PHYS20672 Complex Variables and Vector Spaces: Examples 9

1. Use the appropriate Cauchy integral formula to evaluate the following, where C_1 is a circle with $|z| = 1$ and C_2 is a square with corners at $\pm 2, \pm 2 + 4i$.

$$\text{(a)} \oint_{C_1} \frac{e^{3z}}{z}\,\mathrm{d}z \qquad \text{(b)} \oint_{C_1} \frac{\cos^2(2z)}{z^2}\,\mathrm{d}z \qquad \text{(c)} \oint_{C_1} \frac{\sin^2(2z)}{z^2}\,\mathrm{d}z$$

$$\text{(d)} \oint_{C_2} \frac{z^2}{z-2i}\,\mathrm{d}z \qquad \text{(e)} \oint_{C_2} \frac{z^2}{z^2+4}\,\mathrm{d}z$$

2. Show that

$$\left|\frac{1}{z^2+1}\right| \le \frac{1}{R^2-1} \qquad \text{for } |z| = R > 1.$$

Hence use the estimation lemma to show that

$$\lim_{R\to\infty} \oint \frac{1}{z^2+1}\,\mathrm{d}z = 0 \quad \text{for the circular path } |z| = R,$$

and explain why the same result (zero) will be found for any finite $R > 1$. Verify the result by using Cauchy's integral formula.

3. By writing $z = e^{i\theta}$ (and hence $\mathrm{d}z = ie^{i\theta}\mathrm{d}\theta$), and using formulae such as $\cos\theta = \frac{1}{2}(z + z^{-1})$, convert the following to contour integrals around the unit circle and evaluate using the appropriate Cauchy integral formulae:

$$\text{(a)} \int_0^{2\pi} \cos^4\theta\,\mathrm{d}\theta \qquad \text{(b)} \int_0^{2\pi} \sin^6\theta\,\mathrm{d}\theta$$

$$\text{(c) [harder]} \int_0^{2\pi} \cos^{2n}\theta\,\mathrm{d}\theta, \quad \text{for integer } n \ge 0$$

$$\text{(d)} \int_0^{2\pi} \frac{\cos\theta}{4\cos\theta - 5}\,\mathrm{d}\theta \qquad \text{(e)} \int_0^{2\pi} \frac{\cos 2\theta}{3\cos\theta + 5}\,\mathrm{d}\theta$$

In (c), you should be able to express your answer as $2\pi(2n-1)!!/(2n)!!$, where, e.g., $7!! = 7 \times 5 \times 3 \times 1$ and $8!! = 8 \times 6 \times 4 \times 2$ (though it's fine to leave the answer in terms of a binomial coefficient).

4. Without any calculation, explain why the Taylor expansion of $\tan z$ about $z = \pi/4$ must have a radius of convergence equal to $\pi/4$.

[Harder] Find the first four terms of the Taylor expansion of $\tan z$ about $z = \pi/4$. (Note that $\tan z$ is analytic near $z = \pi/4$, so you can use the usual expression for the coefficients in terms of derivatives of the function.)

5. If z_0 is a non-zero complex number, find the Taylor series of $f(z) = 1/(z - z_0)$ about $z = 0$, and explain why its radius of convergence is $|z_0|$. Check that you can sum the Taylor series to get back $f(z)$.

Use the binomial expansion of $f(z)$ to obtain a series expansion in powers of z^{-1}, which is valid in the region $|z| > |z_0|$. This is an example of a Laurent series, to be discussed in a later lecture.