ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Vector Spaces

25th May 2018, 09.45 a.m. - 11.15 a.m.

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. (a) For a function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, state the Cauchy-Riemann conditions for $f(z)$ to be analytic. Show that they are satisfied for $f(z) = e^{z^2}$.

[6 marks]

(b) Evaluate $\int_C \overline{z}\,\mathrm{d}z$, where the end points are $a = i$ and $b = 1$, and C is a straight-line path connecting the two.

[6 marks]

(c) For the function

$$f(z) = \frac{z}{z^2 - 3z + 1} = \frac{2}{z-2} - \frac{1}{z-1},$$

find the Laurent series about $z = 0$ that is valid in the annulus $1 < |z| < 2$.

[5 marks]

(d) The scalar product of two real functions $u(x)$ and $v(x)$, where $x \in [-1, 1]$, is defined by

$$\langle u|v\rangle = \int_{-1}^{1} u(x)v(x)\,\mathrm{d}x.$$

Let $p_n(x)$ be polynomials of degree n in x. These polynomials form an orthonormal set of functions on the interval $[-1, 1]$. Find p_0, p_1 and p_2.

[8 marks]

2. (a) For an analytic function $f(z) = u + iv$, the Cauchy-Riemann conditions in polar form are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r},$$

where $z = re^{i\theta}$. Use these conditions to show that $u(r, \theta)$ satisfies Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

[6 marks]

(b) For the remaining parts of this question, the function $f(z) = u + iv$ has real part $u(r, \theta) = (r - a^2/r)\cos\theta$, where a is a constant.

(i) Given that $f(z)$ is analytic for $z \neq 0$, use the polar form of the Cauchy-Riemann conditions to find $v(r, \theta)$. Use your result to express f as a function of z.

[9 marks]

(ii) Show that u is a harmonic function for $r \neq 0$.

[3 marks]

(iii) An infinitely long conducting cylinder of radius a is placed in a uniform electric field $\mathbf{E} = E\hat{\mathbf{x}}$ with the axis of the cylinder perpendicular to the xy plane. Demonstrate that $\phi = -Eu$ is the electrostatic potential for this problem by showing that ϕ satisfies appropriate boundary conditions. Sketch the equipotentials and field lines.

[7 marks]

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3. (a) Use Cauchy's residue theorem and a suitable choice of contour to evaluate **one** of the following integrals. If Jordan's lemma is used, show that the conditions for its validity are satisfied.

either (i) $\displaystyle\int_0^{2\pi} \frac{\cos(2\theta)}{5-3\cos\theta}\,\mathrm{d}\theta$

or (ii) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(kx)}{x(1+x^2)}\,\mathrm{d}x$, where k is positive.

[13 marks]

(b) Find the poles of the function $f(z) = \cot(\pi z)/\left(z+\frac{1}{2}\right)^2$. By considering the residues of $f(z)$ at its poles, evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^2}.$$

[12 marks]

4. (a) Give short definitions of (i) the Hermitian conjugate (or adjoint) of a linear operator, (ii) an Hermitian operator, and (iii) a unitary operator.

[4 marks]

(b) State the general properties of the eigenvalues and eigenvectors of (i) an Hermitian operator and (ii) a unitary operator.

[6 marks]

(c) For each of the following matrices, state whether it is Hermitian, unitary, both, or neither:

$$\text{(i)}\ \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix};\ \text{(ii)}\ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix};\ \text{(iii)}\ \begin{pmatrix} 1 & 1+i \\ 1-i & 3 \end{pmatrix};\ \text{(iv)}\ \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

[4 marks]

Calculate the eigenvalues and eigenvectors of the matrix given in part (iv). Verify that they have any properties that should be expected of them.

[6 marks]

(d) A linear operator $\widehat{A}$ on a five-dimensional vector space is both Hermitian and unitary. Its trace is 1. By starting from the eigenvalue equation for $\widehat{A}$, or otherwise, find the eigenvalues of $\widehat{A}$ and their degeneracies.

[5 marks]

END OF EXAMINATION PAPER