<u>TWO HOURS</u>

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Advanced Quantum Mechanics

33rd June 2022, 00.00 a.m. - 12.00 p.m.

Answer **<u>TWO</u>** questions

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

You may assume the following formulae in any question *if proof is not explicitly requested*:

Special integrals

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2a)^n} \sqrt{\frac{\pi}{a}}, \qquad \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}, \quad a > 0.$$

4-vectors
Use notation $x^\mu = (x^0, \mathbf{r})$ for 4-position and $p^\mu = (E/c, \mathbf{p})$ for 4-momentum. In electromagnetism $A^\mu = (\Phi/c, \mathbf{A})$ for 4-potential.

Klein-Gordon equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[(\hat{\mathbf{p}} - \mathbf{V})^2 c^2 + (mc^2 + S)^2\right] \Psi = (i\hbar\partial_t - V_0)^2 \Psi.$$

Dirac equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[\boldsymbol{\alpha} \cdot (\hat{\mathbf{p}} - \mathbf{V})c + \beta(mc^2 + S)\right] \Psi = (i\hbar\partial_t - V_0)\Psi.$$

Standard spinor solutions to the free massive Dirac equation

$$u^{(s)}(E, \mathbf{p}) = \sqrt{E + mc^2} \begin{pmatrix} \phi^s \\ \frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E+mc^2}\phi^s \end{pmatrix}, \quad \phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

Standard matrices

$$\begin{aligned}
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \\
\alpha_i &= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}; \\
\gamma_0 &= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}.
\end{aligned}$$

Space-time metric We use the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -, 1, -1)$.

Laplacian operator in terms of angular momentum operator $\hat{\mathbf{L}}$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\hat{\mathbf{L}}^2}{\hbar^2 r^2}.$$

Solution to Dirac equation in spherical vector potential $(V_0(r)/c, \vec{0})$

$$\begin{aligned}
\psi_{jm}^{\kappa}(\mathbf{r}) &= \begin{pmatrix} f_j^{\kappa}(r)\mathcal{Y}_{jm}^{\kappa}(\theta,\phi) \\ i g_j^{\kappa}(r)\mathcal{Y}_{jm}^{-\kappa}(\theta,\phi) \end{pmatrix} \quad \text{with } F = rf \text{ and } G = rg, \\
\hbar c \left[\frac{d}{dr} - \frac{\kappa}{r}\right] G_j^{\kappa}(r) &= -\left(E - V_0(r) - mc^2\right) F_j^{\kappa}(r), \\
\hbar c \left[\frac{d}{dr} + \frac{\kappa}{r}\right] F_j^{\kappa}(r) &= \left(E - V_0(r) + mc^2\right) G_j^{\kappa}(r).
\end{aligned}$$

1. a) Show that the angular momentum operator $\hat{L}_y$ is the generator of rotations about the y-axis for a spinless particle. Derive an expression in terms of $\hat{L}_y$ for the operator $\hat{U}_\beta$ representing a finite rotation of angle β about the y-axis. Show that the operator $\hat{U}_\beta$ is unitary.

[8 marks]

Write down the corresponding $\hat{U}_\beta$ for a particle with spin, and explain any notation you use.

[2 marks]

b) i) State the selection rules for electric dipole transitions in light atoms. Explain any notation you use.

[3 marks]

ii) Sketch the energy levels of hydrogen (non-relativistic and without spin-orbit coupling) up to the quantum number $n = 3$, and mark at least 4 electric dipole transitions.

[2 marks]

iii) State whether each of the following transitions in the oxygen atom could be an electric dipole transition. Give your reasons.

i. $(2s)^2(2p)^4$, ${}^3P_2 \to (2s)^2(2p)^3(3d)$, ${}^3P_1$

ii. $(2s)^2(2p)^3(3s)$, ${}^3S_1 \to (2s)^2(2p)^3(3d)$, ${}^3P_1$

iii. $(2s)^2(2p)^2(3d)^2$, ${}^5D_1 \to (2s)^2(2p)^3(3s)$, ${}^5S_2$

iv. $(2s)^2(2p)^3(3s)$, ${}^3S_1 \to (2s)^2(2p)^4$, ${}^3P_0$

[2 marks]

c) i) Write down the wave function of a free relativistic spin-1/2 particle of energy E. Define a helicity basis for Pauli spinors and express the wave function for a massless particle in terms of the helicity basis.

[4 marks]

ii) The energy bands of a non-relativistic electron in graphene near the Dirac points with momentum $\mathbf{p} = (p_x, p_y)$ can be obtained from the following Hamiltonian matrices as

$$\hat{H}^\pm = a \begin{pmatrix} 0 & \pm p_x - ip_y \\ \pm p_x + ip_y & 0 \end{pmatrix}$$

where a is a constant. Find the eigenvalues and confirm that $\hat{H}^+$ and $\hat{H}^-$ have the same eigenvalues. Show that this problem can be cast in the same form as the massless Dirac equation in the helicity basis.

[4 marks]

2. a) Consider a non-relativistic electron moving in a two-dimensional xy plane with a perpendicular uniform magnetic field $\mathbf{B}$ described by the 4-potential $(0, \mathbf{A})$ where $\mathbf{A} = (0, xB, 0)$. Write down the time-independent Schrödinger equation of the electron, ignoring its spin ($g = 0$).

[2 marks]

By using a wave function of the form

$$\psi(\mathbf{r}) = e^{ik_y y}\phi(x)$$

with constant k_y, find and solve the equation for $\phi(x)$, giving the energy eigenvalues (Landau levels) of the electron.

[4 marks]

Find the degeneracy per unit area of the Landau levels and briefly describe the Quantum Hall Effect in terms of Landau levels.

[6 marks]

b) A relativistic spin-0 π^- meson of mass m is bound to a nucleus with Z protons by a Coulomb potential

$$V(r) = -Z\alpha\frac{\hbar c}{r}.$$

We assume that the ground-state wave function of the meson can be written in the form

$$\Psi(\mathbf{r}, t) \propto r^{\epsilon} e^{-\beta r} e^{-iEt/\hbar}.$$

Show that

$$\epsilon = \frac{1}{2}\left(-1 \pm \sqrt{1 - (2Z\alpha)^2}\right), \quad \beta = -\frac{\epsilon E}{\hbar c Z\alpha}$$

and

$$E = \pm mc^2\sqrt{1+\epsilon},$$

where the two $\pm$ signs are independent. Be requiring the correct energy for $Z\alpha = 0$, determine both signs.

[10 marks]

Find expressions for ϵ, E and β for small $Z\alpha$, and compare them with the expected non-relativistic results.

[3 marks]

3. a) A one-dimensional harmonic oscillator of mass m and angular frequency ω in its ground state is subject to a small force $F = F_0 e^{-t/\tau}$ for $t > 0$. Find the probability, in the first-order approximation, that the oscillator is in its first excited state in the limit $t \to \infty$. The ground and first excited state wave functions of the oscillator are

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right), \quad \phi_1(x) = \sqrt{\frac{2m\omega}{\hbar}}\, x\phi_0(x)\,.$$

[5 marks]

b) A free spin-1/2 particle is described by the Dirac equation. Define a probability current density $J^\mu = (\rho, \mathbf{j})$ and prove that it satisfies the continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j} = 0.$$

[3 marks]

c) A beam of relativistic electrons with mass m and energy E_p is moving in the positive z direction and encounters an electrostatic step potential at $z = 0$:

$$e\Phi = \begin{cases} 0, & z < 0 \\ V_0, & z \geq 0. \end{cases}$$

i) Calculate the probability current density for the incident, reflected, and transmitted beams. Hence determine the reflection coefficient R and transmission coefficient T. Show that $R + T = 1$.

[10 marks]

ii) State the conditions for total reflection to occur.

[2 marks]

iii) For the case $V_0 > E_p + mc^2$, show that the wave number for the right-moving wave must be negative. Hence show that the reflection coefficient is greater than unity. Briefly state your resolution of this paradox.

[3 marks]

iv) Determine the reflection and transmission coefficients for a beam of massless neutrinos in the case $V_0 > E_p$.

[2 marks]

END OF EXAMINATION PAPER