ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Vector Spaces

26th May 2017, 09.45 a.m. –11.15 a.m.

Answer **ALL** parts of question 1 and **TWO** other questions

---

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. (a) For a function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, state the Cauchy-Riemann conditions for $f(z)$ to be analytic. Show that they are satisfied for $f(z) = z^3$ and verify that $\partial f/\partial x = 3z^2$.

[6 marks]

(b) Evaluate $\int_C \frac{1}{z}\,\mathrm{d}z$, where the end points are $a = -i$ and $b = i$, and C is a semi-circular path which, apart from its end points, lies in the right half plane. Without further explicit integration, obtain the result for a path with the same end points that lies in the left half plane.

[6 marks]

(c) Find the derivative of the function $f(z) = z^2$, starting from the definition of the derivative for functions of a complex variable z. Also show that the derivative of $f(z) = |z|^2$ does not exist for $z \neq 0$.

[6 marks]

(d) The real functions $u_n(x)$ have the form $u_n(x) = e^{-x/2} q_n(x)$, where $q_n(x)$ is a polynomial of degree n in the real variable $x \geq 0$. The scalar product is given by

$$\langle u_m | u_n \rangle = \int_0^\infty u_m(x) u_n(x)\,\mathrm{d}x\,.$$

Given that the functions $\{u_0, u_1, u_2\}$ are orthonormal, find $u_1(x)$.

You may assume that $\int_0^\infty x^n e^{-x}\,\mathrm{d}x = n!$

[7 marks]

2. (a) A function $f(z) = u + iv$, where $z = x + iy$, has real part $u(x, y) = \cosh x \cos y$. Show that u satisfies $\nabla^2 u = 0$. Given that f is analytic and that $f(0) = 1$, find the corresponding imaginary part $v(x, y)$ and show that $f(z) = \cosh z$.

[9 marks]

(b) Consider the mapping $Z = X + iY = \operatorname{arccosh} z$. Show that under the inverse mapping, lines of constant $X = a \neq 0$ map to the ellipses

$$\frac{x^2}{\cosh^2 a} + \frac{y^2}{\sinh^2 a} = 1\,.$$

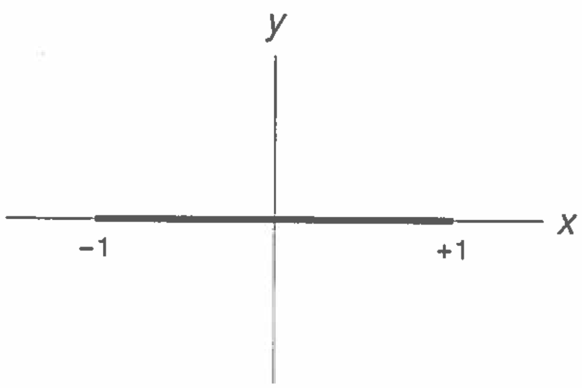
Also show that the line $X = 0$ maps to the segment of the x-axis with $-1 \leq x \leq 1$.

[5 marks]

(c) For the mapping considered in part (b), find the equations of the curves to which lines of constant $Y = b$ are mapped for $-\pi/2 < b < \pi/2$. Sketch curves of constant X and curves of constant Y on the xy-plane.

[5 marks]

(d) The function $Z(x, y) = \operatorname{arccosh}(x + iy)$ can be regarded as the complex potential around an infinitely long charged strip of metal. The cross-section of this strip is indicated by the heavy line in the figure below.



State how the field lines and equipotentials are related to the curves sketched in part (c). Use the Cauchy–Riemann conditions to show that the magnitude of the electric field $\mathbf{E}(x, y)$ is given by

$$E = \left|\frac{\mathrm{d}Z}{\mathrm{d}z}\right|.$$

Hence show that $E \approx 1/(x^2 + y^2)^{1/2}$ at large distances from the metal strip.

[6 marks]

3. (a) Use Cauchy's residue theorem and a suitable choice of contour to evaluate **one** of the following integrals:

either (i) $\displaystyle\int_0^{\infty} \frac{x^a}{(1+x)^2}\, dx$, where $-1 < a < 1$;

or (ii) $\displaystyle\int_0^{2\pi} \frac{1}{5 + 3\cos(2\theta)}\, d\theta$.

[13 marks]

(b) Find the poles of the function $f(z) = \cot(\pi z)/(a^2 + z^2)$, where $a \neq 0$ is a real number. By considering the residues of $f(z)$ at its poles, evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{a^2 + n^2}.$$

[12 marks]

4. (a) State the definitions of (i) an Hermitian operator and (ii) a unitary operator.

[4 marks]

(b) For each of the following matrices, state whether it is Hermitian, unitary, both, or neither:

$$\text{(i)}\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; \quad \text{(ii)}\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \text{(iii)}\ \begin{pmatrix} 0 & 2i \\ -2 & 3 \end{pmatrix}; \quad \text{(iv)}\ \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

[4 marks]

(c) An operator $\widehat{S}$ is defined as follows by its action on square-integrable **even** functions of x, $f(x) = f(-x)$:

$$\langle x|\widehat{S}|f\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} f(y)\, \mathrm{d}y\,.$$

(i) Given the definition of the scalar product of two functions f and g,

$$\langle f|g\rangle = \int_{-\infty}^{\infty} \overline{f}(x) g(x)\, \mathrm{d}x\,,$$

show that $\widehat{S}$ is both Hermitian and unitary on the space of even functions of x.

[8 marks]

(ii) Find the eigenvalues of $\widehat{S}$ by showing that $\widehat{S}^2$ acts as the identity operator when applied to an even function.

[4 marks]

(iii) For $g(x) = e^{-|x|}$, the result of applying $\widehat{S}$ is $\langle x|\widehat{S}|g\rangle = \sqrt{\dfrac{2}{\pi}}\, \dfrac{1}{1+x^2}$.
Use this result to find **one** eigenfunction of $\widehat{S}$, expressing your result as a function of x.

[5 marks]

END OF EXAMINATION PAPER