<u>ONE HOUR THIRTY MINUTES</u>

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Vector Spaces

[redacted] 2024, [redacted] - [redacted]

Answer **<u>ALL</u>** parts of question 1 and **<u>TWO</u>** other questions

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The use of calculators is permitted, as long as they cannot store text or perform algebra, and have no graphing capability.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. a) State the conditions a set of vectors must satisfy to be a basis for a given vector space V^N. For the vector space $\mathbb{R}^3$ with orthonormal basis $\{|e_j\rangle\}_{j=1}^3$, determine whether the following vectors

$$\begin{aligned}|v_1\rangle =&|e_1\rangle - 2|e_2\rangle + 3|e_3\rangle,\\ |v_2\rangle =&2|e_1\rangle + 5|e_2\rangle + 7|e_3\rangle,\\ |v_3\rangle =&-2|e_1\rangle + 4|e_2\rangle - 6|e_3\rangle,\end{aligned}$$

form a basis. Justify your answer.

[4 marks]

b) State whether each of the following operators is unitary, Hermitian, both or neither.

$$\text{(i)}\ \ \mathbf{L} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \text{(ii)}\ \ \mathbf{O} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

[4 marks]

c) An operator $\hat{A}$ is said to be anti-Hermitian if $\hat{A}^\dagger = -\hat{A}$. From the definition of the adjoint, show that the eigenvalues of $\hat{A}$ are pure imaginary.

[6 marks]

d) The contour Γ is given by a semi-circle of radius $|z| = R$ centred at the origin in the upper half plane. Integrate the function $f(z) = |z|^2$ from $z = -R$ to R along Γ. State without proof whether the value of this integral could change if an alternative path is used. Justify your answer.

[6 marks]

e) Using the residue theorem or otherwise evaluate

$$I_C = \oint_C \frac{1}{z}\,\mathrm{d}z,$$

where C is a circular contour with radius $|z| = R$ centred at the origin.
Can the function $g(z) = z^{-1/2}$ be integrated along the same contour? Justify your answer.

[5 marks]

2. All vector spaces may be assumed to be finite dimensional for this question.

a) i) Show that the trace of a linear operator is basis invariant.

[4 marks]

ii) A Hermitian operator $\hat{A}$ on a finite dimensional vector space has eigenvalues λ_j and normalised eigenvectors $|\lambda_j\rangle$. Show that the trace of $\hat{A}$ is given by

$$\mathrm{Tr}(\hat{A}) = \sum_j \lambda_j.$$

[3 marks]

iii) Consider an operator that scales one component of a given orthonormal basis by a real number, while leaving all other components unchanged. Can we be sure this operator is Hermitian? Justify your answer.

[4 marks]

b) Consider the vector space $\mathbb{R}^2$ which has orthonormal basis vectors $\{|x\rangle, |y\rangle\}$. A Hermitian operator ρ can be written as

$$\rho = \frac{3}{8}(|x\rangle\langle x| + |y\rangle\langle y|) + \frac{1}{8}(|x\rangle\langle y| + |x\rangle\langle y|).$$

i) Write the matrix representation of ρ with respect to the $\{|x\rangle, |y\rangle\}$ basis vectors.

[2 marks]

ii) Find the eigenvalues and normalised eigenvectors of ρ.

[7 marks]

iii) Hence or otherwise calculate

$$S(\rho) = -\,\mathrm{Tr}(\rho \log_2 \rho).$$

[5 marks]

3. a) An analytic function $f(z) = u + iv$, where $z = x + iy$, has real part $u(x, y) = x^2 + 2x - y^2$.

i) Show that $u(x, y)$ satisfies Laplace's equation.

[2 marks]

ii) Find $v(x, y)$, the imaginary part of $f(z)$, given that $v(0, 0) = 0$.

[6 marks]

iii) Express $f(z)$ in terms of z.

[3 marks]

b) In the remainder of this question we consider the mapping $w = u + iv = 1/z$, where $z = x + iy$.

i) Show that the line $x = 0$, $y \neq 0$ is mapped into a straight line in the w plane.

[2 marks]

ii) By considering the definition $u = \mathrm{Re}(1/z)$, show that lines of constant $u \neq 0$ are circles in the z plane, described by $(x - x_0)^2 + y^2 = R^2$. Write down expressions for x_0 and R in terms of u.

[4 marks]

iii) By using the results from parts (i) and (ii), sketch lines of constant u in the xy plane for the cases $u = 0$, $\pm\frac{1}{2}$ and $\pm\frac{1}{4}$.

[5 marks]

iv) Describe an electrostatics problem for which $u(x, y)$ would be the potential. For the problem you suggest, specify the conductors and their potentials.

[3 marks]

4. In the following problems involving contour integration, the contour should be sketched and the positions of any poles indicated on your sketch. If Jordan's lemma is used, show that the conditions for its validity are satisfied.

a) Evaluate the following integral,

$$\int_0^\infty \frac{x^a}{x+1}\,\mathrm{d}x,$$

where $-1 < a < 0$.

[11 marks]

b) A function f of the complex variable z is defined by $f(z) = \dfrac{1}{z\cos(\pi z)}$.

i) Find the poles and residues of $f(z)$.

[4 marks]

ii) By considering the integral of $f(z)$ around a suitable contour, show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

[10 marks]

END OF EXAMINATION PAPER