<u>ONE HOUR THIRTY MINUTES</u>

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Advanced Quantum Mechanics

28 May 2019, 09:45 - 11:15

Answer any **<u>TWO</u>** questions

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

You may assume the following formulae in any question *if proof is not explicitly requested*:

Special integrals

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2a)^n} \sqrt{\frac{\pi}{a}}, \qquad \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}, \quad a > 0.$$

4-vectors
Use notation $x^\mu = (x^0, \mathbf{x})$ for 4-position and $p^\mu = (E/c, \mathbf{p})$ for 4-momentum. In electromagnetism $A^\mu = (\Phi/c, \mathbf{A})$ for 4-potential.

Klein-Gordon equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[(\hat{\mathbf{p}} - \mathbf{V})^2 c^2 + (mc^2 + S)^2\right] \Psi = (i\hbar\partial_t - V_0)^2 \Psi.$$

Dirac equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[\boldsymbol{\alpha} \cdot (\hat{\mathbf{p}} - \mathbf{V})c + \beta(mc^2 + S)\right] \Psi = (i\hbar\partial_t - V_0)\Psi.$$

Standard spinor solutions to the free massive Dirac equation

$$u^{(s)}(E, \mathbf{p}) = \sqrt{E + mc^2} \begin{pmatrix} \phi^s \\ \frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E+mc^2}\phi^s \end{pmatrix}, \quad \phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$v^{(s)}(-|E|, -\mathbf{p},) = \sqrt{|E| + mc^2} \begin{pmatrix} \frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{|E|+mc^2}\chi^s \\ \chi^s \end{pmatrix}, \quad \chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \chi^2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Standard matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix};$$

$$\gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}.$$

Space-time metric We use the metric $\eta_{\mu\nu} = \text{diag}(1, -1, -, 1, -1)$.

Laplacian operator in terms of angular momentum operator $\hat{\mathbf{L}}$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\hat{\mathbf{L}}^2}{\hbar^2 r^2}.$$

Solution to Dirac equation in spherical vector potential $(V_0(r)/c, \mathbf{0})$

$$\psi^\kappa_{jm}(\mathbf{r}) = \begin{pmatrix} f^\kappa_j(r)\mathcal{Y}^\kappa_{jm}(\theta, \phi) \\ i g^\kappa_j(r)\mathcal{Y}^{-\kappa}_{jm}(\theta, \phi) \end{pmatrix} \quad \text{with } F = rf \text{ and } G = rg,$$

$$\hbar c \left[\frac{d}{dr} - \frac{\kappa}{r}\right] G^\kappa_j(r) = -\left(E - V_0(r) - mc^2\right) F^\kappa_j(r),$$

$$\hbar c \left[\frac{d}{dr} + \frac{\kappa}{r}\right] F^\kappa_j(r) = \left(E - V_0(r) + mc^2\right) G^\kappa_j(r).$$

1. a) Consider a spinless non-relativistic particle of charge q and mass m moving in a space with a 4-potential $(\Phi/c, \mathbf{A})$.

i) The 4-potential transforms under a gauge transformation according to $\mathbf{A} \to \mathbf{A} + \nabla\lambda$, $\Phi \to \Phi - \partial\lambda/\partial t$, where λ is a function of space and time. Write down the corresponding gauge transformation operator for the wave function of the particle, and demonstrate that the operator is unitary. Show that the time-dependent Schrödinger equation of the particle is invariant under the gauge transformation.

[5 marks]

ii) Now consider the particle moving in a space with a uniform magnetic field $\mathbf{B}$ described by the 4-potential $(0, \mathbf{A})$ where $\mathbf{A} = (0, xB, 0)$. Write down the time-independent Schrödinger equation of the particle. By using a wave function of the form

$$\psi(\mathbf{r}) = e^{ik_y y + ik_z z}\phi(x)$$

with constants k_y and k_z, find and solve the equation for $\phi(x)$, giving the energy eigenvalues of the particle.

[6 marks]

b) i) The two components of the angular momentum operator with quantum number $l = 1$ can be written as

$$\hat{L}_z = \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}, \quad \hat{L}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Write down the eigenvalues and eigenvectors of $\hat{L}_z$. Use the angular momentum commutation relations to derive the $\hat{L}_y$ matrix.

[4 marks]

ii) Consider a charged quantum rotor of angular-momentum quantum number $l = 1$. The rotor is placed in a uniform magnetic field along the z-axis with the Hamiltonian given by

$$\hat{H}_0 = \alpha \hat{L}_z,$$

where α is a positive constant. Initially the rotor is in the ground state of $\hat{H}_0$. At $t = 0$, a weak rotating magnetic field of angular frequency ω is switched on in the xy plane. The Hamiltonian of the rotor is now

$$\hat{H}(t) = \hat{H}_0 + \beta\left(\cos\omega t\, \hat{L}_x + \sin\omega t\, \hat{L}_y\right),$$

where β is a constant ($|\beta| \ll \alpha$). Use first-order perturbation theory to calculate the probability of transition to the first excited state as a function of time. Find the resonant frequency. Determine the transition probability at the resonant frequency and comment on the validity of your result in the long-time limit in this case.

[10 marks]

2. a) List all the excited states of helium for the configurations $(1s)(nl)$ with $n = 1, 2, 3$. Write down the corresponding term symbols ^{2S+1}L for these states. Sketch the corresponding energy levels and mark all the possible electric dipole transitions.

[7 marks]

b) A spin 1/2 particle of mass m, subjected to a modified vector potential, is described by the following Dirac equation:

$$\left[c\boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + imc\omega(\beta\boldsymbol{\alpha})\cdot\mathbf{r} + mc^2\beta\right]\Psi = E\Psi, \text{ where } \Psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix}.$$

i) Show that ϕ and χ satisfy the following equations

$$\begin{aligned}(E^2 - m^2c^4)\phi &= 2mc^2\left[\frac{\hat{\mathbf{p}}^2}{2m} + \frac{m\omega^2}{2}\mathbf{r}^2 - \frac{3}{2}\hbar\omega - 2\frac{\omega}{\hbar}\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}\right]\phi,\\ (E^2 - m^2c^4)\chi &= 2mc^2\left[\frac{\hat{\mathbf{p}}^2}{2m} + \frac{m\omega^2}{2}\mathbf{r}^2 + \frac{3}{2}\hbar\omega + 2\frac{\omega}{\hbar}\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}\right]\chi,\end{aligned}$$

where $\hat{\mathbf{S}} = \hbar\boldsymbol{\sigma}/2$.

[8 marks]

ii) We can represent ϕ as $|n, l, s, j\rangle$ and χ as $|n', l', s, j\rangle$, where these are spin-1/2 eigenstates of a *three-dimensional* harmonic oscillator

$$\left(\frac{\hat{\mathbf{p}}^2}{2m} + \frac{m\omega^2}{2}\mathbf{r}^2\right)|n, l, s, j\rangle = \hbar\omega\left(n + \frac{3}{2}\right)|n, l, s, j\rangle,$$

where l has the following allowed values:

$$l = \begin{cases} 0, 2, \cdots, n & \text{for even } n,\\ 1, 3, \cdots, n & \text{for odd } n.\end{cases}$$

Find the energy eigenvalues for the two equations satisfied by ϕ and χ *separately* in terms of n, n', l, l' and j.

[3 marks]

iii) By considering $n' = n \pm 1$, what are the relations between n and n', and l and l', in order that Ψ is a solution of the given Dirac equation? (Hint: there are two possibilities.)

[5 marks]

iv) The eigenenergies all have the form

$$E = mc^2\sqrt{1 + \frac{2\hbar\omega N}{mc^2}}.$$

Determine the degeneracy of the eigenstates with even N.

[2 marks]

3. a) Show that the angular momentum operator $\hat{L}_x$ is the generator of rotations about the x-axis for a spinless particle. Derive an expression in terms of $\hat{L}_x$ for the operator $\hat{U}_\beta$ representing a finite rotation of angle β about the x-axis. Show that the operator $\hat{U}_\beta$ is unitary.

[8 marks]

Write down the corresponding $\hat{U}_\beta$ for a particle with spin, and explain any notation you use.

[2 marks]

b) Consider a relativistic spinless particle of mass m interacting with a weak scalar potential $S(x) = g\delta(x)$, where g is the strength of the potential. The time-independent Klein-Gordon equation can be written approximately as

$$-\hbar^2 c^2 \frac{d^2\psi(x)}{dx^2} + m^2c^4\psi(x) + 2mc^2 g\delta(x)\psi(x) = E^2\psi(x).$$

i) Show that the generic solutions for the wavefunction can be written as

$$\begin{aligned} \psi_-(x) &= A_- e^{-ikx} + B_- e^{ikx}, \quad x < 0, \\ \psi_+(x) &= A_+ e^{-ikx} + B_+ e^{ikx}, \quad x > 0, \end{aligned}$$

where $A_\pm$ and $B_\pm$ are constants. Express k in terms of the energy E.

[2 marks]

ii) Write down the continuity boundary condition for the wave functions at $x = 0$. By integrating the Klein-Gordon equation across $x = 0$, show that the other boundary condition is given by

$$\lim_{\epsilon \to 0} \left[-\hbar^2 c^2 \left(\left. \frac{d\psi_+(x)}{dx} \right|_{x=+\epsilon} - \left. \frac{d\psi_-(x)}{dx} \right|_{x=-\epsilon} \right) \right] + 2mc^2 g\psi_+(0) = 0.$$

[3 marks]

iii) Assume that the particle is incident from the negative x direction with $E > mc^2$. Solve the resulting boundary conditions to find the transmission amplitude B_+ and the reflection amplitude A_- in terms of k.

[5 marks]

iv) Now consider the case of the bound state. Assume that the energy satisfies $|E| < mc^2$, and let $k = i\kappa$ where $\kappa > 0$. Explain why $A_+ = B_- = 0$ in this case. Solve the boundary conditions and show that a bound state exists only if $g < 0$. Find the bound-state energy E in terms of g. Determine its nonrelativistic limit.

[5 marks]

END OF EXAMINATION PAPER