ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Thermal and Statistical Physics

16th May 2018, 2.00 p.m. – 3.30 p.m.

Answer **ALL** parts of question 1 and **TWO** other questions

---

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. a) Describe what is meant by a function of state. Explain why any function of state for an ideal gas can be written in the form $f(T, V)$.

[5 marks]

b) What are the SI units of the following quantities:

(i) heat, Q;

(ii) surface tension, σ;

(iii) chemical potential, μ;

(iv) partition function, Z;

(v) density of states, $D(k)$?

[5 marks]

c) (i) Draw a schematic diagram indicating the flow of heat and work in a heat pump.

(ii) A heat pump needs to deliver 10 kW of heat to a building to maintain its temperature at 20°C. The heat source is a river at 10°C. How much power must be supplied to the motor if the pump's efficiency is 90% of the maximum achievable?

[5 marks]

d) (i) Distinguish between macrostates and microstates.

(ii) State the definition of Boltzmann entropy, clearly defining any symbols you use.

[5 marks]

e) A particle has two energy states, $\varepsilon_0 = 0$ eV, and $\varepsilon_1 = 1$ eV. It is in thermal equilibrium with a heat bath.

Calculate the temperature the heat bath needs to have so that the average energy of the particle is 0.25 eV.

[5 marks]

2. a) Define the Gibbs free energy for a hydrostatic system and derive the Maxwell relation

$$\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P .$$

[7 marks]

b) Using the result of part a) show that, for any substance, a change of entropy $\mathrm{d}S$ is related to changes of temperature $\mathrm{d}T$ and pressure $\mathrm{d}P$ by

$$\mathrm{d}S = \frac{C_P}{T}\,\mathrm{d}T - \left(\frac{\partial V}{\partial T}\right)_P \mathrm{d}P,$$

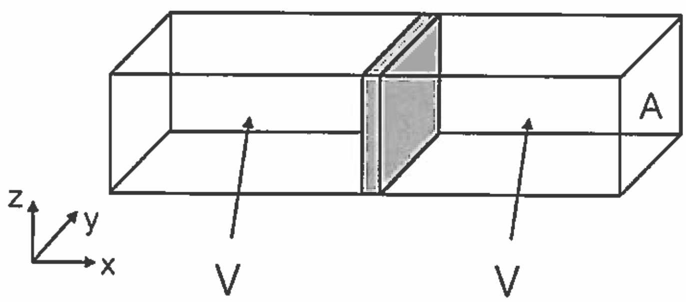
where C_P is the heat capacity at constant pressure of the substance.

[4 marks]

c) Using the result of b) show that $PV^{5/3} = \text{const.}$ for adiabatic changes of a monoatomic ideal gas. [You may use, without derivation, the known value of C_P for the ideal gas.]

[6 marks]

d) The thermodynamic system illustrated in the diagram consists of two subsystems, isolated from the environment. Initially, each subsystem has volume V, and contains one mole of an ideal gas at pressure P, and with adiabatic index $\gamma = C_P/C_V$, where C_V is the heat capacity at constant volume. The partition between the subsystems is adiabatic and impermeable; it has mass m, and it can move along the x-direction (see diagram). The cross section of the container in the $y - z$ plane is A.



The partition is displaced by a distance Δx from the equilibrium point.

(i) Explain briefly why the partition will undergo oscillatory motion.

(ii) Assuming the initial displacement is small, derive an expression for the angular frequency of these oscillations.

(iii) Explain how this setup can be used to determine the adiabatic index γ of a gas.

[2+5+1 marks]

3. a) Briefly state what physical systems are described by the microcanonical and the canonical ensembles, respectively.

For each of the two ensembles write down the probability for a system to be in a given microstate.

[5 marks]

b) A system is composed of two distinguishable non-interacting particles. Each particle can be in one of three energy levels, with energies $\varepsilon_0 = 0,\ \varepsilon_1 = \varepsilon$, and $\varepsilon_2 = -\varepsilon$, where $\varepsilon > 0$.

Determine all macrostates of this system, and the microstates that belong to each macrostate.

[5 marks]

c) The N lattice sites in a paramagnetic material each can have three possible states in which the component of the magnetic moment parallel to an applied field of strength B takes the values $-\mu, 0, \mu$. The material is in thermal equilibrium at temperature T.

(i) Compute the single-particle partition function, and determine the probabilities that a given lattice site will be in each of the three states.

(ii) Compute the mean magnetic moment per lattice site. Use this to show that the magnetisation of the sample is

$$M = \frac{2N\mu \sinh(\beta\mu B)}{V[1 + 2\cosh(\beta\mu B)]},$$

where V is the volume of the sample, and $\beta = 1/(k_B T)$.

(iii) Determine the probability of finding a given lattice site in the state with magnetic moment μ as $T \to 0$ and $T \to \infty$ respectively. For each case calculate the limiting value of the magnetisation M. *(You do not need to determine the functional form in which these limits are approached.)*

[5+5+5 marks]

4. [In this question you may use without proof the integrals

$$\int_{-\infty}^{\infty} e^{-\alpha u^2} du = \sqrt{\frac{\pi}{\alpha}}, \quad \int_{-\infty}^{\infty} u^2 e^{-\alpha u^2} du = \frac{\pi^{1/2}}{2\alpha^{3/2}},$$

valid for $\alpha > 0$. The question can also be answered without use of these integrals.]

a) A classical one-dimensional harmonic oscillator is held in thermal equilibrium at temperature T. *Without explicit calculation* indicate the mean energy of the oscillator, and briefly justify your answer.

[4 marks]

b) A particle of mass m moves in one dimension, its spatial coordinate is $x \geq 0$, and its potential energy is $U(x)$. Starting from the Boltzmann distribution at temperature T show that the probability distribution for x is given by

$$P(x) = Ce^{-\frac{U(x)}{k_B T}} \quad (x \geq 0),$$

and find an expression for the constant C.

[6 marks]

c) Consider a gas in a gravitational field resulting in an acceleration g. The mass of each particle in the gas is m. The temperature T is independent of the height.

Using the result of part b) show that the probability density of finding a particle at height z is

$$P(z) = \frac{mg}{k_B T} e^{-\frac{mgz}{k_B T}}.$$

[5 marks]

d) (i) The random quantity x has a mean of μ and a variance of σ^2. Calculate the variance of the quantity $y = ax + b$, where a and b are constants.

(ii) Consider an ideal gas in thermal equilibrium at temperature T. Each particle in the gas is a source of radiation of frequency f_0 in the rest frame of the particle. The particles move in a container of volume V, with no potential energy. Emitting the radiation does not affect their energy.

Due to the Doppler effect the observed frequency f of light emitted from a particle moving with velocity v_x along the line of sight of an observer is

$$f = f_0 \left(1 + \frac{v_x}{c}\right),$$

where c is the speed of light.

As a function of temperature, calculate the line width defined as the variance $\langle (f - f_0)^2 \rangle$ as seen by the external observer.

[5+5 marks]

END OF EXAMINATION PAPER