PHYS20672 Complex Variables and Vector Spaces: Examples 1

Some of these examples are copied from the exercises in the notes, but have been included here for completeness.

1. Confirm that a vector in Cartesian coordinates, $\mathbf{v} = (v_x, v_y, v_z)^T$, satisfy the definition of a vector space with addition and scalar multiplication defined in the usual way.

2. Consider the set of real 2×2 matrices of the form,

$$\boldsymbol{a} = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$$

   where $\alpha, \beta, \delta, \gamma \in \mathbb{R}$. Show that these matrices form a vector space over the real numbers.

3. Show that the set of all real functions on $[0, L]$, $f : [0, L] \to \mathbb{R}$, form a vector space.

4. Are the following three vectors linearly independent: $\boldsymbol{a} = (2, 3, -1)^T$, $\boldsymbol{b} = (0, 1, 2)^T$, $\boldsymbol{c} = (0, 0, -5)^T$? [Hint: Write $\alpha\boldsymbol{a}+\beta\boldsymbol{b}+\gamma\boldsymbol{c} = \boldsymbol{0}$ and prove that the only solution is $\alpha = \beta = \gamma = 0$.]. Find the decomposition of $(2, -3, 1)^T$ in the basis $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$.

5. Consider the set of all polynomials of degree not exceeding 3, that is functions of the form $f(x) = \sum_{n=0}^{3} \alpha_n x^n$, where $\alpha_n \in \mathbb{C}$.

   (i) Prove that these polynomials form a complex vector space.
   [Hint: You can regard 0 as a polynomial of degree zero.]

   (ii) Write down the additive inverse of the vector $1 + ix + (2 + 3i)x^3$.

   (iii) Find a basis for the space and hence determine its dimension.

   (iv) What changes if only odd functions of x are considered?

   (v) Why is the set of strictly cubic polynomials not a vector space?

6. If $\{\boldsymbol{e}_i\}$ is a basis of V_N, prove that for any $\boldsymbol{u} \in V_N$, the coefficients u_i in the expansion $\boldsymbol{u} = \sum_{i=1}^{N} u_i \boldsymbol{e}_i$ are unique.

7. For the vector space consisting of real all 2×2 matrices, show that

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \boldsymbol{e}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \boldsymbol{e}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (1)$$

   are linearly independent, and can therefore be used as a basis.

8. Given a vector space V_N and the components x_i of the vector $\boldsymbol{x}$ in a fixed basis, the 1-norm is defined as $\|\boldsymbol{x}\|_1 \equiv \sum_{i=1}^{n} |x_i|$. Show that the 1-norm satisfies the conditions for a norm.

9. Consider the vector space V of all functions $f(x) : [0, L] \to \mathbb{R}$. Define the supremum norm as $\max_{x\in[0,L]} |f(x)|$. Show that this is a norm.

10. Show that the sequence $a_n = (n^2 - 1)/n^2$ is Cauchy.

11. Show that the sequence $a_n = n!/n^n$ is Cauchy.
    Hint you may want to use Stirling's formula $n! \approx \sqrt{2\pi n}(n/e)^n$ which is valid for large n.