PHYS 10672 Advanced Dynamics 2024

Section 4 : Rigid Body Dynamics

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Rigid Body Dynamics - FS chapter 10

START OF 11TH LECTURE

4.1 Properties of a rigid body - see FS10.1 and 10.2

The definition of a rigid body is a system of particles in which the distance between them does not change regardless of the forces acting. In 3D three points not in a straight line can be used to define the body. The coordinates of the N particles of a rigid body are

$$\mathbf{r}^{(k)} = r_i^{(k)}\mathbf{e}_i \tag{4.1}$$

where $k = 1, .., N$ labels to the particles. For the three particles defining the body this means that there are 9 components (3×3) with 3 constraints

$$r_{ij} = |\mathbf{r}^{(i)} - \mathbf{r}^{(j)}| \tag{4.2}$$

being constants. Therefore, there are $9 - 3 = 6$ degrees of freedom and these correspond to 3 translations and 3 rotations of the rigid body.

We can define various quantities

$$\begin{aligned}
M &= \sum_{k=1}^{N} m_k = \text{total mass}\,, \\
\mathbf{R} &= \frac{1}{M}\sum_{k=1}^{N} m_k \mathbf{r}^{(k)} = \text{position vector of the centre of mass}\,, \\
\mathbf{P}^{(\mathrm{tot})} &= \sum_{k=1}^{N} m_k \dot{\mathbf{r}}^{(k)} = \text{total linear momentum}\,, \\
\mathbf{L}^{(\mathrm{tot})} &= \sum_{k=1}^{N} m_k \mathbf{r}^{(k)} \times \dot{\mathbf{r}}^{(k)} = \text{total angular momentum}\,, \\
T^{(\mathrm{tot})} &= \frac{1}{2}\sum_{k=1}^{N} m_k |\dot{\mathbf{r}}^{(k)}|^2 = \text{total kinetic energy}\,. \qquad (4.3)
\end{aligned}$$

These quantities are for discrete mass distributions of point particles but they can be adapted to continuous mass distributions using the *continuum limit* where

$$\sum_{k=1}^{N} m_k \to \int dM \,, \tag{4.4}$$

that is converting the sum into an integral. The infinitesimal mass element dM can be represented by a volume density ρdV, surface density σdA or line density $\mu d\ell$. In the case of a volume density we find that

$$M = \int \rho dV \,, \quad \mathbf{R} = \frac{1}{M}\int \rho \mathbf{r} dV \,, \quad \mathbf{P}^{(\text{tot})} = \int \rho \dot{\mathbf{r}} dV \,, \tag{4.5}$$

$$\mathbf{L}^{(\text{tot})} = \int \rho \mathbf{r} \times \dot{\mathbf{r}} dV \,, \quad T^{(\text{tot})} = \frac{1}{2}\int \rho |\dot{\mathbf{r}}|^2 dV \,. \tag{4.6}$$

Now remember from lecture 1 exercises:

$$\begin{aligned} (\mathbf{A}\times\mathbf{B})\times\mathbf{C} &= (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A} \to \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\mathbf{C})\mathbf{A} - (\mathbf{A}\cdot\mathbf{C})\mathbf{B} \,, \\ (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) \,, \end{aligned} \tag{4.7}$$

and from lecture 4

$$\dot{\mathbf{r}}^{(k)}|_S = \dot{\mathbf{r}}^{(k)}|_{S'} + \boldsymbol{\omega}\times\mathbf{r}^{(k)} \,. \tag{4.8}$$

Now define S' to be the frame of the rigid body so that $\dot{\mathbf{r}}^{(k)}|_{S'} \equiv 0$ and hence $\dot{\mathbf{r}}^{(k)}|_S = \boldsymbol{\omega}\times\mathbf{r}^{(k)}$. Substituting into the expression for the total angular momentum from (4.3) one finds that

$$\mathbf{L}^{(\text{tot})} = \sum_{k=1}^{N} m_k \mathbf{r}^{(k)} \times (\boldsymbol{\omega}\times\mathbf{r}^{(k)}) = \sum_{k=1}^{N} m_k \left[|\mathbf{r}^{(k)}|^2 \boldsymbol{\omega} - (\mathbf{r}^{(k)}\cdot\boldsymbol{\omega})\mathbf{r}^{(k)} \right] \,, \tag{4.9}$$

and therefore

$$L_i^{(\text{tot})} = \sum_{k=1}^{N} m_k \left[|\mathbf{r}^{(k)}|^2 \omega_i - r_j^{(k)} \omega_j r_i^{(k)} \right] = I_{ij}\omega_j \,, \tag{4.10}$$

where

$$I_{ij} = \sum_{k=1}^{N} m_k \left[|r^{(k)}|^2 \delta_{ij} - r_i^{(k)} r_j^{(k)} \right] = \int \rho \left(r^2 \delta_{ij} - r_i r_j \right) dV \,, \tag{4.11}$$

is the moment of inertia matrix *about the origin* and the second equality is in the continuum limit.

Now write $\mathbf{r}^{(k)} = \mathbf{R} + \mathbf{r}'^{(k)}$ where $\mathbf{R}$ is the position of the centre of mass and $\mathbf{r}'^{(k)}$ is the position of the kth particle with respect to the centre of mass which must satisfy

$$\sum_{i=1}^{N} m_k \mathbf{r}'^{(k)} \equiv 0 \,, \tag{4.12}$$

This will allow us to remove all the terms linear in $m_k \mathbf{r}'^{(k)}$ in subsequent calculations, facilitating a split into the motion of the centre of mass and the rotation motion. Substituting this in the expressions for $\mathbf{P}^{(\mathrm{tot})}$ and $\mathbf{L}^{(\mathrm{tot})}$ gives

$$\begin{aligned}
\mathbf{P}^{(\mathrm{tot})} &= \sum_{k=1}^{N} m_k \left(\dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{r}'^{(k)}\right) = \left(\sum_{k=1}^{N} m_k\right) \dot{\mathbf{R}} + \boldsymbol{\omega} \times \left(\sum_{k=1}^{N} m_k \mathbf{r}'^{(k)}\right) \\
&= M\dot{\mathbf{R}} \equiv M\mathbf{V} \equiv \mathbf{P}^{(\mathrm{com})} ,
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}^{(\mathrm{tot})} &= \sum_{k=1}^{N} m_k \left(\mathbf{R} + \mathbf{r}'^{(k)}\right) \times \left(\dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{r}'^{(k)}\right) \\
&= \sum_{k=1}^{N} m_k \left[\mathbf{R} \times \dot{\mathbf{R}} + \mathbf{r}'^{(k)} \times \dot{\mathbf{R}} + \mathbf{R} \times \left(\boldsymbol{\omega} \times \mathbf{r}'^{(k)}\right) + \mathbf{r}'^{(k)} \times \left(\boldsymbol{\omega} \times \mathbf{r}'^{(k)}\right)\right] \\
&= \mathbf{R} \times M\dot{\mathbf{R}} + \sum_{k=1}^{N} m_k \mathbf{r}'^{(k)} \times \left(\boldsymbol{\omega} \times \mathbf{r}'^{(k)}\right) = \mathbf{L}^{(\mathrm{com})} + \mathbf{L}^{(\mathrm{rot})} , \qquad (4.13)
\end{aligned}$$

where $L_i^{(\mathrm{com})} = M\epsilon_{ijk} R_j \dot{R}_k = \epsilon_{ijk} R_j P_k^{(\mathrm{com})}$ is the angular momentum of the centre of mass, $L_i^{(\mathrm{rot})} = I'_{ij}\omega_j$ is the rotational angular momentum and

$$I'_{ij} = \sum_{k=1}^{N} m_k \left[|\mathbf{r}'^{(k)}|^2 \delta_{ij} - r_i'^{(k)} r_j'^{(k)}\right] \neq I_{ij} , \qquad (4.14)$$

is the momentum of inertia about the centre of mass. Note that whenever we talk about the moment of inertia it must be *about a point* and with respect to some axes.

We can also substitute in the expression for $T^{(\mathrm{tot})}$ which gives

$$\begin{aligned}
T^{(\mathrm{tot})} &= \frac{1}{2} \sum_{k=1}^{N} m_k |\dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{r}'^{(k)}|^2 \\
&= \frac{1}{2} \sum_{k=1}^{N} m_k \left[|\dot{\mathbf{R}}|^2 + 2\dot{\mathbf{R}} \cdot \boldsymbol{\omega} \times \mathbf{r}'^{(k)} + |\boldsymbol{\omega} \times \mathbf{r}'^{(k)}|^2\right] \\
&= \frac{1}{2} M\dot{\mathbf{R}}^2 + \frac{1}{2} \sum_{k=1}^{N} m_k |\boldsymbol{\omega} \times \mathbf{r}'^{(k)}|^2 = T^{(\mathrm{com})} + T^{(\mathrm{rot})} , \qquad (4.15)
\end{aligned}$$

where $T^{(\mathrm{com})} = \frac{1}{2} M|\dot{\mathbf{R}}|^2$ is the kinetic energy of the centre of mass and

$$T^{(\mathrm{rot})} = \frac{1}{2} \sum_{k=1}^{N} m_k \left(|\boldsymbol{\omega}|^2 |\mathbf{r}'^{(k)}|^2 - |\mathbf{r}'^{(k)} \cdot \boldsymbol{\omega}|^2\right) = \frac{1}{2} \omega_i I'_{ij} \omega_j , \qquad (4.16)$$

is the rotational kinetic energy.

END OF 11TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE

EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 11 Exercises

1. Calculate the moment of inertia matrix for a system of 4 particles of mass $m = 5\,\mathrm{kg}$ held rigidly at corners of a square of side length $L = 80\,\mathrm{cm}$ about the centre of mass. The square is rotating about an axis $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(1,1,0)$ with an angular speed $\omega = 8\,\mathrm{rad\,s^{-1}}$ what is the angular momentum vector and what is the rotational kinetic energy?

2. A rigid body consists of two particles of mass m fixed to the end of a massless rod of length $2L$. The rod is inclined at an angle θ to the vertical and rotates with constant angular velocity ω about a vertical axis through its midpoint.

(a) Find the angular momentum $\mathbf{L}$ of the body about the midpoint of the rod by evaluating the sum $\mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2$. Use a sketch to illustrate the direction of $\mathbf{L}$.

(b) Find the torque on the body needed to maintain this rotation.

(c) Determine the moment of inertia matrix for rotations about the midpoint of the rod - your answer should depend on the azimuthal angle ϕ - and use it to confirm your answer to parts (a) and (b).

START OF 12TH LECTURE

4.2 Calculating moments of inertia - see FS 10.2.1

In the continuum limit, the moment of inertia matrix can be written as

$$I_{ij} = \int dM \left(r^2\delta_{ij} - r_i r_j\right) = \int dM \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & z^2+x^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix} . \tag{4.17}$$

In this section we will learn how to calculate this matrix for various continuum distributions via some examples and a couple of theorems.

4.2.1 Example: cylinder

We will consider a cylinder of radius R and height, h, about its centre of mass as shown in the lecture slides. There are two interesting cases: first a solid cylinder with uniform density where

$$dM = \rho dV = \rho r dr d\theta dz\,, \qquad \rho = \frac{M}{\pi R^2 h}\,, \tag{4.18}$$

and the shell of a cylinder with uniform surface density where

$$dM = \sigma dA = R d\theta dz\,, \qquad \sigma = \frac{M}{2\pi R h}\,. \tag{4.19}$$

We will use coordinates $x = r\cos\theta$ and $y = r\sin\theta$ for the solid case and $x = R\cos\theta$ and $y = R\sin\theta$ for the shell.

Solid cylinder:

$$\begin{aligned}
I_{ij} &= \rho \int_0^R rdr \int_0^{2\pi} d\theta \int_{-\frac{1}{2}h}^{\frac{1}{2}h} dz \begin{pmatrix} r^2\sin^2\theta + z^2 & -r^2\cos\theta\sin\theta & -rz\cos\theta \\ -r^2\cos\theta\sin\theta & r^2\cos^2\theta + z^2 & -rz\sin\theta \\ -rz\cos\theta & -rz\sin\theta & r^2 \end{pmatrix} \\
&= \frac{M}{\pi R^2 h} \int_0^R rdr \begin{pmatrix} \pi r^2 h + \frac{1}{6}\pi h^3 & 0 & 0 \\ 0 & \pi r^2 h + \frac{1}{6}\pi h^3 & 0 \\ 0 & 0 & 2\pi r^2 h \end{pmatrix} \\
&= M \begin{pmatrix} \frac{1}{4}R^2 + \frac{1}{12}h^2 & 0 & 0 \\ 0 & \frac{1}{4}R^2 + \frac{1}{12}h^2 & 0 \\ 0 & 0 & \frac{1}{2}R^2 \end{pmatrix}\,.
\end{aligned} \tag{4.20}$$

Cylindrical shell:

$$\begin{aligned}
I_{ij} &= \sigma R \int_0^{2\pi} d\theta \int_{-\frac{1}{2}h}^{\frac{1}{2}h} dz \begin{pmatrix} R^2\sin^2\theta + z^2 & -R^2\cos\theta\sin\theta & -Rz\cos\theta \\ -R^2\sin\theta\cos\theta & r^2\cos^2\theta + z^2 & -Rz\sin\theta \\ -Rz\cos\theta & -Rz\sin\theta & r^2 \end{pmatrix} \\
&= \frac{M}{2\pi h} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} dz \begin{pmatrix} \pi R^2 + 2\pi z^2 & 0 & 0 \\ 0 & \pi R^2 + 2\pi z^2 & 0 \\ 0 & 0 & 2\pi R^2 \end{pmatrix} \\
&= M \begin{pmatrix} \frac{1}{2}R^2 + \frac{1}{12}h^2 & 0 & 0 \\ 0 & \frac{1}{2}R^2 + \frac{1}{12}h^2 & 0 \\ 0 & 0 & R^2 \end{pmatrix}\,.
\end{aligned} \tag{4.21}$$

4.2.2 Parallel axis theorem - Steiner's theorem

Consider a situation where we have a collection of N particles whose position vector, $\mathbf{r}^{(k)}$, can be separated into the position of the centre of mass, $\mathbf{R}$, and the position relative to the centre of mass, $\mathbf{r}'^{(k)}$, such that

$$\mathbf{r}^{(k)} = \mathbf{R} + \mathbf{r}'^{(k)}\,. \tag{4.22}$$

Hence, we can deduce that

$$|\mathbf{r}^{(k)}|^2 = |\mathbf{R}|^2 + 2\mathbf{R}\cdot\mathbf{r}'^{(k)} + |\mathbf{r}'^{(k)}|^2\,, \quad r_i^{(k)} r_j^{(k)} = R_i R_j + R_i r_j'^{(k)} + R_j r_i'^{(k)} + r_i'^{(k)} r_j'^{(k)}\,. \tag{4.23}$$

Now remember that $\sum_{k=1}^{N} m_k r_i'^{(k)} \equiv 0$ which implies that

$$I_{ij} = M\left(|\mathbf{R}|^2\delta_{ij} - R_iR_j\right) + I'_{ij}\,, \tag{4.24}$$

where I'_{ij} is the moment of inertia tensor about the centre of mass. This is known as the parallel axis theorem which is often attributed as Steiner's Theorem.

As an example of its use let us consider a thin rod in the y-direction of length of L. First calculate the moment of inertia tensor about the centre of mass. This can be done by writing

$$dM = \mu d\ell\,, \quad \mu = \frac{M}{L}\,, \quad \mathbf{r} = (x,y,z) = \ell(0,1,0)\,, \tag{4.25}$$

for $-\frac{1}{2}L \leq \ell \leq \frac{1}{2}L$, from which we can deduce that

$$I'_{ij} = \mu \int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\ell \begin{pmatrix} \ell^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \ell^2 \end{pmatrix} = \frac{1}{12}ML^2\mathrm{diag}(1,0,1)\,. \tag{4.26}$$

We can use this and the parallel axis theorem to calculate the moment of inertia tensor about the end of the rod. We have that

$$\mathbf{R} = \frac{1}{2}L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\,, \tag{4.27}$$

so that

$$|\mathbf{R}|^2\delta_{ij} - R_iR_j = \frac{1}{4}L^2\,\mathrm{diag}(1,0,1)\,, \tag{4.28}$$

and therefpre

$$I_{ij} = \frac{1}{12}ML^2\,\mathrm{diag}(1,0,1) + \frac{1}{4}ML^2\,\mathrm{diag}(1,0,1) = \frac{1}{3}ML^2\,\mathrm{diag}(1,0,1)\,, \tag{4.29}$$

4.2.3 Perpendicular axis theorem

Now consider a lamina, that is, a 2D object for which $dM = \sigma(x,y)dxdy$ and $r_i = (x,y,0)$ so that

$$I_{ij} = \int \sigma(x,y)dxdy \begin{pmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & x^2+y^2 \end{pmatrix} \tag{4.30}$$

It is easy to see that

$$I_{11} + I_{22} = \int \sigma(x,y)dxdy(x^2+y^2) = I_{33}\,, \tag{4.31}$$

which is known as the perpendicular axis theorem.

4.2.4 Other well known examples

Note that

$$I_{ij} = \mathrm{diag}(I_1, .., I_N) \equiv \begin{pmatrix} I_1 & 0 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & I_N \end{pmatrix} . \tag{4.32}$$

(a) Solid, uniform density sphere of radius R about the centre of mass:

$$I_{ij} = \frac{2}{5} M R^2 \delta_{ij} \,, \tag{4.33}$$

and for a equivalent spherical shell:

$$I_{ij} = \frac{2}{3} M R^2 \delta_{ij} \,. \tag{4.34}$$

(b) Solid, uniform density cone of height, h and radius, R about its apex:

$$I_{ij} = \frac{3}{5} M \mathrm{diag}\left(h^2 + \frac{1}{4}R^2, h^2 + \frac{1}{4}R^2, \frac{1}{2}R^2 \right) . \tag{4.35}$$

(c) Solid, uniform density ellipsoid with semi-radii (a, b, c) about its centre of mass:

$$I_{ij} = \frac{1}{5} M \mathrm{diag}\left(b^2 + c^2, c^2 + a^2, a^2 + b^2 \right) . \tag{4.36}$$

END OF 12TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 12 Exercises

1. A cuboid of mass M with sides a, b and c in the x, y and z-directions respectively - see the diagram in the lecture slides. Calculate its moment of inertia for rotations about:

(a) its centre of mass;

(b) one of the corners.

2. A uniform, thin disk of radius R in the $x - y$ plane calculate the moment of inertia matrix and show that it is compatible with the perpendicular axis theorem.

START OF 13TH LECTURE

4.3 Principal Axes - see FS10.3

Let us first review what we have learnt. The moment of inertia matrix

$$I_{ij} = \int dM \left(r^2 \delta_{ij} - r_i r_j \right) , \tag{4.37}$$

is symmetric, that is, $I_{ij} = I_{ji}$, and it can be used to relate the angular momentum and rotational kinetic energy to the angular velocity vector

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = L_i = I_{ij}\omega_j = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} , \tag{4.38}$$

and

$$T = \frac{1}{2}\omega_i I_{ij} \omega_j = \frac{1}{2} \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix} \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} . \tag{4.39}$$

The fact that the angular momentum and the angular velocity are vectors implies that the coordinates in two frames, S and S' are related by $L'_i = O_{ij}L_j$ and $\omega'_i = O_{ij}\omega_j$ where we are are using O_{ij} as the components of the rotation matrix (since L is being used for the angular momentum) which also satisfy $O_{ik}O_{jk} = \delta_{ij}$. If the form $L_i = I_{ij}\omega_j$ is the same all frames then

$$L'_i = I'_{ij}\omega'_j \quad \to \quad O_{ij}L_j = O_{ij}I_{jk}\omega_k = I'_{ij}O_{jk}\omega_k \,, \tag{4.40}$$

where I'_{ij} is the moment of inertia in S'. This expression must be true for any ω_k so it must be that

$$O_{ij}I_{jk} = I'_{ij}O_{jk} \quad \to \quad O_{lk}O_{ij}I_{jk} = I'_{ij}O_{jk}O_{lk} = I'_{ij}\delta_{jl} = I'_{il} \,, \tag{4.41}$$

which implies that $I'_{il} = O_{ij}O_{lk}I_{jk}$. Hence, we have shown that I_{ij} is a rank 2 tensor!

There is a theorem of linear algebra which you touched upon in the course on Mathematics I which we will use to understand the moment of inertia. It states that any real symmetric, square matrix, A, is *diagonalizable*, which means that there exists a matrix P such that

$$P^{-1}AP = D = \mathrm{diag}(d^{(1)}, .., d^{(n)}) \,, \tag{4.42}$$

where n is the rank of the matrix. The matrix P can be constructed from the eigenvectors of A, $\mathbf{E}^{(k)}$ where $k = 1, .., n$ and n is the dimension of the matrix, which are defined by

$$A\mathbf{E}^{(k)} = \lambda^{(k)}\mathbf{E}^{(k)} \,, \tag{4.43}$$

with $\lambda^{(k)}$ being the eigenvalues which are calculated by solving the equation $\det(A - \lambda \mathbb{1}_n)$ for λ where $\mathbb{1}_n$ is the identity matrix with rank n. In particular, one find that

$$P = \left(\mathbf{E}^{(1)}, .., \mathbf{E}^{(n)} \right) \,, \quad D = \mathrm{diag}\left(\lambda^{(1)}, .., \lambda^{(n)} \right) \tag{4.44}$$

where for $n = 3$, that is in 3D,

$$P = \begin{pmatrix} E_1^{(1)} & E_1^{(2)} & E_1^{(3)} \\ E_2^{(1)} & E_2^{(2)} & E_2^{(3)} \\ E_3^{(1)} & E_3^{(2)} & E_3^{(3)} \end{pmatrix} . \tag{4.45}$$

When the matrix is symmetric, one can always choose the eigenvectors to be orthonormal, that is, $\mathbf{E}^{(k)} \cdot \mathbf{E}^{(m)} = \delta^{km}$ corresponding to an *orthogonal diagonalization* where $P^{-1} = P^T$. The matrix P is said to be orthogonal meaning that $PP^T = \mathbb{1}_n$. Note that there are some complications in constructing the eigenvectors when two or more of the eigenvalues are degenerate, that is, two more of them are equal.

Note that the orthogonal diagonalization of a symmetric matrix implies that

$$A = PDP^T = \left(\lambda^{(1)}\mathbf{E}^{(1)} \ldots \lambda^{(n)}\mathbf{E}^{(n)}\right) \begin{pmatrix} \mathbf{E}^{(1)T} \\ \vdots \\ \mathbf{E}^{(n)T} \end{pmatrix} = \sum_{k=1}^{n} \lambda^{(k)}\mathbf{E}^{(k)}\mathbf{E}^{(k)T} . \tag{4.46}$$

From this we can deduce that

$$\mathbf{x}^T A\mathbf{x} = A_{ij}x_i x_j = \sum_{k=1}^{n} \lambda^{(k)} \left(\mathbf{E}^{(k)T}\mathbf{x}\right)^2 , \tag{4.47}$$

for any vector $\mathbf{x}$.

In order to make this explicit let us consider an example for 2×2 matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} . \tag{4.48}$$

One can calculate the eigenvalues by solving $\det(A - \lambda\mathbb{1}_2) = 0$ which yields $\lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0$ and therefore $\lambda = -1, 3$ First, choose $\lambda^{(1)} = -1$ in which case

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} x \\ y \end{pmatrix} , \tag{4.49}$$

which yields $x + y = 0$ and so we can choose

$$\mathbf{E}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \tag{4.50}$$

Now for $\lambda^{(2)} = 3$ we need to solve

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix} , \tag{4.51}$$

which yields $x = y$ and therefore we can define

$$\mathbf{E}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \tag{4.52}$$

Therefore, in order to diagonalize the matrix we can use

$$P = \left(\mathbf{E}^{(2)}\ \mathbf{E}^{(1)}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = P^T\,. \tag{4.53}$$

Note that in this specific case $P^{-1} = P$ which is not the case in general, and that we have chosen to order to the eigenvectors 2 then 1 in the matrix P for no specific reason. One can always do this and it will just permute the eigenvalue order in the diagonalised matrix. We can perform the diagonalization explicitly by computing

$$P^{-1}AP = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.54}$$

There is 3×3 case for you to study in the exercises which is an interesting one since it has degenerate eigenvalues - there are only 2 distinct values for a 3×3 matrix - and this requires a little more care to identify a set of orthonormal vectors.

Now let us return to the rigid body problem. The moment of inertia tensor (note I have started calling it that rather than a matrix, but these terms are interchangeable) is symmetric, so it can be diagonalized according to the procedure discussed above. In particular, by calculating its eigenvalues $I^{(1)}$, $I^{(2)}$, $I^{(3)}$ and define

$$\hat{I}_{ij} = \begin{pmatrix} I^{(1)} & 0 & 0 \\ 0 & I^{(2)} & 0 \\ 0 & 0 & I^{(3)} \end{pmatrix}, \tag{4.55}$$

such that in terms of matrix notation $I = P\hat{I}P^T$. This means that we can always work with a diagonal moment of inertia tensor.

We have that $\mathbf{L} = P\hat{I}P^T\boldsymbol{\omega}$ which can be written as $\mathbf{L}' = \hat{I}\boldsymbol{\omega}'$ where $\mathbf{L}' = P^T\mathbf{L}$ and $\boldsymbol{\omega}' = P^T\boldsymbol{\omega}$. The diagonalization involved the rotation to a coordinate system where the eigenvectors are the basis directions, and these are known as the *Principal Axes* of the rigid body. We can write

$$\boldsymbol{\omega}' = P^T\boldsymbol{\omega} = \begin{pmatrix} \mathbf{E}^{(1)T} \\ \mathbf{E}^{(2)T} \\ \mathbf{E}^{(3)T} \end{pmatrix}\boldsymbol{\omega} = \begin{pmatrix} \mathbf{E}^{(1)T}\boldsymbol{\omega} \\ \mathbf{E}^{(2)T}\boldsymbol{\omega} \\ \mathbf{E}^{(3)T}\boldsymbol{\omega} \end{pmatrix}, \tag{4.56}$$

and

$$T = \frac{1}{2}I_{ij}\omega_i\omega_j = \frac{1}{2}\boldsymbol{\omega}^T I\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}^T P\hat{I}P^T\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}'^T\hat{I}\boldsymbol{\omega}' = \frac{1}{2}\sum_{k=1}^{3} I^{(k)}\left[\mathbf{E}^{(k)T}\boldsymbol{\omega}\right]^2. \tag{4.57}$$

END OF 13TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 13 Exercises

1. Calculate the eigenvalues of the symmetric matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{4.58}$$

2. For each of the eigenvalues of the matrices from question 1 describe the geometry of the eigenspace and attempt to construct a set of mutually orthogonal eigenvectors. You may need to use the Gram-Schmidt orthogonalization process for the second one - look this up on the internet if you don't know what it is!

3. Construct the matrices P - note that the eigenvectors need to be orthogonal, but not necessarily orthonormal - which diagonalizes the matrices A_i for $i = 1, 2$ and calculate $P^{-1}A_iP$.

4. Show that any square matrix $\mathcal{M}$ can be written as $\mathcal{M} = \mathcal{S} + \mathcal{T}$, where $\mathcal{S}$ and $\mathcal{T}$ are its symmetric and antisymmetric components, respectively. Compute $\mathcal{M}_{ij}\mathcal{S}_{ij}$, $\mathcal{M}_{ij}\mathcal{T}_{ij}$ and $\mathcal{S}_{ij}\mathcal{T}_{ij}$.

START OF 14TH LECTURE

From the exercises for lecture 12 we can deduce that the moment of inertia tensor for a uniform density cube of mass M with side length L about one of its corners is

$$I_{ij} = \frac{1}{12}ML^2 \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} = \frac{1}{12}ML^2 A_{ij} \tag{4.59}$$

One can calculate the principal axes by calculating the eigenvectors of the matrix A and the moments of inertia about those axes will be $\frac{1}{12}ML^2\times$ the corresponding eigenvalue.

We can calculate the eigenvalues of A by solving the equation $\det(A - \lambda\mathbb{1}_3) \equiv 0$ which yields

$$242 - 165\lambda + 24\lambda^2 - \lambda^3 = -(\lambda - 11)^2(\lambda - 2) = 0\,, \tag{4.60}$$

Therefore, the eigenvalues are $\lambda = 11$ (twice) and $\lambda = 2$. The eigenvector corresponding to $\lambda^{(1)} = 2$ is the solution to

$$\begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\,, \tag{4.61}$$

which yield $6x - 3y - 3z = 0$ and $-3x + 6y - 3z = 0$ which are solved by $x = y = z$ -

which corresponds to a line in 3D - and so

$$\mathbf{E}^{(1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} . \tag{4.62}$$

The eigenvectors corresponding to $\lambda^{(2)} = \lambda^{(3)} = 11$ satisfy

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 , \tag{4.63}$$

and therefore $x + y + z = 0$ which is a plane in 3D that is perpendicular to $\mathbf{E}^{(1)}$. The two eigenvectors could be any two linearly independent vectors in this plane. They need not be orthogonal, but as previously stated they can be chosen to be so. A combination which spans the space and are mutually orthogonal are

$$\mathbf{E}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} , \quad \mathbf{E}^{(3)} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} . \tag{4.64}$$

The principal axes - that is the eigen-directions - are physically intuitive. The eigenvector $\mathbf{E}^{(1)}$ is in the direction from the corner at the origin and the opposite corner with moment of inertia $I^{(1)} = \frac{1}{6}ML^2$, while the space spanned by the vectors $\mathbf{E}^{(2)}$ and $\mathbf{E}^{(3)}$ is the plane perpendicular to this vector through the other corners of the cube, with moments of inertia $I^{(2)} = I^{(3)} = \frac{11}{12}ML^2$. These are shown in the diagram on the lecture slides.

4.4 Euler's equation - see FS10.5

We have that the moment of inertia tensor relates to the angular velocity and angular momentum vectors, $\mathbf{L} = I\boldsymbol{\omega}$. N2 relates the rate of change of angular momentum to the torque, $\mathbf{M}$. If we define S to be a static, inertial frame and S' to be the body fixed, rotating frame then N2 implies that

$$\begin{aligned} \mathbf{M} &= \dot{\mathbf{L}}|_S = \dot{\mathbf{L}}|_{S'} + \boldsymbol{\omega} \times \mathbf{L} \\ &= I\dot{\omega} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}) , \end{aligned} \tag{4.65}$$

where the derivative on $\boldsymbol{\omega}$ is measured in the body fixed frame, S'. This equation is known as *Euler's equation* and can be written in index notation as

$$M_i = I_{ij}\dot{\omega}_j + \epsilon_{ijk} I_{kl} \omega_j \omega_l . \tag{4.66}$$

We have shown in the previous section that one can always transform to a coordinate system where the moment inertia tensor is diagonal with $I_{ij} = \mathrm{diag}(I_1, I_2, I_3)$ and so the angular momentum vector can be written as

$$\mathbf{L} = I\boldsymbol{\omega} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} , \tag{4.67}$$

and

$$I\dot{\boldsymbol{\omega}} = \begin{pmatrix} I_1\dot{\omega}_1 \\ I_2\dot{\omega}_2 \\ I_3\dot{\omega}_3 \end{pmatrix} . \tag{4.68}$$

In which case we can deduce that Euler's equations can be written as

$$\begin{aligned} M_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\ M_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 \\ M_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \, . \end{aligned} \tag{4.69}$$

4.5 Free rotation of a symmetric top - see FS10.6

Now imagine a situation where $\dot{\boldsymbol{\omega}}$ and $\mathbf{M} = 0$, that is, free rotation. In this case $\boldsymbol{\omega}$ is parallel to $\mathbf{L}$ which means that $\boldsymbol{\omega}$ is an eigenvector of the moment of inertia tensor. This means that time-independent free rotation can take place about one of the principal axes.

Let us set this axis to be the 3-direction without loss of generality and further assume that the rigid body (or "top") as it is often called in this context to be symmetric about this axis so that $I_{11} = I_{22} = I$, but not assume time-independent motion about the other two. Euler's equations for free rotation force $\dot{\omega}_3 \equiv 0$ by design and so we can set $\omega_3 = \omega_{\rm t}$ to be a constant - $\omega_{\rm t}$ being the angular velocity of the top. The other two Euler equations are

$$\dot{\omega}_1 = -\Omega\omega_2 \, , \quad \dot{\omega}_2 = \Omega\omega_1 \, , \tag{4.70}$$

where

$$\Omega = \left(\frac{I_3 - I}{I} \right) \omega_{\rm t} \, , \tag{4.71}$$

suggesting that the motion a more complicated situation than just a time-independent free rotation is possible. By differentiation we can show that $\ddot{\omega}_1 = -\Omega^2\omega_1$ and therefore there is a solution

$$\omega_1 = A\cos(\Omega t + \phi) \, , \quad \omega_2 = A\sin(\Omega t + \phi) \, , \tag{4.72}$$

and so $\omega_1^2 + \omega_2^2 = A^2$ is a constant of motion in addition to $\omega_{\rm t}$.

END OF 14TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE

EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 14 Exercises

1. Calculate the moment of inertia for a thin rod of mass M bent into the shape of a circle of radius R placed in the $x-y$ plane about its centre of mass. Now use the parallel axis theorem to calculate the moment of inertia about a point on the circle, an angle θ measured from the x-axis. Calculate the principal moments of inertia - showing that they should be independent of θ - and the principal axes of the system.

2. Use Euler's equations with no external torque to show that the rotational kinetic energy $T^{(\mathrm{tot})}$ is conserved.

START OF 15TH LECTURE

At the end of the last lecture we showed that the free rotation of a symmetric top can be described by

$$\boldsymbol{\omega} = A\cos(\Omega t + \phi)\mathbf{E}_1 + A\sin(\Omega t + \phi)\mathbf{E}_2 + \omega_{\mathrm{t}}\mathbf{E}_3\,, \tag{4.73}$$

where the $\mathbf{E}_i$ are the principal axes of the spinning top in the body fixed frame. In the notation used in the section on non-inertial frames $\mathbf{e}'_i = \mathbf{E}_i$. We would now like to understand what this means in the lab frame - or in the terminology of section 2 the inertial frame.

First let us consider the angular momentum $L_i = I_{ij}\omega_j$ and the rotational kinetic energy

$$T_{\mathrm{R}} = \frac{1}{2}\omega_i I_{ij}\omega_j = \frac{1}{2}\omega_i L_i = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}\,. \tag{4.74}$$

Since T_R is a constant in this free rotation we have that $\boldsymbol{\omega}\cdot\mathbf{L}$ is a constant. This is true for any free rotation, that is, when the there is no torque applied.

In the particular case under consideration where we can show explicitly that

$$|\boldsymbol{\omega}| = \sqrt{A^2 + {\omega_{\mathrm{t}}}^2}\,, \quad |\mathbf{L}| = \sqrt{I^2A^2 + I_3^2\omega_{\mathrm{t}}^2}\,, \quad \boldsymbol{\omega}\cdot\mathbf{L} = IA^2 + I_3\omega_{\mathrm{t}}^2\,. \tag{4.75}$$

All of these are clearly constant, and therefore the angle between $\mathbf{L}$ and $\boldsymbol{\omega}$ must remain constant. In the Lab frame $\mathbf{L}$ is constant by N2 and we will choose this to be in the $\mathbf{e}_3$ direction, that is, $\mathbf{L} = L\mathbf{e}_3$ and this means that the $\boldsymbol{\omega}$ will rotate, or precess, around the constant vector $\mathbf{L}$ maintaining a constant angle between them. Note that $\mathbf{L}$ is not constant in the body-fixed (non-inertial frame).

We can understand this motion more clearly by understanding the evolution of $\mathbf{E}_3$ in the lab frame. First, we see that

$$\mathbf{L} - I\boldsymbol{\omega} = (I_3 - I)\omega_{\rm t}\mathbf{E}_3\,, \tag{4.76}$$

which indicates that $\mathbf{L}$, $\boldsymbol{\omega}$ and $\mathbf{E}_3$ lie in a plane and we can write

$$\boldsymbol{\omega} = \frac{L}{I}\mathbf{e}_3 - \left(\frac{I_3 - I}{I}\right)\omega_{\rm t}\mathbf{E}_3\,. \tag{4.77}$$

If we define Θ to be the fixed angle between $\mathbf{E}_3$ and $\mathbf{L}$ and $\Phi \equiv \Phi(t)$ to be the angle of rotation around the 3-axis then we can write

$$\mathbf{E}_3 = \cos\Theta\mathbf{e}_3 + \sin\Theta\left(\cos\Phi\mathbf{e}_1 + \sin\Phi\mathbf{e}_2\right)\,, \tag{4.78}$$

and

$$\frac{d\mathbf{E}_3}{dt} = \dot{\Phi}\sin\Theta\left(-\sin\Phi\mathbf{e}_1 + \cos\Phi\mathbf{e}_2\right)\,. \tag{4.79}$$

However, we also have that $\dot{\mathbf{E}}_3 = \boldsymbol{\omega} \times \mathbf{E}_3$ and hence

$$\frac{d\mathbf{E}_3}{dt} = \frac{L}{I}\mathbf{e}_3 \times \mathbf{E}_3 = \frac{L}{I}\sin\Theta\left(\cos\Phi\mathbf{e}_2 - \sin\Phi\mathbf{e}_1\right)\,, \tag{4.80}$$

and so we can read-off $\dot{\Phi} = L/I$ when $\Theta \neq 0$. The special case when $\Theta = 0$ corresponds to a fixed rotation about the principal axis since $\mathbf{E}_3$, $\boldsymbol{\omega}$ and $\mathbf{L}$ are all parallel. Moreover, we can deduce that

$$\mathbf{L}\cdot\mathbf{E}_3 = L\cos\Theta = I_3\omega_{\rm t}\,, \tag{4.81}$$

where the first quality comes from the definition of $\mathbf{E}_3$ (4.78) and the second uses (4.73) in conjunction with $L_i = I_{ij}\omega_j$.

From all this analysis we can deduce that $\boldsymbol{\omega}$ and $\mathbf{E}_3$ precess about $\mathbf{L}$ in the lab frame with constant frequency

$$\Omega_{\rm p} = \dot{\Phi} = \frac{L}{I} = \frac{I_3\omega_{\rm t}}{I\cos\Theta}\,. \tag{4.82}$$

The top is spinning with frequency $\omega_{\rm t}$ about the symmetry axis and this symmetry axis is precessing around $\mathbf{L}$ with frequency $\Omega_{\rm p}$ - this is sometimes known as “wobbling”. We can talk of there being two cones defined by the direction $\boldsymbol{\omega}$: the space cone whose axis is fixed and the body cone which rolls around the surface of the space cone - this is illustrated in the lecture slides. There are two cases: where the body is prolate, whereby $I_3 < I$, which is the case in the diagram and oblate $I_3 > I$ where the space cone is is inside the body code.

4.6 Intermediate axis theorem - see FS10.7

We will now consider a general spinning top with no symmetry rotating around the 3-axis in the coordinate system defined by the principal coordinates with constant angular velocity $\omega_{\rm t}$, but with small perturbations in all directions so that $\omega_1 = \delta\omega_1$, $\omega_2 = \delta\omega_2$ and $\omega_3 = \omega_{\rm t} + \delta\omega_3$. Substituting these into Euler's equations implies that $\dot{\delta\omega_3} = 0$ and

$$\begin{aligned} I_1 \dot{\delta\omega_1} &= (I_2 - I_3)\omega_{\rm t}\delta\omega_2 \,, \\ I_2 \dot{\delta\omega_2} &= (I_3 - I_1)\omega_{\rm t}\delta\omega_1 \,. \end{aligned} \tag{4.83}$$

By differentiating the equation for $\delta\omega_1$ and substituting in for $\dot{\delta\omega_2}$ we can deduce that $\ddot{\delta\omega_1} + \Omega^2\delta\omega_1 = 0$ where

$$\Omega^2 = \frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\omega_{\rm t}^2 \,, \tag{4.84}$$

and similarly for $\delta\omega_2$. We see, therefore, that there are stable perturbations if $I_3 >$ both I_1 and I_2, or if $I_3 <$ both of them, whereas they will be unstable if $I_2 > I_3 > I_1$ or $I_1 > I_3 > I_2$, that is, if the moment of inertia around the spinning axis is the "intermediate" of the three principal moments of inertia. This is known as the intermediate axis theorem.

END OF 15TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 15 Exercises

1. Consider a thin circular plate rotating at an angular speed ω. It is tossed into the air and is seen to wobble slightly. Show that it is precessing with an angular speed $\approx 2\omega$. This is known as Feynman's plate.

2. The Earth is not exactly spherical and is in fact an oblate spheroid - meaning that it is squashed at the poles. The moment of inertia about the north-south axis is 0.33% of that about the symmetric axes. Calculate the period of precession assuming that we the earth is rigid. Why might this not be completely accurate?

3. Think about a tennis racket spinning about it centre of mass in a microgravity environment - that is, one where one can ignore the effects of gravity. About which axes is in the rotation stable?

START OF 16TH LECTURE

4.7 Gyroscopes - see FS10.8

In the previous lectures we have investigated free rotations of a spinning top, that is, when there is no torque applied to the top. We will now include the effects of forces which we will perform in the lab frame (that is, not the body fixed frame that was seen to be very helpful in analysing free rotations). We will also use a number of results that we derived in the section of the course on rotating frames of reference. In particular we will use

$$\begin{aligned}
\boldsymbol{\omega} &= \dot{\phi}\left(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta\right) + \dot{\theta}\mathbf{e}_\phi\,,\\
\dot{\mathbf{e}}_r &= \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\mathbf{e}_\phi\,,\\
\dot{\mathbf{e}}_\theta &= -\dot{\theta}\mathbf{e}_r + \dot{\phi}\cos\theta\mathbf{e}_\phi\,,\\
\dot{\mathbf{e}}_\phi &= -\dot{\phi}\left(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta\right)\,,
\end{aligned} \tag{4.85}$$

for a rotating spherical polar coordinate system.

The physical system that we will investigate is that of a gyroscope that can be modelled as a "heavy spinning top" which experiences two forces: the gravitational weight force of the top $\mathbf{F}_{\text{grav}} = m\mathbf{g}$ where $\mathbf{g} = -g\mathbf{e}_z$ that acts through the centre of mass located at $\mathbf{r} = R\mathbf{e}_r$, and the contact force on the top due to the ground. The point of contact will be assumed to be fixed and the contact force will be assumed to act in "point-like fashion" which means that it will impart no torque. Obviously this is an idealisation which is not true in any realistic situation, but gyroscopes are usually designed to have limited contact/friction so as to be as close as possible to this idealisation. The torque imparted by the gravitational force is

$$\mathbf{M} = \mathbf{r}\times m\mathbf{g} = -mgR\mathbf{e}_r\times\mathbf{e}_z = mgR\sin\theta\mathbf{e}_\phi\,, \tag{4.86}$$

where the polar angles θ and ϕ are used to describe the orientation of the principal axis of the top - the 3-axis - which points in the direction of $\mathbf{e}_r$. Note that the torque only acts in the $\mathbf{e}_\phi$ direction implying that the are two conserved components of the angular momentum. We will assume that the top is symmetric and that the moments of inertia are $I_1 = I_2 = I$ and I_3. The rotational degree of freedom of the top will be denoted ψ - and the angles θ, ϕ and ψ are known as *Euler angles.* A diagram illustrating the system is shown in the lecture slides.

Inclusion of the spin of the top adds an extra term to the angular velocity vector which becomes

$$\boldsymbol{\omega} = \left(\dot{\phi}\cos\theta + \dot{\psi}\right)\mathbf{e}_r - \dot{\phi}\sin\theta\mathbf{e}_\theta + \dot{\theta}\mathbf{e}_\phi\,, \tag{4.87}$$

and hence the angular momentum vector can be written as

$$\mathbf{L} = I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\mathbf{e}_r + I\left(\dot{\theta}\mathbf{e}_\phi - \dot{\phi}\sin\theta\mathbf{e}_\theta\right)\,. \tag{4.88}$$

N2 tells us that $\dot{\mathbf{L}} = \mathbf{M}$ and hence we can deduce that

$$\frac{1}{2}\frac{d}{dt}|\mathbf{L}|^2 = \mathbf{M}\cdot\mathbf{L}. \tag{4.89}$$

In the case where the top evolves such that θ is a constant we have that $\mathbf{M}\cdot\mathbf{L} = 0$ and hence $|\mathbf{L}|$ is conserved. So in this special case the angular momentum vector will evolve so as to maintain a constant modulus - that is, it will rotate.

By differentiation one can compute the rate of change of the angular momentum vector

$$\begin{aligned}\dot{\mathbf{L}} &= I_3\left(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta + \ddot{\psi}\right)\mathbf{e}_r + \left[I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\dot{\phi}\sin\theta + I\left(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta\right)\right]\mathbf{e}_\phi \\ &+ \left[I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\dot{\theta} - I\left(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta\right)\right]\mathbf{e}_\theta\,,\end{aligned} \tag{4.90}$$

from which we can deduce via N2 that

$$\begin{aligned}\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta + \ddot{\psi} &= 0\,, \\ I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\dot{\theta} - I\left(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta\right) &= 0\,, \\ I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\dot{\phi}\sin\theta + I\left(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta\right) &= mgR\sin\theta\,.\end{aligned} \tag{4.91}$$

The first of these equations yields

$$\frac{d}{dt}\left(\dot{\phi}\cos\theta + \dot{\psi}\right) = 0\,, \tag{4.92}$$

which implies that the component of the angular velocity vector along the spin axis of the top, $\omega_3 \equiv \dot{\phi}\cos\theta + \dot{\psi}$, is a constant. One can substitute this into the second equation to yield

$$I_3\omega_3\dot{\theta} - \frac{I}{\sin\theta}\frac{d}{dt}\left(\dot{\phi}\sin^2\theta\right) = 0 \tag{4.93}$$

which implies that

$$I_3\omega_3\cos\theta + I\dot{\phi}\sin^2\theta \equiv bI\,, \tag{4.94}$$

is a constant. We can rearrange this so that

$$\dot{\phi} = \frac{bI - I_3\omega_3\cos\theta}{I\sin^2\theta} = \frac{b - a\cos\theta}{\sin^2\theta}\,, \tag{4.95}$$

where $a = I_3\omega_3/I$. The two constant parameters a and b can be used to define the dynamics of the system. Below we will define $L = I_3\omega_3 = Ia$ and $K = Ib$. The first is the angular momentum of the system about the spin axis (the 3-axis) of the system and the second is due to the conservation of angular momentum in the direction of $\mathbf{e}_\theta$. The final equation of motion can be rewritten as

$$\ddot{\theta} = \sin\theta\left(\frac{1}{2}\beta - a\dot{\phi} + \dot{\phi}^2\cos\theta\right)\,, \tag{4.96}$$

where $\beta = 2mgR/I$.

Now let us consider a case where $\theta = \theta_0$ is a constant where there can be a spinning and precessing solution with constant $\dot{\psi} = \Omega_{\rm S}$ and $\dot{\phi} = \Omega_{\rm P}$ with

$$\Omega_{\rm S} = \frac{I}{I_3}a - \Omega_{\rm P}\cos\theta_0\,, \tag{4.97}$$

and $\Omega_{\rm P}$ is a solution of the quadratic equation

$$\cos\theta_0\Omega_{\rm P}^2 - a\Omega_{\rm P} + \frac{1}{2}\beta = 0\,, \tag{4.98}$$

that is, there are two solutions

$$\Omega_{\rm P} = \frac{a}{2\cos\theta_0}\left[1 \pm \left(1 - \frac{2\beta\cos\theta_0}{a^2}\right)^{\frac{1}{2}}\right]\,, \tag{4.99}$$

when $\cos\theta_0 < a^2/(2\beta) = I_3^2\omega_3^2/(4mgRI)$. If $a^2 > 2\beta$ then there appears to always be such a solution, whereas if $a^2 < 2\beta$ then there exists a critical angle $\theta_{\rm crit} = \cos^{-1}[a^2/(2\beta)]$ where $\theta_0 > \theta_{\rm crit}$ - that is there is no solution with $\theta_0 = 0$. Alternatively, this can be seen as condition on a, or more physically

$$\omega_3 > \frac{\sqrt{4mgRI\cos\theta_0}}{I_3}\,, \tag{4.100}$$

necessary to prevent the top from falling down.

In order to investigate the physical nature of the two solutions it is helpful to investigate the limit where $2\beta\cos\theta_0/a^2 \ll 1$ which is the limit where the gravitational force is relatively unimportant. The leading order behaviour of the positive root is

$$\Omega_{\rm P}^{\rm fast} \approx \frac{a}{\cos\theta_0} = \frac{I_3\omega_3}{I\cos\theta_0} = \frac{L}{I\cos\theta_0}\,, \tag{4.101}$$

whereas that for the negative root is

$$\Omega_{\rm P}^{\rm slow} \approx \frac{\beta}{2a} = \frac{mgR}{I_3\omega_3} = \frac{mgR}{L}\,. \tag{4.102}$$

These two solutions are knowns as the fast and slow solutions respectively because $\Omega_{\rm P}^{\rm fast} > \Omega_{\rm P}^{\rm slow}$ when $\cos\theta_0 > 0$. The first is exactly that which we see in free rotation - which would be expected to come up in the limit where gravity can be ignored - that is sometimes known as the "fast solution", but there is the additional "slow solution". This slow solution is that which you studied in the Dynamics course last semester, for example, on the final problem sheet of that course. Hopefully, it is physically intuitive that a sufficiently fast rotation can lead to a situation where the existence of gravity can be ignored.

In question 2 of the exercises below you are asked to show that the rate of change of the rotational kinetic energy T satisfies $\dot{T}_R = \boldsymbol{\omega} \cdot \mathbf{M}$ for a general rotating system. In the case under consideration here we have that

$$\boldsymbol{\omega} \cdot \mathbf{M} = mgR\dot{\theta}\sin\theta = -mgR\frac{d}{dt}\cos\theta\,. \tag{4.103}$$

Using this we can deduce that $E = T_R + V = T_R + mgR\cos\theta$ is constant and can be interpreted as energy of the system. It can be written as

$$E = \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 + \frac{1}{2}I\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgR\cos\theta\,. \tag{4.104}$$

If we define

$$\bar{E} = \frac{E - \frac{1}{2}I_3\omega_3^2}{I} = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2 + \frac{mgR}{I}\cos\theta\,, \tag{4.105}$$

so that we can define an effective potential such that $\bar{E} = \frac{1}{2}\dot{\theta}^2 + U_{\rm eff}(\theta)$ where

$$U_{\rm eff}(\theta) = \frac{1}{2}\left[\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2 + \beta\cos\theta\right]\,. \tag{4.106}$$

So we see that the general motion of the top can be described by an effective 1D motion in the θ direction described by the effective potential $U_{\rm eff}(\theta)$. Analysis of this effective potential is best done by making the substitution $u = \cos\theta$ and this is the subject of exercise 3 below. To finish the lecture and this section of the course we will just describe the key features.

- The initial state of the system is governed by conserved quantities L, K and E, or alternatively a, b and $\bar{E}$. Some books (and indeed the exercise below) rename $\bar{E}$ to be $\alpha = 2\bar{E}$. The parameter β is set by the characteristics of the top.
- Typically the motion of θ takes place in the range $\theta_1 < \theta < \theta_2$ which are computed by setting $\dot{\theta} = 0$, that is, $\bar{E} = U_{\rm eff}(\theta)$.
- The particle will *nutate* (from "nodding") between θ_1 and θ_2. The possible patterns of *nutation* are governed by signs of the precession frequency $\Omega_{\rm P}$ at θ_1 and θ_2. These patterns and descriptions are illustrated on the lecture slides.
- Uniform precession ($\Omega_{\rm P}$ constant) can be achieved by deriving a condition that $\dot{\theta} = \ddot{\theta} = 0$ at $\theta = \theta_0$ which amounts to $\bar{E} = U_{\rm eff}(\theta_0)$ and $\frac{dU_{\rm eff}}{d\theta}(\theta_0) = 0$. After some algebra this yields back (4.98).
- One can impose that $\theta = 0$ is one of the solutions. In which case this solution is stable if the condition (4.100) is satisfied.

END OF 16TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 16 Exercises

1. Consider a pencil (assumed to be a cylinder) of length $h = 15\,\text{cm}$ and radius $r = 5\,\text{mm}$ which is spinning about the symmetry axis close to the vertical. Calculate the component of the angular velocity vector in the direction of symmetry axis that would be required for it to not fall over.

2. Show that the rate of change rotation kinetic energy of body in rigid rotation can be written as

$$\dot{T}_R = \boldsymbol{\omega} \cdot \mathbf{M}\,, \tag{4.107}$$

where $\boldsymbol{\omega}$ is the angular velocity vector and $\mathbf{M}$ is the applied torque.

3. Consider a heavy spinning top. Make the substitution $u = \cos\theta$ in the effective potential (4.106) and show that

$$\dot{u}^2 = f(u) = (1 - u^2)(\alpha - \beta u) - (b - au)^2\,, \tag{4.108}$$

and

$$\Omega_{\mathrm{P}} = \dot{\phi} = \frac{b - au}{1 - u^2} \tag{4.109}$$

where $\alpha = 2\bar{E}$ and one is only interested in the region $-1 \leq u \leq 1$. There are four parameters governing the system α, β, a and b and the function $f(u)$ is a cubic and therefore there are three roots: $u_1 < u_2 < u_3$.

(i) By investigating the behaviour of the function as $|u| \to \infty$ and $f(\pm 1)$ plot the function and explain why $u_3 \geq 1$. Deduce that the general motion of the heavy spinning top involves nutation between $u_1 \leq u \leq u_2$ unless $u_3 = 1$.

(ii) By considering the sign of $\dot{\phi}$ at u_1 and u_2 delineate the three cases of nutation illustrated in the lectures.

(iii) Derive conditions for θ to be constant and show that in this case

$$u\Omega_{\mathrm{P}}^2 - a\Omega_{\mathrm{P}} + \frac{1}{2}\beta = 0 \tag{4.110}$$

What is the condition that there is a solution with $\theta \equiv \pi/2$?

(iv) If we impose that $\theta = 0$ is a solution then $f(1) = 0$ and $f'(1) = 0$. In this case find the roots of $f(u)$ and derive the condition that the top is stable.

Section 3 in a Nutshell - what to remember

- Moment of inertia is a symmetric tensor, or equivalent a symmetric matrix. For a continuous distribution
$$I_{ij} = I_{ji} = \int dM \left(r^2\delta_{ij} - r_i r_j\right) \,, \tag{4.111}$$
where $dM = \rho dV$ or σdA or $\mu d\ell$ for volume, surface or line densities. For a discrete distribution
$$I_{ij} = \sum_{k=1}^{N} m_k \left(|\mathbf{r}^{(k)}|^2\delta_{ij} - r_i^{(k)} r_j^{(k)}\right) \,. \tag{4.112}$$
Learn and understand techniques for calculating the moment of inertia.

- Parallel axis theorem: $I'_{ij} = I_{ij} + M(|\mathbf{R}|^2 - R_i R_j)$ where M is the total mass, I_{ij} is the moment of inertia abiout the centre of mass and $\mathbf{R}$ is the position vector relative to the centre of mass.

- Perpendicular axis theorem: for a lamina $I_{33} = I_{11} + I_{22}$.

- Rotational angular momentum $L_i = I_{ij}\omega_j$ and rotational kinetic energy $T_{\mathrm{R}} = \frac{1}{2}I_{ij}\omega_i\omega_j$ using the summation convention. These can be written in matrix notation as
$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = L_i = I_{ij}\omega_j = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \,, \tag{4.113}$$
and
$$T = \frac{1}{2}\omega_i I_{ij}\omega_j = \frac{1}{2}\begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix} \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \,. \tag{4.114}$$

- Symmetric square matrix, A can be diagonalised using $P = \left(\mathbf{E}^{(1)}, \ldots, \mathbf{E}^{(n)}\right)$ where $A\mathbf{E}^{(k)} = \lambda^{(k)}\mathbf{E}^{(k)}$ are defines the eigenvectors and eigenvalues of A with $P^{-1}AP = D = \mathrm{diag}(\lambda^{(1)}, \ldots, \lambda^{(n)})$. This can be chosen to an orthogonal diagionalization with $\mathbf{E}^{(k)} \cdot \mathbf{E}^{(m)} = \delta^{km}$ and $P^{-1} = P^T$. If eigenvalues are degenerate then one has to be careful in the choice of eigenvectors in the degenerate surface.

- Since the moment of inertia matrix is symmetric it can be diagonalized and the eigenvectors define the principal axes of the rigid body. The three principal moments of inertia I_1, I_2 and I_3 are the eigenvalues of the moment of inertia matrix.

- Euler's equations: $\mathbf{M} = \dot{\mathbf{L}}|_S = \dot{\mathbf{L}}|_{S'} + \boldsymbol{\omega} \times \mathbf{L} = I\dot{\omega} + \boldsymbol{\omega} \times (I\boldsymbol{\omega})$.

$$\begin{aligned} M_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\ M_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 \\ M_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2\,. \end{aligned} \tag{4.115}$$

- Symmetric free top ($\mathbf{M} = 0$) with spin frequency ω_{t} and $I_3 > I$ will precess with frequency

$$\Omega = \left(\frac{I_3 - I}{I}\right)\omega_{\mathrm{t}}\,. \tag{4.116}$$

- Intermediate axis theorem: If $I_1 < I_2 < I_3$ then rotations are unstable about the $2-$axis.

- Heavy spinning tops, or gyroscopes, can exhibit fast/slow precession and nutation.