipole moment due to the offset, and therefore the polarization,

$$\mathbf{P}_{\text{induced}} = n\alpha\epsilon_0 \mathbf{E}_{\text{ext}}, \quad (3.29)$$

where

$$\alpha \equiv 4\pi R_0^3 \quad (3.30)$$

is the atomic polarization.

3.4.1.3 Overall

The polarization response of a realistic dielectric material will be a combination of the induced and alignment phenomena discussed above; which process dominates will be different for different materials. The overall polarization response of a dielectric material

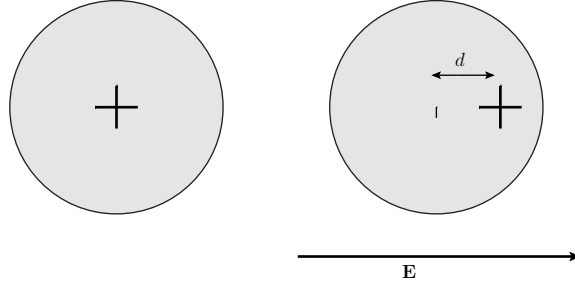


Fig. 3.5. Schematic of how an electric dipole moment can be induced by applying an external electric field.

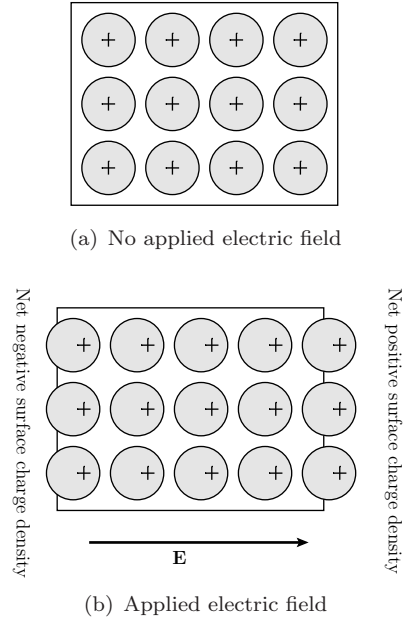


Fig. 3.6. Schematic of the induced polarization mechanism. In (a) we show the distribution of positive charges inside a medium, without applied electric field. In (b) an electric field has been applied, which moves the charge distributions, and creates a net positive/negative surface charge density. This is the collective behavior of the single case in Figure 3.5.

is given by

$$\begin{aligned}
 \mathbf{P} &= \mathbf{P}_{\text{align}} + \mathbf{P}_{\text{induced}} \\
 &= \frac{np_{\text{intrinsic}}^2}{3k_{\text{B}}T} \mathbf{E}_{\text{ext}} + n\alpha\epsilon_0 \mathbf{E}_{\text{ext}} \\
 &= \epsilon_0 n \left(\alpha + \frac{p_{\text{int}}^2}{3k_{\text{B}}T\epsilon_0} \right) \mathbf{E}_{\text{ext}}, \tag{3.31}
 \end{aligned}$$

which is of the form $\mathbf{P} = \chi_E \epsilon_0 \mathbf{E}$ if we identify the electric susceptibility with

$$\chi_E = n \left(\alpha + \frac{p_{\text{int}}^2}{3k_B T \epsilon_0} \right). \quad (3.32)$$

One can measure the strength of the two effects exploiting the different T dependencies but we should note that the actual dipole moments and offsets are very small.

3.5 Electrostatics in dielectrics

We will now study electric fields inside dielectric materials, and explore how the polarization induced by the external electric field affects the electric field inside the material. For simplicity we will only consider static situations. Let us consider a polarization vector

$$\mathbf{P} = P_x(x) \hat{\mathbf{i}}, \quad P_x > 0. \quad (3.33)$$

Electrons will move leftwards out of the dielectric material at one end and into the material at the other end. This creates a shift in the charge, given by

$$\Delta Q = -(P(x + \delta x) - P_x(x)) \delta y \delta z \approx -\frac{\partial P_x}{\partial x} \delta x \delta y \delta z, \quad (3.34)$$

but, the shift in the charge is just given by the elemental volume multiplied by the value of the bound charge density

$$\Delta Q = \rho_{\text{bound}} \delta x \delta y \delta z, \quad (3.35)$$

so that

$$\rho_{\text{bound}} = -\frac{\partial P_x}{\partial x}. \quad (3.36)$$

Repeating the calculation for the other spatial directions, one can see that

$$\rho_{\text{bound}} = -\nabla \cdot \mathbf{P}. \quad (3.37)$$

This bound charge is neutral. The total bound charge is found by integrating the bound charge density within the material, and adding to it the total surface charge density;

$$Q_{\text{bound}} = \int_V \rho_{\text{bound}} dV + \int_S \sigma dS = 0, \quad (3.38)$$

where σ is the surface charge density. Thus,

$$\int_V \nabla \cdot \mathbf{P} dV = \int_S \mathbf{P} \cdot \hat{\mathbf{n}} dS = \int_S \sigma dS, \quad (3.39)$$

where the first equality used the divergence theorem and $\hat{\mathbf{n}}$ is a unit normal of the enclosing surface. Hence, the surface charge density due to the polarization is given by

$$\sigma_P = \mathbf{P} \cdot \hat{\mathbf{n}}. \quad (3.40)$$

Recall the equations of electrostatics, $\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho$, $\nabla \times \mathbf{E} = 0$. We shall split the charge density ρ into two contributions: one from the bound charge ρ_{bound} and another from the free charge ρ_{free} . Thus, Gauss' law reads

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho = \frac{1}{\varepsilon_0} (\rho_{\text{bound}} + \rho_{\text{free}}) = \frac{1}{\varepsilon_0} (-\nabla \cdot \mathbf{P} + \rho_{\text{free}}). \quad (3.41)$$

A trivial rearrangement of this formula reveals that

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{\text{free}}. \quad (3.42)$$

This can be rewritten as a modified Gauss' law

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}}, \quad (3.43)$$

where we define

$$\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad (3.44)$$

which is called the electric displacement vector [Cm^{-2}]. The integral form of Gauss' law becomes

$$\int_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{free}}. \quad (3.45)$$

Note that for a linear isotropic material, $\mathbf{P} = \chi_E \varepsilon_0 \mathbf{E}$, so that

$$\mathbf{D} = (1 + \chi_E) \varepsilon_0 \mathbf{E}. \quad (3.46)$$