

# Mathematics of Waves and Fields

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# Chapter 1

## Partial Differential Equations

### 1.1 Separation of Variables

Suppose we have a PDE whose solution is in the form,  $u(r_1, r_2, \dots, r_n)$  where there are  $n$  co-ordinates  $r_i$ , then we can solve the PDE by separation of variables by assuming a solution of the form,

$$u(r_1, r_2, \dots, r_n) = R_1(r_1)R_2(r_2) \cdots R_n(r_n). \quad (1.1)$$

This will turn a compatible PDE into an ODE.

#### 1.1.1 Specific solutions

We are often most interested in the specific solutions to a wave equation. In order to get a specific solution, constraints/boundary conditions must be provided. The general method is as follows,

1. Use separation of variables;
2. Build superpositions of solutions;
3. Apply boundary conditions and find appropriate constants.

### 1.2 Series Solutions

The general steps to solving an ODE using this method are,

1. Assume a series solution of the form,

$$y = \sum_{n=0}^{\infty} a_n x^n \quad (1.2)$$

2. Obtain the recurrence relation.

## Chapter 2

# Fourier Series

Given a periodic function  $f(x)$  with period  $2L$  in the range  $-L \leq x \leq L$ , the Fourier expansion is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad (2.1)$$

for,

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad (2.2)$$

In order to expand a function it must meet *Dirichlet's Conditions*, so the function must,

1. be single valued,
2. have a finite number of discontinuities,
3.  $\int_{-L}^L |f(x)| dx$  must be finite.

We say that  $\sin \frac{n\pi x}{L}$  and  $\cos \frac{n\pi x}{L}$  form a *complete, orthogonal basis*. And furthermore, the Fourier series allows *an expansion of a function on a set of orthogonal basis functions*.

### 2.1 Exponential Fourier Series

We can further write the Fourier expansion in terms of complex exponentials,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp \left( i \frac{n\pi x}{L} \right) \quad (2.3)$$

for,

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) \exp \left( -i \frac{n\pi x}{L} \right) dx \quad (2.4)$$

From our complex definition of the Fourier series, we can say that 2 complex functions  $u(z)$  and  $v(z)$  are orthogonal on the interval  $a \leq z \leq b$  if,

$$\int_a^b u(z)v(z) dz = 0. \quad (2.5)$$

## Chapter 3

# Fourier Transform

If we wish to analyse non-periodic functions, we can take the limit of our range,  $\lim_{L \rightarrow \infty}(-L, L)$ . Let us write,

$$k_n = \frac{n\pi}{L} \qquad \Delta k = \frac{\pi}{L} \quad (3.1)$$

so,

$$(3.2)$$

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) \exp(-ik_n x) dx. \quad (3.3)$$

Let us write  $F(k) = 2Lac_n$ ,

$$F(k_n) = a \int_{-L}^L f(x) \exp(-ik_n x) dx \quad (3.4)$$

$$\implies f(x) = \frac{1}{2\pi a} \sum_{n=-\infty}^{\infty} F(k_n) \exp(ik_n x) \Delta k. \quad (3.5)$$

In the limit of  $L \rightarrow \infty$ , we obtain,

$$\boxed{F(k) = a \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx} \qquad \text{Fourier transform of } f(x). \quad (3.6)$$

$$\boxed{f(x) = \frac{1}{2\pi a} \int_{-\infty}^{\infty} F(k) \exp(ikx) dk} \qquad \text{Inverse Fourier transform of } F(k). \quad (3.7)$$

We define constant,

$$a = \begin{cases} \text{unity} & \text{Physics} \\ \frac{1}{\sqrt{2\pi}} & \text{Maths} \end{cases}. \quad (3.8)$$

### 3.1 Manipulation of the Fourier transform

Most generally,

$$\boxed{\mathcal{F}\{f(ax-b)\} = e^{-ikb} \frac{1}{a} F\left(\frac{k}{a}\right)}. \quad (3.9)$$

### 3.2 Dirac-Delta Function

Dirac's original approximation of the  $\delta$  function used a function  $\Pi(x)$  which was defined,

$$\Pi(x) = \begin{cases} 1 & -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0 & \text{Elsewhere} \end{cases}. \quad (3.10)$$

Using this definition we can write,

$$\delta(x) = \lim_{k \rightarrow \infty} \{k \Pi(kx)\}. \quad (3.11)$$

It can also be defined using sinc,

$$\delta(x) = \lim_{k \rightarrow \infty} \left\{ \frac{k}{\pi} \frac{\sin kx}{kx} \right\}. \quad (3.12)$$

However, the most commonly used, and most applicable form is,

$$\boxed{\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx} \quad (3.13)$$

We note 4 important properties of the Dirac-Delta,

1.  $\lim_{x \rightarrow 0} \delta(x) = \infty$
2.  $\int_{-\infty}^{\infty} \delta(x) dx = 1$
3.  $\delta(ax + b) = \frac{1}{|a|} \delta\left(x + \frac{a}{b}\right)$
4.  $\delta(x) = \delta(-x)$  i.e., Dirac-Delta is even.

### 3.2.1 Fourier Transform of the Dirac Delta

The Fourier transform of a constant is the Dirac-Delta, i.e.,

$$\mathcal{F}\{1\} = \int_{-\infty}^{\infty} e^{ikx} dx = 2\pi\delta(x), \quad (3.14)$$

and similarly for the inverse Fourier Transform,

$$\mathcal{F}^{-1}\{1\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx = \delta(x). \quad (3.15)$$

Taking the Fourier transform of the Dirac-Delta,

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx = 1 \quad (3.16)$$

and thus clearly,

$$\mathcal{F}\{\delta(x - x_0)\} = \int_{-\infty}^{\infty} \delta(x - x_0) e^{-ikx} dx = e^{-ikx_0}. \quad (3.17)$$

## 3.3 Parseval's Theorem

**Theorem.**

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 dx = a \int_{-\infty}^{\infty} |F(k)|^2 dk} \quad (3.18)$$

where  $a = 1$  for mathematical symmetry, and  $a = \frac{1}{2\pi}$  for physical symmetry.

*Proof.*

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)|^2 dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k) e^{ikx} dk \int_{-\infty}^{\infty} F^*(k') e^{ik'x} dk' dx \\ &= \int_{-\infty}^{\infty} F(k) dk \int_{-\infty}^{\infty} F^*(k') dk' \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} dx}_{\delta(k-k')} \\ &= \int_{-\infty}^{\infty} |F(k)|^2 dk \end{aligned} \quad (3.19)$$

□

### 3.4 Convolution

We define the convolution  $h(x)$  of two functions  $f(x)$  and  $g(x)$  as,

$$\begin{aligned} h(x) &= f(x) * g(x) = \int_{-\infty}^{\infty} f(x-x')g(x') \, dx' \\ &= \int_{-\infty}^{\infty} f(x')g(x-x') \, dx'. \end{aligned} \quad (3.20)$$

If we define the Fourier transforms of  $f(x)$  and  $g(x)$ ,

$$F(k) = \mathcal{F}\{f(x)\} \quad G(k) = \mathcal{F}\{g(x)\} \quad (3.21)$$

then the Fourier transform of the convolution is given by,

$$\mathcal{F}\{h(x)\} = \mathcal{F}\{f(x) * g(x)\} = F(k)G(k). \quad (3.22)$$

*Proof.* Let us define  $\zeta = x - x'$ , then,

$$\begin{aligned} H(k) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dx' f(\zeta)g(x')e^{-ikx} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx' \, d\zeta f(\zeta)g(x')e^{-ik(\zeta+x')} \\ &= \int_{-\infty}^{\infty} g(x')e^{ikx'} \, dx' \int_{-\infty}^{\infty} f(\zeta)e^{ik\zeta} \, d\zeta \end{aligned} \quad (3.23)$$

which clearly corresponds to the product of the two transforms.  $\square$

### 3.5 Wave Packets

In 1 dimension, a forward travelling wave is defined by,

$$\phi(x, t) = e^{-i(kx - \omega t)} \quad (3.24)$$

which satisfies the 1 dimensional wave equation,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}. \quad (3.25)$$

By substituting in the travelling wave, we find,

$$k^2 = \frac{1}{c^2} \omega^2 \implies \omega = ck. \quad (3.26)$$

A plane wave  $\phi(\mathbf{x}, t) = e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$  satisfies the 3 dimensional wave equation,

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}. \quad (3.27)$$

Where we have,

$$\omega = c|k| \quad (3.28)$$

for a plane travelling along  $k$ .

Returning to the 1 dimensional wave, we can sum these travelling waves along the  $+x$  direction,

$$\phi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k) e^{ik(x-ct)} \, dk \quad (3.29)$$

where  $G(k)$  is the Fourier transform of  $\phi(x, 0)$ . This wave satisfies the wave equation as all components of the wave travel at the same velocity  $c$ . We are also able to use the wave equation to describe waves in *non-dispersive media*, i.e., those where the velocity of the waves depends on wavelength,

$$v_p(k) = \frac{\omega(k)}{k}. \quad (3.30)$$

The most general wave can be written as,

$$\boxed{\phi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k) e^{i(kx - \omega(k)t)} \, dk}. \quad (3.31)$$

### 3.5.1 Dispersion

A dispersive wave packet will have the following properties,

- The envelope wave of the wave packet will move with group velocity,

$$v_g = \frac{d\omega}{dk} = v_p + k \frac{dv_p}{dk}. \quad (3.32)$$

- The dispersive effects of the wave are a second order effect. i.e., we must expand any approximations to the second order. We will always assume  $\omega \equiv \omega(k)$ .



## Chapter 4

# Special Functions

### 4.1 Taylor Expansion

The Taylor expansion about  $x_0$  is,

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!} + \dots \\ &= \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x - x_0)^n}{n!} \end{aligned} \quad (4.1)$$

Let us then redefine this as a simple series,

$$f(x) = \sum_{n=0}^{\infty} u_n \quad (4.2)$$

we can then define the convergence criteria,

$$\lim_{n \rightarrow \infty} |r_n| = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1. \quad (4.3)$$

### 4.2 Hermit's Equation

$$\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0, \forall y, n \in \mathbb{Z} \quad (4.4)$$

We can obtain solutions to Hermit's equation by assuming a series solution,

$$y = \sum_{k=0}^{\infty} x^k. \quad (4.5)$$

Substituting this into Hermit's equation,

$$\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - 2x \sum_{k=0}^{\infty} k a_k x^{k-1} + 2n \sum_{k=0}^{\infty} a_k x^k = 0 \quad (4.6)$$

Let us shift  $k$  such that,  $k \rightarrow k+2$ ,

$$\begin{aligned} \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+1} x^k - 2 \sum_{k=0}^{\infty} k a_k x^k + 2n \sum_{k=0}^{\infty} a_k x^k &= 0 \\ \sum_{k=0}^{\infty} \{(k+2)(k+1)a_{k+2} - (2k-2n)a_k\} x^k &= 0 \end{aligned} \quad (4.7)$$

from which we obtain the recurrence relation,

$$a_{k+2} = \frac{2(k-n)}{(k+2)(k+1)} a_n. \quad (4.8)$$

Using this recurrence relation, we are able to form a solution for  $y$ , starting at  $k = 0$  and  $k = 1$  to obtain the even and odd solutions respectively,

$$y = a_0 \left[ 1 - \frac{2n}{2!}x^2 - \frac{2n(4-2n)}{4!}x^4 + \dots \right] + a_1 \left[ x + \frac{(2-2n)}{3!}x^3 + \frac{(2-2n)(6-2n)}{5!}x^5 + \dots \right]. \quad (4.9)$$

Let us note that at  $k = n$  the series will terminate.

Given the solution to Hermit's equation, by considering different values of  $n$ , we are able to obtain *Hermit's Polynomials* which are discussed in the section below.

### 4.2.1 Hermit's Polynomials

We denote Hermit's polynomials by  $y \equiv H_n(x)$ . By simply looking at eq. (4.9), we see that the first three even Hermit polynomials are,

$$H_0(x) = 1 \quad H_2(x) = 1 - 2x^2 \quad H_4(x) = 1 - 4x^2 + \frac{4}{3}x^4, \quad (4.10)$$

and the first 3 odd ones are,

$$H_1(x) = x \quad H_3(x) = x - \frac{2}{3}x^3 \quad H_5(x) = x - \frac{4}{3}x^3 + \frac{4}{5}x^5 \quad (4.11)$$

In physics, we often normalise the Hermit polynomials such that the highest order term is positive and has a coefficient  $2^n$ .

### Orthogonality of Hermit Polynomials

Hermit Polynomials satisfy the orthogonality relation,

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{x^2} dx = \sqrt{\pi} n! 2^m \delta_{nm}. \quad (4.12)$$

This means that Hermit's polynomials can be used as a basis for series expansion of a function. We can further define a normalised Hermit function,

$$\psi_m(x) = \left( \frac{1}{\sqrt{\pi} m! 2^m} \right)^{1/2} H_m e^{\frac{x^2}{2}} \quad (4.13)$$

which satisfies,

$$\int_{-\infty}^{\infty} \psi_m(x) \psi_n(x) dx = \delta_{mn} \quad (4.14)$$

## 4.3 Legendre's Equation

$$\boxed{(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \ell(\ell+1)y = 0 \quad \ell \geq 0, \ell \in \mathbb{Z}} \quad (4.15)$$

We solve Legendre's equation by series expansion, from which we obtain,

$$(1-x^2) \sum_{n=2} n(n-1) a_n x^{n-2} - 2x \sum_{n=1} n a_n x^{n-1} \ell(\ell+1) \sum_{n=0} a_n x^n. \quad (4.16)$$

Let us shift the sums so we only have terms in powers of  $n$ ,

$$\sum_{n=0} \{(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n x^n - 2na_n x^n + \ell(\ell+1)a_n\} x^n = 0. \quad (4.17)$$

From which we can easily obtain a general recurrence relation,

$$a_{n+2} = \frac{n(n+1) - \ell(\ell+1)}{(n+2)(n+1)} a_n \quad (4.18)$$

thus the solution for Legendre's equation has even and odd parts given below,

$$\begin{aligned} y = & a_0 \left[ (1 - \ell(\ell+1)) \frac{x^2}{2!} + (\ell-2)(\ell(\ell+1)(\ell+3)) \frac{x^4}{4!} + \dots \right] & \text{Even} \\ & + a_1 \left[ x - (\ell-1)(\ell+2) \frac{x^3}{3!} + (\ell-3)(\ell-1)(\ell+2) \frac{x^5}{5!} + \dots \right] & \text{Odd} \end{aligned} \quad (4.19)$$

which we can clearly see terminates at  $\ell = n$ , which allows the series to converge.

### 4.3.1 Legendre Polynomials

The steps to finding Legendre polynomials  $P_n(x)$  are as follows,

- Decide whether the polynomial is odd or even, and choose which part of  $y$  you will use.
- Find the coefficients of the polynomial  $y(x)$  in terms of  $a_0$  for even polynomials and  $a_1$  for odd polynomials.
- Set  $y(0) = 1$  to find a value for  $a_1$  or  $a_0$ .
- Evaluate the final polynomial.

#### Orthogonality of Legendre Polynomials

Legendre polynomials are orthogonal over the interval  $|x| \leq 1$ , i.e.,

$$\int_{-1}^1 P_l(x) P_m(x) dx = 0 \quad m \neq l. \quad (4.20)$$

Let us recall eq. (4.15), and rewrite,

$$\frac{d}{dx} \left[ (1-x^2) \frac{\partial P_l}{\partial x} \right] = -l(l+1) P_l(x) \quad (4.21)$$

$$\frac{d}{dx} \left[ (1-x^2) \frac{\partial P_m}{\partial x} \right] = -m(m+1) P_m(x). \quad (4.22)$$

Multiply eq. (4.21) by  $P_m$ , and eq. (4.22) by  $P_l$ , and take them away from each other. We have,

$$\begin{aligned} \text{LHS} = & \int_{-1}^1 \frac{d}{dx} \left[ (1-x^2) \frac{\partial P_l}{\partial x} \right] P_m dx \\ & - \int_{-1}^1 \frac{d}{dx} \left[ (1-x^2) \frac{\partial P_m}{\partial x} \right] P_l dx \end{aligned} \quad (4.23)$$

Evaluating eq. (4.23) by parts, we have,

$$u = P_m \quad \frac{dv}{dx} = \frac{d}{dx} \left[ (1-x^2) \frac{\partial P_l}{\partial x} \right] \quad (4.24)$$

$$\frac{du}{dx} = \frac{\partial P_m}{\partial x} \quad v = (1-x^2) \frac{\partial P_l}{\partial x} \quad (4.25)$$

and similarly for the latter half of the equation. we can then write,

$$\begin{aligned} \text{LHS} = & \underbrace{\left[ (1-x^2) \frac{dP_l}{dx} P_m \right]_{-1}^1}_0 - \int_{-1}^1 (1-x^2) \frac{dP_m}{dx} \frac{dP_l}{dx} dx \\ & - \underbrace{\left[ (1-x^2) \frac{dP_m}{dx} P_l \right]_{-1}^1}_0 - \int_{-1}^1 (1-x^2) \frac{dP_l}{dx} \frac{dP_m}{dx} dx \\ = & 0 \end{aligned} \quad (4.26)$$

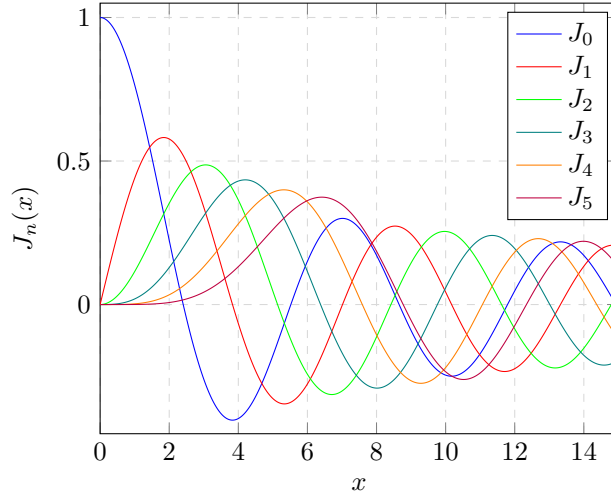


Figure 4.1: The first six Bessel functions.

We then have that, for  $n \neq m$ , the LHS is,

$$[m(m+1) - l(l+1)] P_l(x) P_m(x) = 0 \quad (4.27)$$

Furthermore, we can show,

$$\int_{-1}^1 P_l(x) P_m(x) dx = \frac{2}{2l+1} \delta_{lm}. \quad (4.28)$$

### 4.3.2 Legendre Polynomial Expansion

We can use the Legendre polynomials to perform a Legendre series expansion,

$$f(x) = \sum_{l=0}^{\infty} c_l P_l(x), \quad -1 \leq x \leq 1 \quad (4.29)$$

where the coefficients are given by,

$$c_l = \frac{1}{2} (2l+1) \int_{-1}^1 \int_{-1}^1 f(x) P_l(x) dx. \quad (4.30)$$

## 4.4 Bessel Functions

The most general Bessel equation is given by,

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - m^2) y = 0 \quad (4.31)$$

We can find a general solution by,

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n \quad (4.32)$$

where we impose a boundary condition that  $y(0)$  must be finite. Substituting eq (4.32) into eq. (4.31),

$$x \frac{dx}{dy} = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s} \quad (4.33)$$

$$x^2 \frac{d^2 x}{dy^2} = \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s} \quad (4.34)$$

$$(4.35)$$

$$\begin{aligned}
(x^2 - m^2)y &= \sum_{n=0}^{\infty} a_n x^{n+s+2} - m^2 \sum_{n=0}^{\infty} a_n x^{n+s} \\
&= \sum_{n=2}^{\infty} a_{n-2} x^{n+s} - m^2 \sum_{n=0}^{\infty} a_n x^{n+s}
\end{aligned} \tag{4.36}$$

Putting all these terms together, we can get rid of the  $x^{n+s}$  as we require the equation to be true  $\forall x$ .

$$\sum_{n=0}^{\infty} a_n [(n+s)(n+s-1) + (n+s) - m^2] + \sum_{n=2}^{\infty} a_{n-2} = 0. \tag{4.37}$$

We must first consider  $n = 0$  and  $n = 1$  before we can find the recurrence relation. For  $n = 0$ ,

$$\begin{aligned}
a_0 s(s-1) + a_0 s - m^2 a_0 &= 0 \\
a_0 s^2 - m^2 a_0 & \\
\implies s &= \pm m.
\end{aligned} \tag{4.38}$$

For  $n = 1$ ,

$$\begin{aligned}
a_1 s(s+1) + a_1(s+1) - m^2 a_1 &= 0 \\
a_1(s+1)(s+1) - m^2 a_1 &= 0 \\
\implies a_1 [(s+1)^2 - m^2] &= 0
\end{aligned} \tag{4.39}$$

for which we require  $a_1 = 0$  unless  $s = \pm m = -\frac{1}{2}$ . Otherwise, there will be no odd terms in our solution.

We may now move onto the general recurrence relation,

$$\begin{aligned}
[(n+s)(n+s-1) + (n+s) - m^2] a_n - a_{n-2} &= 0 \\
[(n+s)^2 - m^2] a_n + a_{n-2} &= 0
\end{aligned} \tag{4.40}$$

we then have the recurrence relation,

$$a_n = -\frac{a_{n-2}}{(n+m)^2 - m^2} = -\frac{a_{n-2}}{(2m+n)n} \tag{4.41}$$

which can be further generalised to,

$$a_{2j} = (-1)^j \frac{m!}{2^{2j} j! (m+j)!} a_0 \tag{4.42}$$

where  $j \in \mathbb{Z}^+$ . We can then write the general solution to Bessel's equation,

$$y(x) = \sum_{n=0}^{\infty} a_n x^{m+n} = \sum_{j=0}^{\infty} a_{2j} x^{m+2j}. \tag{4.43}$$

We often rewrite  $y(x)$  as,

$$y(x) = a_0 m! 2^m \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j} 2^m j! (m+j)!} x^{m+2j} \tag{4.44}$$

where we obtain *Bessel's function of the first kind*,

$$J_m(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! (m+j)!} \left(\frac{x}{2}\right)^{m+2j} \tag{4.45}$$

which obeys the orthogonality condition,

$$\int_0^L x J_p\left(\frac{\chi_n x}{l}\right) J_p\left(\frac{\chi_m x}{l}\right) dx = \frac{l^2}{2} [J_{p+1}(\chi_m)]^2 \delta_{mn} \tag{4.46}$$

where,

$$J_p(\chi_n) = 0 \quad n \in \mathbb{Z}^+ \tag{4.47}$$

i.e.,  $\chi_n$  is the  $n$ th zero point of the  $p$ th Bessel function.

For non-integer values of  $m$ , we must use the *gamma function*,

$$J_m(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(m+j+1)} \left(\frac{x}{2}\right)^{x+2j} \quad (4.48)$$

where  $\Gamma$  is defined,

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt. \quad (4.49)$$

We find that,

$$J_m(-x) = (-1)^m J_m(x) \quad (4.50)$$

so the  $m$ th Bessel function is even for even  $m$ , and odd for odd  $m$ .

### Convergence of the solution

By considering,

$$|r_n| = \left| \frac{a_{n+2}x^{n+2}}{a_n x^n} \right| = \left| \frac{x^2}{(n+2+m)} \right| \quad (4.51)$$

which clearly converges, as in the limit  $n \rightarrow \infty$ ,  $|r_n| \rightarrow 0 \forall x$

#### 4.4.1 Modes in a Circular Membrane

Consider a circular membrane clamped at its edges. Using the typical 3D wave equation, we use separation of variables,

$$\phi = F(r, \theta)T(t) \quad (4.52)$$

where  $F(r, \theta) = R(r)\Theta(\theta)$ . We have,

$$\frac{1}{F} \nabla^2 F = \frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = -k^2 \quad (4.53)$$

and applying the Laplacian in polar coordinates,

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial F}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} + k^2 F = 0. \quad (4.54)$$

Let us multiply eq. (4.54) by  $r^2/F$ ,

$$\underbrace{\frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right)}_{n^2} + \underbrace{\frac{1}{\Theta} \frac{d\Theta}{d\theta}}_{-n^2} = 0. \quad (4.55)$$

We then have,

$$\frac{1}{\Theta} \frac{d\Theta}{d\theta} + n^2 = 0 \quad (4.56)$$

$$\frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + k^2 r^2 - n^2 = 0 \quad (4.57)$$

We can clearly see that eq. (4.56) produces a cyclic solution. However, rearranging eq. (4.57),

$$r \frac{d}{dr} \left( r \frac{dR}{dr} \right) + (k^2 r^2 - n^2) R = 0 \quad (4.58)$$

which is Bessel's equation, so,  $R(r) \approx J_n(kr)$ . Given the boundary condition,

$$R(a) = 0 \quad (4.59)$$

we have,

$$J_n(ka) = 0 \quad (4.60)$$

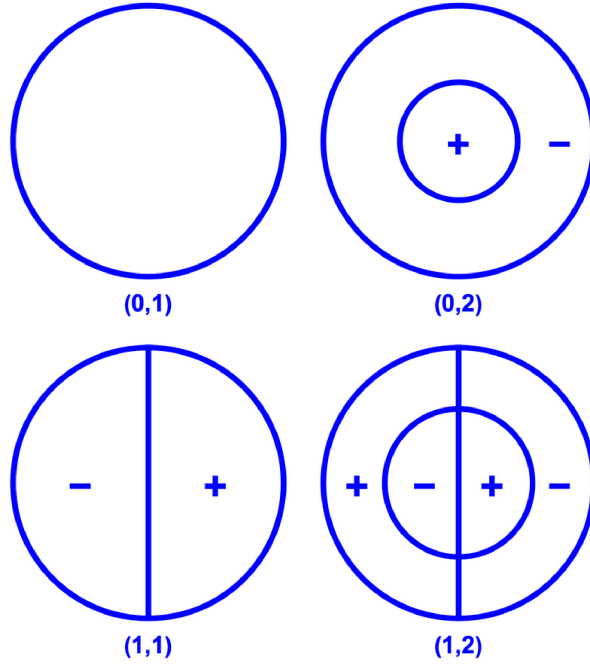


Figure 4.2

so we require,

$$k_{nm}a = \chi_{nm} \quad (4.61)$$

where  $\chi_{nm}$  is the  $m$ th root of the  $n$ th Bessel function of the first kind. We then require the angular frequency of the modes of oscillation of the membrane to be,

$$\omega_{nm} = k_{nm}c = \frac{\chi_{nm}c}{a}. \quad (4.62)$$

Some normal modes of the membrane are shown in figure 4.2.

NOTE: Frequencies of vibration are not integer multiples of the fundamental mode, as it might be with a regular wave.

The final solution for the vibrations of the membrane is given by,

$$\phi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} J_n(k_{nm}r) [A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta)] \cos(\omega_{nm}t) \quad (4.63)$$

## 4.5 Spherical Waves

The spherical wave equation is governed by the Laplacian in spherical polar coordinates. The spherical wave equation is given by,

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}. \quad (4.64)$$

Let us for now consider the solution on the surface,  $r = a$ . Let us use a separation of variables,

$$F(\theta, \phi, t) = Y(\theta, \phi)T(t) \quad (4.65)$$

We have,

$$\frac{1}{Y a^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{Y a^2 \sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = -\frac{l(l+1)}{a^2}. \quad (4.66)$$

Analysing the time equation,

$$\frac{d^2 T}{dt^2} = -\frac{l(l+1)}{a^2 c^2} T \quad (4.67)$$

we have a trivial SHM solution,

$$T = A \sin(\omega t) + B \cos(\omega t) \quad (4.68)$$

where

$$\omega^2 = \frac{c^2}{a^2} l(l+1). \quad (4.69)$$

For the angular solution, we use separation of variables,

$$Y = \Theta(\theta)\Phi(\phi) \quad (4.70)$$

to obtain,

$$\begin{aligned} \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\sin^2 \theta} \frac{d^2 \Phi}{d\phi^2} = -l(l+1)\Theta\Phi \\ \underbrace{\frac{\sin \theta}{\Theta} \frac{d}{d\theta} (\sin \theta) + l(l+1) \sin^2 \theta}_{m^2} + \underbrace{\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2}}_{-m^2} = 0 \end{aligned} \quad (4.71)$$

Let us consider a solution of constant  $\phi$ , and set  $m = 0$ . Our equation is then purely in  $\theta$ ,

$$\frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + l(l+1) \sin^2 \theta = 0. \quad (4.72)$$

We can solve this by a substitution  $x = \cos \theta$ ,

$$\frac{dx}{d\theta} = -\sin \theta \quad \implies \quad d\theta = -\frac{dx}{\sin \theta} \quad (4.73)$$

Our equation then becomes,

$$-\frac{\sin^2 \theta}{\Theta} \frac{d}{dx} \left( -\sin^2 \theta \frac{d\Theta}{dx} \right) + l(l+1) \sin^2 \theta = 0 \quad (4.74)$$

If we note that  $\sin^2 \theta = 1 - x^2$ ,

$$\frac{d}{dx} \left( (1 - x^2) \frac{d\Theta}{dx} \right) + l(l+1)\Theta = 0 \quad (4.75)$$

which is Legendre's polynomials. Our solution is then,

$$\Theta = P_l(x). \quad (4.76)$$

The general solution, considering only  $\theta$  and  $t$  dependence is then,

$$f(\theta, t) = \sum_{l=0}^{\infty} P_l(\cos \theta) (A_l \cos \omega t + B_l \sin \omega t) \quad (4.77)$$

#### 4.5.1 $\phi$ Dependent Solution

Let us now consider solutions where  $m \neq 0$ . Our angular equations become,

$$\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = -m^2 \quad (4.78)$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin^2 \theta \frac{d\Theta}{d\theta} \right) + \left[ l(l+1) - \frac{m^2}{\sin^2 \theta} \right] \Theta = 0 \quad (4.79)$$

Eq. (4.78) is trivial,

$$\Phi = A \sin m\phi + B \cos m\phi, \quad m \in \mathbb{Z}^+. \quad (4.80)$$



Eq. (4.79) can be solved if we consider,

$$x = \cos \theta \quad (4.81)$$

$$\frac{dx}{d\theta} = -\sin \theta \therefore d\theta = \frac{dx}{\sin \theta} \quad (4.82)$$

we can rewrite eq. (4.79) as,

$$\frac{d}{dx} \left( 1 - x^2 \frac{d\Theta}{d\theta} \right) + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] \Theta = 0. \quad (4.83)$$

For  $m = 0$ , we obtain Legendre polynomial solutions. However, for  $m \neq 0$ , we obtain associated Legendre polynomials,

$$\Theta = P_l^m(x), \quad |m| \leq l. \quad (4.84)$$

We then find the total angular solution is,

$$Y(\theta, \phi) = P_l^m(\cos \theta) [A \sin m\phi + B \cos m\phi] \quad (4.85)$$

### 4.5.2 Waves Inside the Sphere ( $r$ dependence)

If we take into account  $r$  dependence, we retain the angular solutions from before. Focusing on the radial and time solutions,

$$\frac{1}{Rr^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{l(l+1)}{r^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = -k^2. \quad (4.86)$$

We see the time solutions are SHM. Focussing on the radial equation,

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + [k^2 r^2 - l(l+1)] R = 0. \quad (4.87)$$

Let us recall the boundary condition  $R(a) = 0$ , where  $a$  is the surface of the sphere. Let  $x = kr$  to rewrite eq. (4.86),

$$\boxed{x^2 \frac{d^2 R}{dx^2} + 2x \frac{dR}{dx} + [x^2 - l(l+1)] R = 0} \quad (4.88)$$

which is the form of the *spherical Bessel equation*. The solution to the spherical Bessel equation are the spherical Bessel functions,

$$R = j_l(kr). \quad (4.89)$$

It can be shown that,

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x). \quad (4.90)$$

Applying the boundary condition,

$$j_l(k_{lm}a) = 0. \quad (4.91)$$

The final solution to the spherical wave equation is given by,

$$f = \sum_{l,m,n}^{\infty} j_l(k_{lm}r) P_l^m(\cos \theta) [\sin(m\phi) + \cos(m\phi)] [A_{lmn} \cos(\omega_{lmn}t) + B_{lmn} \sin(\omega_{lmn}t)] \quad (4.92)$$

# Appendix A

## Examples: Differential Equations

### A.1 1D Wave Equation

The 1 dimensional wave equation for a wavefunction  $\phi$  is given by,

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}}. \quad (\text{A.1})$$

#### A.1.1 Euler/d'Alembert solution

We can find a general solution to eq. (A.1) by using the substitution,

$$v = x - ct \Leftarrow \text{Backward component} \quad (\text{A.2})$$

$$u = x + ct \Leftarrow \text{Forward component}. \quad (\text{A.3})$$

Computing the derivative with respect to  $x$  by the chain rule,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} = \left( \frac{\partial}{\partial v} + \frac{\partial}{\partial u} \right) \phi \quad (\text{A.4})$$

from which the second derivative follows,

$$\frac{\partial^2 \phi}{\partial x^2} = \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right)^2 \phi. \quad (\text{A.5})$$

Similarly for the time component,

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \left( \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right)^2 \phi. \quad (\text{A.6})$$

Applying eqs. (A.5) and (A.6) to eq. (A.1), we find,

$$\begin{aligned} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + 2 \frac{\partial}{\partial u} \frac{\partial}{\partial v} \right) &= \frac{c^2}{c^2} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} - 2 \frac{\partial}{\partial u} \frac{\partial}{\partial v} \right) \\ \Rightarrow \left( \frac{\partial}{\partial u} \frac{\partial}{\partial v} \right) \phi &= 0 \Rightarrow \text{The solution is a sum of backward and forward components.} \end{aligned} \quad (\text{A.7})$$

Thus the general solution to eq. (A.1) is,

$$\boxed{\phi = \phi(x + ct) - \phi(x - ct)}. \quad (\text{A.8})$$

### A.2 Laplace's Equation

Laplace's equation is given by,

$$\boxed{\nabla^2 \phi = 0} \quad (\text{A.9})$$

and can be readily solved using separation of variables.

### A.3 Diffusion Equation

The diffusion equation is given by,

$$\boxed{\nabla^2 T = \frac{1}{\alpha} \frac{\partial T}{\partial t}} \quad (\text{A.10})$$

which reduces to Laplace's equation for a steady-state system. We will use it to analyse the temperature distribution on a circular plate.

Given we have a circular plate, we wish to use the polar form of the Laplacian. We have,

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = \frac{1}{\alpha^2} \frac{\partial T}{\partial t}. \quad (\text{A.11})$$

The circular plate has an insulating boundary, and has a cyclic boundary condition,

$$\left. \frac{\partial T}{\partial r} \right|_{r=a} = 0 \quad (\text{A.12})$$

$$T(r, \theta, t) = T(r, \theta + 2n\pi, t). \quad (\text{A.13})$$

We then solve the diffusion equation by separation of variables,  $T = R(r)\Theta(\theta)\tau(t)$ ,

$$\frac{1}{Rr} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{1}{\Theta r^2} \frac{d^2 \Theta}{d\theta^2} = \frac{1}{\alpha^2 \tau} \frac{d\tau}{dt} = -k^2. \quad (\text{A.14})$$

The *time equation* is,

$$\frac{d\tau}{\tau} = -k^2 \alpha^2 dt \quad (\text{A.15})$$

which has an exponential solution,

$$\tau = Ae^{-k^2 \alpha^2 t}. \quad (\text{A.16})$$

Let us analyse the  $\theta$  and  $r$  dependence in eq. (A.14),

$$\frac{r}{R} \left( r \frac{dR}{dr} \right) + k^2 r^2 = -\frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = m^2 \quad (\text{A.17})$$

The cyclic equation is given by,

$$\frac{d\Theta}{d\theta} = -m^2 \Theta \quad (\text{A.18})$$

which has a trigonometric solution,

$$\Theta = A \cos(m\theta) + B \sin(m\theta) \quad (\text{A.19})$$

where we require  $m \in \mathbb{N}^+$  to satisfy the cyclic boundary condition.

The *radial equation* is given by,

$$\boxed{r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (k^2 r^2 - m^2) R = 0} \quad (\text{A.20})$$

which is Bessel's equation. The radial solution is then given by,

$$R = J_m(kr) \quad (\text{A.21})$$

where  $J_m(kr)$  is an  $m^{\text{th}}$  order Bessel function.