TWO HOURS

UNIVERSITY OF MANCHESTER

Complex Variables and Vector Spaces

26th May 2021,

Answer **TWO** questions

You MUST NOT confer with anyone in answering the questions on this assessment.

The numbers are given as a guide to the relative weights of the different parts of each question.

Solutions must be handwritten and scanned, or handwritten on a tablet, and uploaded to Blackboard **as a single pdf file**.

Order the pages so that the answers to different questions are sequential and make clear on every page which question part is being addressed. Number the pages and write your student ID on the first page.

Ensure the scan is clear. Do not use green or red pens.

One hour of the exam duration has been allowed for accessing the exam and uploading the answers. Multiple submissions are allowed and you should upload your first attempt 30 minutes before the deadline. Only the final submission will be marked.

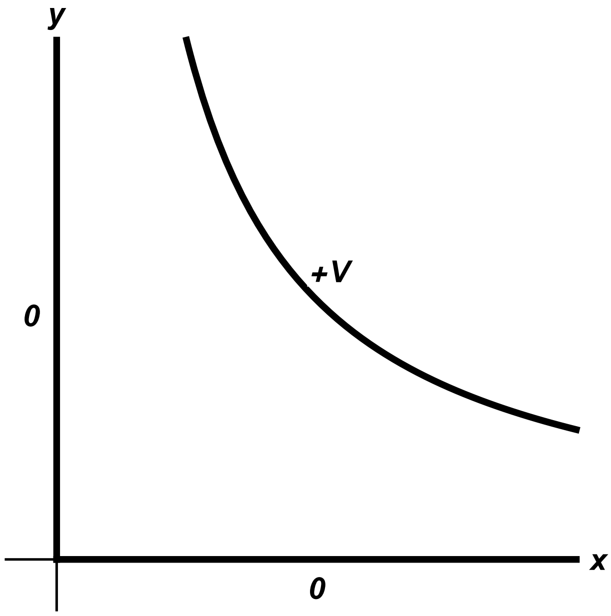
Late penalties will apply to work submitted after the deadline.

If you are a DASS-registered student with extra time, write your own submission time, which will have been communicated to you in advance, on the first page of your solutions, and submit before that deadline.

1. a) An analytic function $f(z)$ has real part $u(x,y) = x^3y - xy^3$. Show that $u(x,y)$ is harmonic, and find the corresponding imaginary part of $f(z)$. Find the form of $f(z)$ as a function of z.

[9 marks]

b) The diagram below shows a cross section through two infinite conducting plates which are perpendicular to the xy plane. One, bent by 90°, lies along the positive x and y axes, and is held at potential $\phi = 0$. The other has the form of a hyperbola given by the equation $y = a^2/x$; this plate is held at a positive potential $\phi = +V$.



i) Using complex variables $z = x + \mathrm{i}\,y$ and $Z = X + \mathrm{i}\,Y$, show that the transformation $Z = z^2/a$ maps these conductors onto two infinite plates parallel to the X axis.

[5 marks]

ii) Write down an expression for the electric field between the plates in the XY plane, and hence find the potential in the region between them, $\phi(X,Y)$. Use the conformal mapping in part (i) to find an expression for the potential in terms of the physical coordinates x and y.

[5 marks]

iii) Find an equation for the field lines in terms of x and y, and sketch some equipotential and field lines for this system.

[6 marks]

2. a) Use Cauchy's residue theorem and a suitable choice of contour (which you should sketch) to evaluate the Fourier transform

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega t}}{\omega^2 - 6\mathrm{i}\,\omega - 25}\,\mathrm{d}\omega,$$

for positive real t. If you make use of Jordan's lemma, show that the conditions for its validity are satisfied.

Find also the value of the integral for negative real t. You should not need to do any calculation, but you should explain your choice of contour.

[12 marks]

b) (i) Find the poles and residues of the function

$$f(z) = \frac{1}{(z-a)^2 \sin(\pi z)},$$

where a is real but is not an integer.

[6 marks]

(ii) Use your results from (i) and a suitable contour integral to show that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n-a)^2} = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}.$$

You should sketch the contour that you use.

[7 marks]

3. a) A Hermitian operator $\widehat{\rho}$ acts on vectors in the N-dimensional complex vector space $\mathbb{C}^N$. The operator has eigenvalues $\{\rho_i\}$ and orthonormal eigenvectors $\{|e_i\rangle\}$. Write down the spectral representation of $\widehat{\rho}$. By considering its action on a general vector $|v\rangle \in \mathbb{C}^N$, verify that the expression you have written down is equivalent to $\widehat{\rho}$.

[6 marks]

b) The state of polarisation of a beam of light may be represented by a matrix

$$\boldsymbol{\rho} = \frac{1}{2}\begin{pmatrix} 1+q & u-iv \\ u+iv & 1-q \end{pmatrix},$$

where (q, u, v) are real parameters that satisfy $0 < q^2 + u^2 + v^2 \le 1$.

i) Show that the eigenvalues of $\boldsymbol{\rho}$ are $(1 \pm p)/2$, where $p^2 = q^2 + u^2 + v^2$. Find the corresponding eigenvectors. You do not need to normalise your vectors.

[10 marks]

ii) Verify that the eigenvalues and eigenvectors found in part (i) have any properties that you would expect on general grounds, given the form of the matrix $\boldsymbol{\rho}$.

[5 marks]

iii) For the special case $q^2 + u^2 + v^2 = 1$, use the spectral representation to explain why $\boldsymbol{\rho}$ can be written in the form

$$\boldsymbol{\rho} = \begin{pmatrix} a\,\overline{a} & a\,\overline{b} \\ b\,\overline{a} & b\,\overline{b} \end{pmatrix},$$

where a and b are the components of a unit vector. No detailed calculation is required.

[4 marks]

END OF EXAMINATION PAPER