ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Vector Spaces

0th mm 2022, aa.aa–bb.bb

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. a) Find the derivative of z^3 by using the definition of the derivative of a function of a complex variable z. Explain briefly why this method cannot be used to differentiate $\overline{z}$.

[6 marks]

b) Evaluate $\int_C z^{-1}, \mathrm{d}z$, where the endpoints of the contour C are $a = 1$ and $b = i$, and the path is a circular arc centred on $z = 0$. Would you obtain a different result by integrating along a straight line from a to b? Explain your answer carefully.

[6 marks]

c) For the function

$$f(z) = \frac{z}{(z-2)(z+1)},$$

derive the Laurent expansion that is valid in the region $1 < |z| < 2$.

[7 marks]

d) Define what is meant by the linear independence of a set of N vectors $\{|a_i\rangle\}$. Determine whether or not the three vectors $|a_1\rangle = (7, 3, 1)$, $|a_2\rangle = (3, 1, 3)$, $|a_3\rangle = (2, 1, -1)$ are linearly independent.

[6 marks]

2. A cylindrical capacitor consists of concentric conducting cylinders with radii a and b, where $b > a$, which are held at potentials 0 and V_0, respectively.

a) Show that the function $Z = \ln z$ maps circles in the z-plane into parallel straight lines in the Z-plane. Illustrate this result by detailed, labelled sketches of the original physical problem and its mapping into the Z-plane.

[7 marks]

b) Write down an expression for the electric field $\boldsymbol{\mathcal{E}}$ in the Z-plane, and hence derive an expression for the complex potential $w = u + iv$ as a function of Z. Use your result to obtain u and v as functions of z.

[6 marks]

c) Use your results from part (b) to obtain the direction and magnitude of the electric field in the cylindrical capacitor.

[3 marks]

d) Consider a general electrostatics problem solved by conformal mapping with complex potential $w(z)$.

Describe how the electric field lines are related to the complex potential. Prove that the magnitude of the electric field, E, is given in terms of w by $E(z) = |\mathrm{d}w/\mathrm{d}z|$. Use this result to relate $E(z)$ to the magnitude of electric field, $\mathcal{E}(Z)$, in the transformed problem in the Z-plane.

[6 marks]

Verify your relation for the case of the cylindrical capacitor by using the function $\mathcal{E}(Z)$ you wrote down in part (b) to derive the result for $E(z)$ obtained in part (c).

[3 marks]

3. Use contour integration to evaluate the following two real integrals. In each case, sketch the contour you use and briefly justify your choice.

a) the principal-value integral $\displaystyle ⨍_{-\infty}^{\infty} \frac{x}{(x-1)(x^2+4)}\,\mathrm{d}x\,.$

[11 marks]

b) $\displaystyle \int_0^{2\pi} \frac{\cos\theta}{13 + 12\cos\theta}\,\mathrm{d}\theta.$

[14 marks]

4. An operator $\widehat{S}$ acts on a two-dimensional complex vector space. With respect to a particular orthonormal basis, it is represented by the matrix

$$\mathbf{S} = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}.$$

A second operator $\widehat{R}(\alpha)$ is defined by

$$\widehat{R}(\alpha) = \exp\big(i\alpha\widehat{S}\big),$$

where α is a real parameter.

a) Show that $\widehat{S}$ is both Hermitian and unitary.

[3 marks]

b) Show that the eigenvalues of $\widehat{S}$ are ± 1 and find its normalized eigenvectors, working in the basis used to define $\mathbf{S}$.

[8 marks]

c) Find $\widehat{S}^2$ and hence show that $\widehat{R}(\alpha)$ can be expressed in the form

$$\widehat{R}(\alpha) = \cos\alpha\,\widehat{I} + i\sin\alpha\,\widehat{S}\,,$$

where $\widehat{I}$ is the identity operator. Show also that $\widehat{R}(\alpha)$ is unitary for all α.

[9 marks]

d) Find the eigenvalues and eigenvectors of $\widehat{R}(\alpha)$. [You should not need to solve another matrix eigenvalue equation for this.] Verify that the eigenvalues and eigenvectors have the properties expected for a unitary operator.

[5 marks]

END OF EXAMINATION PAPER