PHYS 10672 Advanced Dynamics: 2024

Section 1 : Maths and Revision

& Section 2 : Rotating Frames of Reference

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Maths and Revision

START OF 1ST LECTURE

1.1 Einstein summation convention and index notation

First, let us define scalar, vectors and tensors as objects whose components have 0, 1 and n indices.

$$\begin{aligned} A &\equiv \text{Scalar}\,, \\ A_i &\equiv \text{Vector}\,, \\ A_{i_1..i_n} &\equiv \text{Tensor}\,. \end{aligned} \tag{1.1}$$

n is the rank of the tensor and a tensor of rank 1 is a vector. Rank 2 tensors, A_{ij} are a way of representing matrices where the first index labels the row and the second labels the column. We will return to these definitions in the section on Special Relativity and we will see that this kind of tensor is called a *Cartesian or Euclidean tensor*. The indices $i, j, k = 1, .., N$ where N is the dimension of space.

The summation convention is an *opportunity* to reduce the amount of writing you do and is a common thing to do in theoretical physics. It will take some time to get used to it but ultimately is simple and logical. The basis idea is to replace

$$\sum_i A_i B_i = A_i B_i\,, \tag{1.2}$$

that is to remove the summation sign and presume that indices which appear twice are summed. An example of an equation using the summation convention is

$$C_i = B_{ij} A_j\,, \tag{1.3}$$

which is equivalent to

$$C_i = \sum_j B_{ij} A_j\,. \tag{1.4}$$

Important rules to bear in mind are:

- repeated indices must appear twice and only twice;
- free indices on each side of an equation must be the same.

The second of these implies that

$$C_i = B_{ik}A_{kj}\,, \tag{1.5}$$

makes no sense. The k index is repeated and hence summed - no problem there. But on the LHS of the equation has one index, i, while on the RHS there are two i and j. If there were an extra j on the LHS then it would be fine.

Another interesting illustration of the use of the summation convention is

$$(A_iB_i)^2\,. \tag{1.6}$$

The first thing to realise is that this cannot be written as $A_iB_iA_iB_i$ since the index here appears four times - this is against the rules (not also that $A_iB_iC_i$ is meaningless since the index i appears three time). In fact,

$$(A_iB_i)^2 = \left(\sum_i A_iB_i\right)^2 = \left(\sum_i A_iB_i\right)\left(\sum_j A_jB_j\right)\,, \tag{1.7}$$

which can be written as $A_iB_iA_jB_j$ using the summation convention.

One can write vector and matrix calculations in index notation. First, consider the two quantities A_iB_i and A_iB_j. The first is a scalar, with no free indices, and the second is a rank 2 tensor, with 2 free indices and represents a matrix. In fact

$$\begin{aligned}
A_iB_i &= \mathbf{A}^T\mathbf{B} = \mathbf{A}\cdot\mathbf{B} = \begin{pmatrix} A_1 & .. & A_N \end{pmatrix}\begin{pmatrix} B_1 \\ . \\ . \\ B_N \end{pmatrix} = A_1B_1 + ..A_NB_N\,, \\
A_iB_j &= \mathbf{A}\mathbf{B}^T = \begin{pmatrix} A_1 \\ . \\ . \\ A_N \end{pmatrix}\begin{pmatrix} B_1 & .. & B_N \end{pmatrix} = \begin{pmatrix} A_1B_1 & .. & A_1B_N \\ . & & . \\ . & & . \\ A_NB_1 & .. & A_NB_N \end{pmatrix}.
\end{aligned} \tag{1.8}$$

Moreover, matrix multiplication $C = AB$ can be written as

$$C_{ij} = A_{ik}B_{kj}\,, \tag{1.9}$$

(check this for a simple 2x2 matrix multiplication) and $D = BA$ is

$$D_{ij} = B_{ik}A_{kj} \neq C_{ij}\,. \tag{1.10}$$

We will practice using the summation convention in the exercises next week and also in the problem sheets. It requires practice and a bit of thought about what makes sense and what does not. Hopefully, through practice you will get used to it and it will become second nature. It will be used in various parts of the rest of the course and is introduced here to give you the best chance to become familiar with it.

1.2 Vectors and tensors

Now consider two coordinate systems S and S' with basis vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$. These bases will be considered to be orthonormal which means that

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}\,, \quad \mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k\,, \tag{1.11}$$

and

$$\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}\,, \quad \mathbf{e}'_i \times \mathbf{e}'_j = \epsilon_{ijk}\mathbf{e}'_k\,, \tag{1.12}$$

for right-handed coordinate systems. The rank 2 tensor δ_{ij}, known as the *Kronecker-*δ, has the property that it is 1 when $i = j$ and is zero otherwise. As a matrix it can be thought of at the identity matrix

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & .. & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ . & . & . & .. & . \\ 0 & .. & .. & 0 & 1 \end{pmatrix}\,, \tag{1.13}$$

with 1 on the diagonal and zero elsewhere. Note that $\delta_{ii} = \sum_i \delta_{ii} = N$ which is 3 in 3D space. ϵ_{ijk} is known as the *Levi-Civita* symbol. It has the property that it is 1 for cyclic permutations of 123, is -1 for anticyclic permutations of 123 and is zero otherwise, notably when any of the indices are the same. This means that $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$ and $\epsilon_{321} = \epsilon_{132} = \epsilon_{213} = -1$.

Now consider a vector $\mathbf{A}$ which has components A_i in the frame S and A'_i in S' which can be written (using the summation convention - note from now on it will be assumed) as

$$\mathbf{A} = A_i\mathbf{e}_i = A'_j\mathbf{e}'_j\,. \tag{1.14}$$

If we now dot product the 2nd and 3rd expressions with $\mathbf{e}'_k$ then we have that

$$A'_j\mathbf{e}'_j \cdot \mathbf{e}'_k = A_i\mathbf{e}_i \cdot \mathbf{e}'_k\,, \tag{1.15}$$

which implies (using the orthonormality property) that $A'_j\delta_{jk} = L_{ik}A_i$ defining $L_{ik} = \mathbf{e}_i \cdot \mathbf{e}'_k$ to be the rotation matrix which allows the change of coordinate system from S to S'. Hence, we find that

$$A'_k = L_{ik}A_i\,, \tag{1.16}$$

which defines the transformation law for the components of a vector between two frames. This can be thought of as the defining property of a vector.

Now

$$|\mathbf{A}'|^2 = A'_kA'_k = L_{ik}A_iL_{jk}A_j = L_{ik}L_{jk}A_iA_j\,. \tag{1.17}$$

We require that this is $\equiv |\mathbf{A}|^2$ so that the length of the vector is unchanged by the coordinate rotation and, therefore, we find that

$$L_{ik}L_{jk} = \delta_{ij}\,, \tag{1.18}$$

in component notation, or that the matrix L is orthogonal which means that $LL^T = I$ or the transpose is the inverse matrix. Note that the ij component of L^T is L_{ji} and the components of the vector in S' can be related to those in S by $\mathbf{A}' = L^T\mathbf{A}$ and $\mathbf{A} = L\mathbf{A}'$.

A rank n tensor can be defined as a having the property that the components transform according to

$$A'_{i_i..i_n} = L_{i_1j_1}..L_{i_nj_n}A_{j_1..j_n}\,, \tag{1.19}$$

where there are n copies of the rotation matrix L. The rank 2 tensor δ_{ij} and ϵ_{ijk} have the property that they are isotropic meaning that they are the same in all frames which can be formulated as

$$\delta'_{ij} = L_{ik}L_{jm}\delta_{km}\,, \quad \epsilon'_{ijk} = L_{ip}L_{jq}L_{kr}\epsilon_{pqr}\,, \tag{1.20}$$

for all rotations. This also allow us to define the two products

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= \delta_{ij}A_iB_j = A_iB_i \\ \mathbf{A}\times\mathbf{B} &= \mathbf{e}_i\cdot\epsilon_{ijk}A_jB_k\,. \end{aligned} \tag{1.21}$$

or that the i-component of the vector product is

$$(\mathbf{A}\times\mathbf{B})_i = \epsilon_{ijk}A_jB_k\,. \tag{1.22}$$

The reason why we are interested in these quantities is that we can express various physical quantities as scalar and vector *fields* which means that they are functions of position - usually denoted $\mathbf{x}$ or $\mathbf{r}$ - and time t, that is, $A(\mathbf{r},t)$ is a scalar field and $\mathbf{A}(\mathbf{r},t)$ is a vector field. Examples include: scalar fields - density, $\rho(\mathbf{r},t)$ - and vector fields - electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$.

1.3 Newtonian Dynamics

The aim of this section is to remind you of a few things relevant to this course which you should be well aware of at the very least from last semester's courses on Dynamics and Quantum Physics and Relativity - notably the Relativity part of that course.

First, let us consider *Inertial Frames.* These are defined to be frame of reference in which there exists no net force. These frames move relative to each other along straight line and are related by Galilean transformations

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \mathbf{v}t\,, \\ t' &= t\,, \end{aligned} \tag{1.23}$$

whose key feature is that time is universal (as opposed to what happens in Special Relativity as we shall see later in the course).

Now we will define some key quantities which you should be very familiar with - paying attention to whether there are vectors or scalars. In particular,

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \text{Velocity}\,, \\ \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} = \text{Accelertion}\,, \\ \mathbf{p} &= m\mathbf{v} = \text{Linear Momentum}\,, \end{aligned} \tag{1.24}$$

are vector quantities. The speed $v = |\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_i v_i}$ is a scalar and the scalar $\dot{v}$ is not the acceleration

The laws governing dynamics are the well-known Newton's laws of motion:

- N1: there exists at least one inertial frame with respect to which a point mass moves in a straight line;
- N2: force is the rate of change of linear momentum, that is, $\mathbf{F} = \dot{\mathbf{p}} = m\mathbf{a}$;
- N3: if a body A exerts a a force $\mathbf{F}_{AB}$ on body B, then B exerts a force $\mathbf{F}_{BA} = -\mathbf{F}_{AB}$ on A.

One can define the angular momentum vector $\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\mathbf{r}\times\mathbf{v} = m\mathbf{r}\times\dot{\mathbf{r}}$. This allows for the generalisation of N2 to rotational motion where the torque is given by the rate of range of angular momentum, or $\mathbf{M} = \dot{\mathbf{L}}$.

END OF 1ST LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 1 Exercises

1. Check that :
$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$,
and hence show that :
$\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}$ and
$\epsilon_{ijk}\epsilon_{ijk} = 6$.

2. Show that:
$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$ and
$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$.

3. Show that : $\mathbf{M} = \mathbf{r}\times\mathbf{F}$.

START OF 2ND LECTURE

1.4 General motion in 2D polar coordinates

The objective is this section is to derive a formula for the velocity and acceleration vectors which could be used to described motion in a plane using 2D polar coordinates. You will have covered something like this in the course on Dynamics last semester. The basis vectors of 2D polar coordinates are

$$\hat{\mathbf{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \tag{1.25}$$

as shown in the lecture slides. Since $\theta \equiv \theta(t)$, the chain rule allows us to deduce that

$$\frac{d}{dt}\hat{\mathbf{r}} = \dot{\theta}\hat{\boldsymbol{\theta}}, \quad \frac{d}{dt}\hat{\boldsymbol{\theta}} = -\dot{\theta}\hat{\mathbf{r}}, \tag{1.26}$$

that is, the basis vectors depend on time. Now position vector is give by $\mathbf{r}(t) = r(t)\hat{\mathbf{r}}$. Therefore, we can deduce that the velocity vector is given by

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\frac{d}{dt}\hat{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}. \tag{1.27}$$

The acceleration vector can be calculated by further differentiation with respect to time

$$\begin{aligned} \mathbf{a} = \dot{\mathbf{v}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d}{dt}\hat{\mathbf{r}} + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}} \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{\boldsymbol{\theta}}. \end{aligned} \tag{1.28}$$

1.5 Two body interacting system

Now consider the 2-body interacting system that is presented in the lecture slides. The total mass $M = M_1 + M_2$, the centre of mass position is

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}, \tag{1.29}$$

and the relative position of the two particles is $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Our objective is to decouple the motion of the centre of mass and that of the relative positions as was discussed in the course on Dynamics. I am not sure whether this was derived or just stated in that course. Here, the emphasis is on the derivation and we will use it in the section on central forces.

We will use N3 to define a force given by $\mathbf{F} = \mathbf{F}_{21} = -\mathbf{F}_{12}$ and we will assume, having Coulomb or gravitational forces in mind, that this is proportional to the relative position, that is, $\mathbf{F} \propto \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Note both the Coulomb and gravitational forces are centra forces and these will be discussed in section 3 of the course.

By manipulation of the definitions one can deduce expressions for the particle positions in terms of the centre of mass and relative positions

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}\,, \qquad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}\,. \tag{1.30}$$

Now we define the centre of mass velocity, $\mathbf{V} = \dot{\mathbf{R}}$ and the relative velocity $\mathbf{v} = \dot{\mathbf{r}}$ from which we can deduce

$$\mathbf{v}_1 = \mathbf{V} + \frac{m_2}{M}\mathbf{v}\,, \qquad \mathbf{v}_2 = \mathbf{V} - \frac{m_1}{M}\mathbf{v}\,. \tag{1.31}$$

If it is helpful to also define the position of the particles relative to the centre of mass

$$\mathbf{r}'_1 = \mathbf{r}_1 - \mathbf{R} = \frac{m_2}{M}\mathbf{r}\,, \qquad \mathbf{r}'_2 = \mathbf{r}_2 - \mathbf{R} = -\frac{m_1}{M}\mathbf{r}\,, \tag{1.32}$$

and the associated velocities relative to the centre of mass

$$\mathbf{v}'_1 = \frac{m_2}{M}\mathbf{v}\,, \qquad \mathbf{v}'_2 = -\frac{m_1}{M}\mathbf{v}\,. \tag{1.33}$$

At this stage we have not included any physics; what I have derived above are purely definitions. If we apply N2 to the two bodies then

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F} = \mathbf{F}_{21}\,, \qquad m_2\ddot{\mathbf{r}}_2 = -\mathbf{F} = \mathbf{F}_{12}\,, \tag{1.34}$$

from which we can deduce, by adding the two equations, that $M\ddot{\mathbf{R}} = 0$, that the centre of mass velocity is a constant, $\dot{\mathbf{R}} = \mathbf{V} = \text{constant}$ and it is an inertial frame.

Now we aim to separate the centre of mass and relative motion. First, consider

$$\mathbf{F} = m_1\ddot{\mathbf{r}}_1 = m_1\dot{\mathbf{v}}_1 = m_1\left(\dot{\mathbf{V}} + \frac{m_2}{M}\dot{\mathbf{v}}\right) = \mu\ddot{\mathbf{r}}\,, \tag{1.35}$$

where $\mu = m_1 m_2/M$ is the reduced mass of the system. Therefore, we see that Newtons law can be applied to the relative motion if one uses the reduced mass of the system.

One can also derive expressions for the momenta relative to the centre of mass. In particular, one finds that

$$\begin{aligned} \mathbf{p}'_1 &= m_1\mathbf{v}'_1 = \mu\mathbf{v}\,, \\ \mathbf{p}'_2 &= m_2\mathbf{v}'_2 = -\mu\mathbf{v} = -\mathbf{p}'_1\,. \end{aligned} \tag{1.36}$$

The total linear and angular momenta can also be separated. The total linear momentum is given by

$$\mathbf{P}_{\text{tot}} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\left(\mathbf{V} + \frac{m_2}{M}\mathbf{v}\right) + m_2\left(\mathbf{V} - \frac{m_1}{M}\mathbf{v}\right) = M\mathbf{V}\,, \tag{1.37}$$

while the total angular momentum is given by

$$\mathbf{L}_{\text{tot}} = m_1\mathbf{r}_1 \times \mathbf{v}_1 + m_2\mathbf{r}_2 \times \mathbf{v}_2$$

$$
\begin{aligned}
&= m_1\left(\mathbf{R}+\frac{m_2}{M}\mathbf{r}\right)\times\left(\mathbf{V}+\frac{m_2}{M}\mathbf{v}\right)+m_2\left(\mathbf{R}-\frac{m_1}{M}\mathbf{r}\right)\times\left(\mathbf{V}-\frac{m_1}{M}\mathbf{v}\right)\\
&= M\mathbf{R}\times\mathbf{V}+\mu\mathbf{r}\times\mathbf{v}\,. \qquad (1.38)
\end{aligned}
$$

END OF 2ND LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 2 Exercises

1. A particle of mass m moving in 2D is subject to a force $\mathbf{F} = -(G\mu m/r^2)\hat{\mathbf{r}}$ where G is the gravitational constant and μ as dimensions of mass. Show that $\ddot{r} = (h^2/r^3) - (G\mu/r^2)$ where h is a constant with dimensions $[L]^2[T]^{-1}$.

2. Show that in the 2 body system described in lectures:
(i) $T_{\rm tot} = \frac{1}{2}M\mathbf{V}^2 + \frac{1}{2}\mu\mathbf{v}^2$;
(ii) $\dot{\mathbf{L}}_{\rm tot} = 0$.

2

Rotating frames of reference - see FS chapter 8

START OF 3RD LECTURE

2.1 Accelerating frames - see FS8.1

Consider two frames S and S' whose coordinates are related by

$$\mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{R}(t)\,. \tag{2.1}$$

$\mathbf{r}'$ are the coordinates in S' and $\mathbf{r}$ are those in S. The vector $\mathbf{R}$ is the position of the origin of S', which we denoted O', in the frame S. Now differentiate with respect to time and we find that $\ddot{\mathbf{r}}' = \ddot{\mathbf{r}} - \ddot{\mathbf{R}}$ that is there is a connection between the acceleration in the two frames, $\mathbf{a}'$ and $\mathbf{a}$,

$$\mathbf{a}' = \mathbf{a} - \mathbf{A}\,, \tag{2.2}$$

via the relative acceleration, $\mathbf{A} = \ddot{\mathbf{R}}$. This is very simple mathematically, but ultimately has some profound consequences.

Let us apply N2 in S, that is, $\mathbf{F} = m\mathbf{a} = m\left(\mathbf{a}' + \mathbf{A}\right)$ and so one can define a force in $\mathbf{S}'$ which also respects N2

$$\mathbf{F}' = \mathbf{F} - m\mathbf{A} = m\mathbf{a}'\,, \tag{2.3}$$

which requires the definition of $\mathbf{F}_{\text{inertial}} = -m\mathbf{A}$ an apparent or *inertial force* that the observer in the *non-inertial frame*, S', experiences. This is possibly best understood by consideration of a couple examples.

- Consider a ball on a trailer pulled by a car as shown in the lecture slides, ignoring any frictional effects to simplify the understanding. When a force F is applied to the car what happens to the ball? The answer depends on which frame you are in. From the point of view of an observer in the car, the ball experiences an inertial force of $-F$, that is, an apparent force in the opposite direction, and the ball rolls backward along the trailer. But from the point of view of a stationary observer, the ball remains at rest.
- Now imagine a Freely-Falling Lift (FFL) which is an often studied idealisation in the context of gravity. The set-up is illustrated in the lecture slides and comprises a lift falling under gravity, specified by local gravitational acceleration $\mathbf{g} = -g\mathbf{e}_z$, and a particle of mass m inside the lift. In this case there is an inertial force $|\mathbf{F}_{\text{inertial}}| = mg$

upwards on the particle from the point of view of the observer in the lift. This cancels the weight force of mg downwards so that in this frame $\mathbf{F}_{\rm tot} = 0$ - the particle does not appear to experience the effects of gravity. Of course, from the point of view of an observer on the ground the particle is still falling under the effect of gravity. This discussion presumes that gravity is uniform - which is it is not - and demanding the existence of a locally freely falling frame in more general cases is the basis of Einstein's Equivalence Principle and this is the basis of the development of General Relativity.

2.2 Rotating frames - see FS8.2

Now we will move on to discuss another type of non-inertial frame which perhaps has more practical use - *rotating frames of reference*. These are particularly important since we live in a rotating frame, that is, we live on the Earth which we know is rotating at a rate corresponding to a full rotation once a day, that is, $\omega_{\rm E} = 2\pi/(1\,{\rm day}) = 7.3\times 10^{-5}\,{\rm rad\,s^{-1}}$. On short timescales these effects are unimportant, but they can lead to significant phenomena. In order, to understand this we need to develop a detailed formalism.

In order to do this we will use spherical polar coordinates (r,θ,ϕ) defined in terms of the cartesian coordinates (x,y,z) by

$$x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi\,, \quad z = r\cos\theta\,, \tag{2.4}$$

which can be inverted using

$$r = \sqrt{x^2+y^2+z^2}\,, \quad \theta = \cos^{-1}(z/r)\,, \quad \phi = \tan^{-1}(y/x)\,. \tag{2.5}$$

You should have already come across these in other courses, but there is a picture in the lecture slides for you to study and clarify what is going on. The basis vectors for the spherical polar coordinates can be written in terms of Cartesian basis vectors

$$\begin{aligned}
\mathbf{e}_r &= \sin\theta(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) + \cos\theta\,\mathbf{e}_z\,,\\
\mathbf{e}_\theta &= \cos\theta(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) - \sin\theta\,\mathbf{e}_z\,,\\
\mathbf{e}_\phi &= -\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y\,,
\end{aligned} \tag{2.6}$$

and these can be inverted to given the Cartesian basis vectors in terms of the those of the spherical polars.

$$\begin{aligned}
\mathbf{e}_x &= \cos\phi(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) - \sin\phi\,\mathbf{e}_\phi\\
\mathbf{e}_y &= \sin\phi(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) + \cos\phi\,\mathbf{e}_\phi\\
\mathbf{e}_z &= \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta\,.
\end{aligned} \tag{2.7}$$

These definitions will be used in what follows.

Let us consider two frames. The first is a stationary inertial frame, S, with basis vectors $\mathbf{e}_i = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. The other, S' will be the rotating frame with basis vectors

$\mathbf{e}'_i = \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$. What we will do is set $\theta \equiv \theta(t)$ and $\phi \equiv \phi(t)$ and proceed in a similar way as in the case of section 1.4.

It can be shown that

$$\begin{aligned}
\dot{\mathbf{e}}_r &= \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\mathbf{e}_\phi \,, \\
\dot{\mathbf{e}}_\theta &= -\dot{\theta}\mathbf{e}_r + \dot{\phi}\cos\theta\mathbf{e}_\phi \,, \\
\dot{\mathbf{e}}_\phi &= -\dot{\phi}\left(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta\right) \,.
\end{aligned} \tag{2.8}$$

If you think carefully you will realise that these are the equivalent of the (1.26) but for spherical polar coordinates. The first equation of (2.8) is derived in the lectures by calculating

$$\dot{\mathbf{e}}_r = \frac{d}{dt}\left[\sin\theta\left(\cos\phi\,\mathbf{e}_x \sin\phi\,\mathbf{e}_y\right) + \cos\theta\mathbf{e}_z\right] \,, \tag{2.9}$$

using the fact that $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are independent of time and then substituting back from (2.7). Deriving the other two is one of the exercises below.

We would like to relate this to the angular velocity vector $\boldsymbol{\omega} \equiv \boldsymbol{\omega}(\theta, \phi)$. Using the right hand screw rule and $\boldsymbol{\omega} = \omega\mathbf{n}$ for an angular velocity ω about an axis $\hat{\mathbf{n}}$, we can deduce that the general angular velocity associated with a rotation by θ about $\mathbf{e}_\phi$ and ϕ about $\mathbf{e}_z$ is given

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{e}_\phi + \dot{\phi}\mathbf{e}_z = \dot{\phi}\left(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta\right) + \dot{\theta}\mathbf{e}_\phi \,, \tag{2.10}$$

which can be written as $\boldsymbol{\omega} = \omega'_k\mathbf{e}'_k$ where $\omega'_1 = \dot{\phi}\cos\theta$, $\omega'_2 = -\dot{\phi}\sin\theta$ and $\omega'_3 = \dot{\theta}$.

Now calculate

$$\boldsymbol{\omega} \times \mathbf{e}_r = \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_\phi \\ \dot{\phi}\cos\theta & -\dot{\phi}\sin\theta & \dot{\theta} \\ 1 & 0 & 0 \end{vmatrix} = \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\mathbf{e}_\phi = \dot{\mathbf{e}}_r \,. \tag{2.11}$$

Similarly, we can deduce that $\dot{\mathbf{e}}_\theta = \boldsymbol{\omega} \times \mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\phi = \boldsymbol{\omega} \times \mathbf{e}_\phi$ and calculating these is one of the exercises. We can summarise these relations in the simple formula

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i = \omega'_k\mathbf{e}'_k \times \mathbf{e}'_i = \epsilon_{ijk}\mathbf{e}'_j\omega'_k \,. \tag{2.12}$$

END OF 3RD LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 3 Exercises

1. S and S' are two frames related by a Galilean Transformation (GT) ie $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t' + \mathbf{r}'(t')$ and $t = t'$. Show that forces measured in the two frames are equal and hence that N2 and N3 are invariant under GT. Deduce that N1 can be generalised to there being infinitely many inertial frames of reference.

2. Check that $\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}$ if $\mathbf{e}'_i = \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$.

3. Check that $\dot{\mathbf{e}}_\phi = -\dot{\phi}\sin\theta\mathbf{e}_r - \dot{\phi}\cos\theta\mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r + \dot{\phi}\cos\theta\mathbf{e}_\phi$.

4. Verify that $\dot{\mathbf{e}}_\theta = \boldsymbol{\omega} \times \mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\phi = \boldsymbol{\omega} \times \mathbf{e}_\phi$.

5. Reproduce the derivation of $\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i$ in FS Pg 162 and connect with that done in the lectures.

START OF 4TH LECTURE

Consider an arbitrary vector $\mathbf{A} = A_i(t)\mathbf{e}_i = A'_j(t)\mathbf{e}'_j(t)$ written in two frame S and S' with basis vectors $\mathbf{e}_i$ and $\mathbf{e}'_i(t)$, respectively. The basis vectors in S are stationary, but those in the non-inertial frame S' are time dependent. Now calculate the time derivative of $\mathbf{A}$ in the two frames:

$$\frac{d}{dt}\mathbf{A} = \dot{A}_i\mathbf{e}_i = \dot{A}'_j\mathbf{e}'_j + A'_j\dot{\mathbf{e}}'_j = \dot{A}'_j\mathbf{e}'_j + A'_j\boldsymbol{\omega} \times \mathbf{e}'_j\,, \tag{2.13}$$

where we have used the result $\dot{\mathbf{e}}'_j = \boldsymbol{\omega} \times \mathbf{e}'_j$. Hence, we can deduce that

$$\dot{\mathbf{A}}|_S = \dot{\mathbf{A}}|_{S'} + \boldsymbol{\omega} \times \mathbf{A} \tag{2.14}$$

where we have defined the derivatives measured relative to the basis vectors of S and S' as $\dot{\mathbf{A}}|_S = \dot{A}_i\mathbf{e}_i$ and $\dot{\mathbf{A}}|_{S'} = \dot{A}'_i\mathbf{e}'_i$.

We will be interested in a point particle with mass m at point P as shown in the lecture slides and where the coordinates are related by

$$\mathbf{r}(t) = \mathbf{R}(t) + \mathbf{r}'(t)\,. \tag{2.15}$$

This is the same relation that we used to study the effects of accelerating frames and indeed we will re-derive the results found there, but with the addition of the effects of a rotating frame.

We can calculate velocity measured in S, $\mathbf{v}$, in terms of that in S', $\mathbf{v}'$, by differentiation and the application of (2.13) to give

$$\begin{aligned}\mathbf{v} = \dot{\mathbf{r}}|_S &= \dot{\mathbf{R}}|_S + \dot{\mathbf{r}}'|_S \\ &= \dot{\mathbf{R}}|_S + \dot{\mathbf{r}}'|_{S'} + \boldsymbol{\omega} \times \mathbf{r}' \\ &= \dot{\mathbf{R}}|_S + \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}'\,. \end{aligned} \tag{2.16}$$

The first term is the relative velocity of S' in S, and the third term is the velocity induced by the rotation that you might have predicted. A further differentiation yields the acceleration in S, $\mathbf{a}$, in terms of that in S', $\mathbf{a}'$,

$$\mathbf{a} = \dot{\mathbf{v}}|_S \;=\; \ddot{\mathbf{R}}|_S + \dot{\mathbf{v}}'|_S + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times \dot{\mathbf{r}}'|_S$$

$$
\begin{aligned}
&= \ddot{\mathbf{R}}|_S + \dot{\mathbf{v}}'|_{S'} + \boldsymbol{\omega} \times \mathbf{v}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times \big(\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}'\big) \\
&= \ddot{\mathbf{R}}|_S + \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')\,. \qquad (2.17)
\end{aligned}
$$

Note that $\dot{\boldsymbol{\omega}}|_{S'} = \dot{\boldsymbol{\omega}}|_S + \boldsymbol{\omega}\times\boldsymbol{\omega} = \dot{\boldsymbol{\omega}}|_S$ - that is the derivative of $\boldsymbol{\omega}$ is unambiguous between the two frames.

Now apply N2 in the frame S, that is, $\mathbf{F} = m\ddot{\mathbf{r}}|_S = m\mathbf{a}$ and we can deduce that

$$
\mathbf{F} + \mathbf{F}_{\rm inertial} = \mathbf{F}' = m\mathbf{a}'\,, \qquad (2.18)
$$

where

$$
\mathbf{F}_{\rm inertial} = -m\ddot{\mathbf{R}}|_S - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2m\boldsymbol{\omega} \times \mathbf{v}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')\,. \qquad (2.19)
$$

The first term is due to the relative acceleration found in section 2.1 expressed in slightly more flowery language, while the second is only non-zero if the angular velocity vector is varied with time and, typically, we will ignore this. The other two inertial forces are known as the Coriolis force and the Centrifugal force, resepectively,

$$
\mathbf{F}_{\rm cor} = -2m\boldsymbol{\omega} \times \mathbf{v}'\,, \quad \mathbf{F}_{\rm cen} = -m\boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \mathbf{r}'\big)\,. \qquad (2.20)
$$

In the 2D polar coordinate calculation we did in section 1.4 we have that $F_{\rm cen} \propto -r\dot{\theta}^2$ and $F_{\rm cor} \propto 2\dot{r}\dot{\theta}$, so they are not new but they are now fully formulated in a way that they can be calculated in the 3D. One of the main reasons for developing this formalism is to study the effects of the rotation of the Earth, but before doing that in the next lecture, we will study a couple of simple examples. In the context of these two examples, these inertial forces are a "sledgehammer to crack a nut" in the sense that the problems can be solved much more easily by considering the motion in a inertial frame (IF), however t I think it is instructive to understand the ideas by comparing the calculation in the non-inertial frame (NIF) for the same problem. Hopefully, this will convince you that there is nothing weird going on before we apply the NIF calculations in anger.

One can try to make an order of magnitude estimate of the ratio of the two forces

$$
\frac{|\mathbf{F}_{\rm cor}|}{|\mathbf{F}_{\rm cen}|} \propto \frac{|\mathbf{v}'|}{|\boldsymbol{\omega}||\mathbf{r}'|}\,, \qquad (2.21)
$$

which is ~ 1 for $v' \sim 1\,{\rm ms}^{-1}$, $\mathbf{r}' \sim 1\,{\rm m}$ and $\omega \sim 1\,{\rm rad\,s}^{-1}$ and so it is clear that at rotation frequencies $\sim 1\,{\rm Hz}$ in a laboratory setting the Coriolis force will need to be taken into account. If we consider $\omega = \omega_{\rm E} \approx 7.3 \times 10^{-5}\,{\rm rad\,s}^{-1}$ and scales of $r' \sim 1\,{\rm km}$ - as might be relevant in the atmosphere (ie the size of clouds) the ratio is $v'/7.3 \times 10^{-2}\,{\rm m\,s}^{-1}$, and hence the Coriolis force is likely to be important in that regime.

- Particle in circular motion: a particle of mass m is in moving on a circular trajectory of radius R with angular speed ω. as shown in the lecture slides. You know how to formulate this problem in the IF: there is a centripetal acceleration $|\mathbf{a}| = R\omega^2$ inward and this is equated with the force, whatever it might be, that is keeping the particle in circular motion to give $F = mR\omega^2$. Now choosing an NIF which is the particle rest frame, we find that the Coriolis force is zero, but the Centrifugal force can be computed

$$\boldsymbol{\omega} = \omega \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \quad \mathbf{r}' = R \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \quad \to \quad \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') = -R\omega^2 \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} = -R\omega^2 \hat{\mathbf{r}}' , \tag{2.22}$$

where $\hat{\mathbf{r}}'$ is the radial unit vector in the NIF. If we apply N2 in the NIF then there is no acceleration - it is the rest frame of the particle - so $\mathbf{F} - \mathbf{F}_{\text{inertial}} = m\mathbf{a}' = \mathbf{0}$ and therefore $\mathbf{F} = mR\omega^2\hat{\mathbf{r}}'$. So we see that the result is the same in the IF and the NIF! They are just different ways of thinking about the same thing. In this simple example, there was no need to go through the complication of worrying about the NIF, but it is instructive to see the calculation in the two frames to understand that everything is in order.

- Particle on a (frictionless) turntable: now imagine a particle of mass m at the edge of a turntable which rotates with angular speed ω and radius R, and also assume that the ball is added to the table so that it is initially rotating with the turntable. This means that it has an initial velocity component in the y-direction. As shown in the diagram in the lecture slides it also has an initial velocity v_0 in the horizontal direction. Now in the IF it is easy to solve the problem as in the previous example. N2 implies that $\ddot{\mathbf{r}} = \mathbf{0}$ and hence $\mathbf{r}(t) = \mathbf{r}(0) + \dot{\mathbf{r}}(0)t$ where

$$\mathbf{r}(0) = \begin{pmatrix} R \\ 0 \\ 0 \end{pmatrix} , \qquad \dot{\mathbf{r}}(0) = \begin{pmatrix} -v_0 \\ R\omega \\ 0 \end{pmatrix} , \tag{2.23}$$

and we can deduce that $x = R - v_0 t$, $y = R\omega t$, $r^2 = x^2 + y^2 = (R - v_0 t)^2 + (\omega R t)^2$ and $\tan\theta = y/x = R\omega t/(R - v_0 t)$. This corresponds to the particle moving in a straight line from one side of the turntable to another along the trajectory

$$y = \frac{R\omega}{v_0}(R - x) . \tag{2.24}$$

Now let us formulate the problem in the NIF. We have that

$$\boldsymbol{\omega} = \omega \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \mathbf{r}' = \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} , \mathbf{v}' = \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ 0 \end{pmatrix} \to \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') = -\omega^2 \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} , \boldsymbol{\omega} \times \mathbf{v}' = \omega \begin{pmatrix} -\dot{y}' \\ \dot{x}' \\ 0 \end{pmatrix} , \tag{2.25}$$

and therefore

$$\mathbf{F}_{\rm inertial} = \mathbf{F}_{\rm cor} + \mathbf{F}_{\rm cen} = 2m\omega \begin{pmatrix} \dot{y}' \\ -\dot{x}' \\ 0 \end{pmatrix} + m\omega^2 \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} . \tag{2.26}$$

From the point of view of the motion in the NIF, we see that at the initial instant the particle appears to experience a force

$$\mathbf{F}_{\rm inertial}(0) = \begin{pmatrix} mR\omega^2 \\ 2m\omega v_0 \\ 0 \end{pmatrix} , \tag{2.27}$$

where the 2nd component is due to the Coriolis force - so the particle appears to accelerate in the y' direction.

The equations of motion can be deduced by applying N2 in the NIF including the non-inertial forces to give

$$\begin{aligned} \ddot{x}' &= 2\omega\dot{y}' + \omega^2 x' , \\ \ddot{y}' &= -2\omega\dot{x}' + \omega^2 y' . \end{aligned} \tag{2.28}$$

One can solve these equations by considering the complex combination $\mathcal{Z} = x' + iy'$ from which we can deduce that

$$\ddot{\mathcal{Z}} + 2i\omega\dot{\mathcal{Z}} - \omega^2\mathcal{Z} = 0 , \tag{2.29}$$

and search for solutions of the form $\mathcal{Z}(t) = (A + Bt)\exp(-i\omega t)$ - since it is a resonant equation - with initial conditions $\mathcal{Z}(0) = R$ and $\dot{\mathcal{Z}}(0) = -v_0$. We find that

$$\mathcal{Z}(t) = [R + (-v_0 + i\omega R)\, t] \exp(-i\omega t) , \tag{2.30}$$

and so

$$\begin{aligned} x' &= (R - v_0 t)\cos\omega t + \omega R t \sin\omega t , \\ y' &= \omega R t \cos\omega t - (R - v_0 t)\sin\omega t , \end{aligned} \tag{2.31}$$

which yield

$$\begin{aligned} r'^2 &= (R - v_0 t)^2 + (\omega R t)^2 = r^2 , \\ \tan\theta' &= \frac{\omega R t - (R - v_0 t)\tan\omega t}{R - v_0 t + \omega R t \tan\omega t} = \frac{\tan\theta - \tan\omega t}{1 + \tan\theta\tan\omega t} = \tan(\theta - \omega t) . \end{aligned} \tag{2.32}$$

The expression for r^2 is the same as one finds in the IF, while that for the angle is different, but they can be related, as one would expect for a rotating frame $\theta' = \theta - \omega t$. Moreover we can write

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} , \tag{2.33}$$

that the coordinates in S and S' are related by a rotation by $-\omega t$.

Just as an aside using the expression we can find the minimum distance of approach to the centre. By differentiating the expression for r^2 with respect to time one finds that the minimum radius is

$$r_{\min} = \frac{R^2\omega}{\sqrt{R^2\omega^2 + v_0^2}}\,, \tag{2.34}$$

which occurs when $t = t_{\min} = v_0 R/(R^2\omega^2 + v_0^2)$. The particle reaches the radius again at

$$t_{\mathrm{R}} = 2t_{\min} = \frac{2v_0 R}{R^2\omega^2 + v_0^2}\,. \tag{2.35}$$

Now consider the case of a particle at the centre of the turntable given a velocity v_0 outwards. If we, arbitrarily choose this to be in the x-direction, then in the IF the solution is $\mathbf{r}(t) = (v_0 t, 0)$. However, in the NIF we can use the general solution to 2.29 which is

$$\mathcal{Z}(t) = \left[\mathcal{Z}(0) + \left(\dot{\mathcal{Z}}(0) + i\omega\mathcal{Z}(0)\right)t\right]e^{-i\omega t}\,, \tag{2.36}$$

now with $\mathcal{Z}(0) = 0$ and $\dot{\mathcal{Z}}(0) = v_0$. We deduce that $x' = v_0 t\cos\omega t$ and $y' = -v_0 t\sin\omega t$ which can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\,, \tag{2.37}$$

and we find that $r^2 = (v_0 t)^2 = r'^2$, that is the two observers agree how far the particle is from the centre, but the observer in the NIF thinks it is on a curved path and that in the IF thinks it is a straight line.

END OF 4TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 4 Exercises

1. Show that $\frac{1}{m}\mathbf{F}_{\mathrm{cen}} = \omega^2\mathbf{r}'$ when the rotation is perpendicular to the position vector $\mathbf{r}'$ in the rotating frame of reference.

2. A particle of mass m is placed a distance R from the centre of a smooth turntable which rotates with constant angular velocity ω. The particle is given an initial velocity v_0 with respect to the table in a direction perpendicular to the radial direction. What is the direction and magnitude of the inertial force on the particle at this initial point?

3. Derive an expression for the shape of the surface of water in a bucket which rotates with an angular velocity ω about a vertical axis that is through the centre of the bucket.

Hint: consider a small element of water and deduce the force acting upon it in its rest frame. This force must be normal to the water surface.

START OF 5TH LECTURE

2.3 The Earth as a rotating frame - see FS8.3

We now come on to an application of the rotating frame formalism where it is easier to think in terms of the rotating frame - which was not the case in the previous two examples. The motion in the gravitational field of the earth is possibly the most common scenario where this formalism can be used. The set up that we will use is shown in the lecture slides. The Cartesian basis vectors will correspond to x=East, y =North and z will be "up" in the local coordinate system. This is the coordinate system which you would define relative the ground. If λ is the latitude then the angular velocity vector is

$$\boldsymbol{\omega} = \omega \begin{pmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{pmatrix} . \tag{2.38}$$

The rotation speed will be $\omega_{\mathrm{E}} = 2\pi/(1\mathrm{day}) = 7.3 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$ and the radius of the Earth is $R_{\mathrm{E}} = 6371\,\mathrm{km}$. Interestingly we will find that the Coriolis force is the most important in this context, but first let us consider the effect of the centrifugal forces.

2.3.1 Centrifugal force as a modification to g

Consider a pendulum at rest - which means that the Coriolis force will be zero; in this case the centrifugal force will contribute to the force acting on the particle, adding to the gravitational force but in the opposite direction. However, it will not be vertical, so the pendulum will be at rest at an angle to the vertical, α as shown in the lecture slides. To calculate this contribution the "effective g" let us define

$$\mathbf{r}' = \begin{pmatrix} 0 \\ 0 \\ R_{\mathrm{E}} \end{pmatrix} , \tag{2.39}$$

(note that here we are assuming that the height of the pendulum is negligible relative to the radius of the Earth) and so

$$\frac{1}{m}\mathbf{F}_{\mathrm{cen}} = \omega^2\mathbf{r}' - (\mathbf{r}'\cdot\boldsymbol{\omega})\boldsymbol{\omega} = \omega^2 R_{\mathrm{E}} \begin{pmatrix} 0 \\ -\sin\lambda\cos\lambda \\ \cos^2\lambda \end{pmatrix} . \tag{2.40}$$

The effective gravitational acceleration vector is given by

$$\mathbf{g}_{\text{eff}} = \mathbf{g} + \frac{1}{m}\mathbf{F}_{\text{cen}} = \begin{pmatrix} 0 \\ -\omega^2 R_{\text{E}} \sin\lambda\cos\lambda \\ -g + \omega^2 R_{\text{E}}\cos^2\lambda \end{pmatrix} . \tag{2.41}$$

and so

$$|\mathbf{g}_{\text{eff}}| = g\left(1 - \frac{2\omega^2 R_{\text{E}}}{g}\cos^2\lambda + \left(\frac{\omega^2 R_{\text{E}}}{g}\right)^2\cos^2\lambda\right)^{\frac{1}{2}} \approx g\left(1 - \frac{\omega^2 R_{\text{E}}}{g}\cos^2\lambda\right) , \tag{2.42}$$

Moreover, we can calculate α from

$$\tan\alpha = \frac{R_{\text{E}}\omega^2}{g}\sin\lambda\cos\lambda\left(1 - \frac{R_{\text{E}}\omega^2}{g}\cos^2\lambda\right)^{-1} , \tag{2.43}$$

which can be approximated by

$$\alpha \approx \frac{R_{\text{E}}\omega^2}{g}\sin\lambda\cos\lambda . \tag{2.44}$$

We have that $\omega^2 R_{\text{E}}/g \approx 3.5 \times 10^{-3}$ and therefore $\alpha_{\text{max}} \approx 0.1^\circ$ at $\lambda = 45^\circ$ and the modification to g is $\approx 0.2\%$. Neither of these is particularly significant but would have to be taken into account in a high precision measurement.

2.3.2 Particle falling to Earth

Now consider a particle at rest a height h above the surface of the Earth. The initial conditions are

$$\mathbf{r}(0) = \begin{pmatrix} 0 \\ 0 \\ R_{\text{E}} + h \end{pmatrix} , \quad \dot{\mathbf{r}}(0) = \mathbf{0} , \tag{2.45}$$

and the equation of motion include the effects of gravity and the inertial forces

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \omega^2\mathbf{r} - (\mathbf{r}\cdot\boldsymbol{\omega})\boldsymbol{\omega} . \tag{2.46}$$

where we have suppressed the $'$ - we are in the NIF.

If

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} , \tag{2.47}$$

then the Coriolis force is given by

$$\frac{1}{m}\mathbf{F}_{\text{cor}} = -2\boldsymbol{\omega} \times \dot{\mathbf{r}} = -2\omega\begin{pmatrix} \dot{z}\cos\lambda - \dot{y}\sin\lambda \\ \dot{x}\sin\lambda \\ -\dot{x}\cos\lambda \end{pmatrix} . \tag{2.48}$$

The timescale for the Earth's rotation is 1 day and this is much longer than the free fall timescale, that is the time taken to hit the ground in the absence of rotation, $t_{\rm F} = \sqrt{2h/g}$. For this reason it is helpful to define the dimensionless time $T = \omega t$ and if we also define $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}/\omega$ to be the unit vector in the direction of the angular velocity vector, then the equation of motion can be rewritten as

$$\frac{d^2\mathbf{r}}{dT^2} = \frac{\mathbf{g}}{\omega^2} - 2\hat{\boldsymbol{\omega}} \times \frac{d\mathbf{r}}{dT} + \mathbf{r} - (\mathbf{r}\cdot\hat{\boldsymbol{\omega}})\hat{\boldsymbol{\omega}}\,, \tag{2.49}$$

or in component form

$$\begin{aligned}
\frac{d^2x}{dT^2} &= -2\left(\frac{dz}{dT}\cos\lambda - \frac{dy}{dT}\sin\lambda\right) + x\,,\\
\frac{d^2y}{dT^2} &= -2\frac{dx}{dT}\sin\lambda + y - \cos\lambda\left(y\cos\lambda + z\sin\lambda\right)\,,\\
\frac{d^2z}{dT^2} &= -\frac{g}{\omega^2} + 2\frac{dx}{dT}\cos\lambda + z - \sin\lambda\left(y\cos\lambda + z\sin\lambda\right)\,.
\end{aligned} \tag{2.50}$$

Solving these equations in an exact form is difficult, although I will present a solution later on, but this is not necessary since for most realistic situations $T \ll 1$, and therefore it is possible to solve as a power series. This is in turn the solution of a differential equation into the solution of a system of algebraic equations. The power series form that we will use is

$$\mathbf{r}(T) = \mathbf{r}(0) + T\frac{\dot{\mathbf{r}}(0)}{\omega} + \mathbf{A}T^2 + \mathbf{B}T^3 + .. = \begin{pmatrix} 0 \\ 0 \\ R_{\rm E} + h \end{pmatrix} + \mathbf{A}T^2 + \mathbf{B}T^3 + \ldots \tag{2.51}$$

The objective is to substitute this expression in to differentiatial equation and then solve for $\mathbf{A} = (A_1, A_2, A_3)$, $\mathbf{B} = (B_1, B_2, B_3)$. One finds that

$$\begin{aligned}
2A_1 + 6B_1T + .. &= -2(2A_3\cos\lambda - 2A_2\sin\lambda)T + ..\,,\\
2A_2 + 6B_2T + .. &= -2(2A_1T)\sin\lambda - (R_{\rm E}+h)\sin\lambda\cos\lambda + ..\,,\\
2A_3 + 6B_3T + .. &= -\frac{g}{\omega^2} + 2(2A_1T)\cos\lambda + (R_{\rm E}+h)(1-\sin^2\lambda)\,,
\end{aligned} \tag{2.52}$$

and hence we can deduce by comparing the constant terms and terms $\propto T$ on each side of the equation that $A_1 = B_2 = B_3 = 0$ and

$$\begin{aligned}
A_2 &= -\frac{1}{2}\left(R_{\rm E}+h\right)\cos\lambda\sin\lambda\,,\\
A_3 &= -\frac{g}{2\omega^2} + \frac{1}{2}(R_{\rm E}+h)\cos^2\lambda\,,
\end{aligned}$$

$$B_1 = \frac{1}{3}\left[\frac{g}{\omega^2} - (R_\mathrm{E} + h)\right]\cos\lambda\,. \tag{2.53}$$

Hence, we can deduce the power series solution

$$\begin{aligned} x &\approx \frac{1}{3}g\omega t^3\left(1 - \frac{(R_\mathrm{E}+h)\omega^2}{g}\right)\cos\lambda\,, \\ y &\approx -\frac{1}{2}(R_\mathrm{E}+h)\omega^2 t^2\cos\lambda\sin\lambda\,, \\ z &\approx R_\mathrm{E} + h - \frac{1}{2}gt^2\left(1 - \frac{(R_\mathrm{E}+h)\omega^2}{g}\cos^2\lambda\right)\,, \end{aligned} \tag{2.54}$$

which is accurate up to order T^3.

The power series can be simplified by assuming that $(R_E + h)\omega^2/g \ll 1$ - which is equivalent to ignoring the centrifugal force (notice that they have the same form as we found when discussing the static pendulum) -

$$\begin{aligned} x &\approx \frac{1}{3}g\omega t^3\cos\lambda\,, \\ y &\approx 0\,, \\ z &\approx R_\mathrm{E} + h - \frac{1}{2}gt^2\,. \end{aligned} \tag{2.55}$$

One can work out the time to reach the ground $t_\mathrm{G} = \sqrt{2h/g}$ by setting $z = R_\mathrm{E}$ which is unchanged by the rotation at this level of approximation. We also have that $y \approx 0$, that is there is no motion in the Northerly direction, but there is a motion the Easterly direction

$$\Delta x = \frac{1}{3}g\omega\left(\frac{2h}{g}\right)^{\frac{3}{2}}\cos\lambda\,. \tag{2.56}$$

The physical interpretation in the NIF is that the Coriolis force, acting perpendicular to the motion and the angular velocity vector, has caused a deviation in the trajectory in the Easterly direction. In the IF the particle initially has a velocity in the Easterly direction since the Earth rotates West to East, but the key physics point is that the angular momentum is conserved, so that as it falls toward the Earth the radius measured from the centre is reduced and the particle's velocity in this direction must be increased, and therefore the particle falls in front of the tower as it also moves in the Easterly direction. The calculation that we have done has shown that it is sufficient to ignore the centrifugal force (as we would have anticipated from our earlier calculations showing that the change in g is less than 1%), and that one can ignore the corrections to the time taken for the particle to hit the ground so long this is much less than the rotational timescale - 1 day in the case of the Earth's rotation.

Putting some numbers into the expression for Δx: $\lambda = 53^\circ$ (Manchester) $g = 9.8\,\mathrm{m\,s^{-2}}$, $h = 100\,\mathrm{m}$ (for example, the height of the Beetham Tower) and $\omega = \omega_{\mathrm{E}} = 7.3 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$, we fin that $t_G = (200/9.8)^{1/2}\mathrm{s} \approx 4.5\,\mathrm{s}$, and

$$\Delta x = \frac{1}{3} \times 0.6 \times 9.8\,\mathrm{ms^{-2}} \times 7.3 \times 10^{-5}\,\mathrm{rad\,s^{-1}} \times (4.5\mathrm{s})^3 \approx 1.3\,\mathrm{cm}\,. \tag{2.57}$$

One can actually find an exact solution to the equations (2.50) which you are asked to derive - with some hints - on the problem sheet. The form of the solution is

$$\begin{aligned} x &= \left(\frac{g}{\omega^2} - R_{\mathrm{E}} - h\right) f_1(\omega t)\cos\lambda\,, \\ y &= \cos\lambda\sin\lambda\left[\left(\frac{g}{\omega^2} - R_{\mathrm{E}} - h\right) f_2(\omega t) - \frac{1}{2}gt^2\right]\,, \\ z &= R_{\mathrm{E}} + h - \frac{1}{2}gt^2 \\ &\quad - \cos^2\lambda\left[\left(\frac{g}{\omega^2} - R_{\mathrm{E}} - h\right) f_2(\omega t) - \frac{1}{2}gt^2\right]\,, \end{aligned} \tag{2.58}$$

where $f_1(x) = \sin x - x\cos x$ and $f_2(x) = \cos x - 1 + x\sin x$. One of the examples below is to show that one can re-derive the power series solution by expanding the functions for $f_1(x)$ and $f_2(x)$. In the end this solution is not of much practical use in the context of the rotation of the earth since the $\omega t_{\mathrm{G}} \ll 1$ and hence the power series calculation is sufficient, but it could be of interest in a system where g was much smaller and the rotation period much shorter.

END OF 5TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 5 Exercises

1. The work done by a force is the line integral $W = \int \mathbf{F}\cdot d\mathbf{r}$ along the path $\mathbf{r}(t)$. Show that the work done by the Coriolis force is zero.

2. Derive the power series solution for a particle at rest falling in a gravitational field on the Earth as presented in the lectures from the exact solution also given in lectures in the approximation where ωt is small. You will need to use $\cos x = 1 - x^2/2 + ..$ and $\sin x = x - x^3/6 + ...$‘

3. Consider a particle with initial position $\mathbf{r}_0$ and initial velocity $\mathbf{v}_0$ moving under the action of a time and space independent force $\mathbf{F}_0$. Find a power series solution for $\mathbf{r}(t)$ in the dimensionless quantity $T = \omega t$ upto third order. Show that for the special case of $\mathbf{v}_0 = 0$ and $\mathbf{r}_0 = (R_{\mathrm{E}} + h)\mathbf{e}_z$ this yields the power series derived in lectures.

Section 2 in a Nutshell - what to remember

- For two coordinate systems related by $\mathbf{r}' = \mathbf{r} - \mathbf{R}$ there is an inertial force $\mathbf{F}_{\text{inertial}} = -m\ddot{\mathbf{R}}$.
- $\dot{\mathbf{e}}_i' = \boldsymbol{\omega} \times \mathbf{e}_i'$ for rotating coordinate system.
- $\dot{\mathbf{A}}|_S = \dot{\mathbf{A}}|_{S'} + \boldsymbol{\omega} \times \mathbf{A}$.
- General expression for the inertial force in an accelerating and rotating frame:

$$\mathbf{F}_{\text{inertial}} = -m\ddot{\mathbf{R}}|_S - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2m\boldsymbol{\omega} \times \mathbf{v}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')\,. \tag{2.59}$$

- $\mathbf{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}'$.
- $\mathbf{F}_{\text{cen}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$.
- Coriolis force is perpendicular to the motion and the angular velocity vector.
- In the context of the rotation of the Earth the centrifugal force is responsible for a 0.2% modification to g and a static pendulum will not be vertical - $\alpha_{\text{max}} \approx 0.1°$.
- To first approximation, a particle falling from a height h above the surface of the earth an latitude λ will deviate a distance $\Delta x = \frac{1}{3} g\omega \left(\frac{2h}{g}\right)^{\frac{3}{2}} \cos\lambda$ in the Easterly direction.