Electromagnetism

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Contents

Chapter 1

Mathematical Preliminaries

1.1 Vector and Scalar Fields

A field is a function of space. A scalar field $\phi(\mathbf{r})$ is a field with a magnitude at some vector $\mathbf{r}$. A vector field $\mathbf{v}(\mathbf{r})$ has a magnitude and direction at some vector $\mathbf{r}$. Vectorrs are defined in terms of a basis, such that,

$$\mathbf{v} = \sum_i v_i \mathbf{e}_i \tag{1.1}$$

where v_i is the magnitude in the direction of $\mathbf{e}_i$. A unit vector is defined,

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} \tag{1.2}$$

and the magnitude is defined,

$$|\mathbf{v}| = \sqrt{\sum_i v_i^2}. \tag{1.3}$$

The dot product between $\mathbf{a}$ and $\mathbf{b}$ is defined,

$$\mathbf{a} \cdot \mathbf{b} = \sum_i a_i b_i = ab\cos\theta \tag{1.4}$$

and the cross product is defined,

$$(\mathbf{a} \times \mathbf{b})_i = \sum_j \sum_k \epsilon_{ijk} a_j b_k. \tag{1.5}$$

We should know these vector identities,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{c} \times \mathbf{a}) \tag{1.6}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{1.7}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}). \tag{1.8}$$

1.2 Vector Calculus

The **grad** operator acting on a scalar field $\phi(\mathbf{r})$ is defined,

$$\boldsymbol{\nabla}\phi = \sum_i \frac{\partial \phi}{\partial r_i} \mathbf{e}_i. \tag{1.9}$$

The **div** operator acting on a vector field $\mathbf{v}$ is defined,

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \sum_i \frac{\partial v_i}{\partial r_i} \tag{1.10}$$

furthermore,

$$\boldsymbol{\nabla} \cdot \mathbf{v} > 0 \implies \text{Source} \qquad \boldsymbol{\nabla} \cdot \mathbf{v} < 0 \implies \text{Sink.} \tag{1.11}$$

The **curl** of a vector field is such that,

$$\boldsymbol{\nabla} \times \mathbf{v}. \tag{1.12}$$

The **Laplacian** ∇^2 can act on both scalar and vector fields,

$$\nabla^2 \phi = \sum_i \frac{\partial^2 \phi}{\partial r_i^2}, \tag{1.13}$$

$$\nabla^2 \mathbf{v} = \sum_i \frac{\partial^2 v_i}{\partial r_i^2}. \tag{1.14}$$

We should keep in mind the following vector identities,

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = 0 \tag{1.15}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) = 0 \tag{1.16}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \nabla^2(\mathbf{v}). \tag{1.17}$$

1.2.1 Integral Theorems

The fundamental theorem of calculus states that,

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = f(b) - f(a). \tag{1.18}$$

The divergence theorem states,

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{v}\,\mathrm{d}V = \int_S \mathbf{v} \cdot \mathrm{d}\mathbf{S}\,, \tag{1.19}$$

where $\mathrm{d}\mathbf{S} = \mathrm{d}S\,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit vector pointing out of the surface $\mathrm{d}S$.

Stoke's Theorem states that,

$$\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{S} = \oint_L = \mathbf{v} \cdot \mathrm{d}\ell \tag{1.20}$$

where the direction of $\mathrm{d}\mathbf{S}$ is determined by the right hand rule.

1.2.2 Curvilinear coordinates

The volume element in cyllindrical polar coordinates is,

$$\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z\,. \tag{1.21}$$

The volume element in spherical polar coordinates is given by,

$$\mathrm{d}V = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,. \tag{1.22}$$

1.3 Dirac-δ function

The Dirac-δ function is defined,

$$\delta(x-a) = \begin{cases} \infty & x = a \\ 0 & \text{Otherwise} \end{cases}. \tag{1.23}$$

We can also say,

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\,\mathrm{d}x = f(a) \tag{1.24}$$

which implies,

$$\int_{-\infty}^{\infty} \delta(x-a)\,\mathrm{d}x = 1. \tag{1.25}$$

The Dirac-δ function in 3 dimensions is denoted,

$$\delta^{(3)}(\mathbf{r} - \mathbf{a}) = \delta(x - a_1)\delta(y - a_2)\delta(z - a_3). \tag{1.26}$$

Example: Point charges

The Dirac-δ function can be used to model the charge density due to a distribution of point charges,

$$\rho(\mathbf{r}) = \sum_i q_i \delta^{(3)}(\mathbf{r} - \mathbf{r_i}). \tag{1.27}$$

The total charge is,

$$\begin{aligned} Q = \int_V \rho \, \mathrm{d}V &= \sum_i \int_V q_i \delta^{(3)}(\mathbf{r} - \mathbf{r_i}) \\ &= \sum_i q_i \int_V \delta(x - x_i)\delta(y - y_i)\delta(z - z_i) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &= \sum_i q_i \end{aligned} \tag{1.28}$$

1.4 Lagrange's and Laplace's Equation

Poisson's equation is a differential equation of the form,

$$\nabla^2 f = -4\pi g. \tag{1.29}$$

Laplace's equation is of the form,

$$\nabla^2 f = 0. \tag{1.30}$$

1.4.1 Laplace's solution for a spherically symmetric field

If we have a scalar field soley dependent on r such that $f = f(r)$, eq. (1.30) becomes,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}f}{\mathrm{d}r}\right) = 0. \tag{1.31}$$

Integrating eq. (1.31),

$$r^2 \frac{\mathrm{d}f}{\mathrm{d}r} = A \rightarrow \frac{\mathrm{d}f}{\mathrm{d}r} = \frac{A}{r^2}. \tag{1.32}$$

The solution to eq. (1.32) is trivial,

$$f(r) = B - \frac{A}{r}. \tag{1.33}$$

1.4.2 Uniqueness of Poisson's Equation

Theorem. *If there exists a solution to Poisson's equation subject to boundary conditions, it is the unique solution for those boundary conditions.*

Proof. Suppose f_1 and f_2 are solutions to Poisson's equation in a region V, bounded by a surface S. We define,

$$h = f_1 - f_2 \tag{1.34}$$

so inside V,

$$\nabla^2 h = 0 \tag{1.35}$$

and on S,

$$h = 0. \tag{1.36}$$

Let us consider,

$$\boldsymbol{\nabla} \cdot (h \boldsymbol{\nabla} h) = h \nabla^2 h + |\boldsymbol{\nabla} h|^2, \tag{1.37}$$

and integrating eq. (1.37) over V,

$$\int_V \boldsymbol{\nabla} \cdot (h \boldsymbol{\nabla} h) \, \mathrm{d}V = \int_V (h \nabla^2 h + |\boldsymbol{\nabla} h|^2) \, \mathrm{d}V \, . \tag{1.38}$$

Now applying divergence theorem and eq. (1.35) on V to eq. (1.38),

$$\int_S h\boldsymbol{\nabla}h \cdot \mathrm{d}\mathbf{S} = \int_V |\boldsymbol{\nabla}h|^2 \, \mathrm{d}V \,. \tag{1.39}$$

Applying eq. (1.36) on S,

$$\int_V |\boldsymbol{\nabla}h|^2 \, \mathrm{d}V = 0. \tag{1.40}$$

The integral vanishes, and $|\boldsymbol{\nabla}h|^2$ is positive, so $\boldsymbol{\nabla}h = 0$. So, h is constant in V, and must be 0 due to the boundary conditions $\implies f_1 = f_2$. $\square$

1.4.3 General solution to Poisson's equation

The general solution to Poisson's equation is,

$$f(\mathbf{r}) = \int_V \mathrm{d}V' \, \frac{g(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{1.41}$$

We can solve the integral in eq. (1.41) by considering an angle θ' between $\mathbf{r}$ and $\mathbf{r}'$,

$$\begin{aligned} |\mathbf{r} - \mathbf{r}'|^2 &= \mathbf{r}^2 + \mathbf{r}'^2 - 2\mathbf{r} \cdot \mathbf{r}' \\ &= \mathbf{r}^2 + \mathbf{r}'^2 - 2rr' \cos\theta', \end{aligned} \tag{1.42}$$

and thus the integral can be solved by evaluating it over all space, preferably using polar co-ordinates.

Chapter 2

Maxwell's Equations in a Vaccuum

2.1 Currents and the Continuity Equation

If we have a charge density $\rho(\mathbf{r}, t)$, and a current density $\mathbf{j}(\mathbf{r}, t)$, then the charge and current respectively are defined,

$$Q = \int \mathrm{d}q = \int \rho(\mathbf{r}, t)\,\mathrm{d}V \tag{2.1}$$

$$I = \int \mathrm{d}I = \int \mathbf{j}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{S}\,. \tag{2.2}$$

The conservation of charge implies, $\dot{Q} = -I$. Hence,

$$\int_V \dot{\rho}\,\mathrm{d}V = \dot{Q} = -\int_S \mathbf{j} \cdot \mathrm{d}\mathbf{S} = \int_V \boldsymbol{\nabla} \cdot \mathbf{j}\,\mathrm{d}V\,. \tag{2.3}$$

Therefore,

$$\int_V \{\dot{\rho} + \boldsymbol{\nabla} \cdot \mathbf{j}\}\,\mathrm{d}V = 0. \tag{2.4}$$

Eq. (2.4) must hold over any volume, therefore,

$$\boxed{\dot{\rho} + \boldsymbol{\nabla} \cdot \mathbf{j}} \tag{2.5}$$

which is the *continuity equation*.

We can define the current density in terms of the local velocity of the *positive charge carriers*, $\mathbf{j} = \rho\mathbf{v}$. Hence,

$$I = -nev_{\mathrm{drift}} \int \mathrm{d}S \tag{2.6}$$

where

$$v_{\mathrm{drift}} = \mathbf{v} \cdot \hat{\mathbf{n}} \tag{2.7}$$

which is the velocity of the positive charge carriers through the surface, n is the number density, and e is the electron charge.

2.2 Integral Forms of Maxwell's Equations

Let us begin with some elementary definitions. The Lorentz force is defined,

$$\mathbf{F} = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \tag{2.8}$$

The electromagnetic energy density in a vaccuum is defined,

$$u = \frac{1}{2}\epsilon_0|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2. \tag{2.9}$$

The electromotive force is defined,

$$\varepsilon = \int \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell}\,. \tag{2.10}$$

2.2.1 Gauss' Law

Definition. *The flux of electric field through a closed surface S is proportional to the amount of charge in a volume V that the surface encloses.*

The integral form is given by,

$$\boxed{\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\epsilon_0} \int_V \rho \, \mathrm{d}V} \tag{2.11}$$

and the differential form,

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho}. \tag{2.12}$$

Magnetic Monopoles

Magnetic monopoles do not exist, whether the magnetic field is time varying or static. Thus,

$$\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{2.13}$$

2.2.2 Faraday-Lenz Law

Definition. *The electromotive force induced in a closed loop is equal to the rate of change of magnetic flux linked in the loop such that the current flow will oppose the change in flux.*

This can be summarised,

$$\varepsilon = -\frac{\mathrm{d}}{\mathrm{d}t} \Phi_B. \tag{2.14}$$

From the definition of EMF, we can then obtain,

$$\boxed{\boldsymbol{\nabla} \times \mathbf{E} + \dot{B} = 0}. \tag{2.15}$$

2.2.3 Ampere's Law and Displacement Current

Definition. *The circulation of magnetic field through a closed loop L is proportional to the current passing through the surface that the loop encloses.*

Mathematically, in integral form

$$\oint_L \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \int_S \mathbf{j} \cdot \mathrm{d}\mathbf{S} \tag{2.16}$$

and in differential form,

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}. \tag{2.17}$$

However, eqs. (2.16) and (2.17) only work in the case where charge density is time independent. To factor in time-dependence, we must add a *displacement current* to the equations, in the form,

$$\mathbf{j}_{\text{disp}} = \epsilon_0 \dot{\mathbf{E}}. \tag{2.18}$$

The equations then become,

$$\boxed{\oint_L \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \left(\mathbf{j} + \epsilon_0 \dot{\mathbf{E}} \right) \cdot \mathrm{d}\mathbf{S}} \tag{2.19}$$

$$\boxed{\boldsymbol{\nabla} \times \mathbf{B} - \mu_0 \epsilon_0 \dot{\mathbf{E}} = \mu_0 \mathbf{j}}. \tag{2.20}$$

2.3 The time independent forms of Maxwell's equations

$$\begin{aligned} &\text{Differential} && \text{Integral} && (2.21)\\ &\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} && \oint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0} && (2.22)\\ &\boldsymbol{\nabla}\times\mathbf{E} = 0 && \oint_L \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = 0 && (2.23)\\ &\boldsymbol{\nabla}\cdot\mathbf{B} = 0 && \oint_S \cdot \mathrm{d}\mathbf{S} = 0 && (2.24)\\ &\boldsymbol{\nabla}\times\mathbf{B} = 0 && \oint_L \mathbf{B}\cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \sum I_j && (2.25)\end{aligned}$$

We have that eq. (2.24) $\implies$ electric fields are irrotational, and the continuity equation gives $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$ $\implies$ current flow is incompressible.

If we define $\mathbf{A}$ as the vector magnetic potential, and ϕ as the scalar electrostatic potential, we obtain,

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \qquad \mathbf{E} - \boldsymbol{\nabla}\phi, \tag{2.26}$$

from which we can write,

$$\boldsymbol{\nabla}^2\phi = -\frac{\rho}{\epsilon_0} \qquad \boldsymbol{\nabla}^2\mathbf{A} = -\mu_0\mathbf{j} \tag{2.27}$$

which are Poisson's eqations which can be readily solved.

NOTE: $\mathbf{A}$ has a guage degree of freedom, meaning that the magnetic field is invariat under a translation,

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\psi \tag{2.28}$$

for any choice of ψ. This can be removed by choosing the Coloumb guage,

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{2.29}$$

2.3.1 Electrostatics

The solution to the electrostatic Poisson's equation is of the form given by eq. (1.41), so,

$$\boxed{\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \mathrm{d}V' \, \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}} \tag{2.30}$$

and the solution is,

$$\boxed{\mathbf{E} = -\boldsymbol{\nabla}\phi = -\frac{1}{4\pi\epsilon_0}\int_V \mathrm{d}V' \, \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}}. \tag{2.31}$$

Coloumb's Law

A distribution of point charges is given by,

$$\rho(\mathbf{r}) = \sum_i q_i\delta^{(3)}\left(\mathbf{r}-\mathbf{r}_i\right). \tag{2.32}$$

So an electrostatic potential can be written,

$$\begin{aligned}\rho(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int_v \frac{\sum_i q_i\delta^{(3)}(\mathbf{r}'-\mathbf{r}_i')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' \\ &= \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\mathbf{r}-\mathbf{r}'|}.\end{aligned} \tag{2.33}$$

Then the electric field is,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3}. \tag{2.34}$$

Let us consider charges denote bed $i = 1$ and $i = 2$. The force on charge 1 due to charge 2,

$$\mathbf{F}_{12} = q_i\mathbf{E}_2(\mathbf{r}_1) = \frac{q_1q_2}{4\pi\epsilon_0}\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3} = -\mathbf{F}_{21}. \tag{2.35}$$

2.3.2 Magnetostatics

The solution is as previous,

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \mathrm{d}V'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}}. \tag{2.36}$$

Computing $\mathbf{B}$,

$$\begin{aligned}\mathbf{B} &= \frac{\mu_0}{4\pi}\int_V \mathrm{d}V'\,\boldsymbol{\nabla}\times\left(\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\\ &= \frac{\mu_0}{4\pi}\int_V \mathrm{d}V'\,\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'}\right)\times\mathbf{j}(\mathbf{r}')\\ &= \frac{\mu_0}{4\pi}\int_V \mathrm{d}V'\,\frac{\mathbf{j}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3},\end{aligned} \tag{2.37}$$

and we have that $\mathbf{j}\,\mathrm{d}V = I\,\mathrm{d}\boldsymbol{\ell}$,

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{4\pi}\int \frac{\mathrm{d}\boldsymbol{\ell}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}} \tag{2.38}$$

which is the Biot-Savarte law.

2.4 Electric and Magnetic Dipoles

We can define the electric dipole moment as,

$$\mathbf{p}(\mathbf{r}) = \int \mathrm{d}V\,;(\mathbf{r}'-\mathbf{r})\,\rho(\mathbf{r}) \tag{2.39}$$

where we define the charge density of two point charges, seperated by a distance $2a$ along some axis,

$$\rho(\mathbf{r}) = \int \mathrm{d}V\,(\mathbf{r}'-\mathbf{r})q\left[\delta^{(3)(\mathbf{r}'-\mathbf{a})-\delta(\mathbf{r}'+\mathbf{a})}\right] \tag{2.40}$$

which evaluates to,

$$\boxed{\mathbf{p} = 2q\mathbf{a}}. \tag{2.41}$$

We are able to then write the electric field due to a dipole as,

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{r^3}\left(3(\hat{\mathbf{r}}\cdot\mathbf{p})\hat{\mathbf{r}}-\mathbf{p}\right)}. \tag{2.42}$$

The magnetic equivalent is given by,

$$\boxed{\mathbf{m}(\mathbf{r}) = \frac{I}{2}\oint_L(\mathbf{r}'-\mathbf{r})\times\mathrm{d}\boldsymbol{\ell}} \tag{2.43}$$

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^2}} \qquad \boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi r^2}\left[3(\hat{\mathbf{r}}\cdot\mathbf{m})\hat{\mathbf{r}}-\mathbf{m}\right]}. \tag{2.44}$$

2.5 Structure of Maxwell's Equations

2.5.1 Potential formulation of Maxwell's equations

We have previously redefined the electric and magnetic fields in terms of potentials in eq. (2.26) for electro and magnetisostatics, and we will now apply it to electro and magnetodynamics. Using this formulation allows for the simplification of many problems in electromagnetism. We have that,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{2.45}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \dot{\mathbf{B}} = 0 \tag{2.46}$$

we find,

$$\boldsymbol{\nabla} \left(\mathbf{E} + \dot{\mathbf{A}} \right) = 0 \tag{2.47}$$

which we can satisfy by choosing,

$$\mathbf{E} = -\dot{\mathbf{A}} - \boldsymbol{\nabla}\phi. \tag{2.48}$$

As in the satic case, the choice of ϕ and $\mathbf{A}$ is not unique, thus they both have a guage freedom. We can fix their values by choosing the Lorenz guage,

$$\boxed{\frac{1}{c^2}\dot{\phi} + \boldsymbol{\nabla} \cdot \mathbf{A} = 0}. \tag{2.49}$$

The time independent version of eq. (2.49) is the Coloumb guage.

In the case that the Lorenz guage is not satisfied for some ϕ or $\mathbf{A}$, we can perform a guage transformation,

$$\phi \longrightarrow \phi' = \phi - \dot{\Psi} \tag{2.50}$$

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\Psi \tag{2.51}$$

so that we obtain,

$$\frac{1}{c^2}\dot{\phi}' + \boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{c^2} + \boldsymbol{\nabla} \cdot \mathbf{A} - \left(\frac{1}{c^2}\ddot{\Psi} - \boldsymbol{\nabla}^2\Psi \right) \tag{2.52}$$

and choose Ψ to be a solution of,

$$\frac{1}{c^2}\ddot{\psi} - \boldsymbol{\nabla}^2\Psi = \frac{1}{c^2}\dot{\phi} + \boldsymbol{\nabla} \cdot \mathbf{A}. \tag{2.53}$$

Redifining Maxwell's Laws

We can redefine Maxwell's laws in terms of potentials. These are listed below,

$$-\boldsymbol{\nabla}^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{1}{\epsilon_0}\rho \qquad \text{Gauss' Law} \tag{2.54}$$

$$-\boldsymbol{\nabla}^2\mathbf{A} + \frac{1}{c^2}\ddot{\mathbf{A}} = \mu_0\mathbf{j} \qquad \text{Ampere-Maxwell Law} \tag{2.55}$$

Let us define the differential "wave" operator,

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 \tag{2.56}$$

from whcih we can redefine eqs. (2.55) and (2.54),

$$\Box\phi = \frac{\rho}{\epsilon} \tag{2.57}$$

$$\Box\mathbf{A} = \mu_0\mathbf{j}. \tag{2.58}$$

2.5.2 Lorentz Transformations

We note that $\Box$ is invariant under Lorentz transformations, and by extension, Maxwell's equations are also Lorentz invariant. However, the 3-current $\mathbf{j}$ is not. We can then define a new 4-current,

$$\mathcal{J}^2 = c^2\rho^2 - |\mathbf{j}|^2. \tag{2.59}$$

Chapter 3

Electromagnetic Effects in Simple Materials

3.1 Conductors

In a conductors, there are a significant fraction of conduction electrons present. This means that in a conductor, the resistance to current flow R is very low, and the corresponding conductivity σ is very high. *Ohm's law* is an empirical law which states,

$$I = \frac{1}{R}V = \frac{1}{R}\oint \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} \tag{3.1}$$

We define the conductivity,

$$\sigma\, \mathrm{d}\mathbf{S} = \frac{1}{R\, \mathrm{d}\boldsymbol{\ell}}. \tag{3.2}$$

Using eq. (3.1) and the definition of current, we are able to deduce a relationship between the electric field, conductivity, and current,

$$\mathbf{j} = \sigma \mathbf{E}, \tag{3.3}$$

which is a form of Ohm's law which can be used in the context of Maxwell's equations.

Using the continuity equation and Gauss' law, we have,

$$\dot{\rho} + \boldsymbol{\nabla}\cdot\mathbf{j} = \dot{\rho} + \sigma\boldsymbol{\nabla}\cdot\mathbf{E} = \dot{\rho} + \frac{\sigma}{\epsilon_0} = 0. \tag{3.4}$$

Integrating eq. (3.4),

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0)e^{-\frac{t}{t_R}} \tag{3.5}$$

where we define

$$t_R = \frac{\epsilon_0}{\sigma} \tag{3.6}$$

which can be interpreted as the relaxation time scale for the movement of conduction electrons to smooth out non-uniformities in the charge density.

3.2 Method of Images

This exploits the uniqueness theorem which states,

Theorem. *The solution to a set of differential equations under a given set of boundary conditions is the solution.*

We will use the result that at a perfect conducter, the electric field is 0, which is also the case half-way between a charge and an anti-charge.

Consider a point charge q a distance a above an infinite, perfectly conducting plane at $z = 0$. The plane is grounded such that $\phi(x, y, 0) = 0$. We further have that $\mathbf{E} = 0$. We wish to solve Poisson's equation given these boundary conditions. Let us consider an "image charge" at $z = -a$, with charge $-q$. If we choose to ignore the plane, we thus have,

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{r}-\mathbf{a}|} - \frac{1}{|\mathbf{r}+\mathbf{a}|}\right] \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+a)^2}}\right]. \end{aligned} \tag{3.7}$$

Substituting in the boundary conditions, we find $\phi(x, y, 0) = 0$. Thus, the solution satisfies the boundary condition, and by the uniqueness theorem is the only solution. We have thus reformulated the problem to be much simpler to understand and compute.

3.3 Capacitance and Relative Permitivity

We understand capacitance to be the ratio of charge and potential difference within a capacitor, $C = Q/V$. For a parallel plate capacitor, this is given by,

$$C = \frac{\varepsilon_0 A}{d} \tag{3.8}$$

and whose energy cnan be calculated as,

$$U = \frac{1}{2}\varepsilon_0 \int |\mathbf{E}|^2 \, \mathrm{d}V = \frac{1}{2}V\left(\Delta V\right)^2. \tag{3.9}$$

If we were to add a material between the two plates of the capacitor, we would observe a voltage drop. We can define the relative permitivity to be,

$$\varepsilon_r = \frac{C}{C_{\text{vacuum}}} \tag{3.10}$$

which is also known as the dielectric constant. Dielectrics are a type of electrical insulator which can undergo polarisation. An ideal dielectric has the following properties,

1. *Homogenous* - Properties are the same throughout
2. *Isotropic* - Same in every direction
3. $\mathbf{P} \propto \mathbf{E}$ - Behaves linearly
4. *Stationary* - No time dependence

3.4 Polarisation

Definition. *When the constituents of the substance align in some preferred direction associated with an electric field.*

Dipoles will attempt to rotate to align with the applied electric field in order to minimise their action. If a field is applied to a material, two things will occur,

1. intrinsic dipoles will algin to minimise action,
2. atoms and molecules can be polarised, inducing a dipole moment.

We can define a macroscopic vector field knonw as the polarisation,

$$\mathbf{P} = n\mathbf{p} \tag{3.11}$$

where n is the number density of atoms or molecules, and $\mathbf{p}$ is the average dipole moment. The polarisation will in general be a function of the applied electric field, known as a constitutive relation. A linear isotropic material will have a constitutive relation,

$$\mathbf{P} = \chi_E \varepsilon_0 \mathbf{E} \tag{3.12}$$

where χ_E is the electric susceptibility, which can be defined in terms of the realtive permitivity,

$$\chi_E = 1 - \varepsilon_r. \tag{3.13}$$

We can generalise eq. (3.12) to include anistropic direction dependancies and non-linear responses by,

$$P_i = \varepsilon_0 \sum_j \chi_{ij}^{(1)} E_j + \varepsilon_0 \sum_{j,k} \chi_{ijk}^{(2)} E_j E_k \tag{3.14}$$

where the matrix $\chi_{ij}^{(1)}$ encodes anisotropic responses, and the tensor $\chi_{ijk}^{(2)}$ encoces quadratic responses of the polarisation vector.

3.4.1 Mechanisms for Polarisation

Alignment of instrinsic dipoles

Atoms of molecules with an intrinsic dipole moment $\mathbf{p}_{\text{int}}$ subject to an external magnetic field $\mathbf{E}_{\text{ext}}$ have a polarisation,

$$\mathbf{P}_{\text{align}} = \frac{np_{\text{int}}^2}{3k_B T}\mathbf{E}_{\text{ext}} \tag{3.15}$$

from which we can infer the electric susceptibility χ_E,

$$\chi_E = \frac{np_{\text{int}}^2}{4k_B T \varepsilon_0}. \tag{3.16}$$

Induced

When atoms are subject to an external electric field, negative electrons are shifted relative to the positive nucleus, creating a dipole. This mechanism is known as atomic polarisation. If the external electric field causes an offset d for a distribution of radius R_0, then the force due to the offset is given by,

$$qE_{\text{ext}} = \frac{q^2 f}{4\pi\varepsilon_0 d^2} \tag{3.17}$$

where f is the fractional charge offset, $f \approx \frac{d^3}{R_0^3}$. Hence,

$$qd \approx 4\pi\varepsilon_0 R_0^3 E_{\text{ext}}. \tag{3.18}$$

The polarisation is then,

$$\mathbf{P}_{\text{ind}} = n\alpha\varepsilon_0 \mathbf{E}_{\text{ext}} \tag{3.19}$$

where

$$\alpha = 4\pi R_0^3 \tag{3.20}$$

is the atomic polarisation. We can infer the electric susceptibility as,

$$\chi_E = n\alpha. \tag{3.21}$$

Combined mechanism

Obtaining the combined mechanism is trivially,

$$\boxed{\mathbf{P}_{\text{align}} + \mathbf{P}_{\text{ind}} = \varepsilon_0 n \left(\alpha + \frac{p_{\text{int}}^2}{3k_B T\varepsilon_0}\mathbf{E}_{\text{ext}}\right)} \tag{3.22}$$

from which we can identify the combined electric susceptibility as,

$$\boxed{\chi_E = n\left(\alpha + \frac{p_{\text{int}}^2}{3k_B T\varepsilon_0}\right)}. \tag{3.23}$$

3.5 Electrostatics in Dielectrics

We will be studying the electric fields inside dielectric materials. Let us consider a polarisation vector,

$$\mathbf{P} = P_x\left(x\right)\hat{\mathbf{i}}, \quad P_x > 0. \tag{3.24}$$

We have electrons moving left out of the material at one end, and into at the other end. This results in a shift in charge, given by,

$$\Delta Q = -\left(P(x+\delta x) - P_x(x)\right)\delta y \delta z \approx -\frac{\partial P_x}{\partial x}\delta x \delta y \delta z. \tag{3.25}$$

However, te shift in charge is given by the elemental volume, multiplied by the value of the bound charge dnesity,

$$\Delta Q = \rho_{\text{bound}}\delta x \delta y \delta z \tag{3.26}$$

such that,

$$\rho_{\text{bound}} = -\frac{\partial P_x}{\partial x}. \tag{3.27}$$

We can generalise this to 3 dimensions,

$$\rho_{\text{bound}} = -\boldsymbol{\nabla}\cdot\mathbf{P} = 0 \tag{3.28}$$

which is 0 because this bound charge is neutral. We can obtain the total bound charge by adding it to the total surface charge density,

$$Q_{\text{bound}} = \int_V \rho_{\text{bound}}\,\mathrm{d}V + \int_S \sigma\,\mathrm{d}S = 0. \tag{3.29}$$

Thus,

$$\int_V \boldsymbol{\nabla}\cdot\mathbf{P}\,\mathrm{d}V = \int_S \mathbf{P}\,\mathrm{d}\mathbf{S} = \int_S \sigma\,\mathrm{d}S \tag{3.30}$$

from which we can obtain the surface denstiy due to polarisation,

$$\sigma_P = \mathbf{P}\cdot\hat{\mathbf{n}}. \tag{3.31}$$

We can then write Gauss' law by splitting the free and bound charges,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\left(\rho_{\text{bound}} + \rho_{\text{free}}\right) = \frac{1}{\varepsilon_0}\left(-\boldsymbol{\nabla}\cdot\mathbf{P} + \rho_{\text{free}}\right). \tag{3.32}$$

Rearranging eq. (3.32),

$$\boldsymbol{\nabla}\cdot\left(\varepsilon_0\mathbf{E} + \mathbf{P}\right) = \rho_{\text{free}}. \tag{3.33}$$

We can write this as a modified form of Gauss' law,

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_{\text{free}}} \tag{3.34}$$

where we define the electric displacement vector as,

$$\boxed{\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}} \tag{3.35}$$

which for a linear isotropic material becomes,

$$\boxed{\mathbf{D} = \left(1+\chi_E\right)\varepsilon_0\mathbf{E}}. \tag{3.36}$$

The integral form of Gauss' law is then given by,

$$\boxed{\int_S \mathbf{D}\cdot\mathrm{d}\mathbf{S} = Q_{\text{free}}}. \tag{3.37}$$

The benefit of this electric displacement field is that it is a field whose lines are completely straight and continuous in a dielectric parallel plate capacitor.

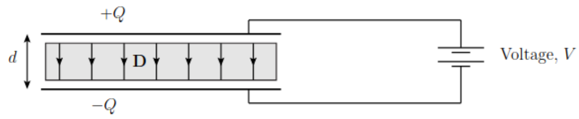


Figure 3.1: The boundary between two dielectric materials. Each region has a distinct relative permitivity, denoted by $\varepsilon_r^{(1)}$ and $\varepsilon_r^{(2)}$.

3.5.1 Properties of the D field

Consider a parallel plate capacitor with a dielectric, as in fig. 3.1. The surface charge of the plate is given by $\sigma_P \frac{Q}{A}$. By eq. (3.37), we have,

$$\oint \mathbf{D} \cdot \mathrm{d}\mathbf{S} = D_z A = Q. \tag{3.38}$$

We can then write that the z component of the electric field is,

$$E_z = \frac{Q}{(1+\chi_E)\varepsilon_0 A} \tag{3.39}$$

and the z component of the polarisation field is,

$$P = \frac{\chi_E Q}{(1+\chi_E)A}. \tag{3.40}$$

The potential difference between the plates is,

$$\Delta\phi = E_z d = \frac{Qd}{(1+\chi_E)\varepsilon_0 A} \tag{3.41}$$

so the capacitance is,

$$\begin{aligned} C = \frac{Q}{\Delta\phi} &= (1+\chi_E)\frac{\varepsilon_0 A}{d} \\ &= (1+\chi_E)C_0 \end{aligned} \tag{3.42}$$

where C_0 is the capacitance in vacuum. Eq. (3.42) holds at constant Q with out a battery connected. From this, we can conclude that the electric field is lowered by $(1+\chi_E) = \varepsilon_r$ due to the dieelectric. So, we can rewrite the electric displacement vector as,

$$\boxed{\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E}}. \tag{3.43}$$

Energy in a capacitor

We know trivially,

$$u = \frac{1}{2}C\Delta\phi^2 = \frac{\frac{1}{2}A\varepsilon_r\varepsilon_0\left(E_z d^2\right)}{d} = \frac{1}{2}AdD_zE_z \tag{3.44}$$

in the z direction for a parallel plate capacitor. However, generally,

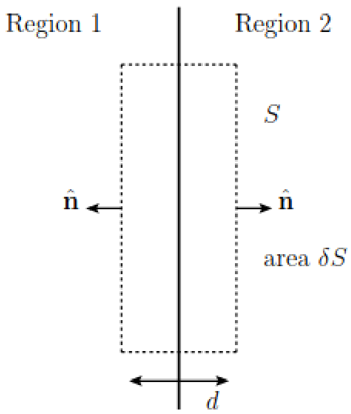
$$u = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{V}\,\mathrm{d}V\,. \tag{3.45}$$

3.5.2 Interfaces between Dielectrics

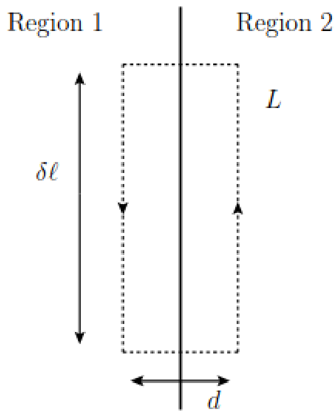
Electric diplacement between two materials

Consider fig. 3.2 (a), where we have a gaussian surface S enclosing the boundary between the two materials, with thickness d and area δS. When we take $d \to 0$, let us assume no free charges at the boundary. We then have,

$$\begin{aligned} \oint \mathbf{D}\cdot\mathrm{d}\mathbf{S} = 0 = -D_\perp^{(1)}\delta S + D_\perp^{(2)}\delta S \\ \implies D_\perp \text{ is continous at the boundary.} \end{aligned} \tag{3.46}$$



(a) Surface S enclosing an interface between dielectrics.



(b) Loop L enclosing an interface between dielectrics.

Figure 3.2

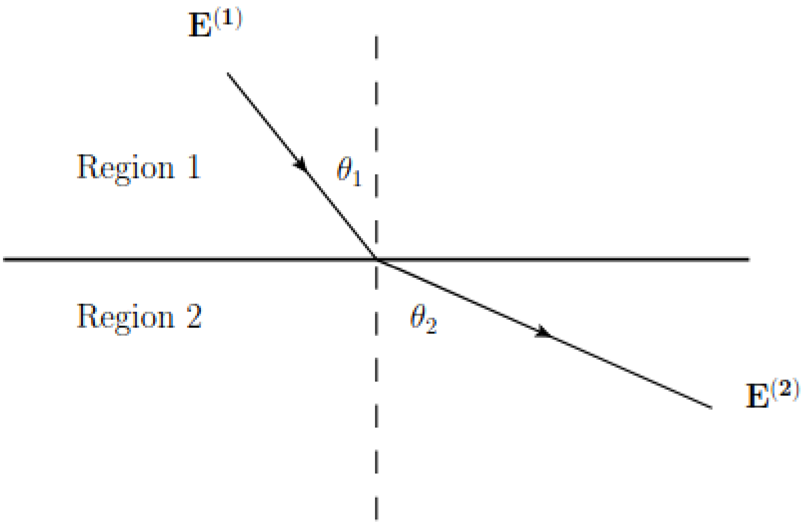


Figure 3.3

Electric field between two materials

Let us now consider fig. 3.2 (b). Let us consider a loop L enclosing the boundary, with length dℓ and width d. We assume no surface charges. Then, as $d \to 0$,

$$\begin{aligned} \oint \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = 0 = E_{\parallel}^{(1)} \delta\ell + E_{\parallel}^{(2)} \delta\ell \\ \implies E_{\parallel} \text{ is continuous across the boundary.} \end{aligned} \tag{3.47}$$

The general case

Let us consider fig. 3.3. The electric field in the two regions is,

$$\mathbf{E}^{(1)} = E_1 \begin{pmatrix} \sin\theta_1 \\ \cos\theta_1 \end{pmatrix} \qquad \mathbf{E}^{(2)} = E_2 \begin{pmatrix} \sin\theta_2 \\ \cos\theta_2 \end{pmatrix} \tag{3.48}$$

If $E_{\parallel}$ is continous,

$$E_1 \sin\theta_1 = E_2 \sin\theta_2. \tag{3.49}$$

If $D_{\perp}$ is continuous,

$$D_1 \cos\theta_1 = D_2 \cos\theta_2. \tag{3.50}$$

Dividing eq. (3.50) by eq. (3.49),

$$\frac{D_1}{E_1} \cot \theta_1 = \frac{D_2}{E_2} \cot \theta_2 \tag{3.51}$$

and by eq. (3.43),

$$\boxed{\varepsilon_r^{(1)} \cot \theta_1 = \varepsilon_r^{(2)} \cot \theta_2}. \tag{3.52}$$

Appendix A

Proofs

A.1 Loop integral over perfect differential

Proof.

$$\oint_L \mathrm{d}\boldsymbol{\ell} \cdot \boldsymbol{\nabla} f \underbrace{=}_{\text{Stoke's theorem}} \int_S \mathrm{d}\mathbf{S} \cdot \left(\underbrace{\boldsymbol{\nabla} \times \boldsymbol{\nabla} f}_{0} \right) = 0 \tag{A.1}$$

□

Appendix B

Examples

B.1 Coloumb's law for two, straight, parallel wires

The force on an infinitesimal line element is,

$$\mathbf{F}_{12} = I_1 \, \mathrm{d}\boldsymbol{\ell} \times \mathbf{B}_2 \left(\mathbf{r}_1\right). \tag{B.1}$$

The magnetic field due $\mathbf{B}_2$ due to wire 2 is given by,

$$\mathbf{B}_2 \left(\mathbf{r}\right) = \frac{\mu_0 I_2}{2\pi \left|\mathbf{r} - \mathbf{r}_2\right|} \hat{\boldsymbol{\theta}}, \tag{B.2}$$

thus,

$$\mathrm{d}\mathbf{F}_{12} = \frac{\mu_0 I_1 I_2}{2\pi d} \, \mathrm{d}\ell \, \hat{\mathbf{z}} \times \hat{\boldsymbol{\theta}}, \tag{B.3}$$

where $\hat{\mathbf{z}} \times \hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}$, thus,

$$\frac{\mathrm{d}F_{12}}{\mathrm{d}\ell} = \pm \frac{\mu_0 I_1 I_2}{2\pi d} \tag{B.4}$$

where we have an attractive force if both currents are travelling in the same direction.

B.2 Coloumb's law for two parallel current carrying wires

If we define current $I \, \mathrm{d}\boldsymbol{\ell} = q\mathbf{v}$, then the magnetic force is given by,

$$I \, \mathrm{d}\boldsymbol{\ell} = q\mathbf{v}. \tag{B.5}$$

We have that the force on wire 1 due to wire 2 is,

$$\begin{aligned} \mathbf{F}_{12} &= \oint_{L_1} I_1 \, \mathrm{d}\boldsymbol{\ell}_1 \times \mathbf{B}_2(\mathbf{r}_1) \\ &= \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \mathrm{d}\boldsymbol{\ell}_1 \times \oint_{L_2} \mathrm{d}\boldsymbol{\ell}_1 \times \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3} \\ &= -\frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \mathrm{d}\boldsymbol{\ell}_1 \cdot \mathrm{d}\boldsymbol{\ell}_2 \left(\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3} \right) \end{aligned} \tag{B.6}$$

We were able to rewrite the integral using the vector identity,

$$\mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{v}_3) = -(\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{v}_3 + \mathbf{v}_2 \left(\mathbf{v}_1 \cdot \mathbf{v}_3\right) \tag{B.7}$$

and the second term in eq. (B.7) in our case goes to 0 as we are performing a loop integral over a perfect differential.