ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Advanced Quantum Mechanics

29th May 2018, 09.45 a.m. - 11.15 a.m.

Answer any **TWO** questions

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

You may assume the following formulae in any question *if proof is not explicitly requested*:

Special integrals

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2a)^n} \sqrt{\frac{\pi}{a}}, \qquad \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}, \quad a > 0.$$

4-vectors
Use notation $x^\mu = (x^0, \mathbf{x})$ for 4-position and $p^\mu = (E/c, \mathbf{p})$ for 4-momentum. In electromagnetism $A^\mu = (\Phi/c, \mathbf{A})$ for 4-potential.

Klein-Gordon equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[(\hat{\mathbf{p}} - \mathbf{V})^2 c^2 + (mc^2 + S)^2\right] \Psi = (i\hbar\partial_t - V_0)^2 \Psi.$$

Dirac equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[\boldsymbol{\alpha} \cdot (\hat{\mathbf{p}} - \mathbf{V})c + \beta(mc^2 + S)\right] \Psi = (i\hbar\partial_t - V_0)\Psi.$$

Standard spinor solutions to the free massive Dirac equation

$$u^{(s)}(E, \mathbf{p}) = \sqrt{E + mc^2} \begin{pmatrix} \phi^s \\ \frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E+mc^2}\phi^s \end{pmatrix}, \quad \phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$v^{(s)}(-|E|, -\mathbf{p},) = \sqrt{|E| + mc^2} \begin{pmatrix} \frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{|E|+mc^2}\chi^s \\ \chi^s \end{pmatrix}, \quad \chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \chi^2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Standard matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix};$$

$$\gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}.$$

Space-time metric We use the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -, 1, -1)$.

Laplacian operator in terms of angular momentum operator $\hat{\mathbf{L}}$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\hat{\mathbf{L}}^2}{\hbar^2 r^2}.$$

Solution to Dirac equation in spherical vector potential $(V_0(r)/c, \mathbf{0})$

$$\psi^\kappa_{jm}(\mathbf{r}) = \begin{pmatrix} f^\kappa_j(r)\mathcal{Y}^\kappa_{jm}(\theta,\phi) \\ i g^\kappa_j(r)\mathcal{Y}^{-\kappa}_{jm}(\theta,\phi) \end{pmatrix} \quad \text{with } F = rf \text{ and } G = rg,$$

$$\hbar c\left[\frac{d}{dr} - \frac{\kappa}{r}\right] G^\kappa_j(r) = -\left(E - V_0(r) - mc^2\right) F^\kappa_j(r),$$

$$\hbar c\left[\frac{d}{dr} + \frac{\kappa}{r}\right] F^\kappa_j(r) = \left(E - V_0(r) + mc^2\right) G^\kappa_j(r).$$

1. a) Consider a free relativistic spin-1/2 particle of mass m.

   i) Write down the proper definition for the probability current density of the particle and show that it satisfies the continuity equation.

   [5 marks]

   ii) Show that the orbital angular momentum $\hat{\mathbf{L}} = \mathbf{r} \times \hat{\mathbf{p}}$ is not a constant of motion and that

   $$[\hat{H}, \hat{\mathbf{L}}] = -i\hbar c \boldsymbol{\alpha} \times \hat{\mathbf{p}}.$$

   [4 marks]

   iii) Write down the total angular momentum operator of the particle and show that it is a constant of motion.

   [7 marks]

   iv) Write down the wave function of the particle in a state with definite momentum and spin. For a massless particle, define a helicity basis and express the wave function in terms of the helicity basis.

   [4 marks]

   b) Write down the Dirac Hamiltonian of an electron in a uniform magnetic field B in the z direction. Derive the Hamiltonian in the non-relativistic limit and find the g factor of the intrinsic spin of the electron.

   [5 marks]

2. a) Consider the weak Zeeman effects for transitions $^1D_2 \to {}^1P_1$ in helium. The Landé g factor is defined as

$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}.$$

Determine the g factor for the energy levels, and draw a level diagram accurately showing the weak Zeeman shifts. In the diagram mark all the transitions consistent with electric dipole selection rules. State how the spectral line $^1D_2 \to {}^1P_1$ changes due to the weak Zeeman effects.

[4 marks]

b) Find the generator of rotations about the z-axis for a spinless particle. Hence, derive the operator $\hat{U}_\alpha$ representing a finite rotation of angle α about the z-axis. Show that $\hat{U}_\alpha$ is unitary.

[9 marks]

c) Consider a spinless non-relativistic particle of charge q and mass m.

i) Write down the electromagnetic gauge transformation for the scalar and vector potentials Φ and $\mathbf{A}$ in terms of a differentiable function $\lambda(\mathbf{r}, t)$.

[2 marks]

ii) The particle is placed in a laser field which is modeled by a homogeneous oscillating electric field in the z direction, $E_z = E_0 \cos\omega t$. In two different gauges, the electric field may be considered as arising purely from a scalar potential Φ, or purely from a vector potential $\mathbf{A}$. Write down the corresponding potentials, and find a function $\lambda(\mathbf{r}, t)$ which transforms from the first to the second. Write down the Hamiltonian of the particle for each case. Given that $\psi_1(\mathbf{r}, t)$ is a solution in the first case, show that

$$\psi_2(\mathbf{r}, t) = \exp\left(iq\lambda/\hbar\right) \psi_1(\mathbf{r}, t),$$

is a solution in the second case.

[10 marks]

3. a) A hydrogen atom in its ground state ψ_{100} is subject to a uniform electric field in the z direction,

$$E(t) = \begin{cases} 0, & t \leq 0, \\ E_0 e^{-t/\tau}, & t > 0, \end{cases}$$

where E_0 and $\tau (> 0)$ are constants.

i) Find, at time $t\,(> 0)$, the probability $P(t)$ that the hydrogen atom ends up in the excited state ψ_{210} using the first-order approximation. You may use the following wave functions of hydrogen,

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}, \quad \psi_{210}(\mathbf{r}) = \frac{r}{\sqrt{32\pi a_0^5}} e^{-r/2a_0} \cos\theta,$$

where a_0 is the Bohr radius.

[8 marks]

ii) Find the small t behavior of $P(t)$. Determine the limit $P(\infty)$ and discuss the conditions for its validity.

[4 marks]

b) A spinless particle of charge e is bounded by an infinite electrostatic spherical potential well of radius R,

$$e\Phi(r) = \begin{cases} 0, & r \leq R, \\ \infty, & r > R. \end{cases}$$

i) Write down the time-independent Klein-Gordon equation for the eigenstates of the particle.

[2 marks]

ii) By considering an eigenstate of total angular momentum with quantum number l, find the eigenvalue equation for the radial part of the wave function of the particle.

[4 marks]

iii) Using suitable boundary conditions, find the normalized ground-state wave function and the ground-state energy of the particle. Discuss the non-relativistic limit for the ground-state energy.

[7 marks]

END OF EXAMINATION PAPER