<u>ONE HOUR THIRTY MINUTES</u>

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Advanced Quantum Mechanics

32nd January 2023, 2.00 p.m. - 3.30 p.m.

Answer **<u>TWO</u>** questions

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

You may assume the following formulae in any question *if proof is not explicitly requested*:

Special integrals

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2a)^n} \sqrt{\frac{\pi}{a}}, \qquad \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}, \quad a > 0.$$

4-vectors
Use notation $x^\mu = (x^0, \mathbf{r})$ for 4-position and $p^\mu = (E/c, \mathbf{p})$ for 4-momentum. In electromagnetism $A^\mu = (\Phi/c, \mathbf{A})$ for 4-potential.

Klein-Gordon equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[(\hat{\mathbf{p}} - \mathbf{V})^2 c^2 + (mc^2 + S)^2\right] \Psi = (i\hbar\partial_t - V_0)^2 \Psi.$$

Dirac equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[\boldsymbol{\alpha} \cdot (\hat{\mathbf{p}} - \mathbf{V})c + \beta(mc^2 + S)\right] \Psi = (i\hbar\partial_t - V_0)\Psi.$$

Standard spinor solutions to the free massive Dirac equation

$$u^{(s)}(E, \mathbf{p}) = \sqrt{E + mc^2} \begin{pmatrix} \phi^s \\ \frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E+mc^2}\phi^s \end{pmatrix}, \quad \phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

Standard matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix};$$

$$\gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}.$$

Space-time metric We use the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

Laplacian operator in terms of angular momentum operator $\hat{\mathbf{L}}$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\hat{\mathbf{L}}^2}{\hbar^2 r^2}.$$

Solution to Dirac equation in spherical vector potential $(V_0(r)/c, \vec{0})$

$$\psi^\kappa_{jm}(\mathbf{r}) = \begin{pmatrix} f^\kappa_j(r)\mathcal{Y}^\kappa_{jm}(\theta,\phi) \\ ig^\kappa_j(r)\mathcal{Y}^{-\kappa}_{jm}(\theta,\phi) \end{pmatrix} \quad \text{with } F = rf \text{ and } G = rg,$$

$$\hbar c\left[\frac{d}{dr} - \frac{\kappa}{r}\right] G^\kappa_j(r) = -\left(E - V_0(r) - mc^2\right) F^\kappa_j(r),$$

$$\hbar c\left[\frac{d}{dr} + \frac{\kappa}{r}\right] F^\kappa_j(r) = \left(E - V_0(r) + mc^2\right) G^\kappa_j(r).$$

1. a) State the selection rules for electric-dipole transitions in light atoms and explain any notation that you use. State whether each of the following transitions in nitrogen [(i),(ii)] or carbon [(iii)] could be electric-dipole transitions, and justify your answers:

i) $(2s)^2(2p)^3\ {}^4S_{3/2} \rightarrow (2s)^2(2p)^2(3d)\ {}^4P_{5/2}$
ii) $(2s)^2(2p)^2(3s)\ {}^4D_{3/2} \rightarrow (2s)^2(2p)^2(3d)\ {}^4P_{5/2}$
iii) $(2s)^2(2p)(3d)\ {}^1D_2 \rightarrow (2s)^2(2p)^2\ {}^1S_0$

[4 marks]

b) Show that the momentum operator $\hat{\mathbf{p}}$ is the generator of infinitesimal translations in space. Derive an expression in terms of $\hat{\mathbf{p}}$ for the operator $\hat{U}_{\mathbf{a}}$ representing a finite translation $\mathbf{r} \rightarrow \mathbf{r} - \mathbf{a}$. Show that $\hat{U}_{\mathbf{a}}$ is unitary.

[7 marks]

c) Write down the operator $\hat{U}_{\alpha,\beta,\gamma}$ representing a rotation with Euler angles (α, β, γ) for a spin-1/2 particle with zero orbital angular momentum. Determine the corresponding Wigner D-matrix. Use this Wigner D-matrix to show that a 2π rotation of an eigenstate of $\hat{S}_z$ about the x-axis changes the sign of the state.

[9 marks]

d) Consider a free relativistic electron of mass m. Show that the orbital angular momentum operator $\hat{\mathbf{L}}$ is not a constant of the motion.

[5 marks]

2. a) A spin-1/2 system subject to a static field is described by the Hamiltonian

$$\hat{H}_0 = \frac{\hbar\gamma}{2}\sigma_z.$$

At time $t = 0$ a weak rotating field in the xy plane is switched on and the Hamiltonian becomes $\hat{H} = \hat{H}_0 + \hat{V}(t)$, where

$$\hat{V}(t) = \frac{\hbar\Omega}{2}(\sigma_x \cos\omega t + \sigma_y \sin\omega t).$$

Both γ and Ω are independent of t. Initially, at time $t = 0$, the spin system is in the ground-state of H_0. Assuming Ω to be weak and taking $\hat{V}(t)$ as a perturbation, use first-order time-dependent perturbation theory to derive the transition rate (probability per unit time) for the system to make a transition to the excited state of $\hat{H}_0$ in the large-t limit. You may find the asymptotic behaviour of the following function useful:

$$t\frac{\sin^2(xt/2)}{(xt/2)^2} = t\,\mathrm{sinc}^2(xt/2) \to 2\pi\delta(x), \quad \text{as } t \to \infty.$$

[12 marks]

b) i) Write down the time-dependent Schrödinger equation for a non-relativistic spinless particle of mass m and charge q in the 4-potential $A^\mu = (\Phi/c, \mathbf{A})$. Under a gauge transformation the 4-potential transforms as:

$$\mathbf{A} \to \mathbf{A} + \nabla\lambda(\mathbf{r}, t), \qquad \Phi \to \Phi - \frac{\partial\lambda(\mathbf{r}, t)}{\partial t}.$$

Write down the corresponding gauge transformation operator for the wave function of the particle. Show that the time-dependent Schrödinger equation transforms as expected using this gauge transformation operator.

[6 marks]

ii) Consider the particle moving in a uniform magnetic field $\mathbf{B}$ that points in the z-direction and is described by the 4-potential $(0, \mathbf{A})$. Show that the field can be described by the vector potential $\mathbf{A} = (-yB, 0, 0)$. Using the time-independent Schrödinger equation, find the eigenvalues (Landau levels) of the Hamiltonian for the motion in the xy plane and express the eigenfunctions in terms of the eigenfunctions $\Phi_n(y)$ of a simple harmonic oscillator.

[7 marks]

3. a) i) Define the action of the parity operator $\hat{P}$ on an arbitrary state represented by the position space wave function $\psi(\mathbf{r},t)$. Calculate the eigenvalues of $\hat{P}$ and comment on the symmetry of its eigenstates.

[5 marks]

ii) By considering a matrix element of the momentum operator $\hat{\mathbf{p}}$ of the form

$$\langle\phi|\hat{\mathbf{p}}|\psi\rangle = \int d\mathbf{r}\,\phi^*(\mathbf{r},t)\,\hat{\mathbf{p}}\,\psi(\mathbf{r},\mathbf{t}),$$

for arbitrary $|\phi\rangle$ and $|\psi\rangle$, show that the passive parity transformation of $\hat{\mathbf{p}}$ is

$$\hat{P}^\dagger\,\hat{\mathbf{p}}\,\hat{P} = -\hat{\mathbf{p}}.$$

[5 marks]

b) Consider a spinless particle of charge e in a vector potential field $(e\Phi(r)/c, \mathbf{0})$, where

$$e\Phi(r) = \begin{cases} 0, & \text{for } r \le R \\ \infty & \text{for } r > R, \end{cases}$$

defines an infinite electrostatic spherical potential of radius R.

i) Write down the time-independent Klein-Gordon equation for the eigenstates of the particle.

[3 marks]

ii) By considering an eigenstate of total angular momentum with quantum number l, find the eigenvalue equation for the radial part $R(r)$ of the wave function of the particle.

[4 marks]

iii) Using suitable boundary conditions, find the ground-state wave function and the ground-state energy of the particle. The ground-state wave function does not need to be normalised and you might find it helpful to make the substitution $R(r) = u(r)/r$. Briefly discuss the non-relativistic limit for the ground-state energy.

[8 marks]

END OF EXAMINATION PAPER