ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Integral Transforms

17th May 2013, 14:00 - 15:30

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. (a) For a function $f(z) = u(x, y) + iv(x, y)$ (where $z = x + iy$), state the Cauchy-Riemann equations. Show that they are satisfied for $f(z) = 1/z$ (for $z \neq 0$).

[7 marks]

(b) Evaluate $\int_C |z|^2 \, dz$, where the end-points are $a = 0$ and $b = 1 + i$, and C is the path consisting of two straight-line segments parallel to the axes passing through the point $c = 1$. Should you expect the same answer for a parabolic path between the same end-points?

[7 marks]

(c) For the function $f(z) = \dfrac{1}{z(z-2)}$, find the Laurent series about the point $z = 0$ which is valid for $0 < |z| < 2$.

[6 marks]

(d) The following table gives the value of the function $f(z) = z^7 + 5z^2 + 2$ at the points $z = \mathrm{e}^{2n\pi i/8}$. How many roots has $f(z)$ within the unit circle?

[5 marks]

n	0	1	2	3
$f(z)$	8.00	$2.71 + 4.29i$	$-3.00 - 1.00i$	$1.29 - 5.71i$
n	4	5	6	7
$f(z)$	6.00	$1.29 + 5.71i$	$-3.00 + 1.00i$	$2.71 - 4.29i$

2. A function $w(z)$ has real part $u(x, y) = \sin x \cosh y$. Verify that this is a harmonic function. Given that $w(z)$ is analytic and vanishes at $z = 0$, find the corresponding imaginary part and show that $w = \sin z$.

[9 marks]

Now consider the mapping $w(z) = \arcsin z$. Show that under the inverse mapping, lines of constant $u = a$, for $-\pi/2 < a < \pi/2$, map to the curves

$$\frac{x^2}{\sin^2 a} - \frac{y^2}{\cos^2 a} = 1.$$

Show also that the lines $u = \pi/2$ and $u = -\pi/2$ map to the segments of the x axis $1 \le x < \infty$ and $-1 \ge x > -\infty$ respectively.

Find the curves to which lines of constant $v = b$ map (for any b), and sketch both sets of curves in the xy-plane. (Hint: it may help to show first that all the constant-u curves cross the x axis at $|x| \le 1$, while all the constant-v curves cross at $|x| \ge 1$.)

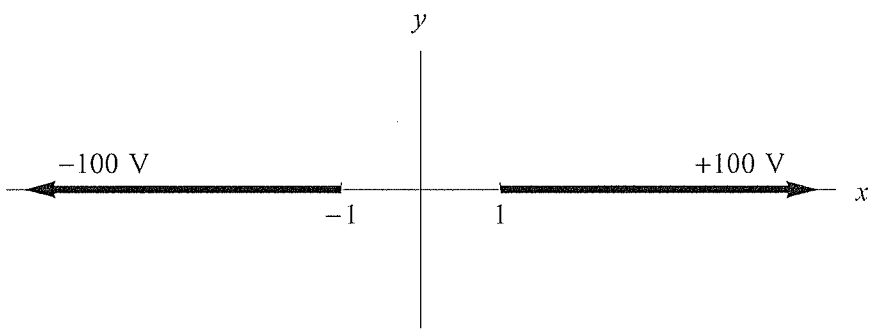
[10 marks]

Consider a pair of semi-infinite metal sheets placed as shown in the diagram below, the left-hand one being held at $-100\,\text{V}$ and the right-hand one at $100\,\text{V}$. Use the mapping above to argue that the electrostatic potential in the surrounding space is

$$\phi(x, y) = \frac{200\,\text{V}}{\pi}\text{Re}[\arcsin(x + iy)],$$

and relate it to your sketch.

[6 marks]



3. Use Cauchy's residue theorem and a suitable choice of contour to calculate <u>TWO</u> of the following integrals. If Jordan's lemma is used, ensure that all the conditions for its validity are satisfied.

(a)

$$\int_0^{2\pi} \frac{1}{5 + 4\cos 2\theta}\, \mathrm{d}\theta$$

(b)

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2 + 4}\, \mathrm{d}x$$

(c)

$$\int_{-\infty}^{\infty} \frac{\cos(kx)}{2 + x}\, \mathrm{d}x \qquad \text{for } k > 0$$

[25 marks]

4. (a) Find the residues at the poles of the following functions:

$$i)\ z^{-5}\sin^2 z, \qquad ii)\ z^3 \mathrm{e}^{1/z}, \qquad iii)\ \frac{1}{\sin z}.$$

[10 marks]

(b) Prove the following Laplace transforms (a may be taken as real).

$$i)\ \text{L.T.}\,[\mathrm{e}^{\alpha t}] = \frac{1}{s - \alpha} \quad \text{for } s > \mathrm{Re}[\alpha], \qquad ii)\ \text{L.T.}\,[\sin at] = \frac{a}{s^2 + a^2} \quad \text{for } s > 0.$$

[4 marks]

State the convolution theorem for inverse Laplace transforms. Using the fact that L.T. $[\theta(t - t_0)] = \mathrm{e}^{-st_0}/s$, where $\theta(t - t_0)$ is the Heaviside step function, find the inverse Laplace transform of

$$\frac{\mathrm{e}^{-3s}}{s(s^2 + 4)}.$$

[6 marks]

Hence find the solution to the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 4y = \theta(t - 3),$$

subject to the initial conditions $y(0) = 0$ and $y'(0) = 2$.

[5 marks]

END OF EXAMINATION PAPER