ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Integral Transforms

Tuesday 14th May, 2015
09:45 a.m. – 11.15 a.m.

Answer **ALL** parts of question 1 and **TWO** other questions

---

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

P.T.O.

1. (a) For a function $f(z) = u(x, y) + iv(x, y)$ (where $z = x + iy$), state the Cauchy-Riemann equations. Choose one of them and show that it is satisfied for $f(z) = z^{-2}$ $(z \neq 0)$.

[6 marks]

(b) Evaluate $\int_C |z|^2 \, \mathrm{d}z$, where C is the closed path which runs from $a = 0$ to $b = 1$ and then to $c = 1 + i$ in two straight-line segments, and returns from c to a along the line $y = x^2$.

[8 marks]

(c) Show by explicit integration that $\oint_C \frac{1}{z-a} \, \mathrm{d}z = 2\pi i$, where C is a unit circle about $z = a$.

[5 marks]

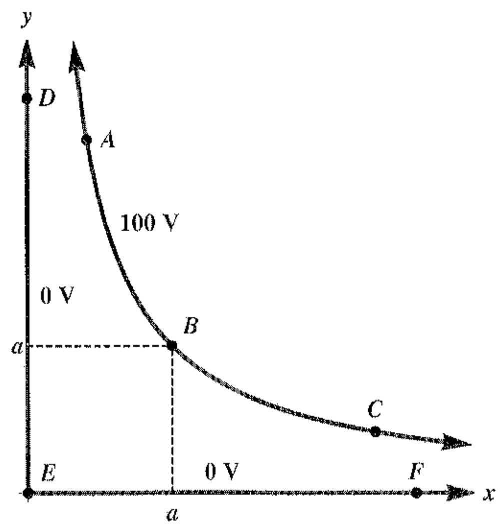
(d) Sketch the following regions of, or paths in, the complex plane:
i) $1 < |z - 1 - i| < 2$; ii) $0 < \mathrm{Arg}[z + i] < \pi/4$; iii) $|z| = |z + i|$.

[6 marks]

2. (a) An analytic function $f(z)$ has real part $u(x, y) = \cos x \cosh y$. Show that $u(x, y)$ is harmonic, and find the corresponding imaginary part of $f(z)$, if $f(0) = 1$. Express $f(z)$ in terms of z.

[12 marks]

(b) The diagram below shows a cross section through two infinite conducting metal plates which are perpendicular to the xy plane. One, bent by 90°, lies along the positive x and y axes, and is held at 0 V. The other has the form of a hyperbola, symmetric about the line $x = y$ and passing through the point (a, a); this plate is held at 100 V. The diagram shows the location of three points on each plate; the exact coordinates of points A, C, D and F do not matter.



If the plane is interpreted as the complex plane, and we consider the function $f(z) = \left(\frac{z}{a}\right)^2$, show that the two lines are lines of constant $\mathrm{Im}[f(z)]$, and find the corresponding values.

[6 marks]

Hence show that under the mapping $Z = \left(\frac{z}{a}\right)^2$, the two lines map to straight lines in the Z plane, and indicate the position of the images of the marked points A-F. Use this to show that the potential between the plates in the original problem is $\phi(x, y) = 100xy/a^2$. Sketch some equipotential and field lines.

[7 marks]

3. Use Cauchy's residue theorem and a suitable choice of contour to do <u>TWO</u> of the following integrals. If Jordan's lemma is used, ensure that all the conditions for its validity are satisfied.

(a)

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\, d\theta\,,$$

(b)

$$\int_0^{\infty} \frac{1}{x^{\alpha}(1+x)}\, dx \qquad \text{for non-integer } 0 < \alpha < 1\,,$$

(c)

$$\int_{-\infty}^{\infty} \frac{x\sin(kx)}{(1+x^2)}\, dx \qquad \text{for } k > 0\,.$$

[25 marks]

4. (a) Find the Taylor or Laurent series, as appropriate, of the function $\dfrac{z}{(z+1)(z-2)}$:

i) about $z = 0$, for the region $|z| < 1$;
ii) about $z = 2$, for the region $|z - 2| > 3$.

[8 marks]

(b) Find the first three terms of the Taylor-Laurent series about $z = 0$ of $z^{-2}\operatorname{cosec} z$ (you may use standard Taylor series for trigonometric functions). What is the nature of the singularity and the value of the residue? What is the radius of convergence of the series?

[6 marks]

(c) Show that, if $F(s)$ is the Laplace transform of $f(t)$,

$$\text{L.T.}\left[\frac{\mathrm{d}^3 f}{\mathrm{d}t^3}\right] = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)\,,$$

and

$$\text{L.T.}\,[t \sin t] = \frac{2s}{(s^2+1)^2}\,.$$

Hence solve the differential equation

$$\frac{\mathrm{d}^3 f}{\mathrm{d}t^3} = t \sin t$$

subject to the boundary conditions $f(0) = 0$, $f'(0) = -2$, $f''(0) = 0$.

[You may use without proof the result that the inverse Laplace transform of $\dfrac{s^2-1}{(s^2+1)^2}$ is $t\cos t$.]

[11 marks]

END OF EXAMINATION PAPER

Physical Constants and Measured Values

Symbol	Description	Numerical Value
c	Speed of light in vacuum	299 792 458 m s^{-1}, exactly
μ_0	Permeability of vacuum	$4\pi \times 10^{-7}$ N A^{-2}, exactly
ϵ_0	Permittivity of vacuum where $c = 1/\sqrt{\epsilon_0\mu_0}$	8.854×10^{-12} C^2 N^{-1} m^{-2}
h	Planck constant	6.626×10^{-34} J s
$\hbar$	$h/2\pi$	1.055×10^{-34} J s
G	Gravitational constant	6.674×10^{-11} m^3 kg^{-1} s^{-2}
e	Elementary charge	1.602×10^{-19} C
eV	Electron volt	1.602×10^{-19} J
α	Fine structure constant, $\frac{e^2}{4\pi\epsilon_0\hbar c}$	$\frac{1}{137.0}$
m_e	Electron mass	9.109×10^{-31} kg
$m_e c^2$	Electron rest-mass energy	0.511 MeV
μ_B	Bohr magneton, $\frac{e\hbar}{2m_e}$	9.274×10^{-24} J T^{-1}
R_∞	Rydberg energy, $\frac{\alpha^2 m_e c^2}{2}$	13.61 eV
a_0	Bohr radius $\frac{1}{\alpha}\frac{\hbar}{m_e c}$	0.5292×10^{-10} m
Å	Ångström	10^{-10} m
m_p	Proton mass	1.673×10^{-27} kg
$m_p c^2$	Proton rest-mass energy	938.272 MeV
$m_n c^2$	Neutron rest-mass energy	939.565 MeV
μ_N	Nuclear magneton, $\frac{e\hbar}{2m_p}$	5.051×10^{-27} J T^{-1}
fm	Femtometer or Fermi	10^{-15} m
b	Barn	10^{-28} m^2
u	Atomic mass unit, $\frac{1}{12}m(^{12}\mathrm{C}$ atom)	1.661×10^{-27} kg
N_A	Avogadro constant, atoms in mole	6.022×10^{23} mol^{-1}
T_t	Triple-point temperature	273.16 K exactly
k	Boltzmann constant	1.381×10^{-23} J K^{-1}
R	Molar gas constant, $N_A k$	8.314 J mol^{-1} K^{-1}
σ	Stefan-Boltzmann constant, $\frac{\pi^2}{60}\frac{k^4}{\hbar^3 c^2}$	5.670×10^{-8} W m^{-2} K^{-4}
M_E	Mass of Earth	5.972×10^{24} kg
R_E	Mean radius of Earth	6.371×10^6 m
g	Standard acceleration due to gravity	9.806 65 m s^{-2}, exactly
atm	Standard atmosphere	101 325 Pa, exactly
$M_\odot$	Solar mass	1.989×10^{30} kg
$R_\odot$	Solar radius	6.955×10^8 m
$L_\odot$	Solar luminosity	3.84×10^{26} W
$T_\odot$	Solar effective temperature	5.78×10^3 K
AU	Astronomical unit, mean Earth-Sun distance	1.496×10^{11} m
pc	Parsec	3.086×10^{16} m
	Year	3.156×10^7 s