ADVANCED DYNAMICS PHYS10672 2024 : Prof R.A. Battye

EXAMPLES SHEET 1 : Maths/Revision and Non-inertial frames

1. Index notation and summation convention

Use the expression for matrices in index notation and the summation convention to show that
(a) $(AB)^T = B^T A^T$,
(b) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

2. Motion in 2D polar coordinates

A particle of mass m moves in two dimensions and initially moves in the tangential direction with a velocity $R\omega$ where R is the distance from the origin. It experiences a force in the tangential direction, $\hat{\theta}$, whose magnitude is $2m\omega\dot{r}$. Show that $r = R\cosh\omega t$.

3. Cartesian frames moving parallel to each other

Let S_1 and S_2 be two Cartesian frames moving relative to one another with parallel axes. The motion of the particle in S_1 is described by the position vector

$$\mathbf{r}_1(t) = (6a_1t^2 - 4a_2t)\mathbf{e}_1 - 3a_3t^3\mathbf{e}_2 + 3a_4\mathbf{e}_3\,,$$

and in S_2 by

$$\mathbf{r}_2(t) = (6a_1t^2 + 3a_2t)\mathbf{e}_1 - (3a_3t^3 - 11a_5)\mathbf{e}_2 + 4a_6t\mathbf{e}_3$$

where the a_i are constants for $i = 1, .., 6$.

(a) What is the velocity of S_1 relative to S_2?

(b) Calculate the acceleration that the particle experiences in the two Cartesian frames.

(c) If S_1 is an inertial frame is S_2 also an intertial frame?

4. Coriolis force

An ant crawls with constant velocity v outward along the spoke of a wheel which is rotating with constant angular velocity ω about the vertical axes. Show that, when the ant is at a distance ℓ from the centre of the wheel, the minimum frictional force needed to keep the ant moving has magnitude $F_{\min} = m\omega\sqrt{4v^2 + \ell^2\omega^2}$.

5. Simple solution for the motion of a particle on the Earth

The equations of motion for a particle moving on Earth as described in the lectures are

$$\begin{aligned}
\ddot{x} &= -2\omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda) + \omega^2 x\,, \\
\ddot{y} &= -2\omega\dot{x}\sin\lambda + \omega^2 y - \omega^2\cos\lambda(y\cos\lambda + z\sin\lambda)\,, \\
\ddot{z} &= -g + 2\omega\dot{x}\cos\lambda + \omega^2 z - \omega^2\sin\lambda(y\cos\lambda + z\sin\lambda)\,,
\end{aligned}$$

where ω is angular rotation of the Earth and λ is the latitude. By ignoring the terms due to the centrifugal force and assuming that $\dot{x}, \dot{y} \ll \dot{z}$, deduce that a particle initially at rest, a distance h above the surface of the Earth moves a distance

$$\Delta x = \frac{1}{3} g\omega \left(\frac{2h}{g}\right)^{3/2} \cos\lambda \,,$$

in the easterly direction before it hits the ground. Explain why this very simple approximation gives a good estimate, and indeed why it is the same to that found in lectures from the power series solution when $(R_{\rm E} + h)\omega^2/g \ll 1$.

6. Exact solution for motion on the Earth

In this question we will derive the exact solution to the system of equations in question 5. We do this by first defining

$$\begin{aligned} \alpha &= y\cos\lambda + z\sin\lambda \\ \beta &= -y\sin\lambda + z\cos\lambda \,, \end{aligned}$$

and deriving equations of motion for x, α and β. Show that the solution for α is

$$\alpha = \sin\lambda \left(R_{\rm E} + h - \frac{1}{2} g t^2\right) \,,$$

and that the complex valued quantity $\mathcal{Z} = \beta + ix$ satisfies

$$\ddot{\mathcal{Z}} + 2i\omega\dot{\mathcal{Z}} - \omega^2 \mathcal{Z} + g\cos\lambda = 0 \,.$$

Hence, deduce the exact solution presented on the lecture slides.

Note that the solution to the differential equation

$$\ddot{\mathcal{Z}} + 2i\omega\dot{\mathcal{Z}} - \omega^2 \mathcal{Z} = 0 \,,$$

is $\mathcal{Z} = (A + Bt)\exp[-i\omega t]$.

7. Larmor precession

The force on a particle with mass m and charge q moving with velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and in a magnetic field $\mathbf{B}$ is given by the Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \,.$$

Consider the motion of this particle from the viewpoint of a rotating frame of reference. Show that with an appropriate choice of angular velocity, the acceleration of the particle in the rotating frame can be written in the form

$$m\mathbf{a}' = q\mathbf{E}_{\rm eff} \,,$$

where $\mathbf{E}_{\text{eff}}$ is the effective electric field in this frame which depends on the particle's position but not its velocity. Hint: you will need to rewrite $\mathbf{v}'$ in terms of the velocity in the rotating frame.

In classical physics an electron can move around a proton in an elliptical orbit with angular velocity $\boldsymbol{\Omega}$. Describe the effect of a weak magnetic field on $\boldsymbol{\Omega}$. You should assume that the field is weak enough that terms of order $|\mathbf{B}|^2$ can be neglected. Hint: First ask how does the orbit look in a frame rotating so as to eliminate the magnetic field. Then ask how it will appear in the inertial frame. Note that $\boldsymbol{\Omega}$ need not be aligned with the direction of the magnetic field.

8. Bead on a hoop

A bead of mass m is constrained to move on a hoop of radius R. The hoop rotates with a constant angular speed ω about an axis which coincides with the diameter of the hoop held in the vertical direction.

(a) Obtain the equations of motion for the bead by applying Newton's 2nd law in the rest frame of the hoop, ignoring the effects of friction.

(b) Find the critical angular speed, Ω, below which the bottom of the hoop provides a stable equilibrium position for the bead.

(c) Find the stable equilibrium position for $\omega > \Omega$.

9. Impact of Coriolis force on naval gun trajectories

Early in World War I, a naval engagement took place near the Falkland Islands. During it the British gunners were surprised to see their accurately aimed salvos falling 100 yards to the left of the German ships. Suggest a reason for this. Back up your suggestion with a rough order-of-magnitude estimate, or a more detailed calculation using the equations of motion if you are able. Note that the Falkland Islands are at 50°S and the muzzle speed of the shells was $\approx 500\,\text{ms}^{-1}$.

10. Conserved quantity and EOM in a rotating frame

The equation of motion for a particle moving under gravity in a rotating frame with a constant angular velocity ω is

$$\ddot{\mathbf{r}}' = \mathbf{g} - 2\omega \times \dot{\mathbf{r}}' + \omega^2 \mathbf{r}' - (\mathbf{r}' \cdot \omega)\omega\,.$$

(a) Derive and solve the equation of motion for $\mathbf{r}_{||} = \mathbf{r}' \cdot \hat{\omega}$ where $\hat{\omega}$ is a unit vector in the direction of ω.

(b) By evaluating the dot product of this equation with $\dot{\mathbf{r}}'$ show that

$$\frac{1}{2}|\dot{\mathbf{r}}'|^2 - \mathbf{g} \cdot \mathbf{r}' - \frac{1}{2}\mathbf{r}'^T P \mathbf{r}'\,,$$

is a constant where

$$P = I - \hat{\omega}\hat{\omega}^T\,,$$

is a matrix with components $P_{ij} = \delta_{ij} - \omega_i \omega_j$. Show that $P^2 = P$. .