1. (a) $\underline{F} = \frac{F_0}{R^2}\begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} = -\underline{\nabla} U$ $\qquad \underline{\nabla}\times\underline{F} = \frac{F_0}{R^2}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \frac{F_0}{R^2}\begin{pmatrix} x-x \\ y-y \\ z-z \end{pmatrix} = \underline{0}$

$\Downarrow$

Conservative!

$$\frac{\partial U}{\partial x} = \frac{F_0}{R^2} yz \ ; \ \frac{\partial U}{\partial y} = \frac{F_0}{R^2} zx \ ; \ \frac{\partial U}{\partial z} = \frac{F_0}{R^2} xy$$

$\Downarrow$

$$U = \frac{F_0}{R^2} xyz + f(y,z) \Rightarrow \frac{\partial U}{\partial y} = \frac{F_0}{R^2} zx + \frac{\partial f}{\partial y} \ \& \ \frac{\partial U}{\partial z} = \frac{F_0}{R^2} xy + \frac{\partial f}{\partial z} \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \Rightarrow f = \text{const}$$

$$\Rightarrow U(x,y,z) = \frac{F_0}{R^2} xyz + \text{const}.$$

(b) $\underline{F} = \frac{F_0}{R}\begin{pmatrix} y+z \\ z+x \\ x+y \end{pmatrix} = -\underline{\nabla} U$ $\qquad \underline{\nabla}\times\underline{F} = \frac{F_0}{R}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = \frac{F_0}{R}\begin{pmatrix} 1-1 \\ 1-1 \\ 1-1 \end{pmatrix} = \underline{0}$

$\Downarrow$

conservative

$$\frac{\partial U}{\partial x} = -\frac{F_0}{R}(y+z) \ ; \ \frac{\partial U}{\partial y} = -\frac{F_0}{R}(z+x) \ ; \ \frac{\partial U}{\partial z} = -\frac{F_0}{R}(x+y)$$

$\Downarrow$

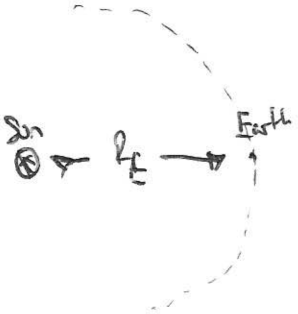
$$U = -\frac{F_0}{R}(y+z)x + f(y,z) \Rightarrow \frac{\partial U}{\partial y} = -\frac{F_0}{R} x + \frac{\partial f}{\partial y} \ \& \ \frac{\partial U}{\partial z} = \frac{F_0}{R} x + \frac{\partial f}{\partial z}$$

$$\Rightarrow \frac{\partial f}{\partial y} = -\frac{F_0}{R} z \ \& \ \frac{\partial f}{\partial z} = -\frac{F_0}{R} x \Rightarrow f = -\frac{F_0}{R} zy + g(z) \ \& \ \frac{\partial g}{\partial z} = 0$$

$$\Rightarrow U = -\frac{F_0}{R}(xy + yz + zx) + \text{const}.$$

NB Minus signs added

2.



By Newton shell theorem $g = -\frac{G_N M_\odot}{R_E^2} - \frac{G_N M(<R_E)}{R_E^2}$

where $M(<R_E) = \frac{4}{3}\pi R_E^3 \rho$

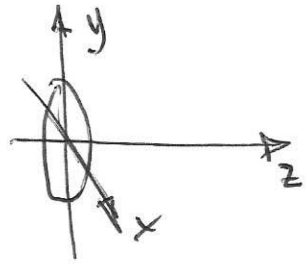
$$\Rightarrow g = -\frac{G_N M_\odot}{R_E^2} - \frac{4\pi G_N R_E \rho}{3}$$

$$R_E \omega^2 = G_N\left(\frac{M_\odot}{R_E^2} + \frac{4\pi\rho R_E}{3}\right) \Rightarrow \omega^2 = G_N\left(\frac{M_\odot}{R_E^3} + \frac{4\pi\rho}{3}\right)$$

$$\Rightarrow \left|\frac{2\pi}{T}\right|^2 = \left|\frac{2\pi}{T_0}\right|^2 + \frac{4\pi G_N \rho}{3} \quad \text{where } T_0 = 1\text{yr} \Rightarrow \frac{1}{T^2} - \frac{1}{T_0^2} = \frac{G_N \rho}{3\pi}$$

Define $T = T_0(1+\epsilon)$ then $-2\epsilon \simeq \frac{G_N \rho T_0^2}{3\pi} \Rightarrow \rho = -\frac{6\pi\epsilon}{G_N T_0^2}$

$$|\epsilon| = \frac{1}{365} \Rightarrow \rho = \frac{6\pi \times 1/365}{6.67 \times 10^{-11}\, kg^{-1}\, m^3\, s^{-2} \times (365 \times 24 \times 3600\, s)^2} \simeq 7.8 \times 10^{-7}\, kg\, m^{-3}$$
$$= 780\, g\, cm^{-3}$$

3. 

$x' = R\cos\theta,\ y' = R\sin\theta,\ z' = 0$

$d\Phi = -\frac{G_N dM}{|\underline{r} - \underline{r}'|}$ $\quad dM = \mu\, dl = \mu R\, d\theta$

where $\mu = M/2\pi R$

$$\Rightarrow d\Phi = -\frac{G_N M}{2\pi} \frac{d\theta}{[(x - R\cos\theta)^2 + (y - R\sin\theta)^2 + z^2]^{1/2}}$$

$$\Rightarrow \Phi(x,y,z) = -\frac{G_N M}{2\pi} \int_0^{2\pi} \frac{d\theta}{[(x - R\cos\theta)^2 + (y - R\sin\theta)^2 + z^2]^{1/2}}$$

$$z = 0 \Rightarrow \Phi(x,y,0) = -\frac{G_N M}{2\pi} \int_0^{2\pi} \frac{d\theta}{(r^2 + R^2 - 2Rx\cos\theta - 2Ry\sin\theta)^{1/2}}$$

Define $x = r\cos\alpha,\ y = r\sin\alpha$

$$\Rightarrow \Phi(x,y,0) = -\frac{G_N M}{2\pi} \int_0^{2\pi} \frac{d\theta}{[r^2 - 2Rr\cos(\theta - \alpha) + R^2]^{1/2}} = -\frac{G_N M}{2\pi r} \int_0^{2\pi} \left(1 - \frac{2R}{r}\cos(\theta-\alpha) + \frac{R^2}{r^2}\right)^{-1/2}$$

Now $(1+x)^{-1/2} \simeq 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \dots$

$$\Rightarrow \Phi(x,y,0) = -\frac{G_N M}{2\pi r} \int_0^{2\pi} \left(1 - \frac{R}{r}\cos(\theta-\alpha) - \frac{R^2}{2r^2} + \frac{3}{8}\frac{4R^2}{r^2}\cos^2(\theta-\alpha) + \dots\right) d\theta$$

Now $\int_0^{2\pi} \cos(\theta-\alpha)\, d\theta = \left[\sin(\theta-\alpha)\right]_0^{2\pi} = \underline{\underline{0}}$ & $\int_0^{2\pi} \cos^2(\theta-\alpha)\, d\theta = \frac{1}{2}\int_0^{2\pi} (1 + \cos[2(\theta-\alpha)])\, d\theta$

$= \underline{\underline{\pi}}$

$$\Rightarrow \Phi(x,y,0) = -\frac{G_N M}{r}\left(1 - \frac{R^2}{2r^2} + \frac{3R^2}{4r^2} + \dots\right) = \underline{\underline{-\frac{G_N M}{r}\left(1 + \frac{R^2}{4r^2}\right) + \dots}}$$

4. $$\left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{L^2}(E-U) - u^2$$

$$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} = -\frac{2\mu}{L^2}\cdot\frac{du}{d\theta}\frac{dU}{du} - 2u\frac{du}{d\theta} \Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{\mu}{L^2}\frac{dU}{du}$$

$$U = -\frac{\alpha}{r} = -\alpha u \Rightarrow \frac{dU}{du} = -\alpha \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{\mu\alpha}{L^2}$$

$$\Rightarrow u = A\cos\theta + B\sin\theta + \frac{\mu\alpha}{L^2}$$

$$\Rightarrow u(\pi/2) = B + \frac{\mu\alpha}{L^2} \quad \& \quad \frac{du}{d\theta}(\pi/2) = -A.$$

$$\left[\frac{du}{d\theta}(\pi/2)\right]^2 = \frac{2\mu}{L}\left[E + \alpha u(\pi/2)\right] - \left[u(\pi/2)\right]^2 = \frac{2\mu}{L^2}\left(E + \frac{\alpha^2\mu}{L^2}\right) - \frac{\alpha^2\mu^2}{L^4} = \left(\frac{\alpha\mu}{L^2}\right)^2\left(1 + \frac{2EL^2}{\alpha^2\mu}\right)$$

$$= \left(\frac{\alpha\mu\varepsilon}{L^2}\right)^2 \text{ where } \varepsilon^2 = 1 + \frac{2EL^2}{\alpha^2\mu}$$

$$\Rightarrow \frac{du}{d\theta}(\pi/2) = -\frac{\alpha\mu\varepsilon}{L^2}$$

Hence $B=0$ & $A = \frac{\alpha\mu}{L^2}\varepsilon \Rightarrow u = \frac{\mu\alpha}{L^2}(1+\varepsilon\cos\theta) = u_0(1+\varepsilon\cos\theta) \Rightarrow r = \frac{r_0}{1+\varepsilon\cos\theta}$

5. $$U_{eff} = \frac{L^2}{2\mu r^2} - \frac{\alpha}{r^n} \Rightarrow \frac{dU_{eff}}{dr} = -\frac{L^2}{\mu r^3} + \frac{n\alpha}{r^{n+1}} \quad \& \quad \frac{d^2U_{eff}}{dr^2} = \frac{3L^2}{\mu r^4} - \frac{n(n+1)\alpha}{r^{n+2}}$$

$$\frac{dU_{eff}}{dr}(r_0) = 0 \Rightarrow \frac{r_0^{n+1}}{r_0^3} = \frac{n\mu\alpha}{L^2} \Rightarrow r_0^{n-2} = \frac{n\mu\alpha}{L^2} \Rightarrow r_0 = \left(\frac{n\mu\alpha}{L^2}\right)^{\frac{1}{n-2}}$$

$$\frac{d^2U_{eff}}{dr^2}(r_0) = \frac{1}{r_0^4}\left(3L^2 - \frac{n(n+1)\alpha}{r_0^{n-2}}\right) = \frac{1}{r_0^4}\left(\frac{3L^2}{\mu^2} - \frac{n(n+1)\alpha}{n\mu\alpha/L^2}\right) = \frac{L^2}{\mu r_0^4}(2-n)$$

$\Rightarrow n>2$ has $\frac{d^2U_{eff}}{dr^2}<0$ and hence unstable but for $n<2$ $\frac{d^2U_{eff}}{dr^2}>0 \Rightarrow$ stable.

6. $\alpha = G_N M \mu \Rightarrow U = -\frac{\alpha}{r}\left(1 + \frac{3\alpha}{c^2\mu r}\right) \Rightarrow \frac{dU}{du} = -\alpha\left(1 + \frac{6u\alpha}{\mu c^2}\right)$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{L^2}\left\{E + \alpha u\left[1 + \frac{3u\alpha}{c^2\mu}\right]\right\} - u^2$$

$$\& \quad \frac{d^2u}{d\theta^2} + u = \frac{\mu\alpha}{L^2}\left(1 + \frac{6u\alpha}{\mu c^2}\right) \Rightarrow \frac{d^2u}{d\theta^2} + \left(1 \mp \frac{6\alpha^2}{L^2c^2}\right)u = \frac{\mu\alpha}{L^2} = u_0$$

$$\Rightarrow u = A \sin\left[\left(1 \mp \frac{6\alpha^2}{L^2c^2}\right)^{1/2}\theta\right] + B\cos\left[\left(1 \mp \frac{6\alpha^2}{L^2c^2}\right)^{1/2}\theta\right] + \frac{u_0}{1 + \frac{6\alpha^2}{L^2c^2}}$$

A & B would be fixed by BC's as in Q 4

In one full rotation the position where u is max/min will move by $\Delta\theta$ where

$$\left(1 \mp \frac{6\alpha^2}{L^2c^2}\right)^{1/2} 2\pi = 2\pi \mp \Delta\theta \Rightarrow \Delta\theta = \frac{6\pi(G_N M)^2}{\ell^2c^2} \quad \text{where } \ell = \frac{L}{\mu}.$$

$$\simeq \left(1 + \frac{3\alpha^2}{L^2c^2} + \cdots\right)2\pi$$

7.(a) $z^2 = R^2(x^2+y^2)$ & $z = \frac{1}{B}(1 - AX) \Rightarrow (1 - AX)^2 = (BR)^2(x^2+y^2)$

$$\Rightarrow [(BR)^2 - A^2]x^2 + (BR)^2y^2 + 2AX = 1$$

(i) $(BR)^2 - A^2 = (BR)^2 \Rightarrow A = 0$

(ii) $|BR| > |A|$

(iii) $|BR| = |A|$

(iv) $|BR| < |A|$

(b) $|PF_1| + |PF_2| = 2a \Rightarrow [(x-f)^2+y^2]^{1/2} + [(x+f)^2+y^2]^{1/2} = 2a$ where $f = \sqrt{a^2-b^2}$

$$\Rightarrow (x-f)^2 + y^2 = 4a^2 - 4a[(x+f)^2+y^2]^{1/2} + (x+f)^2 + y^2$$

$$\Rightarrow 4a[(x+f)^2+y^2]^{1/2} = 4a^2 + 4xf$$

$$\Rightarrow a^2[x^2 + 2xf + f^2 + y^2] = a^4 + 2a^2xf + x^2f^2$$

$$\Rightarrow (a^2 - f^2)x^2 + a^2y^2 = a^4 - a^2f^2 \Rightarrow b^2x^2 + a^2y^2 = a^2b^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \text{ellipse!}$$

(c) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\underset{\substack{\uparrow \\ f^2}}{x^2 = a^2 - b^2}$ & $y = l > 0 \Rightarrow \frac{l^2}{b^2} = 1 - \frac{1}{a^2}(a^2 - b^2) \Rightarrow \underline{\underline{l = \frac{b^2}{a}}}$

(ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\underset{\substack{\uparrow \\ f^2}}{x^2 = a^2 + b^2}$ & $y = l > 0 \Rightarrow \frac{l^2}{b^2} = 1 - \frac{1}{a^2}(a^2 + b^2) \Rightarrow \underline{\underline{l = \frac{b^2}{a}}}$

(iii) $y^2 = 4ax$

$x = a = f$ & $y = l > 0 \Rightarrow l^2 = 4a^2 \Rightarrow \underline{\underline{l = 2a}}$

8 (a) $r_{min} = a(1-\epsilon) = 1838\,km$ & $r_{max} = a(1+\epsilon) = 1861\,km \Rightarrow a = 1848.5\,km$

$T = \frac{2\pi a^{3/2}}{\sqrt{G_N M}} \Rightarrow G_N M = \frac{4\pi^2 a^3}{T^2}$

$M = \frac{4\pi^2 \times (1.8485 \times 10^6\,m)^3}{6.67 \times 10^{-11}\,kg^{-1} m^3 s^{-2} \times (119 \times 60\,s)^2} = \underline{7.3 \times 10^{22}\,kg}$

(b) $\mu = 2000\,kg$ $\quad r_{min} = R_E + 1100\,km = 7500\,km$
$\quad r_{max} = R_E + 4400\,km = 10800\,km$ $\Bigg\} \Rightarrow a = 9000\,km$

(i) $\epsilon = \left(1 - \frac{b^2}{a^2}\right)^{1/2} = \left(1 + \frac{2L^2 E}{\alpha^2 \mu}\right)^{1/2} \Rightarrow \frac{2L^2 E}{\alpha^2 \mu} = -\frac{b^2}{a^2} \Rightarrow \frac{2r_0 E}{\alpha} = -\frac{b^2}{a^2} \Rightarrow E = -\frac{\alpha}{2a}$

$E_{orbit} = -\frac{G_N M_E \mu}{2a} \Rightarrow$ Energy required $= G_N M_E \mu \left(\frac{1}{R_E} - \frac{1}{2a}\right)$

$= 6.67 \times 10^{-11}\,kg^{-1} m^3 s^{-2} \cdot 6 \times 10^{24}\,kg \times 2000\,kg \left(\frac{1}{6400\,km} - \frac{1}{18000\,km}\right)$

$= 6.67 \times 6 \times 2 \left(\frac{1}{64} - \frac{1}{18}\right) \times 10^{-11+24+3-6}\,J$

$\underline{= 8 \times 10^{10}\,J}$

(ii) At perigee / apogee : $\frac{1}{2}\mu v^2 - \frac{G_N M_E \mu}{r_{min}/r_{max}} = -\frac{G_N M_E \mu}{2a}$

$$\Rightarrow v^2 = G_N M_E \left(\frac{2}{r_{min}/r_{max}} - \frac{1}{a} \right)$$

$$= 6.67\times10^{-11} \cancel{kg^{-1}} m^3 s^{-2} \times 6\times10^{24} \cancel{kg} \left(\frac{2}{7500\text{km or } 10800\text{km}} - \frac{1}{9000\text{km}} \right)$$

$$= 6.67\times6\times10^{13-6} m^2 s^{-2} \begin{cases} \left(\frac{2}{7.5} - \frac{1}{9}\right) \\ \left(\frac{2}{10.8} - \frac{1}{9}\right) \end{cases}$$

$$= \begin{cases} 6.2\times10^7 \ m^2 s^{-2} \\ 2.9\times10^7 \ m^2 s^{-2} \end{cases}$$

$$\Rightarrow V = \underset{\text{perigee}}{\underset{\uparrow}{7.8 \text{ km sec}^{-1}}} \ \& \ \underset{\text{apogee}}{\underset{\uparrow}{5.4 \text{ km sec}^{-1}}}$$

(iii) velocity is purely tangential at perigee or apogee

$$\Rightarrow L = \mu v r = 2000\,kg \times 7.8\times10^3\,ms^{-1} \times 7.5\times10^6\,m = \underline{1.2\times10^{14}\,kg\,m^2 s^{-1}}.$$

9. $\mu\ddot{\underline{r}} = -\frac{\alpha}{r^2}\hat{\underline{r}}$; $\underline{L} = \mu\,\underline{r}\times\dot{\underline{r}}$ NB $\dot{\underline{L}} = \mu(\dot{\underline{r}}\times\dot{\underline{r}} + \underline{r}\times\ddot{\underline{r}}) = \underline{0}$

(a) $\underline{A} = \dot{\underline{r}}\times\underline{L} - \alpha\hat{\underline{r}}$

$$\frac{dA}{dt} = \ddot{\underline{r}}\times\underline{L} + \dot{\underline{r}}\times\cancel{\dot{\underline{L}}} - \frac{\alpha}{r}\dot{\underline{r}} + \frac{\alpha\dot{r}\underline{r}}{r^2} = -\frac{\alpha}{r^2}\hat{\underline{r}}\times(\underline{r}\times\dot{\underline{r}}) - \frac{\alpha}{r}\dot{\underline{r}} + \frac{\alpha\dot{r}}{r}\hat{\underline{r}}$$

$$\hat{\underline{r}}\times(\hat{\underline{r}}\times\dot{\underline{r}}) = (\hat{\underline{r}}\cdot\dot{\underline{r}})\hat{\underline{r}} - (\hat{\underline{r}}\cdot\hat{\underline{r}})\dot{\underline{r}} = \dot{r}\hat{\underline{r}} - \dot{\underline{r}}$$

$$\Rightarrow \frac{dA}{dt} = -\frac{\alpha}{r}(\dot{r}\hat{\underline{r}} - \dot{\underline{r}}) - \frac{\alpha}{r}\dot{\underline{r}} + \frac{\alpha\dot{r}}{r}\hat{\underline{r}} = 0 \Rightarrow \underline{\underline{A} \text{ is constant.}}$$

(b) $\underline{A} = \dot{\underline{r}} \times \underline{L} - \alpha \hat{\underline{r}}$

$$\Rightarrow |A|^2 = \dot{\underline{r}} \times \underline{L} \cdot \hat{\underline{r}} \times \underline{L} + \alpha^2 - 2\alpha\, \hat{\underline{r}} \cdot (\dot{\underline{r}} \times \underline{L})$$

$$= \dot{r}^2 L^2 - \cancel{(\dot{\underline{r}} \cdot \underline{L})}^2 + \alpha^2 - 2\alpha\, \underline{L} \cdot \hat{\underline{r}} \times \dot{\underline{r}}$$

$$= |\dot{\underline{r}}|^2 L^2 + \alpha^2 - \frac{2\alpha}{\mu r} L^2$$

$$= \alpha^2 + \frac{2L^2}{\mu}\left(\frac{1}{2}\mu \dot{r}^2 - \frac{\alpha}{r}\right) = \alpha^2 + \frac{2L^2 E}{\mu}$$

$$\Rightarrow |A| = \left(\alpha^2 + \frac{2L^2 E}{\mu}\right)^{1/2} \equiv \alpha \varepsilon$$

(c) $\underline{r} \cdot \underline{A} = r|A|\cos\theta = r\alpha\varepsilon\cos\theta$

$\quad \|$

$$\underline{r} \cdot (\underline{v} \times \underline{L}) - \frac{\alpha r^2}{r} = \underline{L} \cdot (\underline{r} \times \underline{v}) - \alpha r = \frac{L^2}{\mu} - \alpha r$$

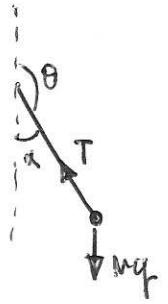
$$\Rightarrow \frac{L^2}{\mu} = \alpha r (1 + \varepsilon\cos\theta) \Rightarrow r = \frac{\frac{L^2}{\mu\alpha}}{1 + \varepsilon\cos\theta} = \frac{r_0}{1 + \varepsilon\cos\theta} \quad \text{where } r_0 = \frac{L^2}{\mu\alpha}$$

(d) $\underline{A} = \underline{v} \times \underline{L} - \alpha \hat{\underline{r}} \equiv$ constant.

$$\underline{v} \times \underline{L} = \mu\, \dot{\underline{r}} \times (\underline{r} \times \dot{\underline{r}}) = \mu\left(|\dot{\underline{r}}|^2 \underline{r} - (\underline{r} \cdot \dot{\underline{r}})\dot{\underline{r}}\right)$$

@ pericentre $\underline{r} \cdot \dot{\underline{r}} = 0 \Rightarrow \underline{A} = \left(\frac{\mu |\dot{\underline{r}}|^2}{r} - \alpha\right)\hat{\underline{r}}$ which is in the direction between the pericentre & the centre of mass.

10. (a)



$$m\ddot{\underline{r}} = -T\hat{\underline{r}} - mg\hat{\underline{z}}$$

$$\underline{r} = l\hat{\underline{r}} \Rightarrow m\ddot{\underline{r}} = -\frac{T}{l}\underline{r} - mg\hat{\underline{z}}$$

$$\Rightarrow \ddot{\underline{r}} = -\frac{T}{ml}\underline{r} - g\hat{\underline{z}}$$

NB $\theta + \alpha = \pi \Rightarrow \sin\theta = \sin\alpha$

$\cos\theta = -\cos\alpha$

(b) $\underline{r}^2 = \ell^2 \hat{\underline{r}}\cdot\hat{\underline{r}} = \ell^2 \Rightarrow \dot{\underline{r}}\cdot\underline{r} = 0$

Hence $\dot{\underline{r}}\cdot\ddot{\underline{r}} = -\frac{T}{m\ell}\underline{r}\cdot\dot{\underline{r}} - g\hat{\underline{z}}\cdot\dot{\underline{r}} = -g\hat{\underline{z}}\cdot\dot{\underline{r}}$

$\Rightarrow \frac{1}{2}\dot{\underline{r}}^2 + g\hat{\underline{z}}\cdot\underline{r} \equiv \text{constant} \equiv \frac{E}{M}$

(c) $\underline{r} = \ell\hat{\underline{r}} \Rightarrow \dot{\underline{r}} = \ell\dot{\hat{\underline{r}}} = \ell\underline{\Omega}\times\hat{\underline{r}}$

$\Rightarrow \dot{\underline{r}}^2 = \ell^2(\underline{\Omega}\times\hat{\underline{r}})(\underline{\Omega}\times\hat{\underline{r}}) = \ell^2\left[\Omega^2 - (\hat{\underline{r}}\cdot\underline{\Omega})^2\right]$

$\Omega^2 = \dot{\phi}^2 + \dot{\theta}^2$; $\hat{\underline{r}}\cdot\underline{\Omega} = \dot{\phi}\cos\theta$

$\Rightarrow \dot{\underline{r}}^2 = \ell^2\left(\dot{\phi}^2 + \dot{\theta}^2 - \dot{\phi}^2\cos^2\theta\right) = \ell^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$

(d) $\ddot{\underline{r}} = \ell\left[\dot{\underline{\Omega}}\times\hat{\underline{r}} + \underline{\Omega}\times\dot{\hat{\underline{r}}}\right] = \ell\left[\dot{\underline{\Omega}}\times\hat{\underline{r}} + \underline{\Omega}\times(\underline{\Omega}\times\hat{\underline{r}})\right]$

$\underline{\Omega}\times(\underline{\Omega}\times\hat{\underline{r}}) = (\hat{\underline{r}}\cdot\underline{\Omega})\underline{\Omega} - \Omega^2\hat{\underline{r}} = \dot{\phi}\cos\theta\left[\dot{\phi}(\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}}) + \dot{\theta}\hat{\underline{\phi}}\right] - (\dot{\phi}^2 + \dot{\theta}^2)\hat{\underline{r}}$

$= -(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\hat{\underline{r}} - \dot{\phi}\sin\theta\cos\theta\,\hat{\underline{\theta}} + \dot{\theta}\dot{\phi}\cos\theta\,\hat{\underline{\phi}}$

$\underline{\Omega} = \dot{\phi}\hat{\underline{z}} + \dot{\theta}\hat{\underline{\phi}}$ where $\hat{\underline{z}} = \cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}} \equiv \text{const}$ & $\hat{\underline{\phi}} = -\sin\phi\,\hat{\underline{x}} + \cos\phi\,\hat{\underline{y}}$

$\Downarrow$

$\dot{\hat{\underline{\phi}}} = \dot{\phi}\left(-\cos\phi\,\hat{\underline{x}} - \sin\phi\,\hat{\underline{y}}\right) = -\dot{\phi}\left(\sin\theta\,\hat{\underline{r}} + \cos\theta\,\hat{\underline{\theta}}\right)$

$\Rightarrow \dot{\underline{\Omega}} = \ddot{\phi}\hat{\underline{z}} + \ddot{\theta}\hat{\underline{\phi}} + \dot{\theta}\dot{\hat{\underline{\phi}}}$

$= (\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta)\hat{\underline{r}} - (\ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta)\hat{\underline{\theta}} + \ddot{\theta}\hat{\underline{\phi}}$

$\Rightarrow \dot{\underline{\Omega}}\times\underline{r} = (\ddot{\phi}\sin\theta + \dot{\theta}\dot{\phi}\cos\theta)\hat{\underline{\phi}} + \ddot{\theta}\hat{\underline{\theta}}$

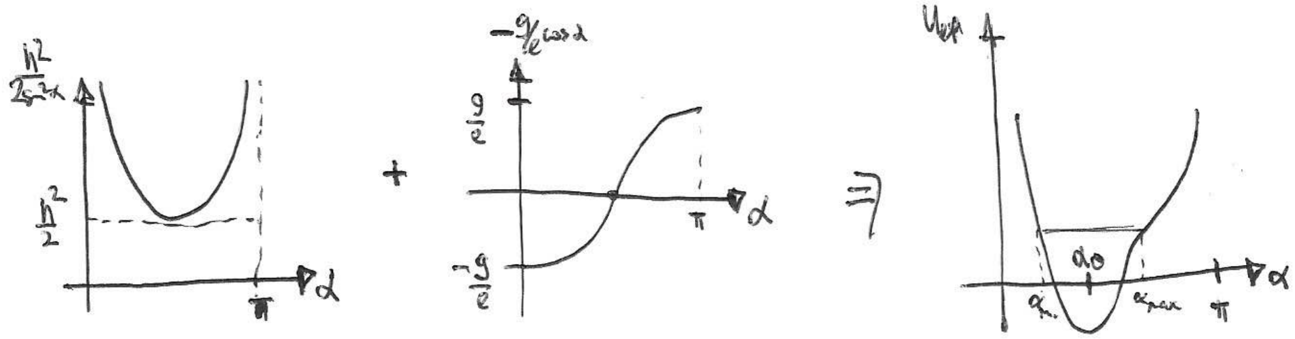
$\Rightarrow \ddot{\underline{r}}\cdot\hat{\underline{\phi}} = \ell\left[\ddot{\phi}\sin\theta + \dot{\theta}\dot{\phi}\cos\theta + \dot{\theta}\dot{\phi}\cos\theta\right] = 0 \Rightarrow \ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta = 0$

$\Rightarrow \dot{\phi}\sin^2\theta \equiv h = \text{const}$

Hence, $\frac{1}{2}\ell^2\left[\dot{\theta}^2 + \sin^2\theta\left(\frac{h}{\sin^2\theta}\right)^2\right] + g\ell\cos\theta \equiv \frac{E}{M} = \text{const}$

$\Rightarrow \frac{E}{M} = \frac{1}{2}\dot{\theta}^2 + \frac{h^2}{2\sin^2\theta} + \frac{g}{\ell}\cos\theta = \frac{1}{2}\dot{\alpha}^2 + \frac{h^2}{2\sin^2\alpha} - \frac{g}{\ell}\cos\alpha$

$$U_{eff}(\alpha) = \frac{h^2}{2\sin^2\alpha} - \frac{g}{l}\cos\alpha$$



$\Rightarrow$ elliptical orbits