ADVANCED DYNAMICS PHYS10672 2024 : Prof R.A. Battye

EXAMPLES SHEET 2 : Gravitation and conservative/central forces

1. Force fields and potentials

(a) A force field is

$$\mathbf{F}(x,y,z) = \frac{F_0}{R^2}\left[yz\mathbf{e}_x + zx\mathbf{e}_y + xy\mathbf{e}_z\right].$$

Show that $\mathbf{F}$ is a conservative force field and determine the potential $U(x,y,z)$.

(b) A point mass is moving under the influence of the force field

$$\mathbf{F}(x,y,z) = \frac{F_0}{R}\left[(y+z)\mathbf{e}_x + (z+x)\mathbf{e}_y + (x+y)\mathbf{e}_z\right].$$

Is this force field conservative? If so, calculate the corresponding potential $U(x,y,z)$.

2. Dust-filled solar system

If the Solar System was embedded in a uniform dust cloud of density ρ_0, what would be the force law on a planet at a distance r from the centre of the Sun?

Treating the Earth's orbit as circular, estimate the density of dust required to change the orbit period of 1 year by 1 day.

3. Gravitational potential due to a thin ring

Show that the gravitational potential at a point (x,y,z) due to a thin circular ring of radius R and mass M which is orientated in the x-y plane and centred at the origin is given by

$$\Phi(x,y,z) = -\frac{G_{\rm N}M}{2\pi}\int_0^{2\pi}\frac{d\theta}{\sqrt{(x-R\cos\theta)^2+(y-R\sin\theta)^2+z^2}}.$$

Deduce that the potential in the plane of the ring a distance $r(\gg R)$ from the centre is

$$\Phi(r) \approx -\frac{G_{\rm N}M}{r}\left(1+\frac{R^2}{4r^2}\right).$$

4. Binet's equation

We showed in lectures that the trajectory of system with reduce mass μ with energy E and angular momentum L under the influence of central potential $U(r)$ satisfies

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{L^2}\left[E - U\right] - u^2\,,$$

where $u(\theta) = 1/r(\theta)$. Show that u satisfies a 2nd-order differential equation and solve this equation for $U = -\alpha/r$ for a trajectory with an initial condition $u(\pi/2) = u_0 = \alpha\mu/L^2$ and $\frac{du}{d\theta} < 0$.

5. Orbits in a general potential

A particle of mass m moves under the influence of a central potential

$$U(r) = -\frac{\alpha}{r^n}\,,$$

where α and $n > 0$ with angular momentum L . Find the radius of circular orbits and show that they are stable for $n < 2$ and unstable for $n > 2$.

6. Equation for orbits in an approximation for General Relativity

Orbits within the framework of Einstein's General Relativity can be modelled using a potential per unit mass

$$-\frac{G_{\mathrm{N}}M}{r}\left(1 + \frac{3G_{\mathrm{N}}M}{c^2 r}\right)$$

for motion around a body of mass M. Using the approach from the lectures or that in question 4 (which is equivalent), derive the solution for $u(\theta)$ in this case and deduce that the periapsis rotates by an angle

$$\Delta\theta = 6\pi\frac{G_{\mathrm{N}}^2 M^2}{l^2 c^2}\,,$$

each turn of the ellipse where l is the angular momentum per unit mass.

7. Conic sections

(a) The equation for a cone centred at the origin in Cartesian coordinates (X, Y, Z) is $Z^2 = R^2(X^2 + Y^2)$ where R is a constant. Consider the intersection of the plane $AX + BZ = 1$ with the cone, and derive the conditions for it to be a (i) a circle, (ii) an ellipse, (iii) a parabola and (iv) a hyperbola.

(b) Show that you can derive the equation for an ellipse defining it as the set of points, P where the sum of the distances to focus point, F_1 and F_2, is equal to $2a$ where a is the semi-major axis. ie $|PF_1|+|PF_2|=2a$. Note that the positions of the focus points of the ellipse are $(\pm\sqrt{a^2-b^2},0)$ and b is the semi-minor axis.

(c) The semi-latus rectum of a conic section defined as the length of the chord from the focus to the section parallel to the directrix. Calculate this for the: (i) ellipse; (ii) hyperbola and (iii) parabola.

8. Satellite orbits

(a) Before the Moon landing, the Apollo 11 space vehicle was put into an elliptical orbit around the Moon. The period of the orbit was 119 minutes. The maximum and minimum distances firm the centre of the Moon were 1861 km and 1838 km. Find the mass of the Moon from this information.
(b) A satellite of mass 2000 kg is in an elliptical orbit around the Earth. At perigee it has a altitude of 1100 km and at apogee it is 4100 km.

(i) What is the minimum energy needed to put the satellite in orbit?
(ii) How fast is it travelling at perigee and apogee?
(iii) What is its angular momentum about the centre of the Earth?

9. Laplace-Runge-Lenz (LRL) vector

(a) Show that the LRL vector

$$\mathbf{A}=\mathbf{v}\times\mathbf{L}-\alpha\frac{\mathbf{r}}{r}\,,$$

is a constant of motion for a system with reduced mass μ in a central potential $U(r)=-\alpha/r$ with angular momentum $\mathbf{L}$. Remember from the lectures that $\dot{r}=\hat{\mathbf{r}}\cdot\dot{\mathbf{r}}=\hat{\mathbf{r}}\cdot\mathbf{v}$ and the exercises that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$.

(b) Show that

$$|\mathbf{A}|=\sqrt{\alpha^2+\frac{2L^2}{\mu}E}\equiv\alpha\varepsilon$$

where ε is the eccentricity of the conic section. Remember that $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$.

(c) Calculate the inner product $\mathbf{r}\cdot\mathbf{A}=rA\cos\theta$ and use the result in (b) to derive the general equation for conic sections

$$r=\frac{r_0}{1+\varepsilon\cos\theta}\,,$$

where $r_0=L^2/(\mu\alpha)$ is the semi-latus rectum.

(d) Show that the constant LRL vector points from the centre of mass to the pericentre (the point of closet approach).

10. Pendulum in multiple directions

(a) Initially consider a pendulum with a bob of mass m on a light inextensible string of length ℓ ignoring the effects of the Coriolis and centrifugal forces that is able to move in all 3 directions. Set up the system such that the bob of the pendulum is at position $\mathbf{r}$ and the string, which starts at the origin, has length ℓ, such that $|\mathbf{r}| = \ell$. Explain why

$$\ddot{\mathbf{r}} = -\frac{T}{m\ell}\mathbf{r} - g\hat{\mathbf{z}}\,,$$

where T is the magnitude of the tension force and g is the local gravitational field strength.

(b) Show that $\mathbf{r} \cdot \dot{\mathbf{r}} = 0$ and hence deduce a constant of motion.

(c) $\boldsymbol{\Omega}$ is the angular velocity vector

$$\boldsymbol{\Omega} = \dot{\phi}\,(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta) + \dot{\theta}\mathbf{e}_\phi\,,$$

for the angular coordinate system (θ, ϕ) describing the pendulum. Show that

$$\dot{\mathbf{r}}^2 = \ell^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\,.$$

(d) Calculate $\ddot{\mathbf{r}} \cdot \mathbf{e}_\phi$ and show that

$$\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta = 0\,,$$

and deduce that the energy per unit mass of the system is

$$\frac{E}{m} = \frac{1}{2}\dot{\theta}^2 + \frac{h^2}{2\sin^2\theta} + \frac{g}{\ell}\cos\theta\,,$$

where h is a constant. Hence, explain why oscillations of the pendulum will lead to elliptical orbits.