ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Integral Transforms

21st May 2010, 2.00 p.m. - 3.30 p.m.

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. (a) For a function $f(z) = u(x, y) + iv(x, y)$ (where $z = x + iy$), state the Cauchy-Riemann equations. Show that they are satisfied for $f(z) = \cos z$.

[6 marks]

(b) Show by explicit integration that $\oint_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$, where C is a unit circle about $z = a$.

[5 marks]

(c) For the function $f(z) = \frac{1}{z(z-2)}$, find the Laurent series about the point $z = 2$ which is valid for $0 < |z-2| < 2$.

[7 marks]

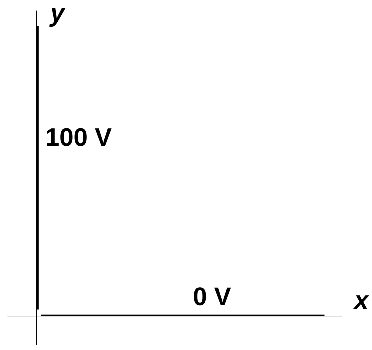
(d) Calculate the Laplace transform of $\theta(t-\tau)\sin(2\pi t/\tau)$ where $\tau > 0$ and the step function $\theta(t)$ is 0 for $t < 0$ and 1 for $t > 0$.

[7 marks]

2. (a) An analytic function $f(z)$ has imaginary part $v(x,y) = ye^x \cos y + xe^x \sin y$. Show that $v(x,y)$ is harmonic, and find the corresponding real part of $f(z)$. Express $f(z)$ in terms of z.

[13 marks]

(b)



Two semi-infinite metal sheets are at right angles to each other (not quite touching); one is held at 0 Volts and the other is held at 100 Volts. The diagram above shows a cross-section of the plates in the xy-plane.

Use the method of conformal mapping, with the transformation $Z = \ln z$, to show that the system maps into a parallel plate capacitor with plate separation of $\pi/2$. Find the potential in terms of $\{X, Y\}$ and hence in terms of $\{x, y\}$, verifying that it obeys the boundary conditions in the original geometry. Sketch some equipotentials and field lines.

[12 marks]

3. Use Cauchy's residue theorem and a suitable choice of contour to calculate TWO of the following integrals. If Jordan's lemma is used, ensure that all the conditions for its validity are satisfied.

(a)

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\,\mathrm{d}\,\theta$$

(b)

$$\int_0^{\infty} \frac{1}{x^{\alpha}(1+x)}\,\mathrm{d}\,x \qquad \text{for non-integer } 0 < \alpha < 1$$

(c)

$$\int_{-\infty}^{\infty} \frac{x\sin(kx)}{(1+x^2)}\,\mathrm{d}\,x \qquad \text{for } k > 0$$

[25 marks]

4. (a) Find the position of, and residues at, the poles of $f(z) = 1/(z^3\cos(\pi z))$. Hence show that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots = \frac{\pi^3}{32}$$

[13 marks]

(b) Use Laplace transforms to solve the following differential equation, subject to the initial conditions $y(0) = 0$ and $\mathrm{d}y/\mathrm{d}t|_{t=0} = 0$:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 4y = t\sin(2t)$$

You may make use of the following table, where $\overline{f}(s)$ is the Laplace transform of $f(t)$:

$f(t)$	$\sin(\alpha t)$	$\sin(\alpha t) - \alpha t\cos(\alpha t)$
$\overline{f}(s)$	$\dfrac{\alpha}{s^2+\alpha^2}$	$\dfrac{2\alpha^3}{(s^2+\alpha^2)^2}$

[12 marks]

END OF EXAMINATION PAPER