Lagrangian Dynamics

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Chapter 1

Lagrangian Mechanics

1.1 Introduction

We have up until now solved dynamical problems using Newton's laws. However, they are not applicable to all scenarios. The laws are the following,

1. A body acted on with no forces moves with uniform velocity.
   - Einstein's general relativity is a counterexample to this law.
2. The rate of change of momentum of a body is give by the total force acting on that body.
   - In quantum mechanics, momentum is probabilistic.
3. Every action force has an equal and opposite reaction force.
   - This is not true in collective phenomenon, e.g. confiment of quarks in a proton.

Lagrangian/Hamiltonian mechanics is able to solve all systems where Newtonian mechanics is applicable[1], as well as alleviating Newtonian mechanics of its shortcomings.

1.2 Review of classical mechanics and mathematicl preliminaries

Consider a particle of mass m moving in 1 dimension in an arbitrary, explicitly time-independent potential $V(x)$. The particle will feel a force F due to the potential such that,

$$F = -\frac{\partial V}{\partial x} \tag{1.1}$$

and has an acceleration such that,

$$F = m\ddot{x}. \tag{1.2}$$

The kinetic energy of the particle is given by,

$$T = \frac{1}{2}m\dot{x}^2 \tag{1.3}$$

and the total energy,

$$E = T + V \tag{1.4}$$

is conserved. The momentum,

$$p = m\dot{x} \tag{1.5}$$

is not conserved.

[1] Except for cases with friction, which will be omitted in this course.

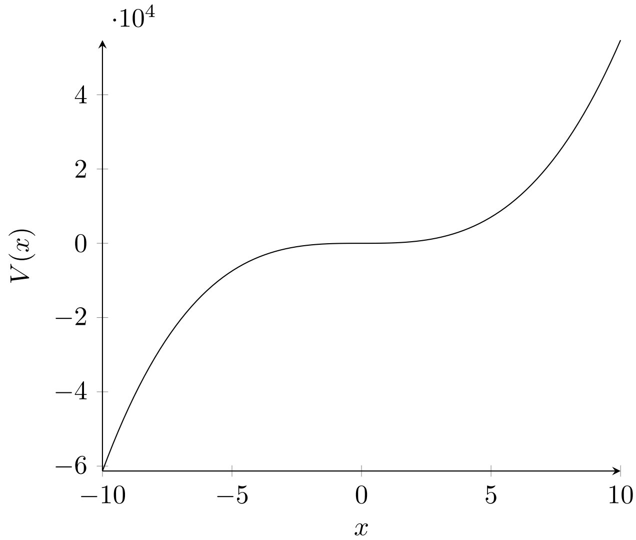


Figure 1.1: Arbitrary potential

1.2.1 Potential confusion

We have stated that the potential is *explicitly* time independent, however, we may have reason to suspect this is not the case. There are two ways to looks at this,

1. **NO** $\rightarrow$ The potential does not depend on t as we have not specified it to be so. Mathematically, we can express this via the partial derivative,

$$\frac{\partial V}{\partial t} = 0. \tag{1.6}$$

2. **YES** $\rightarrow$ The potential does depend on time, as the particle may move to a different position, and thus be in an area of different potential, i.e. the time dependence is *implicit*. Mathematically, this is expressed by considering the total derivative,

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial V}{\partial \dot{x}}\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} \\ &= \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial V}{\partial x}\dot{x}. \end{aligned} \tag{1.7}$$

We can extend this concept further to kinetic energy T and total energy E, whose total derivatives with time are,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = m\dot{x}\ddot{x} \tag{1.8}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(T+V) = \dot{x}\left(\frac{\partial V}{\partial x} + m\ddot{x}\right). \tag{1.9}$$

We can get some very interesting physics from eq. (1.9), and where *Lagrange's approach* to mechanics falls in. Lagrange's method says to assume two solutions to eq. (1.9),

$$\dot{x} = 0 \qquad\qquad \frac{\partial V}{\partial x} + m\ddot{x} = 0 \tag{1.10}$$

where the latter is what we wish to investigate.

1.3 Lagrangian and Lagranges Equation

Let us define the *Lagrangian*,

$$\boxed{L = T - V}. \tag{1.11}$$

In the 1 dimensional case,

$$
\begin{aligned}
L(x,\dot{x}) &= \frac{1}{2}m\dot{x}^2 - V(x) \\
\frac{\partial L}{\partial x} &= \underbrace{-\frac{\partial V}{\partial x} = m\ddot{x}}_{\text{Newton's seconod law}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right).
\end{aligned}
\tag{1.12}
$$

From eq. (1.12), we obtain *Lagrange's equation*,

$$
\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}}.
\tag{1.13}
$$

1.3.1 Generalised approach to solving a dynamical system

1. Work out the degrees of freedom of a system, that is,

$$\{q_i\} \qquad \text{Set of generalised coordinates} \tag{1.14}$$

$$\{\dot{q}_i\} \qquad \text{Set of generalised velocities.} \tag{1.15}$$

2. Write down the Lagrangian,

$$L\left(\{q_i\}, \{\dot{q}_i\}, t\right) = T - V. \tag{1.16}$$

3. Derive the equations of motion using Lagrange's equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left(\frac{\partial L}{\partial \dot{q}_i}\right)}_{\text{Generalised Momentum}} = \overbrace{\frac{\partial L}{\partial q_i}}^{\text{Generalised Force}} \tag{1.17}$$

1.3.2 Generalised Coordinates

Before we proceed to talking about generalised coordinates, lets find the Lagrangian in plane polar coordinates. We require the following velocities to compute the Lagrangian,

$$v_r = \dot{r} \qquad v_\theta = r\dot{\theta} \qquad v^2 = \dot{r}^2 + r^2\dot{\theta}^2. \tag{1.18}$$

The kinetic energy is then,

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2), \tag{1.19}$$

and the Lagrangian,

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r,\theta). \tag{1.20}$$

Since there are two coordinates in the Lagrangian, we have 2 equations of motion, given by,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}, \tag{1.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}. \tag{1.22}$$

We find that each coordinate (degree of freedom) is treated democratically. If we were to do this in spherical polar coordiantes, we would find that we would get 3 equations of motion, etc. Instead of being restricted to a set coordinate system, let us describe the lagrangian for a *generalised coordinate system*. Below are the generalised quantities we should be aware of:

$$\text{Generalised coordinates} \qquad q_i(t) \tag{1.23}$$

$$\text{Generalised velocities} \qquad \dot{q}_i(t) \tag{1.24}$$

$$\text{Generalised momenta} \qquad p_i(t) = \frac{\partial L}{\partial \dot{q}_i} \tag{1.25}$$

for which the generalised Lagrangian equation is given by eq. (1.17).

Chapter 2

The Calculus of Variations

2.1 The Principle of Least Action

We define the action S,

$$\boxed{S = \int L \, \mathrm{d}t} \tag{2.1}$$

and we assert that a system evolved over time such that the action is minimised. This is a fundamental law of physics. We have that $S = S[x(t)]$, indicating it is a function of a function.

Our general problem is that we wish to find a function $x(t)$ which minimises,

$$S[x(t)] = \int_{t_1}^{t_2} L(x(t), \dot{x}(t), t) \, \mathrm{d}t \tag{2.2}$$

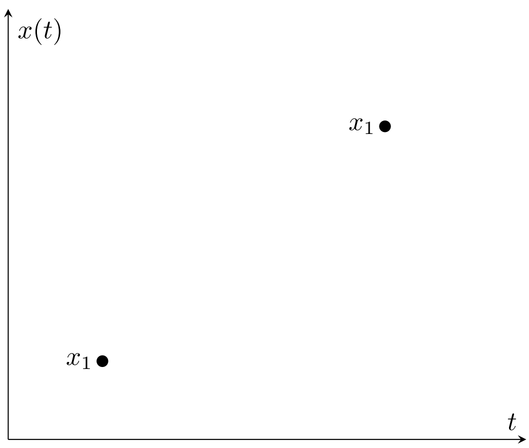
subject to the boundary conditions,

$$x(t_1) = x_1 \tag{2.3}$$
$$x(t_2) = x_2 \tag{2.4}$$

for single values functions of time.

Let us suppose a function $x(t)$ minimises S. Let us now consider a small displacement $\delta x(t)$ on $x(t)$, such that we have a modified path $x(t) + \delta x(t)$. This modified path must obey the boundary conditions, so

$$\delta x(t_1) = 0 \tag{2.5}$$
$$\delta x(t_2) = 0. \tag{2.6}$$



We must find an $x(t)$ such that $\delta x(t)$ does not change the value of S. We can write the modified Lagrangian,

$$L\left(x(t)+\delta x(t), \dot{x}(t)+\frac{\mathrm{d}}{\mathrm{d}t}\delta x(t), t\right) = L + \delta L. \tag{2.7}$$

We can then calculate the modified action,

$$\delta S = \int_{t_1}^{t_2} \delta L \, \mathrm{d}t = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x}\delta x(t) + \frac{\partial L}{\partial \dot{x}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta x(t)\right)\right] \mathrm{d}t\,. \tag{2.8}$$

We can integrate the second term of eq. (2.8) by parts,

$$\begin{aligned}
\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta x(t)\right)\mathrm{d}t &= \underbrace{\left[\frac{\partial L}{\partial x}\delta x(t)\right]_{t_1}^{t_2}}_{\text{0 by BCs}} - \int_{t_1}^{t_2} \delta x(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\mathrm{d}t \\
&= -\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x(t)\,\mathrm{d}t\,\mathrm{d}t \\
\implies \delta S &= \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\delta x(t)\,\mathrm{d}t = 0.
\end{aligned} \tag{2.9}$$

We can interpret eq. (2.9) in 2 ways,

1. $\delta x(t)$ is carefully chosen such that the integrand is non-zero, but integrates to 0. However, since $\delta x(t)$ is arbitrary, we can choose it to not be so;
2. The integrand is 0 everywhere.

Given the second conclusion, we have that,

$$\delta x(t) = 0 \qquad \boxed{\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0} \tag{2.10}$$

and we have thus derived the Euler-Lagrange equations.

A furhter conclusion is that the principle of least action directly implies that $x(t)$ obeys Lagrange's equation, which directly implies Newton's laws.

Principle of Least Action - More Generally

Any function defined,

$$f = f(y_i(x), y_i'(x), x) \tag{2.11}$$

with an integral,

$$S = \int_{x_1}^{x_2} f \, \mathrm{d}x \tag{2.12}$$

is minimised by,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_i'}\right) = \frac{\partial f}{\partial y_i}. \tag{2.13}$$

Chapter 3

Hamiltonian Mechanics

We are motivated to reformulate our approach to mechanics to directly include the conservation of momentum. However, because of the nature of partial derivatives, simply rewriting the Lagrangian to be $L = L(q, p, t)$ violates the principle of least action. We require a new quantity, which we will call the Hamiltonian, which we write $H = H(\{q_i\}, \{p_i\}, t)$ which we can obtain by the *Legendre Transform*.

Let us begin by writing,

$$\mathrm{d}L = \frac{\partial L}{\partial q}\,\mathrm{d}q + \frac{\partial L}{\partial \dot{q}}\,\mathrm{d}\dot{q} + \frac{\partial L}{\partial t}\,\mathrm{d}t\,. \tag{3.1}$$

We want to find a function $H(q, p, t)$ where,

$$p \equiv \frac{\partial L}{\partial \dot{q}} \tag{3.2}$$

where,

$$\mathrm{d}H = \frac{\partial H}{\partial q}\,\mathrm{d}q\,\frac{\partial H}{\partial p} + \frac{\partial H}{\partial t}\,\mathrm{d}t\,. \tag{3.3}$$

We have that the Legendre transformation of L is H, so,

$$H = p\dot{q} - L. \tag{3.4}$$

From eq. (3.4),

$$\mathrm{d}H = p\,\mathrm{d}\dot{q} + \dot{q}\,\mathrm{d}p - \mathrm{d}L\,. \tag{3.5}$$

and by eq. (3.1),

$$\begin{aligned} \mathrm{d}H = p\,\mathrm{d}\dot{q} + \dot{q}\,\mathrm{d}p - \frac{\partial L}{\partial q}\,\mathrm{d}q - \underbrace{\frac{\partial L}{\partial \dot{q}}\,\mathrm{d}\dot{q}}_{p\mathrm{d}\dot{q}} - \frac{\partial L}{\partial t}\,\mathrm{d}t \\ \dot{q}\,\mathrm{d}p - \frac{\partial L}{\partial q}\,\mathrm{d}q - \frac{\partial L}{\partial t}\,\mathrm{d}t\,. \end{aligned} \tag{3.6}$$

By comparing the coefficients of eqs. (3.3) and (3.6), we obtain *Hamilton's equations*,

$$\frac{\partial H}{\partial p} = \dot{q} \qquad \frac{\partial H}{\partial q} = -\dot{p} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{3.7}$$

$$\tag{3.8}$$

Most generally, for a system of N degrees of freedom $\{q_i\}$, and Lagrangian, $L(\{q_i\}, \{\dot{q}_i\}, t)$, the hamiltonian is defined,

$$\boxed{H(\{q_i\}, \{p_i\}, t) = \sum_i p_i\dot{q}_i - L(\{q_i\}, \{\dot{q}_i\}, t)} \tag{3.9}$$

for,

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{3.10}$$

and Hamilton's equations,

$$\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i} \tag{3.11}$$

$$\boxed{\frac{\partial H}{\partial q_i} = \dot{p}_i} \tag{3.12}$$

$$\boxed{\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}, \tag{3.13}$$

which produce $2N$ 1$^{\text{st}}$ order differential equations, in contrast to the N 2$^{\text{nd}}$ order differential equations produces by Lagrange's method. Let us further state that,

$$\frac{\partial H}{\partial t} = \frac{\mathrm{d}H}{\mathrm{d}t}. \tag{3.14}$$

3.1 Euler-Lagrange Equations of the Seecond Kind

Equations eq. (2.10) are known as the Euler-Lagrange equations of the first kind. Let us consider a general function $f(y(x), y'(x), x)$ and its total derivative in x,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''. \tag{3.15}$$

If y is a solution of the Euler-Lagrange equations, then eq. (2.13) holds, and,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\partial f}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)y' + \frac{\partial f}{\partial y'}y'' \\ &= \frac{\partial f}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'}\right)\end{aligned} \tag{3.16}$$

where the latter terms were reduced by the reverse product rule. By trivial rearranging, eq. (3.16) becomes the Euler-Lagrange equations of the second kind,

$$\boxed{\frac{\partial f}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(f - y'\frac{\partial f}{\partial y'}\right)}. \tag{3.17}$$

Applying eq. (3.17) to the Lagrangian,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left(L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right)}_{-H} = \underbrace{\frac{\partial L}{\partial t}}_{-\frac{\partial H}{\partial t}} \\ \implies &\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}.\end{aligned} \tag{3.18}$$

3.2 Theory of (Galilean) Relativity

The postulates of Galilean relativity state that there is no preferred,

1. Origin of time;
2. Position in space;
3. Inertial frame of reference;
4. Oritentation in space;

which hold in classical physics. Let us analyse the consequences of these postulates in the Hamiltonian. Let us first define our system:

Consider an isolated system of N particles of mass m_j with vector coordinates $\mathbf{r}_j$ and momenta $\mathbf{p}_j$.

For a system of N particles in 3 dimensions, we have $3N$ degrees of freedom and $6N$ independent variables. The total momentum can be defined,

$$\mathbf{P} = \sum_j \mathbf{p}_j. \tag{3.19}$$

We can then summarise the consequences of the postulates,

1. Equations of motion are unchanged by a displacement in time, i.e., $t \to t + \delta t$. This requires the Hamiltonain to be explicitly time independent,

$$\frac{\partial H}{\partial t} = 0 = \frac{\mathrm{d}H}{\mathrm{d}t} \tag{3.20}$$

   implying H is a conserved quantity.

2. Equations of motion are unchagned by a displacement of the entire system in space. We must then requiure $H(q_i, p_i)$ to only depend on relative position,

$$\mathbf{r}_j - \mathbf{r_k} = \mathbf{r}_{jk} \equiv \mathbf{r}_j \tag{3.21}$$

4. Equations of motion are invariant under spacial rotations. We require H to only depend on scalar products of $\mathbf{r}_{jk}$.

We will analyse these in more detail in the sections below.

3.2.1 Time Dependence

Consider a genreal functions, $F(q_i, p_i, t)$. The general equation for change of the function with time is,

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}t} &= \sum_i \left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i \right) + \frac{\partial F}{\partial t} \\ \text{Hamilton's Equations} \implies &= \underbrace{\sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} \right)}_{[F,H]} + \frac{\partial F}{\partial t} \end{aligned} \tag{3.22}$$

Above, we have defined the *possoin bracket*, denoted as $[F, H]$. Let us note that $[F, G] = -[G, F]$ for any function F and G. Let us then rewrite eq. (3.22) fully,

$$\boxed{\frac{\partial F}{\partial t} = [F, H] + \frac{\partial F}{\partial t}} \tag{3.23}$$

$\implies$ *The Hamiltonian generates a displacement in time..*

If $\frac{\partial F}{\partial t} = 0$, and $[F, H]$, then we say F commutes and is a *conserved quantity.* Furthermore, the Poisson bracket for the set $\{\{q_i\}, \{q_i\}\}$ is given by,

$$[q_\alpha, p_\beta] = \delta_{\alpha\beta}. \tag{3.24}$$

3.2.2 Spatial Dependence

Let us denote the generalised coordinates q_i as q_{sj}, where s represents the spacial direction, and j denotes the particle. Let us consider a displacement of N particles in the x direction,

$$q_{xj} \to q_{xj} + \delta x, \forall j \in \mathbb{N} \tag{3.25}$$

Our function F then undergoes a displacement,

$$F \to F + \delta F \tag{3.26}$$

where we can approximate δF by Taylor expansion,

$$\delta F = \sum_i \frac{\partial F}{\partial q_{xj}} \delta x. \tag{3.27}$$

Let us assume that this displacement is generated by the total x momentum, $P_x = \sum_k p_{xk}$. Let us study the behaviour of the Poisson bracket,

$$\begin{aligned} [F, P_x] &= \sum_j \sum_s \left(\frac{\partial F}{\partial q_{sj}} \underbrace{\frac{\partial P_x}{\partial P_{sj}}}_{\delta_{sx}} - \frac{\partial F}{\partial P_{sj}} \underbrace{\frac{\partial P_x}{\partial q_{sj}}}_{0} \right) \\ &= \sum_j \frac{\partial F}{\partial q_{xj}} \\ \implies \delta F &= [F, P_x]\, \delta x \end{aligned} \tag{3.28}$$

which in words, states, P_x *generates a displacement in the* x*-direction of the entire system.*

P_x can be applied to any observable, so let us apply it to the Hamiltonian. We have,

$$\Delta H = [H, P_x]\, \delta x \tag{3.29}$$

Spatial displacements leave the Hamiltonian unchanged, so we have,

$$[H, P_x] = 0. \tag{3.30}$$

We furthermore have the relation in eq. (3.23), when applied to P_x,

$$\frac{\mathrm{d}P_x}{\mathrm{d}t} = [P_x, H] = -[H, P_x] = 0 \tag{3.31}$$

which reveals linear momentum is a conserved quantity.

3.2.3 Spatial Orientation

Generator of rotations about an axis is the component of angular momentum along that axis. Angular momentum is conserved.

3.2.4 Independence of Reference Frame

A boost is a change of intertial frame by a constant velocity. We wish to find the generator of boosts. Let us call this quantity B,

$$\boxed{B = \sum_i (m_i x_i - t p_i)} \tag{3.32}$$

which is the position of where the centre of mass was at $t = 0$, multiplied by the total mass, $M = \sum_i m_i$. We wish to use this quantity as it is conserved. Let us consdier the possion bracket of an arbitrary function F with B,

$$\begin{aligned} [F, B] &= \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i} \frac{\partial F}{\partial p_i} \right) \\ &= sum_i \left(\frac{\partial F}{\partial q_i} (-t) - \frac{\partial F}{\partial q_i} (m_i) \right) \end{aligned} \tag{3.33}$$

Let us now set $F = q_\alpha$. A small displacement to this generalised coordinate can be written as, $\delta q_k = -t\varepsilon$, where ε is a small velocity. The transformation caused by the generator B in the x-coordinate is then,

$$x_\alpha \to x_\alpha - \varepsilon t. \tag{3.34}$$

Similarly, let us consider $F = p_\alpha$. The small displacement in this generalised momentum is, $p_\alpha = -m_\alpha \varepsilon$. So, the transformation caused by the generator B in the x component of momentum is,

$$p_{\alpha x} \to p_{\alpha x} - m\varepsilon \tag{3.35}$$

In summary, B *generates boosts in the* x *direction of velocity* ε.

Boost generator behaviour with the Hamiltonian

Given that B is a conservative quantity, we must write,

$$\frac{\mathrm{d}B}{\mathrm{d}t} = [H, B] + \frac{\partial B}{\partial t} = 0. \tag{3.36}$$

Rearranging this,

$$[H, B] = -\frac{\partial B}{\partial t} = \sum_i p_i = P. \tag{3.37}$$

In order for eq. (3.37) to hold, we require $H \propto P^2$, i.e., kinetic energy depending quadratically on momenta is a consequence of the principle of relativity.

Appendix A

Misc. Forumlae

A.1 Total Derivative

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_i \frac{\partial f}{\partial r_i} \frac{\mathrm{d}r_i}{\mathrm{d}t} \tag{A.1}$$

for $r_i = r_i(t)$.