ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Thermal and Statistical Physics

19th May 2017, 2.00 p.m. – 3.30 p.m.

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. a) Explain what is meant by *extensive* and *intensive* quantities in thermodynamics. Give two examples for each kind of thermodynamic quantity.

[5 marks]

b) Determine whether the following differentials are exact:

(i) $xy\,dx + (x^2 + y + 1)dy$,

(ii) $xyz(dx + dy + dz)$.

[5 marks]

c) Write down the fundamental thermodynamic relation for a gas.

The internal energy of a certain gas, of volume V and entropy S, is given by

$$E = cS^{4/3}V^{-1/3},$$

where c is a constant. Show that the pressure is given by $P = E/(3V)$.

[5 marks]

d) Define Boltzmann entropy and explain the statistical basis of the second law of thermodynamics.

[5 marks]

e) A system consists of three particles: two particles of type A and one of type B. Particles of type A can be distinguished from those of type B, but the particles of type A are indistinguishable from each other. Each particle can be in one of two different energetic states, $\varepsilon_0 = 0$ and $\varepsilon_1 > 0$.

Show that the canonical partition function of the system is of the form

$$Z = a + be^{-\beta\varepsilon_1} + ce^{-2\beta\varepsilon_1} + de^{-3\beta\varepsilon_1},$$

and find the coefficients a, b, c and d.

[5 marks]

2. a) A heat engine runs in a cycle of reversible transformations. The working substance is an ideal gas. During the cycle the engine interacts with only two different heat baths.

Explain why this means that all processes in the engine must either be isothermal or adiabatic.

[3 marks]

b) A Carnot cycle consists of reversibly running an ideal gas through four processes:

1. isothermal expansion from V_1 to V_2 at temperature T_H;
2. adiabatic expansion from V_2 to V_3;
3. isothermal compression from V_3 to V_4 at temperature T_C;
4. adiabatic compression from V_4 to V_1.

(i) Draw the cycle on a $V-P$ diagram. Indicate the types of the four processes clearly on the diagram.

(ii) Draw the cycle on an $S-T$ diagram, again indicating the types of the four processes.

[5 marks]

c) For the cycle in b) and with n moles of an ideal gas as the working substance, calculate the amount of heat added or extracted from the system during each of the isothermal processes. For each isothermal process, state clearly whether heat is added to the system or extracted from it.

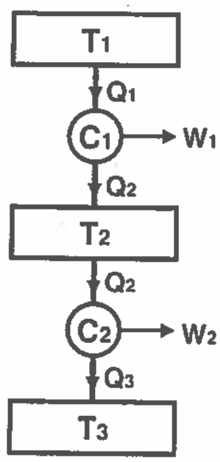
[5 marks]

d) From your result in c), derive an expression for the efficiency of the Carnot cycle in terms of T_C and T_H.

You may use without proof the relation $TV^{\gamma-1}=$ const, valid for adiabatic processes on an ideal gas ($\gamma = C_P/C_V$).

[6 marks]

e) A heat engine consists of two Carnot engines, C_1 and C_2, operating in series as shown in the figure. The temperatures of the heat reservoirs are such that $T_1 \geq T_2 \geq T_3$.



Both engines, C_1 and C_2, if operated in isolation, have the same efficiency, η.

(i) For given T_1 and T_3, find T_2.

(ii) Show that the total efficiency, η_{tot}, of this composite engine is $\eta_{tot} = 2\eta - \eta^2$.

[6 marks]

3. a) Two metal blocks of 1 kg each, one at 80° C and the other at 0° C, are brought into thermal contact, and are left to reach equilibrium. The specific heat capacity of the metal is 375 J kg^{-1} K^{-1}, independent of temperature.

(i) Calculate the total change in entropy.

(ii) By what factor does the number of microstates in the universe increase?

[5 marks]

b) Define the Helmholtz free energy, F, and show explicitly how it may be used to derive the Maxwell relation

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T$$

for a gas.

[6 marks]

c) By considering the entropy, $S = S(T, V)$, as a function of T and V, show that

$$dS = \frac{C_V}{T} dT + \left(\frac{\partial P}{\partial T}\right)_V dV,$$

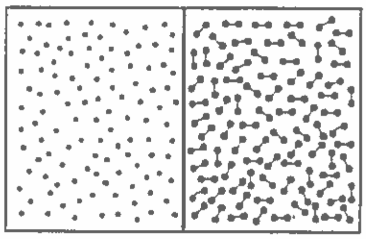
where C_V is the heat capacity at constant volume.

[4 marks]

d) State the equipartition theorem for a classical system with quadratic degrees of freedom.

[3 marks]

e) The thermodynamic system illustrated in the figure consists of two subsystems, isolated from the environment. The first subsystem contains 2 moles of an ideal *monoatomic* gas at $T_1 = 245$ K. The second subsystem consists of 3 moles of a *diatomic* ideal gas at $T_2 = 350$ K. The partition between the systems is initially adiabatic, rigid and impermeable.



(i) Calculate the internal energies of the two subsystems in the initial state. (You may ignore vibrations in the diatomic gas.)

(ii) The partition is now made heat conducting, and the system is left to equilibrate. Calculate the final temperature.

(iii) By how much does the entropy change during this process?

[7 marks]

4. a) A system consists of an even number, N, of distinguishable spins, which can each either be in the up state or in the down state.

(i) What is the entropy of the macrostate in which n spins are in the up state, and $N - n$ spins in the down state?

(ii) State the value of n which maximises the entropy.

[3 marks]

b) Consider a one-particle system with energy levels ε_i $(i = 1, 2, 3, \dots)$.

Define the average single-particle energy, $\langle E_1 \rangle$, in the canonical ensemble and show that

$$\begin{aligned} \langle E_1 \rangle &= -\left(\frac{\partial \ln Z_1}{\partial \beta}\right), \\ \langle (E_1 - \langle E_1 \rangle)^2 \rangle &= \left(\frac{\partial^2 \ln Z_1}{\partial \beta^2}\right), \end{aligned}$$

where Z_1 is the single-particle partition function.

[9 marks]

c) The position, x, of a classical particle in one dimension lies in the interval $-L \leq x \leq L$. The particle moves in a potential given by

$$V(x) = \begin{cases} 0 & -L \leq x \leq 0 \\ a & 0 < x \leq L, \end{cases}$$

where a is a constant.

Write down the single-particle energy, $H_1(x, p)$, and compute the single-particle partition function in the canonical ensemble for a general value of a.

You may use the integral $\displaystyle\int_{-\infty}^{\infty} e^{-\alpha u^2} du = \sqrt{\frac{\pi}{\alpha}}$, valid for $\alpha > 0$.

[6 marks]

d) Consider again the particle in part c).

(i) For general values of a and $\beta = 1/(k_B T)$ compute the probability to find the particle in the interval $0 < x \leq L$.

(ii) Compute the limiting value of this probability for $a \to \infty$, and comment on your result.

(iii) For finite a, compute the limiting value of the probability for $\beta \to 0$, and comment on your result.

[7 marks]

END OF EXAMINATION PAPER