PHYS 10672 Advanced Dynamics 2024

Section 5 : Special Relativity

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Special Relativity - FS chapters 11,12 and 13

START OF 17TH LECTURE

5.1 Brief recap

The objective of this part of the present course is to enhance your understanding of Special Relativity building on the course on Quantum Physics and Relativity in semester one. In this section we will just quickly review some of the key points you learnt in that course.

The postulates that Einstein made were that:

- all laws of nature are same for all inertial observers;
- the speed of light (in vacuum) is the same for all inertial observers.

These ideas led him to realise that space and time are not absolute, but are related by Lorentz transformations. If we have two inertial frames S and S' moving relative to each other along the x-direction then the coordinates are related by

$$\begin{aligned} ct' &= \gamma(ct - \beta x)\,, \\ x' &= \gamma(x - \beta ct)\,, \\ y' &= y\,, \\ z' &= z\,, \end{aligned} \tag{5.1}$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. Notice that this has been specified in terms of the coordinates ct, x, y and z all of which have the same dimensions and the dimensionless variables β and γ. It is a simple consequence of these definitions, which you are asked to check in the exercises, that

$$c^2t^2 - \mathbf{x}^2 = c^2t^2 - x^2 - y^2 - z^2\,, \tag{5.2}$$

is Lorentz invariant, meaning that is the same in all inertial frames, and in particular in S and S'.

There are important physical consequences of this which include:

- moving clocks run slower by a factor of γ, known as time dilation;
- moving objects look shorter by a factor of γ along the direction of motion, known as length contraction;

- basic quantities in mechanics are modified at high speeds, for example†,

$$E = \gamma mc^2 \,, \quad \mathbf{p} = \gamma m\mathbf{v} \,, \quad K = (\gamma - 1)mc^2 \,; \tag{5.3}$$

- electricity and magnetism are unified into electromagnetism (you are presently studying a course called *Electricity and Magnetism* and next year you will study one called *Electromagnetism* where you will learn about this unification);
- the standard relative velocity (or velocity addition) law is modified to

$$V = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \,, \tag{5.4}$$

  which becomes $V = v_1 + v_2$ when v_1 and $v_2 \ll c$.

5.2 Group theory

The study of Special Relativity is underpinned by a number of abstract constructs that tell us what is well defined from a mathematical point of view. One of them is group theory and in this short section we will define this concept. In order to write things more succinctly is often helpful to use the following mathematical symbols:

- $\forall$ - "for all";
- $\in$ - "is a member of";
- $\exists$ - "there exists";
- $\therefore$ - "therefore".

Using these we can define a group $(G, \bullet)$ which is a set of elements $G = \{a, b, c, \dots\}$ and a composition (or operation) $\bullet$ which have the following properties.

- *Closure:* $\forall\, a, b \in G, c = a \bullet b \in G$.
- *Associativity:* $\forall\, a, b, c \in G, a \bullet (b \bullet c) = (a \bullet b) \bullet c$.
- *Identity:* $\exists\, e \in G$ such that $e \bullet a = a \bullet e = a \;\forall a \in G$.
- *Inverse:* $\forall\, a \in G \,\exists\, a^{-1} \in G$ such that $a^{-1} \bullet a = a \bullet a^{-1} = e$.

A group is said to be Abelian if it is commutative, that is, $\forall\, a, b \in G\, a \bullet b = b \bullet a$.

The real numbers are defined to be $\mathbb{R} = \{\text{distance from a point of line}\}$ and the complex numbers are $\mathbb{C} = \{a + ib \text{ such that } a, b \in \mathbb{R}\}$. If we define $\mathbb{R}\backslash\{0\}$ and $\mathbb{C}\backslash\{0\}$ to be the sets without the zero element, then rather obviously $(\mathbb{R}, +)$ $(\mathbb{R}\backslash\{0\}, \times)$ are Abelian groups as are $(\mathbb{C}, +)$ and $(\mathbb{C}\backslash\{0\}, \times)$.

Sets of matrices can be group, but they are not Abelian since matrix multiplication is

† Note that we use K to signify the kinetic energy in this part of the course

not commutative. We define the General Linear group over the real numbers covering all n-dimensional linear transformations

$$\mathbb{GL}(n,\mathbb{R}) = \{\text{Real } n \times n \text{ matrices with determinant} \neq 0\}\,. \tag{5.5}$$

In this course, we will be interested in some subsets of this group, in particular:

- $O(n) = \{M \in \mathbb{GL}(n,\mathbb{R}) : MM^T = M^TM = I\}$ - the group of orthogonal $n \times n$ matrices;
- $SO(n) = \{M \in O(n) : \det M = 1\}$ - the special orthogonal group which are the rotation matrices.

$SO(2)$ is an example of a special orthogonal group. It comprises the set of 2D rotation matrices

$$SO(2) = \left\{ R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,:\, 0 \leq \theta < 2\pi \right\}\,. \tag{5.6}$$

It is easy to show that $R(\theta)R(\phi) = R(\theta+\phi)$ hence it is closed, matrix multiplication is associative. $R(0) = I$ and $R(\theta)R(-\theta) = I$ implying that there is an identity and an inverse for each component so we have to checked the conditions for it to be a group. Similarly, the group $SO(3)$ is the group of 3D rotation matrices.

The orthogonal matrices maintain the inner (dot) product of two vectors, and hence the length of vectors and the angles between them. Let us consider two n-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ which are rotated into $\mathbf{a}'$ and $\mathbf{b}'$ by a matrix $M \in O(n)$

$$\mathbf{a}' = M\mathbf{a}\,, \quad \mathbf{b}' = M\mathbf{b}\,. \tag{5.7}$$

From this we can proceed in two ways - using matrices or using the summation convention. Using the matrix approach we have that

$$\mathbf{a}' \cdot \mathbf{b}' = \mathbf{a}'^T\mathbf{b}' = \mathbf{a}^T M^T M\mathbf{b} = \mathbf{a}^T\mathbf{b} = \mathbf{a}\cdot\mathbf{b}\,, \tag{5.8}$$

while using the summation convention

$$\mathbf{a}' \cdot \mathbf{b}' = a_i'b_i' = M_{ij}a_jM_{ik}b_k = \delta_{jk}a_jb_k = a_jb_j = \mathbf{a}\cdot\mathbf{b}\,, \tag{5.9}$$

since $M_{ij}M_{ik} = \delta_{jk}$ or $M^TM = I$ are the defining properties of an orthogonal matrix.

END OF 17TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 17 Exercises

1. Show that $K = \frac{1}{2}mv^2 + \frac{3}{8c^2}mv^4 + ..$ and $v = c\sqrt{1 - \left(\frac{mc^2}{E}\right)^2}$ for a relativistic particle.
2. Prove that $c^2t^2 - x^2 - y^2 - z^2$ is invariant under Lorentz transformations.
3. (i) What are the identity elements for the groups $(\mathbb{R}, +)$ and $(\mathbb{R}\backslash\{0\}, \times)$?
(ii) Are $(\mathbb{Z}, +)$, $(\mathbb{Z}, \times)$ and $(\mathbb{N}, +)$ groups?
(iii) Prove that the set

$$C_N = \{\exp\left[2\pi i\alpha/N\right] \,:\, \alpha = 0, 1, .., N-1\}$$

is a group under multiplication.
4. Write down the 3D rotation matrices corresponding to rotations about the 3 coordinate axes. Show that $LL^T = I$ in each case.

START OF 18TH LECTURE

5.3 Measuring distances

Let us consider how distances are calculated. In ordinary 3D Euclidean space we can calculate the distance between two points $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, .y_3)$ in 3D to be

$$|\mathbf{y} - \mathbf{x}| = \left[(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2\right]^{\frac{1}{2}} \ . \tag{5.10}$$

If $\mathbf{y} = \mathbf{x} + d\mathbf{x}$ where $d\mathbf{x}$ is an infinitesimal vector connecting the two points then

$$ds^2 = |d\mathbf{x}|^2 = dx_1^2 + dx_2^2 + dx_3^2 = \delta_{ij}dx^i dx^j \ , \tag{5.11}$$

where ds is the distance and we have written it in terms of the Kronecker-δ in the final expression. Notice that we have put the indices i and j in the "up" position and this is to fit in with what we will discuss in subsequent sections. For the moment forget this subtlety. We see that the Kronecker δ is not only the unique isotropic tensor of rank 2, it is also represents the *metric* of n-dimensional Euclidean space - which means it plays a role in calculating the distance between two points. - noting that we can easily generalise the idea above from $n = 3$ to any n.

We can further generalise this idea to cover any space

$$ds^2 = g_{ij}(\mathbf{x})dx^i dx^j \ , \tag{5.12}$$

where the metric $g_{ij}(\mathbf{x})$ could in principle depend on position and $g_{ij} = \delta_{ij}$ is the special case of Euclidean space. If $\mathbf{x} = (x, y, z)$ then

$$ds^2 = dx^2 + dy^2 + dz^2 \ , \tag{5.13}$$

and the non-zero coefficients of the Kroncker-δ, 1 when $i = j$, are the coefficients in front of dx, dy and dz. Moreover, there are no cross terms, that is terms like $dxdy$, since $\delta_{ij} = 0$ if $i \neq j$.

Last semester you learnt that space and time are intimately related in Special Relativity. We can adapt the concept of distance between two points, to the *interval* between two events. If we use $X^\mu = (ct, \mathbf{x}) = (ct, x, y, z)$ to be the components of what we will call the position *4-vector* in the next section and use Roman indices, for example, $i, j, \cdots = 1, 2, 3$ and Greek indices, $\mu, \nu, \cdots = 0, 1, 2, 3$ with the summation convention, and ignoring the issue of "up" and "down" indices, then the spacetime interval can be written as

$$ds^2 = g_{\mu\nu}(X^\mu) dX^\mu dX^\nu \,. \tag{5.14}$$

The case of $g_{\mu\nu}$ being a function of spacetime position is complicated, and comes in when one studies General Relativity. The equivalent of δ_{ij} in the Minkowski spacetime relevant to Special Relativity is the *Minkowski metric*

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{5.15}$$

and the interval, sometimes called the line element, is

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \,. \tag{5.16}$$

We note that we have chosen to use what is called a $(+ - - -)$ signature referring to the ordering of the pluses and minuses in the definition of $\eta_{\mu\nu}$. Some books and other courses which you might study could use the $(- + + +)$ signature where the signs of the elements in $\eta_{\mu\nu}$ are reversed. This can lead to other sign convention related changes in what we will discuss in the subsequent sections - so look out for this in any book or internet source you might read on this topic.

5.4 4-vectors in Special Relativity - see FS12

First, let us consider two Euclidean vectors $\mathbf{a} = a_i \mathbf{e}_i$ and $\mathbf{b} = b_i \mathbf{e}_i$ where a_i and b_i are the components with respect to some orthonormal basis with $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. We can compute the inner product of the two vectors to be

$$\mathbf{a} \cdot \mathbf{b} = a_i \mathbf{e}_i \cdot b_j \mathbf{e}_j = a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} a_i b_j = a_i b_i \,, \tag{5.17}$$

where the summation is over all the coordinates $i = 1, .., n$. Remember that we showed that this was also invariant under rotations $\mathbf{a}' = L\mathbf{a}$ and $\mathbf{b}' = L\mathbf{b}$ since $LL^T = I$ and

$L \in SO(3)$ - the orthogonal group of rotations in 3D. An example of this is the distance between two points

$$ds^2 = \delta_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2 \,. \tag{5.18}$$

It is logical that rotations of the coordinate system do not effect the distance between two points and indeed also not the angle between two vectors.

Special Relativity works in Minkowski spacetime (as opposed to Euclidean 3-space) and we can define the equivalent 4-vectors which we will denote by $\tilde{\mathbf{A}} = A^\mu \tilde{\mathbf{e}}_\mu$ where A^μ are the components of the 4 -vector and $\tilde{\mathbf{e}}_\mu$ are the basis vectors. Continue to ignore the fact that there are "up" and "down" indices, which will be explained in the next lecture, but remember that there is a summation over the $\mu = 0, 1, 2, 3$ - it is an Lorentz index. We can try to calculate the inner product of two 4-vectors $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$

$$\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}} = A^\mu \tilde{\mathbf{e}}_\mu \cdot B^\nu \tilde{\mathbf{e}}_\nu \,. \tag{5.19}$$

We need to decide what is the equivalent of the orthonormality relation used in the Euclidean spaces, and hopefully it is obvious by analogy to that case that it is

$$\tilde{\mathbf{e}}_\mu \cdot \tilde{\mathbf{e}}_\nu = \eta_{\mu\nu} \,. \tag{5.20}$$

In fact, there are deep mathematical reasons for this to be the case even in more general cases - that is replacing the Minkowski metric with a more general spacetime position dependent one, $g_{\mu\nu}(X^\mu)$. In this case we have that

$$\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}} = \eta_{\mu\nu} A^\mu B^\nu = A^0 B^0 - A^i B^i = A^0 B^0 - \mathbf{A} \cdot \mathbf{B} \,, \tag{5.21}$$

remembering that the indices $\mu, \nu = 0, 1, 2, 3$ and $i = 1, 2, 3$.

The canonical example of a 4-vector is the spacetime position vector whose components are $X^\mu = (ct, x, y, z)$. We can calculate the length of this 4-vector in Minkowski spacetime

$$\eta_{\mu\nu} X^\mu X^\nu = c^2 t^2 - \mathbf{x}^2 = c^2 t^2 - x^2 - y^2 - z^2 \,, \tag{5.22}$$

and also the spacetime interval for infinitesimal distance between two events

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu = c^2 dt^2 - d\mathbf{x}^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \,. \tag{5.23}$$

However, there are many more and we will discuss below three important 4-vectors all of which are related is some way.

- If we define the proper time between two time-like separated intervals to be $d\tau$ then

$$c^2 d\tau^2 = ds^2 = c^2 dt^2 - d\mathbf{x}^2 \,, \tag{5.24}$$

  then the 4-velocity is defined by

$$U^\mu = \frac{dX^\mu}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau} \right) . \tag{5.25}$$

It is one of the exercises to show that

$$\gamma = (1-\beta^2)^{-\frac{1}{2}} = \frac{dt}{d\tau}\,, \tag{5.26}$$

We also find that $g_{\mu\nu}U^\mu U^\nu = c^2$.

- The 4-momentum is a 4-vector defined by

$$P^\mu = \left(\frac{E}{c}, \mathbf{p}\right) = \left(\frac{E}{c}, p_x, p_y, p_z\right)\,, \tag{5.27}$$

and

$$P^\mu = m\frac{dX^\mu}{d\tau} = mU^\mu\,, \tag{5.28}$$

for a particle of mass m. It is part of the exercises to show that this is compatible with $E = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v}$.

- In the vibrations and waves course you have studied plane wave solutions of the wave equation

$$\frac{1}{c^2}\frac{\partial^2 f}{dt^2} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} = 0\,, \tag{5.29}$$

of the form

$$f(t,\mathbf{x}) = f_0 \exp\left[i\left(\omega t - \mathbf{k}\cdot\mathbf{x}\right)\right]\,, \tag{5.30}$$

with dispersion relation $\omega = c|\mathbf{k}|$. We would expect $\phi = \omega t - \mathbf{k}\cdot\mathbf{x}$ to be Lorentz invariant since if this were not to be the case the waves would have different shapes in different frames. Therefore, we can define the wave 4-vector to be

$$k^\mu = \left(\frac{\omega}{c}, \mathbf{k}\right)\,, \tag{5.31}$$

which in Quantum Mechanics - via wave-particle duality - can be related the 4-momentum $P^\mu = \hbar k^\mu$ and the solution to the wave equation is

$$f = f_0 \exp\left[i\eta_{\mu\nu}k^\mu X^\nu\right]\,, \tag{5.32}$$

with dispersion relation $\eta_{\mu\nu}k^\mu k^\nu = 0$.

END OF 18TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 18 Exercises

1. The line element for 2D Euclidean space is $ds^2 = dx^2 + dy^2$. Convert this to 2D polar coordinates using $x = r\cos\theta$ and $y = r\sin\theta$ and deduce the metric in 2D polar coordinates.

2. Show that $\gamma \equiv \left[1 - \frac{1}{c^2}\left(\frac{d\mathbf{x}}{dt}\right)^2\right]^{-\frac{1}{2}} = \frac{dt}{d\tau}$. Hence, deduce that the 4-velocity can be written as $U^\mu = \gamma(c, \mathbf{v})$ where $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ is the 3-velocity, and $P^\mu = m\gamma(c, \mathbf{v})$. Convince yourself that this gives rise to relations that you are familiar with from QPR.

3. The 4-acceleration is defined as $a^\mu = \frac{dU^\mu}{d\tau}$ where U^μ is the 4-velocity and τ is the proper time. Show that it is orthogonal (under the metric of Minkowski spacetime) to the 4-velocity.

START OF 19TH LECTURE

5.5 Covariant and contravariant indices

In this section we will explain the difference between “up” and “down” indices. If we write the Minkowski metric with “down” indices

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{5.33}$$

then the convention is to write the inverse matrix with “up” indices

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{5.34}$$

so that $\eta_{\mu\alpha}\eta^{\alpha\nu} = \delta^\nu_\mu$ and δ^ν_μ is the Kronecker-δ with Lorentz indices. It has the specific properties that it is 1 if $\mu = \nu$ and is zero otherwise. It plays the role of the identity matrix in this context.

In this specific case, thinking of them as matrices, $\eta^{-1} = \eta$ and $\eta^2 = I$, but in more general cases - that we will not study here - they can be different. Our objective is to become familiar with these ideas and the associated use in the context of the summation convention for Lorentz indices.

The components of a 4-vector with an “up” index, A^μ, is known as a contravariant vector whereas one with a lowered index, A_μ, is known as covariant. There is a consistent mathematical basis for these definitions with contravariant vectors living in what is known as a *vector space* and the covariant vectors being members of the associated *dual space*. However, we will not concern ourselves with these mathematical niceties. The key thing from our point of view is that the metric and inverse metric can be seen as index lowering and and raising operators. If we define

$$A_\mu = \eta_{\mu\nu} A^\nu \,, \tag{5.35}$$

then

$$\eta^{\mu\alpha} A_\alpha = \eta^{\mu\alpha} \eta_{\alpha\beta} A^\beta = \delta^\mu_\beta A^\beta = A^\mu \,. \tag{5.36}$$

So if the components of $A^\mu = (A^0, \mathbf{A}) = (A^0, A^1, A^2, A^3)$ and $A_\mu = (A_0, A_1, A_2, A_3)$ then $A^0 = A_0$ and $A^i = -A_i$ for $i = 1, 2.3$. With this definition we see that

$$\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}} = \eta_{\mu\nu} A^\mu B^\nu = A_\mu B^\mu = A^\mu B_\mu \,. \tag{5.37}$$

The summation convention is in operation in this new context, but there are extra things to add to the standard rules:

- Summed indices must be "up" and the other "down";
- There must be the same number of "up" and "down" indices on each side of an equation, and in all terms added together in an equation.

It is probably best to illustrate these points with respect to some correct and incorrect examples. First, the following examples obey the rules

$$\begin{aligned} A^\mu &= B^{\mu\nu} C_\nu + D^\mu \,, \\ A^\mu &= X^\mu Y_\alpha Z^\alpha Y^\beta Z_\beta = X^\mu (Y_\alpha Z^\alpha)^2 \,, \\ A^\mu{}_\nu &= B^\mu{}_{\nu\rho} C^\rho \,, \end{aligned} \tag{5.38}$$

while these do not

$$A^\mu = B^{\mu\nu\rho} C_\rho \,, \quad A_{\mu\nu} = B_{\mu\nu}{}^\rho C^\rho \,, \quad A^\nu = B^\nu + C_\rho \,. \tag{5.39}$$

The first of these is inconsistent since there is only one free index on the LHS, but there are two on the RHS, the second has both of the summed indices (ρ) in the "up" position and finally the third one makes no sense since one is adding the components of a contravariant and covariant vectors. As with the case of the standard summation convention is it helpful to try to learn how to use this new set of rules by practice and there are exercises and problem sheet questions to assist with this.

One might rightly ask the question why we do not routinely distinguish between contravariant and covariant indices in the case for Euclidean space. The metric on Euclidean space is δ_{ij} so $A_i = \delta_{ij} A^j = A^i$ and therefore in this special case the covariant and contravariant components are equal. This is is not true in more general situations and in particular in Special Relativity so you have been able to happily avoid this mathematical complication until now.

Finally, in this section let us consider the examples of 4-vectors that we discussed in the previous section. The spacetime position vector is usually written as $X^\mu = (ct, \mathbf{x}) = (ct, x, y, z)$ and therefore

$$X_\mu = \eta_{\mu\nu} X^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix}, \tag{5.40}$$

and so $X_\mu = (ct, -\mathbf{x}) = (ct, -x, -y, -z)$. Similarly, $U^\mu = \gamma\,(c, \mathbf{v}) \to U_\mu = \gamma\,(c, -\mathbf{v})$, $P^\mu = \left(\frac{E}{c}, \mathbf{p}\right) \to P_\mu = \left(\frac{E}{c}, -\mathbf{p}\right)$ and $k^\mu = \left(\frac{\omega}{c}, \mathbf{k}\right) \to k_\mu = \left(\frac{\omega}{c}, -\mathbf{k}\right)$ where $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ is the 3-velocity.

5.6 Lorentz and Poincare symmetries - see FS11.2

In the previous sections we discussed the inner product in Minkowski spacetime and 4 vectors and how this leads us to the concepts of contravariant and covariant indices. This was done by analogy to the standard inner product. Another property of the inner product which we highlighted was that it was invariant under $O(3)$ transformations. In this section we will explain how Lorentz symmetry can replace the orthogonal symmetry of Euclidean spacetime, leading to the concept of Lorentz invariants, and that the Lorentz symmetry can be extended to Poincaré symmetry. In the next section of the course we will discuss how these symmetries are related to conservation laws.

Let us first return to the Lorentz transformation in the x-direction from (5.1) and write it as a matrix equation acting on the contravariant components of the spacetime position vector

$$X'^\mu = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \Lambda^\mu{}_\nu X^\nu \,. \tag{5.41}$$

The matrix $\Lambda^\mu{}_\nu$ is an element of the Lorentz symmetry group, $SO(1,3)$, which we will discuss in more detail below. This symmetry applies to all 4-vectors, for example, the 4-velocity, 4-momentum and 4-wavevector to transform between frames moving with a velocity $v = \beta c$.

Euclidean	Minkowski
$A'_i = L_{ij}A_j$	$A'^{\mu} = \Lambda^{\mu}{}_{\nu}A^{\nu}$
$L_{ik}L_{jl}\delta_{kl} = \delta_{ij}$	$\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\eta_{\alpha\beta} = \eta_{\mu\nu}$
$L^T I L = I$	$\Lambda^T \eta \Lambda = \eta$
$\mathrm{detL} = 1$	$\det\Lambda = 1$
$L \in SO(3)$	$\Lambda \in SO(1,3)$
$ds^2 = \delta_{ij}dx_i dx_j$	$ds^2 = \eta_{\mu\nu}dX^{\mu}dX^{\nu}$

The Lorentz transformation can be be seen as a "rotation" in Minkowski spacetime in a similar way to the members of the special orthogonal group $SO(3)$ being rotations in 3D Euclidean space. In the table above we make list of some equivalent mathematical statements which hopefully illustrate how things are generalized. Some of the entries in the table warrant further discussion. Notably that I have written $L_{ik}L_{jl}\delta_{kl} = \delta_{ij}$ instead of $L_{ik}L_{jk} = \delta_{ij}$ and $L^T I L = I$ instead of $L^T L = I$ in order to make it clear that there are connections with the Minkowski case under the transformation $\delta_{ij} \to \eta_{\mu\nu}$ and $I \to \eta$. That complication notwithstanding, there is a clear connection between the two columns.

We have defined the action of the group $SO(1,3)$ to be $A'^{\mu} = \Lambda^{\mu}{}_{\nu}A^{\nu}$ on the contravariant components of a 4-vector. We can take this expression and convert each of the contravariant components to be covariant with appropriate Minkowski metrics acting as lowering operators

$$\eta^{\mu\gamma}A'_{\gamma} = A'^{\mu} = \Lambda^{\mu}{}_{\nu}\eta^{\nu\beta}A_{\beta}\,, \tag{5.42}$$

and then we multiply each side by $\eta_{\alpha\mu}$ and noting that $\eta_{\alpha\mu}\eta^{\mu\gamma} = \delta_{\alpha}{}^{\gamma}$ yielding

$$A'_{\alpha} = (\eta_{\alpha\mu}\Lambda^{\mu}{}_{\nu}\eta^{\nu\beta})A_{\beta}\,, \tag{5.43}$$

so that we see that $\eta_{\alpha\mu}\Lambda^{\mu}{}_{\nu}\eta^{\nu\beta}$ is the Lorentz transformation for the covariant components. This can be simplified: the term in the brackets is the components of the matrix $\eta\Lambda\eta^{-1}$. But remember from the table that $\Lambda^T\eta\Lambda = \eta$ which implies $\eta\Lambda = (\Lambda^T)^{-1}\eta$ and, therefore, $(\Lambda^T)^{-1} = \eta\Lambda\eta^{-1}$. We can rewrite (5.43) as

$$A'_{\alpha} = [(\Lambda^T)^{-1}]_{\alpha}{}^{\beta}A_{\beta} = (\Lambda^{-1})^{\beta}{}_{\alpha}A_{\beta}\,. \tag{5.44}$$

So we see that the covariant components transforms with the inverse Lorentz transformation. We note that for a Lorentz boost the inverse involves making the transformation $v \to -v$ in $\Lambda^{\mu}{}_{\nu}$.

The inner product of two 4-vectors is Lorentz invariant. Consider two 4-vectors $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ which are transformed to $\tilde{\mathbf{A}}'$ and $\tilde{\mathbf{B}}'$, then we have that $A'_{\mu} = (\Lambda^{-1})^{\alpha}{}_{\mu}A_{\alpha}$ and $B'^{\mu} = \Lambda^{\mu}{}_{\beta}B^{\beta}$ then

$$\tilde{\mathbf{A}}' \cdot \tilde{\mathbf{B}}' = (\Lambda^{-1})^{\alpha}{}_{\mu}A_{\alpha}\Lambda^{\mu}{}_{\beta}B^{\beta} = \delta^{\alpha}_{\beta}A_{\alpha}B^{\beta} = A_{\alpha}B^{\alpha} = \tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}}\,. \tag{5.45}$$

Lorentz invariant quantities such as the inner product of two 4-vectors are important since they have the same value in all frames. Examples of these include

$$ds^2 = dX_\mu dX^\mu = c^2 dt^2 - d\mathbf{x}^2\,, \quad X_\mu X^\mu = c^2t^2 - \mathbf{x}^2\,, \quad k_\mu X^\mu = \omega t - \mathbf{k}\cdot\mathbf{x}\,. \tag{5.46}$$

Other physically important Lorentz invariants are $U_\mu U^\mu = c^2$, which is an identity,

$$P_\mu P^\mu = \left(\frac{E}{c}\right)^2 - \mathbf{p}^2 = (mc)^2\,, \tag{5.47}$$

where the final equality is attained by shift to the particles rest frame and

$$k_\mu k^\mu = \left(\frac{\omega}{c}\right)^2 - \mathbf{k}^2\,, \tag{5.48}$$

where $k_\mu k^\mu = 0$ becomes the dispersion relation for massless particles, $\omega = c|\mathbf{k}|$.

The group $SO(1,3)$ - where the 1 is the number of plus signs in the signature and 3 is the number of minus signs - is known as the Lorentz group. It includes 3 spatial rotations and 3 Lorentz boosts (rotations). In principle, $\Lambda^\mu{}_\nu \equiv \Lambda^\mu{}_\nu(\theta,\hat{\mathbf{n}},\boldsymbol{\beta})$ corresponding to a rotation by angle θ about an axis defined by the unit vector $\hat{\mathbf{n}}$, and velocity $\mathbf{v} = \boldsymbol{\beta}c$. One can consider general rotations and boosts:

- <u>Rotations</u>

$$\Lambda^0{}_0 = 1\,, \quad \Lambda^0{}_i = \Lambda^i{}_0 = 0\,, \quad \Lambda^i{}_j = (\delta^i_j - \hat{n}^i\hat{n}_j)\cos\theta - \epsilon^i{}_{jk}\hat{n}^k\sin\theta + \hat{n}^i\hat{n}_j\,, \tag{5.49}$$

  where $\hat{n}_i = \hat{n}^i$, that is, $\hat{\mathbf{n}}$ is a Euclidean vector;
- <u>Lorentz boosts</u>

$$\Lambda^0{}_0 = \gamma\,, \quad \Lambda^0{}_i = -\gamma\beta_i\,, \quad \Lambda^i{}_0 = -\gamma\beta^i\,, \quad \Lambda^i{}_j = \delta^i_j + (\gamma-1)\frac{\beta^i\beta_j}{|\boldsymbol{\beta}|^2}\,, \tag{5.50}$$

  where $\beta_i = \beta^i$.

Two examples of this are for rotations about the x-axis, that is, $\hat{\mathbf{n}} = (1,0,0)$, we have that

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}\,, \tag{5.51}$$

and for a Lorentz boost with velocity v at an angle θ to the x-axis, that is, $\mathbf{v} = v(\cos\theta,\sin\theta,0)$, we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\frac{\gamma v}{c}\cos\theta & -\frac{\gamma v}{c}\sin\theta & 0 \\ -\frac{\gamma v}{c}\cos\theta & 1+(\gamma-1)\cos^2\theta & (\gamma-1)\cos\theta\sin\theta & 0 \\ -\frac{\gamma v}{c}\sin\theta & (\gamma-1)\cos\theta\sin\theta & 1+(\gamma-1)\sin^2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{5.52}$$

As a closing comment in this section one can extend the Lorentz symmetry group to include the spatial and translation invariance that we will discuss in the next lecture. The associated symmetry group, known as the *Poincaré* group, has ten independent generators - the 6 from the Lorentz group corresponding to rotations and Lorentz boosts discussed above plus the 3+1 additional generators from 3 spatial translation and 1 time translation. The maximal symmetry of Minkowski spacetime and is denoted $\mathbb{R}^{1,3} \times SO(1,3)$.

END OF 19TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 19 Exercises

1. The covariant components of some dimensionless 4-vector are $A_\mu = (1, 2, 0, -1)$ in a frame S. What are the contravariant components of the vector in S? A frame S' is moving with a speed $v = \sqrt{3}\,c/2$ in the y-direction relative to S. What are the contravariant and covariant components in S'. By explicitly calculating $A_\mu A^\mu$ in the two frames, demonstrate that it is Lorentz invariant in this specific case.

2. U^μ is the 4-velocity of a fluid, $\gamma_{\mu\nu} = \frac{U_\mu U_\nu}{c^2} - \eta_{\mu\nu}$ and the energy-momentum tensor is

$$T_{\mu\nu} = \rho \frac{U_\mu U_\nu}{c^2} + P\gamma_{\mu\nu}\,, \tag{5.53}$$

where ρ is the energy density and P is the pressure. Ignoring the physical context, while noting that there is one, and treating this as just a exercise in manipulating 4-vectors and Lorentz indices, show that $U^\mu \gamma_{\mu\nu} = 0$, $\gamma_{\mu\nu}\gamma^{\nu\alpha} = -\gamma_\mu{}^\alpha$ and calculate $U^\mu U^\nu T_{\mu\nu}$, $\gamma^{\mu\nu}T_{\mu\nu}$ and $U^\mu \gamma^{\nu\alpha} T_{\mu\nu}$.

3. Show that $\det\Lambda = 1$ for the example elements of the Lorentz group presented in the lecture.

4. A particle is at rest in a frame S. What are the contravariant components of the 4-velocity in S. What are the contravariant and covariant components in a frame, S' comoving with a velocity $\mathbf{v}$ relative to S. Remember that in the lectures it was stated that $\Lambda^0{}_0 = \gamma$ and $\Lambda^i{}_0 = -\gamma\beta^i$ and that the inverse Lorentz transformation is given by $\mathbf{v} \to -\mathbf{v}$.

START OF 20TH LECTURE

5.7 Symmetries in physics and conservation laws - see FS11.1

In previous lectures we have discussed the concept of groups and that the symmetries of the Euclidian space are related to orthogonal groups. Right at the end of the last lecture we discussed the Lorentz and Poincaré symmetries of the Minkowski spacetime which underlies Special Relativity. In this section we will discuss some implications that the existence of these symmetries has for physical laws.

Let us first start with the simplest symmetry, that of time translation invariance. This means that the physical law in question is invariant under the transformation $t \to t' = t + t_0$ where t_0 is a constant, that is, that events can be measured by identical clocks offset by a constant amount,such as in different time zones on the Earth. An example of this Newton's 2nd law: $m\ddot{\mathbf{r}} = \mathbf{F}$ when the force has no explicit dependence of time (eg. if $\mathbf{F} = -k\mathbf{r}$) which is invariant since $dt' = dt$ because t_0 is a constant. No doubt the property is so trivial that it will have escaped your notice, but in fact it has some profound implications

A related symmetry is that of spatial translation invariance where $\mathbf{r} \to \mathbf{r}' = \mathbf{r} + \mathbf{r}_0$ and $\mathbf{r}_0$ is a constant. An example of a physical law where this is relevant is Newtonian gravity. Consider two particles at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. In earlier sections of the course we showed that this could be represented by

$$\ddot{\mathbf{r}} = -G_{\mathrm{N}} M_{\mathrm{tot}} \frac{\mathbf{r}}{|\mathbf{r}|^3}\,, \tag{5.54}$$

where M_{tot} is the total mass of the system and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the vector separating the two particles. Under the translation $\mathbf{r}_1 \to \mathbf{r}_1 + \mathbf{r}_0$ and $\mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_0$ and hence $\mathbf{r} \to \mathbf{r}_1 + \mathbf{r}_0 - (\mathbf{r} + \mathbf{r}_0) = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}$. Hence, we see that Newton's law of gravity is *form covariant* under spatial translations meaning that the form of the law is unchanged.

Another symmetry that we have discussed is rotational invariance. In this case we consider transformations in Euclidean space of the form $\mathbf{r} \to \mathbf{r}' = L\mathbf{r}$ where $L \in SO(3)$ is a rotation matrix with the properties $LL^T = I$ or $L_{ik}L_{jk} = \delta_{ij}$ and $\det L = 1$. Let us again consider Newtonian gravity in this context. If $\mathbf{r}' = L\mathbf{r}$ then

$$\ddot{\mathbf{r}}' = L\ddot{\mathbf{r}} = -G_{\mathrm{N}} M_{\mathrm{tot}} \frac{L\mathbf{r}}{|\mathbf{r}|^3} = -G_{\mathrm{N}} M_{\mathrm{tot}} \frac{\mathbf{r}'}{|\mathbf{r}'|^3}\,, \tag{5.55}$$

since $|\mathbf{r}'| = |\mathbf{r}|$. This shows that Newtonian gravity is also form covariant under rotational transformations as it was for spatial translations.

Let us further consider rotation invariance by considering the properties of the cross product under rotations. Lets us define $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ and assume that $\mathbf{A}' = L\mathbf{A}$ and $\mathbf{B}' = L\mathbf{B}$. If we further define $\mathbf{C}' = \mathbf{A}' \times \mathbf{B}'$ then

$$C_i' = \epsilon_{ijk} A_j' B_k' = \epsilon_{ijk} L_{jp} L_{kq} A_p B_q\,. \tag{5.56}$$

Previously have stated that the Levi-Civita symbols is an isotropic tensor, which means that it is the same in all frames, and hence

$$\epsilon'_{ijk} = \epsilon_{ijk} = L_{ia}L_{jb}L_{kc}\epsilon_{abc}\,, \tag{5.57}$$

which can be substituted into (5.56) and therefore

$$C'_i = L_{ia}L_{jb}L_{jp}L_{kc}L_{kq}\epsilon_{abc}A_pB_q = L_{ia}\delta_{bp}\delta_{cq}\epsilon_{abc}A_pB_q\,. \tag{5.58}$$

Therefore, we can write

$$C'_i = L_{ia}\epsilon_{abc}A_bB_c = L_{ia}C_a\,. \tag{5.59}$$

This means that $L\mathbf{C} = (L\mathbf{A}) \times (L\mathbf{B})$, that is, the form of the cross product is invariant under the rotations.

Now consider the motion of a particle of mass m and charge q in a magnetic field $\mathbf{B}$. The equation of motion for the particle is

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}\,. \tag{5.60}$$

If $\mathbf{r}' = L\mathbf{r}$ then

$$m\ddot{\mathbf{r}}' = qL(\dot{\mathbf{r}}' \times \mathbf{B}) = q(L\dot{\mathbf{r}}') \times (L\mathbf{B}) = q\dot{\mathbf{r}}' \times \mathbf{B}'\,, \tag{5.61}$$

where $\mathbf{B}' = L\mathbf{B}$ is the rotated magnetic field. This implies that the Lorentz force law is form covariant under rotational transformations.

Another symmetry which you may have come across in the study of Newtonian Mechanics is Galilean invariance - the term Galilean invariance referring to Galileo. This is symmetry where $\mathbf{r} \to \mathbf{r}' = \mathbf{r} + \mathbf{v}t$ where $\mathbf{v}$ is a constant velocity. It is an invariance which inertial frames have in Newtonian mechanics. In particular, we see that $\dot{\mathbf{r}}' = \dot{\mathbf{r}} + \mathbf{v}$ and $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}'$, that is Newton's 2nd law is Galilean invariant. Just as an aside the Lorentz transformation can become the Galilean transformation in the limit $c \to \infty$,

$$\begin{aligned} t' &= \gamma\left(t - \beta\frac{x}{c}\right) \to t\,, \\ x' &= \gamma(x - \beta ct) \to x - vt\,, \\ y' &= y\,, \\ z' &= z\,, \end{aligned} \tag{5.62}$$

where we have written this in a dimensionful way, so as to facilitate sending a dimensionful quantity to ∞. We note that it is unusual to take a limit of a dimensionful parameter as opposed to saying that something is large/small relative to another quantity with the same dimensions, that is, taking limits of a dimensionless parameter.

This discussion of symmetries has been building up to Noether's theorem. This can be stated and proved mathematically within specific well-defined theories, but we will just state it colloquially here. It says that "there is a conserved quantity associated with every symmetry of a physical system" and conversely that the existence of a conserved quantity implies a symmetry. We have discussed that Euclidean space has $\mathbb{R}^3 \times SO(3)$

symmetry due spatial and rotational invariance, and that Newtonian Mechanics is time-translation invariance. These symmetries imply the conservation of energy, momentum and angular momentum:

- time translation invariance implies energy conservation;
- spatial translation invariance implies momentum conservation;
- rotational invariance implies angular momentum conservation.

We have also discussed the Poincaré symmetry of Minkowski spacetime which amongst other things leads to conservation of 4-momentum.

5.8 Spacetime diagrams and the light cone - see FS13

In this section we will investigate the role of causality in Special Relativity and the relationship to what is known as the light-cone. Causality refers to the idea that events which are separated by distances further than the speed of light multiplied by their separation in time cannot affect each other since even light would not be able to have travelled between them,

Let us first imagine two events separated by Δt and Δx in time and space in some frame S. The Lorentz transformation implies that the two events are separated by

$$c\Delta t' = \gamma(c\Delta t - \beta\Delta x)\,, \tag{5.63}$$

in another frame S' moving at a speed v relative to S. If we consider the case where $\Delta t > 0$ and $\beta > 0$ then the events would be in the opposite order, that is $\Delta t' < 0$ if

$$\frac{\beta\Delta x}{c} > \Delta t \quad \rightarrow \quad \Delta x > \frac{c\Delta t}{\beta}\,, \tag{5.64}$$

but we have that $\beta < 1$, that is, speed must be less than the speed of light and therefore $\Delta x > c\Delta t$. This means that the distance between the two events must be separated by a distance further than light can travel in the time between the two events in S. This means that causality is respected by Special Relativity and, in particular, by the Lorentz transformation.

On the lecture slides there is a picture to explain the concept of causality in the context of what is known as a spacetime diagram in $1+1$ dimension (ie 1 space and 1 time) where events are plotted in the $(ct{,}x)$ plane. The lines are lines $x = \pm ct$ centred on the two events A and B which is known as the light cone. We see that B is in the causal future of A since it is in the future light cone of A, while A and C are both in the causal past of B. A and C are said to be causally disconnected since C is outside the past or future light cone of A.

In the $1 + 1$ situation we are discussing we have that $c^2\tau^2 = c^2t^2 - x^2$ where τ is the proper time is a Lorentz invariant. In the spacetime diagram we find that different events lie along lines which are $\tau =$ constant which are hyperbolae in the spacetime diagram. In the diagram in the lecture slides events A and A' lie on the same hyperbola, as do B and B', so that they take place at the same value of τ and can be related by a Lorentz transformation. However, A and B cannot be related.

Now let us imagine an event which takes place at $t = 0$ and $x = 0$ in some frame. The causal past of this event is the region with $t < 0$ bounded by the lines $x = \pm ct$ as shown in the diagram on the lecture slides, whereas the causal future is the equivalent region in the upper half plane $t > 0$. Events in the causal past future of an event are said to be time-like separated and $\Delta\tau^2 = \Delta t^2 - (\Delta x/c)^2 > 0$†. By analogy we define events which are causally disconnected, or acausal, to be space-like and have $\Delta\tau^2 < 0$. These definitions divide up the space time diagram into time-like regions delineated by the lines $x = \pm ct$ which correspond to the paths of light rays and these are referred as null with $\Delta\tau = 0$.

The description we have given so far is focussed on the case of $1 + 1$ dimensions we are typically concerned with higher spatial dimensions, $2 + 1$ and $3 + 1$. Obviously, this makes things more difficult to draw, but it is not difficult to see that most of the features discussed above go through without much change. The lecture slides include a $2 + 1$ dimensional case where it becomes evident where the term "light-cone" comes from. In this case the hyperbolae become hyperboloids and the points, which represent events in $1 + 1$ dimensions, become what are referred to as world lines. Hopefully, it is clear that one can generalise this all to higher dimensions, but it is not possible to represent that in a simple picture.

END OF 20TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 20 Exercises

1. There is a conservation law for electric charge density ρ and current density $\mathbf{j}$ which can be written as

$$\dot{\rho} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0\,. \tag{5.65}$$

Show that this equation can be written as $\partial_\mu j^\mu = 0$, and hence is Lorentz invariant, where $\partial_\mu = \frac{\partial}{\partial X^\mu}$ and j^μ is a 4-vector with contravariant components $(c\rho, \mathbf{j})$. Using the divergence theorem discussed in the Maths II course show that

$$Q = \int \rho\, d^3\mathbf{x}\,, \tag{5.66}$$

† The term like-like and the sense (> 0) refers to the coefficient of the Δt^2 term in the spacetime interval and hence is related to our choice of spacetime signature $(+ - - - -)$. If we had used $(- + + +)$ which is used in some books then it would have been $\Delta\tau^2 < 0$.

is a constant if the current flowing into the region of integration is zero.

2. Consider an infinitesimal coordinate transformation $X^\mu \to X'^\mu = X^\mu + \xi^\mu$ where ξ^μ is a function of X^μ and will be considered small, so that $\mathcal{O}(\xi^2)$ can be ignored. Show that the spacetime interval ds^2 is invariant under these transformations if $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0$ - this is known as Killing's equation - and hence that $\Box \xi^\alpha = 0$ where $\Box = \partial_\mu \partial^\mu$. Solve this equation and deduce the number of independent constants needed to specify ξ^α and relate these to the symmetries of the Lorentz group using the expressions for $\Lambda^\mu{}_\nu$ given in lectures when $|\boldsymbol{\beta}| \ll 1$ and $\theta \ll 1$.

3. Define the bivector $M^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu$. How many independent components does it have? Construct the 3-vectors $L^i = \frac{1}{2}\epsilon^i{}_{jk} M^{jk}$ and $N^i = -M^{0i}/c$ and explain their physical significance in the context of a particle moving at a constant velocity $\mathbf{v}$.

4. Three events, A, B and C, take place at $(t, x) = (5\times 10^{-8}\,\mathrm{sec}, 3\,\mathrm{m})$, $(7\times 10^{-8}\,\mathrm{sec}, 15\,\mathrm{m})$ and $(9 \times 10^{-8}\,\mathrm{sec}, 24\,\mathrm{m})$ in some frame S. Which of these events can be related by a Lorentz transformation?

START OF 21ST LECTURE

5.9 Propagation of massless particles

In this section we will discuss the propagation of massless or nearly massless particles within Special Relativity. Massless particles have $E = c|\mathbf{p}|$ and of course $E = hc/\lambda = hf$. The de-Broglie wavelength is $\lambda_{\mathrm{dB}} = h/|\mathbf{p}|$ allowing us to consider nearly massless particles that is, those with energy much larger than the rest mass energy. These can be expressed in terms of the wavenumber $|\mathbf{k}| = 2\pi/\lambda$ and the angular frequency $\omega = 2\pi f$ as $\mathbf{p} = \hbar\mathbf{k}$ and $E = \hbar c|\mathbf{k}| = \hbar\omega$.

Now consider a massless particle with energy $E = c|\mathbf{p}|$ then the contravariant components of the 4-momentum are $P^\mu = (|\mathbf{p}|, \mathbf{p})$. In 2 spatial dimensions - for a particle moving in the $x - y$ plane - this can be written in some frame S as

$$P^\mu = p\,(1, \cos\theta, \sin\theta) = \frac{E}{c}\,(1, \cos\theta, \sin\theta)\ , \tag{5.67}$$

where θ is the angle relative to the x-axis and $p = |\mathbf{p}|$. The 4-momentum is a 4-vector and therefore we can apply a Lorentz transformation to it moving it to a moving frame, for example, one moving at a velocity v in the x-direction

$$P'^\mu = \begin{pmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \Lambda^\mu{}_\nu P^\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}\ . \tag{5.68}$$

Using this we can deduce that

$$E' = \gamma \left(E - vp_x\right) \, , \quad p'_x = \gamma\left(p_x - \frac{vE}{c^2}\right) \, , \quad p'_y = p_y \, , \quad p'_z = p_z \, . \tag{5.69}$$

If we write $(p'_x, p'_y) = p'(\cos\theta', \sin\theta')$ then we can deduce that

$$\begin{aligned} E' &= \gamma\left(E - vp\cos\theta\right) \, , \\ p'\cos\theta' = p'_x &= \gamma\left(p\cos\theta - \frac{vE}{c^2}\right) \, , \\ p'\sin\theta' = p'_y &= p\sin\theta \, . \end{aligned} \tag{5.70}$$

The first of these equations implies that

$$p' = \gamma p\left(1 - \frac{v}{c}\cos\theta\right) \, , \tag{5.71}$$

which we we will see below leads to a Doppler shift in the frequency, whereas the second two equations lead to the aberration of light.

Before we discuss the physical consequences let us clarify the physical picture that we are trying to understand and this is summarised using a picture in the lecture slides. There are two frames S and S' which are moving relative to each other with a velocity v in the x-direction. We will consider a source at rest at some position $(x_{\rm s}, y_{\rm s})$ in S with the observer located at some point $(x_{\rm o}, y_{\rm s})$ (that is, the coordinate system has been orientated so that the source and observer have the same y-coordinate) moving at a velocity v in the positive x-direction. We will assume that $x_{\rm o} > x_{\rm s}$, $v > 0$ then corresponds to the two moving apart while if $v < 0$ they are moving toward each other. The source is assumed to be emitting radiation isotropically with $\cos\theta = 1$ corresponding to a photon emitted to the right in the diagram, and $\cos\theta = -1$ to a photon moving to the left. In the frame S', the observer is at rest and the source appears to be moving away from it - in the -ve x' direction.

If we assume that the photons are emitted with a momentum $p = hf/c$ then

$$\frac{f'}{f} = \gamma\left(1 - \frac{v}{c}\cos\theta\right) \, . \tag{5.72}$$

If $v > 0$ and $\cos\theta = 1$ then the source is moving away and this is called the "longitudinal relativistic Doppler effect" and

$$\frac{f'}{f} = \gamma\left(1 - \frac{v}{c}\right) = \left(\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}\right)^{\frac{1}{2}} \approx 1 - \frac{v}{c} + \ldots \, . \tag{5.73}$$

This leads to a "redshifting" of the photon with f decreased and λ increased. If $v < 0$ the source is moving toward and the photon is "blueshifted" with f increased and λ decreased.

The apparent angle θ' measured in S' can be deduced to satisfy

$$\begin{aligned}\cos\theta' &= \frac{\cos\theta - \frac{v}{c}}{1-\frac{v}{c}\cos\theta}\,,\\ \sin\theta' &= \frac{\sin\theta}{\gamma\left(1-\frac{v}{c}\cos\theta\right)}\,,\end{aligned} \tag{5.74}$$

and it is an exercise to check that $\cos^2\theta' + \sin^2\theta' = 1$ irrespective of the value of v. In the limit $v \to -c$ - which is an observer moving toward the source at a speed close to that of light - one finds that $\cos\theta' \to 1$ and $\sin\theta' \to 0$. We see that this leads to a "relativistic headlight effect".

Before we do that let us summarise that

$$\frac{f'}{f} = \gamma\left(1 - \frac{v}{c}\cos\theta\right)\,, \quad \cos\theta' = \frac{\cos\theta - \frac{v}{c}}{1-\frac{v}{c}\cos\theta}\,. \tag{5.75}$$

If we assume that $\theta' = \pi/2$, that is the photon is coming from the transverse direction as measured in the observer's rest frame, S', then we have that $\cos\theta = v/c$ which means that $f' = f/\gamma$ which is the equivalent of time dilation since $\Delta t' = \gamma\Delta t$. This is known as the transverse Doppler effect.

Now return to the observer moving toward the source at a speed close to light, that is, $v = -c + \Delta v$. After some algebraic manipulation one can deduce that

$$\cos\theta' = 1 - \frac{\Delta v}{c}\tan^2\left(\frac{1}{2}\theta\right) + \mathcal{O}\left[\left(\frac{\Delta v}{c}\right)^2\right]\,, \tag{5.76}$$

which implies that the light rays emitted from an isotropic source experience *aberration* into the direction of the observer. This is illustrated in the cartoon in the slides of a spaceship moving toward the observer. When $v \approx 0$ the source appears like an isotropic source - we are assuming that somehow light is being reflected isotropically off the spaceship toward the observer. As $|v|$ increases toward $-c$ the number of rays pointing leftward in the diagram is reduced and the number pointing rightward are increased, and, for example, when $v = -0.99c$ nearly all the rays pointing rightward.

A further attempt to visualise this phenomena is shown on the next slide. This is a picture of a field of stars and galaxies for $v = 0$ where as one goes to $v = -0.5c$ each of the sources become brighter as flux is focused via the "relativistic headlight effect". When $v = -0.99c$ all the light sources in the field are focussed into a smaller area in the centre. The idea that we are trying to understand is that which we see depicted in films such as Star Wars when the "Millennium Falcon makes the jump to light speed". Of course, one cannot "jump to light speed", but if one were able to accelerate to toward the speed of light then one would see something similar to what is shown in the film - if you have seen it - all the stars rush toward you and get much brighter. Don't take this too seriously, but have it in mind!

In fact there are real world applications of this phenomena. This first that we will discuss is in the Cosmic Microwave Background (CMB) which presumably has been discussed in the Introductory Astronomy and Cosmology course last semester. It is an isotropic background of black-body radiation with an average temperature $T \approx 2.725\,\mathrm{K}$ leftover from the Big Bang. It is interesting in that it is an isotropic source and in fact we are expected to be moving relative to it due to the inhomogeneous distribution of matter in our local neighbourhood. The CMB comes from all directions in the sky and is usually represented within the Aitoff projection and on the slides there is a picture of what one might expect for $v = 0$ - the colours represent the temperature differences from the average with blue slightly colder $\sim 50\,\mu\mathrm{K}$ and the red slight warmer. Again there are hypothetical cases where the observer is moving with $v = -0.5c$ and $v = -0.99c$. Hopefully you can see that, as the $|v|$ moves toward $-c$ there is more structure in the direction of motion and less in the opposite direction. In reality these are extreme examples, but CMB has been measured with exquisite precision and, remarkably, a motion of $\sim 300\,\mathrm{km\,sec}^{-1}$ has been detected.

Let us try to assess the focussing of flux caused by an isotropic source emitting $\dot{N}$ photons per unit time in some frame S and therefore the number per unit time and solid angle is

$$\frac{dN}{d\Omega dt} = \frac{\dot{N}}{4\pi}\,. \tag{5.77}$$

Now imagine another frame S' to move along the x-direction as before but with the y-direction rotated into the page by an azimuthal angle which is unaffected by the Lorentz transformation, that is, $\phi' = \phi$. We would like to calculate the flux of photons in S'

$$\frac{dN}{d\Omega' dt} = \frac{dN}{d(\cos\theta')d\phi'} = \frac{d(\cos\theta)}{d(\cos\theta')}\frac{dN}{d\Omega dt}\,, \tag{5.78}$$

hence

$$\frac{dN}{d\Omega' dt} = \frac{\dot{N}}{4\pi}\frac{\left[1-\left(\frac{v}{c}\right)^2\right]}{\left(1+\frac{v}{c}\cos\theta'\right)^2}\,, \tag{5.79}$$

and it is one of the exercises to show that this integrates to the same value in each of the frames. For $\theta' \approx 0$ we have that

$$\frac{dN}{d\Omega' dt} = \frac{\dot{N}}{4\pi}\left(\frac{1-\frac{v}{c}}{1+\frac{v}{c}}\right)\,, \tag{5.80}$$

which is $\sim \dot{N}c/(2\pi\Delta v)$ when $v = -c + \Delta v$. So we see that the flux $\to \infty$ as $\Delta v \to 0$.

It is believed that there is a supermassive black hole at the centre of all galaxies. They can be so massive (up to $10^9\,M_\odot$) that there is an accretion disk of fast moving particles that form jet emitted along by a magnetic field as illustrated in the slides. The phenomena can be observed using high resolution radio observations where we see the jet moving towards us and increasing the expected flux since the speeds at which the particles are sent out along the jet are moving close to the speed of light when they emit photons.

END OF 21ST LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 21 Exercises

1. Using the expressions for $\cos\theta'$ and $\sin\theta'$ presented in lectures show that these satisfy $\cos^2\theta' + \sin^2\theta' = 1$.

2. Invert the expression for $\cos\theta'$ and $\sin\theta'$ in terms of v/c, $\cos\theta$ and $\sin\theta$ to be expressions for $\cos\theta$ and $\sin\theta$ in terms of v/c, $\cos\theta'$ and $\sin\theta'$. What is the physical interpreation of the symmetry you find between the expressions in the two frames?

3. Show that the number of particles emitted in a unit time is conversed by integrating the expression for $d\dot{N}/d\Omega'$ over the solid angle $d\Omega'$.

START OF 22ND LECTURE

5.10 Relativistic collisions - See FS7.2 and previous course on QPR

In the final lecture of the course we will go back over some of the ideas on relativistic collisions you studied in the course on Quantum Physics and Relativity. In some sense there is nothing conceptually new in the material covered, but we will make use of some of the new techniques that we have learnt as part of this course. There is nothing to stop you from doing what you would have done last semester, but it is often more compact to use the 4-vector notation.

Let us start by considering a general scattering process as depicted in the lecture slides involving 2 particles, 1 and 2 (the initial state), eventually becoming A and B (the final state) after some interactions - which are unimportant here. This is called 2-2 scattering process and can be represented by $1 + 2 \to A + B$. We will focus on what are called the "kinematic" constraints, that is, those imposed by the conservation of 4-momentum. This is represented by

$$\tilde{\mathbf{P}}_1 + \tilde{\mathbf{P}}_2 = \tilde{\mathbf{P}}_A + \tilde{\mathbf{P}}_B \,, \tag{5.81}$$

which implies that the energy is conserved $E_1 + E_2 = E_A + E_B$, as is the 3-momentum $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_A + \mathbf{p}_B$.

The *on-shell conditions* imply that $\tilde{\mathbf{P}}_i^2 = (m_i c)^2$ for $i = 1, 2, A, B$ and in particular

$\tilde{\mathbf{P}}_i^2 = 0$ for massless particles. By taking the inner product of each side of (5.114) we find that

$$\tilde{\mathbf{P}}_A \cdot \tilde{\mathbf{P}}_B = \tilde{\mathbf{P}}_1 \cdot \tilde{\mathbf{P}}_2 + \frac{1}{2}\Delta c^2\,, \tag{5.82}$$

where $\Delta = m_1^2 + m_2^2 - m_A^2 - m_B^2$ is the change in the mass squared.

One might ask what information is required to solve such a problem? Let us assume that the initial state is known and is specified by E_1, E_2, $\mathbf{p}_1$, $\mathbf{p}_2$ - the masses m_1 and m_2 could be inferred from the on shell conditions. The unknown final state is specified by E_A, E_B, $\mathbf{p}_A$, $\mathbf{p}_B$, m_A and m_B corresponding to $2n+4$ unknown in n spatial dimensions. The conservation of 4-momentum imposes $n+1$ conditions and the on shell conditions a further 2 (or one could knock the masses off the list of unknowns) and so there are $n+3$ constraints on the $2n+4$ unknowns. This implies that the final state cannot be uniquely calculated with extra information even when $n = 1$. If we know m_A and m_B in $n = 1$ then everything can be calculated, but even more information is needed when $n = 2$ and $n = 3$.

Another interesting point is if one cannot distinguish collinear photons within this kinematic framework. For example, the 4-momenta of two photons in x-direction are

$$P_1^\mu = \frac{E_1}{c}(1,1,0,0)\,, \quad P_2^\mu = \frac{E_2}{c}(1,1,0,0)\,, \tag{5.83}$$

which implies that the total 4-momentum is

$$P_{\text{tot}}^\mu = P_1^\mu + P_2^\mu = \left(\frac{E_1 + E_2}{c}\right)(1,1,0,0)\,. \tag{5.84}$$

This cannot be distinguished from that of a single photon with one with energy $E_1 + E_2$ within this kinematic framework. Note that this only applies to collinear photons, but it can be clearly generalised to N collinear photons, and $\tilde{\mathbf{P}} \cdot \tilde{\mathbf{P}} \equiv 0$ for such a system.

Now let us consider a situation where the total 4-momentum for is

$$P_{\text{tot}}^\mu = \left(\frac{E}{c}, p, 0, 0\right)\,, \tag{5.85}$$

which can always be achieved by a choice of coordinate system - we can always choose the coordinates so that the 3-momentum is in the x-direction. Now make a Lorentz transformation along the x-direction so that

$$E' = \gamma(E - vp)\,, \quad p' = \gamma\left(p - \frac{vE}{c^2}\right) = 0\,, \tag{5.86}$$

which requires $v/c = cp/E < 1$ and this defines the Centre of Momentum frame, that is, one where $P_{\text{tot}}^{\prime\mu} = (E'/c, 0, 0, 0)$. In this frame we can define

$$\tilde{\mathbf{P}}_{\text{tot}} \cdot \tilde{\mathbf{P}}_{\text{tot}} = \eta_{\mu\nu} P_{\text{tot}}^\mu P_{\text{tot}}^\nu = (m_{\text{inv}} c)^2 > 0\,, \tag{5.87}$$

where $m_{\rm inv}$ is the invariant mass of the system. If this is non-zero - that is if the system is not a single photon, or collinear photons - then $v/c < 1$ and we can define this frame using the Lorentz transformation.

The invariant mass can be helpful in performing calculations. It is defined by

$$(m_{\rm inv}c)^2 = \left(\frac{1}{c}\sum_i E_i\right)^2 - \left(\sum_i \mathbf{p}_i\right)^2 , \tag{5.88}$$

where the summations are over all the particle. It is conserved in an elastic collision and is also the same in all frames. In the lecture slides there is a picture of an elastic scattering process - similar to the 2-2 scattering described earlier - where two particles collide in the lab frame and go out after the collision as two other particles. In the Centre of Momentum frame both the incoming momentum and the outgoing momenta cancel. Since the invariant mass is the same in the two frames it is often helpful to compare it in the Lab Frame before the collision and in the Centre of Momentum frame afterwards.

In the Centre of Momentum frame we have that

$$m_{\rm inv}c = \frac{1}{c}\sum_i E_i = \sum_i \sqrt{|\mathbf{p}_i|^2 + (m_i c)^2} \geq c\sum_i m_i\,, \tag{5.89}$$

where the inequality is saturated when $\mathbf{p}_i = 0\;\forall i$. This state is often called "at threshold" and it is useful to check whether the particle reaction process is kinematically allowed, that is, there enough energy and momentum to produce the required particles at rest†.

As a somewhat general example of the kind of things we have been talking about let us consider the decay of a massive particle into to particles A and B. This is equivalent to the $2-2$ scattering but without the second particle in the initial state - it would be termed a $1-2$ process. We can always choose to work in a frame where the particle is at rest in the initial state. The 4-momentum of this state is $P^\mu = (m_1 c, 0, 0, 0)$ with $m_1 > 0$. The conservation of 4-momentum implies

$$\tilde{\mathbf{P}} = \tilde{\mathbf{P}}_A + \tilde{\mathbf{P}}_B\,, \tag{5.90}$$

from which we can deduce that

$$2\tilde{\mathbf{P}}_A\cdot\tilde{\mathbf{P}}_B = \tilde{\mathbf{P}}^2 - \tilde{\mathbf{P}}_A^2 - \tilde{\mathbf{P}}_B^2 = \Delta c^2\,, \tag{5.91}$$

where $\Delta = m_1^2 - m_A^2 - m_B^2$ using the "on-shell conditions". Substituting $\tilde{\mathbf{P}}_A = \tilde{\mathbf{P}} - \tilde{\mathbf{P}}_\mathbf{B}$ and $\tilde{\mathbf{P}}_B = \tilde{\mathbf{P}} - \tilde{\mathbf{P}}_\mathbf{A}$ yields

$$(\tilde{\mathbf{P}} - \tilde{\mathbf{P}}_A)\cdot\tilde{\mathbf{P}}_A = \frac{1}{2}\Delta c^2\,, \quad (\tilde{\mathbf{P}} - \tilde{\mathbf{P}}_B)\cdot\tilde{\mathbf{P}}_B = \frac{1}{2}\Delta c^2\,, \tag{5.92}$$

† There are number of other criteria defining whether particles are produced including a number of conservation laws and indeed whether a particular reaction is dynamically allowed by the specific particle physics theory under consideration. However, if it is not kinematically allowed then it can't happen no matter what!

and $\tilde{\mathbf{P}} \cdot \tilde{\mathbf{P}}_A = m_1 E_A$ and $\tilde{\mathbf{P}} \cdot \tilde{\mathbf{P}}_B = m_1 E_B$ where E_A and E_B are the energies of the outgoing particles. This leads to the final result

$$\frac{E_A}{c^2} = \frac{\frac{1}{2}\Delta + m_A^2}{m_1}\,, \quad \frac{E_B}{c^2} = \frac{\frac{1}{2}\Delta + m_B^2}{m_1}\,, \tag{5.93}$$

which has been deduced entirely using 4-vectors and with minimal algebra. We note that the final state has zero 3-momentum and $\mathbf{p}_A = -\mathbf{p}_B$ and that

$$m_1 c = \frac{E_A + E_B}{c} = \sqrt{|\mathbf{p}_A|^2 + (m_A c)^2} + \sqrt{|\mathbf{p}_B|^2 + (m_B c)^2} \geq m_A c + m_B c\,, \tag{5.94}$$

which is the threshold argument applied to this specific case.

Now let us consider two more specific real world examples one based on particle physics at CERN and the other which is relevant to cosmology. There are some other examples in the exercises at the end of the lectures and also some others in the example sheets. Some of these might be familiar from last semester, but you are now armed with a more sophisticated understanding of the problems and the hope is that you will try to use the 4-vector approaches to these problems which are often more elegant, requiring less algebra, but they do require more insight. If you still feel the need to work in terms of the energy and 3-momentum, you can so long as you get the right answer, but possibly prepare yourself for heavy algebra.

- *Higgs Boson production at the Large Electron-Positron (LEP) collider at CERN*: The LEP was a predecessor of the LHS which collided electrons, e^-, and positrons, e^+, the ant-particle of the electron which is identical except that the charge is positive rather than negative. An interesting process is the collision of e^+e^- to give a Higgs particle and a Z-boson, $e^+e^- \to hZ$. If we assume that the collision of the e^+e^- at energy of $E = 103\,\mathrm{GeV}$ then we have that

$$P_1^\mu = \left(\frac{E}{c}, p, 0, 0\right)\,, \quad P_2^\mu = \left(\frac{E}{c}, -p, 0, 0\right)\,, \tag{5.95}$$

where $E^2 = (pc)^2 + (m_\mathrm{e} c^2)^2$ and the lab frame coincides with the centre of momentum frame in this particular case. If we label the Higgs particle A and the Z particle B, with $M_\mathrm{Z} = 91\,\mathrm{GeV}/c^2$ then

$$P_A^\mu = \left(\frac{E_h}{c}, \mathbf{p}'\right)\,, \quad P_B^\mu = \left(\frac{E_Z}{c}, -\mathbf{p}'\right)\,. \tag{5.96}$$

Comparing the invariant mass before and after the collision then we have that

$$2E = E_h + E_Z = \sqrt{(cp')^2 + (m_h c^2)^2} + \sqrt{(cp')^2 + (m_Z c^2)^2} \geq m_h c^2 + m_Z c^2\,, \tag{5.97}$$

where the quality is when $\mathbf{p}' = 0$, that is the particles, are produced at rest. Hence, we see that in such collisions $m_h c^2 < 2E - m_Z c^2 = (206 - 91)\,\mathrm{GeV} = 115\,\mathrm{GeV}$, and the highest mass Higgs boson that could have been produced has $m_\mathrm{h} = 115\,\mathrm{GeV}/c^2$.

Note that the LHC has since found that the Higgs particle has a mass $\approx 125\,\mathrm{GeV}/c^2$. This could be have been possible if the energy of LEP was $E = 108\,\mathrm{GeV}$!

- *Gresisen-Zatsepin-Kuzmin (GZK) cut-off for cosmic rays*: Cosmic rays are high energy particles - typically protons and neutrons - emitted by various astrophysical processes. We discussed the CMB in the previous section. Just to remind you it is a Black-Body radiation background emitted in the Big Bang with $T = 2.725\,\mathrm{K}$ corresponding to a photon number density of $n_\gamma \approx 400\,\mathrm{cm}^{-3}$. The typical process that we are interested in is that of a CMB photon scattering off a cosmic ray proton creating a neutral pion, that is,

$$\gamma_{\mathrm{CMB}} + p \to p + \pi^0 \,. \tag{5.98}$$

This is depicted in the lecture slides with the CMB photon and the proton moving toward each other in the initial state. First, let us calculate the photon energy required for the particles to be produced in the final state to be produced at "threshold" when the proton in the final state is stationary, $p = 0$. This can be done by calculating the invariant mass in both states

$$(m_{\mathrm{inv}}c)^2 = \left(m_p c + \frac{E_\gamma}{c}\right)^2 - \left(\frac{E_\gamma}{c}\right)^2 = [(m_\pi + m_{\mathrm{p}})c]^2 \,, \tag{5.99}$$

which implies that

$$E_\gamma = m_\pi c^2 \left(1 + \frac{m_\pi}{2m_{\mathrm{p}}}\right) \approx 145\,\mathrm{MeV}\,, \tag{5.100}$$

where $m_\pi c^2 \approx 135\,\mathrm{MeV}$. However, CMB photons have an energy of $E_\gamma \approx k_{\mathrm{B}} T_{\mathrm{CMB}} \approx 6 \times 10^{-4}\,\mathrm{eV}$ and therefore they are too low energy to interact with a proton at rest. Now repeat the calculation for non-zero p for the proton in the initial state. In this case the invariant mass of the initial state is

$$(m_{\mathrm{inv}}c^2)^2 = 2E_\gamma(E_{\mathrm{p}} + pc) + (m_{\mathrm{p}}c^2)^2 \,, \tag{5.101}$$

which we equate with $(m_{\mathrm{p}} + m_\pi)^2 c^4$ in order to see if the reaction can now take place from which we deduce that

$$E_{\mathrm{p}} + pc = \sqrt{(m_{\mathrm{p}}c^2)^2 + (pc)^2} + pc = \hat{E} = \frac{m_\pi c^2(2m_{\mathrm{p}}c^2 + m_\pi c^2)}{2E_\gamma} \approx 2\times 10^{20}\,\mathrm{eV}\,. \tag{5.102}$$

So we can deduce that

$$pc = \frac{\hat{E}^2 - (m_{\mathrm{p}}c^2)^2}{2\hat{E}} \approx \frac{1}{2}\hat{E} \approx 10^{20}\,\mathrm{eV}\,. \tag{5.103}$$

What this means is that cosmic ray protons with an $E_{\mathrm{p}} \approx pc > 10^{20}\mathrm{eV}$ - which are remarkably abundant from astrophysical sources - will be significantly slowed down by the production of particles such as pions due to the interaction with CMB photons known as the GZK cut-off, once such interactions are kinematically allowed. A more detailed calculation shows that cosmic rays with energies greater than this are heavily suppressed and this suppression is quantitively confirmed by observations.

END OF 22ND LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 22 Exercises

1. In the process of Compton scattering a photon is incident on an electron at rest. The scattered photon moves at an angle ϕ to the incoming direction. Show that

$$\tilde{\mathbf{P}}_\gamma \cdot \tilde{\mathbf{P}}_{\mathrm{e}} = \tilde{\mathbf{P}}_{\gamma'} \cdot (\tilde{\mathbf{P}}_\gamma + \tilde{\mathbf{P}}_e)\,, \tag{5.104}$$

where $\tilde{\mathbf{P}}_\gamma$ and $\tilde{\mathbf{P}}_{\gamma'}$ are the 4-momenta of the incoming and outgoing photons and $\tilde{\mathbf{P}}_{\mathrm{e}}$ is that of the static electron. Use this to derive the shift in the photon wavelength

$$\Delta\lambda = \frac{h}{m_{\mathrm{e}}c}(1 - \cos\phi)\,, \tag{5.105}$$

during the scattering process.

2. Two photons are in head-collision. What is the condition on the energies of the incoming photons, E_1 and E_2 for the collision to produce a e^+e^- pair?

Section 5 in a Nutshell - what to remember

- Definition of a group. $(G, \bullet)$ which is a set of elements $G = \{a, b, c, \dots\}$ and a composition (or operation) $\bullet$ which have the following properties.
  - *Closure:* $\forall\, a, b \in G, c = a \bullet b \in G$.
  - *Associativity:* $\forall\, a, b, c \in G, a \bullet (b \bullet c) = (a \bullet b) \bullet c$.
  - *Identity:* $\exists\, e \in G$ such that $e \bullet a = a \bullet e \,\forall a \in G$.
  - *Inverse:* $\forall\, a \in G \,\exists\, a^{-1} \in G$ such that $a^{-1} \bullet a = a \bullet a^{-1} = e$.
- $O(n) = \{M \in \mathbb{GL}(n, \mathbb{R}) : MM^T = M^TM = I\}$ - the group of orthogonal $n \times n$ matrices;
- $SO(n) = \{M \in O(n) : \det M = 1\}$ - the special orthogonal group which are the rotation matrices.

- Minkowski metric and inverse

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{5.106}$$

- Inner product of two 4 vectors

$$\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}} = \eta_{\mu\nu} A^\mu B^\nu = A^0 B^0 - A^i B^i = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}\,, \tag{5.107}$$

- Common 4-vectors

$$U^\mu = \frac{dX^\mu}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau} \right) = \gamma(c, \mathbf{v})\,. \tag{5.108}$$

  and $\eta_{\mu\nu} U^\mu U^\nu = c^2$.

$$P^\mu = m\frac{dX^\mu}{d\tau} = mU^\mu \tag{5.109}$$

  and $P_\mu P^\mu = (mc)^2$.

$$k^\mu = \left(\frac{\omega}{c}, \mathbf{k} \right)\,, \tag{5.110}$$

  and $k_\mu k^\mu = 0$.

- Contravariant and covariant indices $A_\mu = \eta_{\mu\nu} A^\nu$ and $A^\mu = \eta^{\mu\nu} A_\nu$.
- Lorentz invariants:

$$\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}} = \eta_{\mu\nu} A^\mu B^\nu = A_\mu B^\mu = A^\mu B_\mu\,. \tag{5.111}$$

- Lorentz transformation:

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu\,, \quad A'_\alpha = [(\Lambda^T)^{-1}]_\alpha{}^\beta A_\beta = (\Lambda^{-1})^\beta{}_\alpha A_\beta\,. \tag{5.112}$$

- Doppler effect and aberration:

$$\begin{aligned} p' &= \gamma p \left(1 - \frac{v}{c}\cos\theta \right)\,, \\ \cos\theta' &= \frac{\cos\theta - \frac{v}{c}}{1 - \frac{v}{c}\cos\theta}\,, \\ \sin\theta' &= \frac{\sin\theta}{\gamma\left(1 - \frac{v}{c}\cos\theta\right)}\,, \end{aligned} \tag{5.113}$$

- Conservation of 4-momentum in 2-2 scatter

$$\tilde{\mathbf{P}}_1 + \tilde{\mathbf{P}}_2 = \tilde{\mathbf{P}}_A + \tilde{\mathbf{P}}_B \,, \tag{5.114}$$

  which implies that the energy is conserved $E_1 + E_2 = E_A + E_B$, as is the 3-momentum $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_A + \mathbf{p}_B$.
- The *on-shell conditions* imply that $\tilde{\mathbf{P}}_i^2 = (m_i c)^2$ for $i = 1, 2, A, B$ and in particular $\tilde{\mathbf{P}}_i^2 = 0$ for massless particles.
- Invariant mass

$$(m_{\rm inv} c)^2 = \left(\frac{1}{c}\sum_i E_i\right)^2 - \left(\sum_i \mathbf{p}_i\right)^2 \,, \tag{5.115}$$

  and at threshold

$$m_{\rm inv} c = \frac{1}{c}\sum_i E_i = \sum_i \sqrt{|\mathbf{p}_i|^2 + (m_i c)^2} \geq c \sum_i m_i \,. \tag{5.116}$$