Mathematics 2

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Chapter 1

Partial Differentiation and Multiple Integration

The differential function of a function $f = f(x, y)$ is given by,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y\,. \tag{1.1}$$

1.1 Taylor Series

The Taylor series for a function $f(x_1, x_2, \ldots, x_m)$ about a point $(x_1^0, x_2^0, \ldots, x_m^0)$ is,

$$f(x_1, x_2, \ldots, x_m) = f(x_1^0, x_2^0, \ldots, x_m^0) + \sum_{k=1}^{\infty} \frac{1}{k!}\left((x_1 - x_1^0)\frac{\partial f}{\partial x_1} + \cdots + (x_m + x_m^0)\frac{\partial f}{\partial x_m}\right)^k f(x_1, x_2, \ldots, x_m), \tag{1.2}$$

where the power on the bracket applies to the amount of times each differential is applied to f. Each partial derivative is evaluated about $(x_1^0, x_2^0, \ldots, x_m^0)$.

1.2 Multiple Integration

We are able to integrate over multiple variables. This allows for finding quantities such as a volume or area.

$$V = \int f(x, y, z)\,\mathrm{d}\tau = \int\int\int f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\left(\int\left(\int f(x, y, z)\,\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z\,. \tag{1.3}$$

1.2.1 Areas and volumes in different coordinate systems

2D Plane Polar Coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta \tag{1.4}$$

3D Cylindrical Polar Coordinates

$$\text{Curved Surface: } \mathrm{d}A = r\,\mathrm{d}\theta\,\mathrm{d}z \qquad \text{Top Surface: } \mathrm{d}A = r\,\mathrm{d}\theta\,\mathrm{d}r \qquad \text{Volume: } \mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \tag{1.5}$$

3D Spherical Polar Coordinates

$$\mathrm{d}A = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \qquad \mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{1.6}$$

There is an important identity required when working with polar co-ordinates,

$$\sin\theta\,\mathrm{d}\theta = -\,\mathrm{d}(\cos\theta)\,. \tag{1.7}$$

The limits for the volume of a sphere are,

$$\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{R} r^2\,\mathrm{d}r\,. \tag{1.8}$$

1.2.2 Moment of Inertia

The moment of inertia is given by,

$$I = \int r_\perp^2 \,\mathrm{d}m\,, \tag{1.9}$$

which can be rewritten as,

$$I = \rho \int r_\perp^2 \,\mathrm{d}V\,, \tag{1.10}$$

where ρ is the density of the volume. For non-trivial axis, we use,

$$r_\perp = |\hat{\mathbf{a}} \times \mathbf{p}|, \tag{1.11}$$

where $\hat{\mathbf{a}}$ is the vector in the direction of of the axis, and $\mathbf{p}$ is a generic vector such that,

$$\mathbf{p} = r_i \mathbf{e}_i. \tag{1.12}$$

1.2.3 The Jacobian

When converting between co-ordinate frames, we cannot always simply replace variables. We must use,

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |J(u,v,w)|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\,. \tag{1.13}$$

Where, we can define,

$$x = f(u,v,w) \qquad y = g(u,v,w) \qquad z = h(u,v,w) \tag{1.14}$$

and then, the Jacobian is for a 2D system, the determinant of,

$$\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix} \tag{1.15}$$

and for a 3D system,

$$\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix}. \tag{1.16}$$

Chapter 2

Fields

A field is an entity whose value depends on position. A field may be either **scalar** or **vector**. The direction of vector fields may also depend on position. **Scalar fields** are often represented by **contour lines**, which conenct the x and y points. **Vector fields** are often visualised using field lines. We can calculate the equations for these field lines by considering a vector field, $V = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}}$, and solving for,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{V_y}{V_x}. \tag{2.1}$$

2.1 Gradient $\boldsymbol{\nabla}$

We can define the infinitesimal change in a scalar field as,

$$d\psi = \boldsymbol{\nabla}\psi \cdot \mathrm{d}\mathbf{r}\,, \tag{2.2}$$

where we define the **gradient** operator as,

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r_i}\mathbf{e}_i. \tag{2.3}$$

There are some properties of grad which we must be aware of. For a scalar field, $f(x, y, z)$,

- $\boldsymbol{\nabla}f$ is a vector field.
- $\boldsymbol{\nabla}f$ represents the maximum rate of change of f.
- $\boldsymbol{\nabla}f$ is perpendicular to the contours of constant f.
- The unit vector normal to a level surface is

$$\frac{\boldsymbol{\nabla}f}{|\boldsymbol{\nabla}f|}. \tag{2.4}$$

There are two main types of questions where the grad operator is used,

1. Finding the unit vector at (x_0, y_0, z_0) which is normal to a level surface, $f(x, y, z)$.

$$\text{Obtain } \boldsymbol{\nabla}f(x, y, z) \to \mathbf{n} = \boldsymbol{\nabla}f(x_0, y_0, z_0) \to \hat{\mathbf{n}} = \frac{\boldsymbol{\nabla}f(x_0, y_0, z_0)}{|\boldsymbol{\nabla}f(x_0, y_0, z_0)|} \tag{2.5}$$

2. Find the rate of increase at $f(x_0, y_0, z_0)$ in the direction between, (x_1, y_1, z_1) and (x_2, y_2, z_2).

$$\text{Eval. } \boldsymbol{\nabla}f(x_0, y_0, z_0) \to \mathrm{d}\mathbf{s} = (\Delta x, \Delta y, \Delta z) \to \hat{\mathbf{u}} = \frac{\mathrm{d}\mathbf{s}}{|\,\mathrm{d}\mathbf{s}\,|} \to \frac{\mathrm{d}f}{\mathrm{d}s} = \boldsymbol{\nabla}f(x_0, y_0, z_0) \cdot \hat{\mathbf{u}}. \tag{2.6}$$

where ϕ and ψ are scalar fields and K is a position independent constant scalar.

2.1.1 Grad in polar coordinates

$$\text{Cylindrical:} \qquad \boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}, \tag{2.7}$$

$$\text{Spherical:} \qquad \boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\psi}}. \tag{2.8}$$

2.2 Lagrangian Multipliers

These are used when wanting to find the minimum or maximum of a field when a field f is constrained by some function g. If g is a constant constraint, then we can say that,

$$\mathrm{d}g = 0 \tag{2.9}$$

and for a minimum or maximum of f, similarly,

$$\mathrm{d}f = 0. \tag{2.10}$$

We recall that the change in a scalar field along an elemental path is,

$$\mathrm{d}\psi = \boldsymbol{\nabla}\psi \cdot \mathrm{d}\mathbf{s}\,. \tag{2.11}$$

Thus, $\boldsymbol{\nabla}f$ and $\boldsymbol{\nabla}g$ are both perpendicular to d**s**. Thus, $\boldsymbol{\nabla}f$ and $\boldsymbol{\nabla}g$ are parallel or anti parallel to each other. We can then state the relation,

$$\boldsymbol{\nabla}f = \lambda\boldsymbol{\nabla}g \tag{2.12}$$

which brings about a system of equations,

$$\begin{aligned} \frac{\partial f}{\partial x} - \lambda\frac{\partial g}{\partial x} = 0 \\ \frac{\partial f}{\partial y} - \lambda\frac{\partial g}{\partial y} = 0 \end{aligned} \tag{2.13}$$

We will want to include the function of g when finding the solutions to this equation.

1. Find x,y, and z in terms of λ.
2. Substitute these into the expression for g.
3. Substitute the value found for g back into the expressions for x,y, and z to find the coordinates of the maximum and minimum rate of change.

2.3 Div

$$\mathrm{Div}(\mathbf{F}) \equiv \boldsymbol{\nabla}\cdot\mathbf{F} \tag{2.14}$$

The divergence describes the flux through an area of a vector field. A field with 0 divergence is known as a *solenoidal field* The total flux of a vector field through a volume composed of i surfaces is given by the **divergence theorem**,

$$\begin{gathered} \sum_i \int_{S_i} \mathbf{F}\cdot\mathrm{d}\mathbf{S}_i = \int_V (\boldsymbol{\nabla}\cdot\mathbf{F})\,\mathrm{d}V \\ \boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{|\mathbf{v}|}\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} \end{gathered} \tag{2.15}$$

Whether the flux is +ive, −ive or 0 will give us different information.

When $|\mathbf{F}| \equiv \text{const}$, we have,

$$\boldsymbol{\nabla}\cdot\mathbf{F} = 0. \tag{2.16}$$

For *increasing* $|\mathbf{F}|$, the total flux is given by,

$$\int (F_R - F_L)\,\mathrm{d}s \tag{2.17}$$

$$F_R > F_L \therefore \boldsymbol{\nabla}\cdot\mathbf{F} > 0 \implies \text{Net flow } \textit{out of the volume.} \tag{2.18}$$

The field is known as a *source* of flux.

For *decreasing* $|\mathbf{F}|$,

$$F_L > F_R \therefore \boldsymbol{\nabla}\cdot\mathbf{F} < 0 \implies \text{Net flow } \textit{into the volume.} \tag{2.19}$$

The field is then known as a sink of flux.

For non-trivial fields, the values of Div still apply.

2.3.1 Div in polar coordinates

For cylindrical polar coordinates,

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial\theta}A_\theta + \frac{\partial}{\partial z}(A_z)\,. \tag{2.20}$$

For spherical polar coordiantes,

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(A_\phi) \tag{2.21}$$

2.3.2 Gauss' Law

$$\begin{aligned}\int \mathbf{E}\cdot\mathrm{d}\mathbf{S} &= \int_V (\boldsymbol{\nabla}\cdot\mathbf{E})\,\mathrm{d}V = \frac{1}{\epsilon_0}\int \mathrm{d}Q\\ \boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{1}{\epsilon_0}\frac{\mathrm{d}Q}{\mathrm{d}V} = \frac{\rho}{\epsilon_0}\end{aligned} \tag{2.22}$$

2.4 Curl

$$\mathrm{Curl}(\mathbf{F}) \equiv \boldsymbol{\nabla}\times\mathbf{F} \tag{2.23}$$

We can interpret the physical nature of curl if we imagine dropped a ball into a field. If $(\boldsymbol{\nabla}\times\mathbf{F}) \neq 0$, then the ball will experience a torque. A field with 0 curl is known as an *irrotational field.*

Curl in polar coordinates

For cylindrical polar coordinates,

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r}\begin{vmatrix}\hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ A_r & rA_\theta & A_z\end{vmatrix} \tag{2.24}$$

For spherical polar coordinates,

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r\sin\theta}\begin{vmatrix}\hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix} \tag{2.25}$$

2.5 The Laplacian Operator

This is defined,

$$\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}. \tag{2.26}$$

Such that,

$$\nabla^2 = \sum_i \frac{\partial^2}{\partial r_i^2}. \tag{2.27}$$

For a scalar field ψ, the Laplacian is,

$$\nabla^2\psi = \sum_i \frac{\partial^2\psi}{\partial r_i^2}. \tag{2.28}$$

For a vector field, $\mathbf{F}$, the Laplacian is given by,

$$\nabla^2\mathbf{F} = \nabla^2 A_i\mathbf{e}_i. \tag{2.29}$$

2.5.1 Laplacian in Polar Coordinates

In cylindrical polar coordinates,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}. \tag{2.30}$$

In spherical polar coordinates,

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{2.31}$$

2.6 Maxwell's Equations

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{Gauss' Law} \tag{2.32}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \qquad \text{No magnetic monopoles} \tag{2.33}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \text{Faraday's Law} \tag{2.34}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \qquad \text{Ampere's Law} \tag{2.35}$$

The wave equation for electromagnetic waves can then be derived,

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{2.36}$$

2.7 Helmholtz Decomposition

If you have a smooth, rapidly varying field that vanishes faster than $\frac{1}{r}$ as $r\to\infty$,

$$\mathbf{v} = \boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}\phi. \tag{2.37}$$

We know that,

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0 \qquad \boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi = 0 \tag{2.38}$$

$$\text{"B-mode"} \qquad \text{"E-Modes"} \tag{2.39}$$

2.8 Surface Integrals

When we integrate over a surface, we usually integrate over a vector, given as,

$$\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S \tag{2.40}$$

where $\hat{\mathbf{n}}$ is a unit vector which is perpendicular and away from the surface. If our surface is described by a function f, we can compute $\hat{\mathbf{n}}$ as,

$$\hat{\mathbf{n}} = \frac{\boldsymbol{\nabla}f}{|\boldsymbol{\nabla}f|}. \tag{2.41}$$

The elemental area $\mathrm{d}S$ can be rewritten in Cartesian coordinates if we consider that the elemental surface is inclined at some angle, θ to the $x-y$ plane. We then say,

$$\begin{aligned}\mathrm{d}x\,\mathrm{d}y &= \mathrm{d}S\cos\theta = \mathrm{d}S\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{k}})\\ &= \mathrm{d}\mathbf{S}\cdot\hat{\mathbf{k}}.\end{aligned} \tag{2.42}$$

2.8.1 Solid Angles

We often wish to integrate over a solid angle. The infinitesimal element of a solid angle is given by,

$$\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, . \tag{2.43}$$

Surface integrals over solid angles have the infinitesimal area element,

$$\mathrm{d}\Omega = \frac{\hat{\mathbf{r}} \cdot \mathrm{d}\mathbf{S}}{r^2}. \tag{2.44}$$

2.8.2 Flux

The flux through of a vector field $\mathbf{A}$ through a surface S is given by,

$$\mathrm{FLUX} = \int_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} \, . \tag{2.45}$$

Often, it is a lot easier to do this using the Divergence theorem covered in the next section.

2.9 Divergence Theorem

Theorem. *The total flux of a vector* $\mathbf{A}$ *through a closed surface* S *is related to the divergence of a vector field inside a volume* V *by,*

$$\int_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \int_V \boldsymbol{\nabla} \cdot \mathbf{A} \, \mathrm{d}V \, . \tag{2.46}$$

This theorem can also be applied to scalar fields,

$$\int_S \psi \, \mathrm{d}\mathbf{S} = \int_V \boldsymbol{\nabla} \psi \, \mathrm{d}V \, . \tag{2.47}$$

2.9.1 Applications of Divergence Theorem

Maxwell's First Equation

Gauss' law can be given as follows,

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\epsilon_0} \int_V \rho \, \mathrm{d}V \, . \tag{2.48}$$

Re-writing this in terms of the divergence theorem,

$$\begin{aligned} \int_V \boldsymbol{\nabla} \cdot \mathbf{E} \, \mathrm{d}V &= 1\epsilon_0 \int_V \rho \, \mathrm{d}V \\ \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, \end{aligned} \tag{2.49}$$

which is Maxwell's first equationn of electromagnetism.

The Continuity Equation

This is a conservation equation, given by,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \tag{2.50}$$

where ρ is an amount of some quantity, q, per unit volume, and $\mathbf{J}$ is the flux per unit time of q.

2.10 Line Integrals

For a curve C defined by a function f and a small length element $\mathrm{d}l$, the scalar line integral is defined by,

$$\int_C f\,\mathrm{d}l\,. \tag{2.51}$$

For 2D line integrals, we can define,

$$(\mathrm{d}l)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 \tag{2.52}$$

and,

$$l = \int_C \mathrm{d}l = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x\,. \tag{2.53}$$

For a curve defined by parameters, i.e., $x = g(t)$, $y = h(t)$, we write,

$$l = \int_{t_0}^{t_1} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}x\,. \tag{2.54}$$

In polar co-ordinates,

$$\mathrm{d}l = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta\,. \tag{2.55}$$

2.10.1 Vector line integrals

For a vector $\mathbf{A}$, we define the samll length element vector as,

$$\mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{i}}\,\mathrm{d}u\,\hat{\mathbf{j}} \qquad \text{Cartesian} \tag{2.56}$$

$$\mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{\mathbf{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} \qquad \text{Polar.} \tag{2.57}$$

The line integral is then given by,

$$l = \int_C \mathbf{A}\cdot\mathrm{d}\mathbf{l}\,. \tag{2.58}$$

Appendix A

Vector Calculus Identities

A.1 Grad Identities

$$\boldsymbol{\nabla}(\psi + \phi) = \boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\phi \tag{A.1}$$
$$\boldsymbol{\nabla}(K\psi) = K\boldsymbol{\nabla}\psi \tag{A.2}$$
$$\boldsymbol{\nabla}(\psi\phi) = \psi\boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla}\psi \tag{A.3}$$

A.2 Curl Identities

$$\boldsymbol{\nabla} \times (\mathbf{A} + \mathbf{B}) = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times \mathbf{B} \tag{A.4}$$
$$\boldsymbol{\nabla} \times (\phi\mathbf{A}) = \phi(\boldsymbol{\nabla} \times \mathbf{A}) + (\boldsymbol{\nabla}\phi) \times \mathbf{A} \tag{A.5}$$
$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{A}) + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} \tag{A.6}$$

A.3 Div Identities

$$\boldsymbol{\nabla} \cdot (\mathbf{A} + \mathbf{B}) = \boldsymbol{\nabla} \cdot \mathbf{A} + \boldsymbol{\nabla} \cdot \mathbf{B} \tag{A.7}$$
$$\boldsymbol{\nabla} \cdot (\phi\mathbf{A}) = (\boldsymbol{\nabla}\phi) \cdot \mathbf{A} + \phi(\boldsymbol{\nabla} \cdot \mathbf{A}) \tag{A.8}$$
$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) \tag{A.9}$$

A.4 Combination Identities

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = 0 \tag{A.10}$$
$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0 \tag{A.11}$$
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2\mathbf{A}. \tag{A.12}$$

Appendix B

Vector Calculus Proofs

B.1 $\boldsymbol{\nabla} \cdot (g\mathbf{F}) = (\boldsymbol{\nabla} g) \cdot \mathbf{F} + g(\boldsymbol{\nabla} \cdot \mathbf{F})$

Proof.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (g\mathbf{F}) &= (gF_1)_x + (gF_2)_y + (gF_3)_z \\
&= g_x F_1 + g_y F_2 + g_z F_3 + g\left[(F_1)_x + (F_2)_y + (F_3)_z\right] \\
&= \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} + g \begin{pmatrix} (F_1)_x \\ (F_2)_y \\ (F_3)_z \end{pmatrix} \\
&= (\boldsymbol{\nabla} g) \cdot \mathbf{F} + g(\boldsymbol{\nabla} \cdot \mathbf{F})
\end{aligned}$$

$\square$