ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Advanced Quantum Mechanics

[redacted] 2024, [redacted] - [redacted]

Answer **TWO** questions

The use of calculators is permitted, as long as they cannot store text and have no graphing capability.

The numbers are given as a guide to the relative weights of the different parts of each question.

You may assume the following formulae in any question *if proof is not explicitly requested*:

Special integrals

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2a)^n}\sqrt{\frac{\pi}{a}}, \qquad \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}, \quad a>0.$$

4-vectors
Use notation $x^\mu = (x^0, \mathbf{r})$ for 4-position and $p^\mu = (E/c, \mathbf{p})$ for 4-momentum. In electromagnetism $A^\mu = (\Phi/c, \mathbf{A})$ for 4-potential.

Klein-Gordon equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[(\hat{\mathbf{p}} - \mathbf{V})^2 c^2 + (mc^2 + S)^2\right]\Psi = (i\hbar\partial_t - V_0)^2\Psi.$$

Dirac equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[\boldsymbol{\alpha}\cdot(\hat{\mathbf{p}} - \mathbf{V})c + \beta(mc^2 + S)\right]\Psi = (i\hbar\partial_t - V_0)\Psi.$$

Standard spinor solutions to the free massive Dirac equation

$$u_s(E,\mathbf{p}) = \sqrt{E+mc^2}\begin{pmatrix}\phi_s \\ \frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E+mc^2}\phi_s\end{pmatrix}, \quad \phi_1 = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \phi_2 = \begin{pmatrix}0\\1\end{pmatrix};$$

Standard matrices

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix};$$

$$\alpha_i = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}, \quad \beta = \begin{pmatrix}I_2 & 0\\ 0 & -I_2\end{pmatrix}, \quad \Sigma_i = \begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix};$$

$$\gamma_0 = \begin{pmatrix}I_2 & 0\\ 0 & -I_2\end{pmatrix}, \quad \gamma_1 = \begin{pmatrix}0 & \sigma_x\\ -\sigma_x & 0\end{pmatrix}, \quad \gamma_2 = \begin{pmatrix}0 & \sigma_y\\ -\sigma_y & 0\end{pmatrix}, \quad \gamma_3 = \begin{pmatrix}0 & \sigma_z\\ -\sigma_z & 0\end{pmatrix}.$$

Space-time metric We use the metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

Laplacian operator in terms of angular momentum operator $\hat{\mathbf{L}}$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{\hat{\mathbf{L}}^2}{\hbar^2 r^2}.$$

Solution to Dirac equation in spherical vector potential $(V_0(r)/c, \vec{0})$

$$\psi^\kappa_{jm}(\mathbf{r}) = \begin{pmatrix} f^\kappa_j(r)\mathcal{Y}^\kappa_{jm}(\theta,\phi) \\ i g^\kappa_j(r)\mathcal{Y}^{-\kappa}_{jm}(\theta,\phi)\end{pmatrix} \quad \text{with } F = rf \text{ and } G = rg,$$

$$\hbar c\left[\frac{d}{dr} - \frac{\kappa}{r}\right] G^\kappa_j(r) = -\left(E - V_0(r) - mc^2\right) F^\kappa_j(r),$$

$$\hbar c\left[\frac{d}{dr} + \frac{\kappa}{r}\right] F^\kappa_j(r) = \left(E - V_0(r) + mc^2\right) G^\kappa_j(r).$$

1. a) Show that the angular momentum operator $\hat{L}_x$ is the generator of infinitesimal rotations about the x-axis for a spinless particle. Derive an expression in terms of $\hat{L}_x$ for the operator $\hat{U}_\beta$ that represents a finite rotation of angle β about the x-axis.

[8 marks]

b) For a spinless particle with non-zero orbital angular momentum, write down the operator $\hat{U}_{\alpha,\beta,\gamma}$ representing a rotation with Euler angles (α, β, γ).

Within the space of states with angular momentum quantum number $l = 1$, the operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ may be written in matrix representation as

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Show that for the choice of Euler angles $(0, 0, \gamma)$ the $\hat{L}_x$ eigenstate

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

can be transformed into an eigenstate of $\hat{L}_y$ for an appropriate value of γ. Give the corresponding Wigner D-matrix.

[7 marks]

c) A charged quantum rotor of angular momentum quantum number $l = 1$ is placed in a uniform magnetic field oriented along the z-axis. The Hamiltonian is given by

$$\hat{H}_0 = \alpha\hat{L}_z,$$

where α is a positive constant. Initially the rotor is in the ground state of $\hat{H}_0$. At time $t = 0$ a weak magnetic field is switched on in the x-direction, which subsequently decays in time. The Hamiltonian of the rotor is now

$$\hat{H}(t) = \hat{H}_0 + \beta\hat{L}_x \exp(-t/\tau),$$

where β and τ are positive constants, with $\beta \ll \alpha$.

i) Using first-order perturbation theory, calculate the transition probability $P(t)$ to the first excited state of the rotor as a function of time.

[7 marks]

ii) Determine the long-time limit $P(\infty)$ and find a condition on τ to ensure its validity.

[3 marks]

2. a) Consider a free relativistic spin-1/2 particle of mass m. Define the Dirac four-current $j^\mu = (c\rho, \mathbf{j})$ and show that the Dirac equation implies the continuity equation,

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j} = 0.$$

[4 marks]

b) Now consider a relativistic electron moving in the x-y plane with a uniform applied magnetic field in the z-direction, given by $\mathbf{B} = (0, 0, B)$.

i) Show that this magnetic field can be described by the vector potential $\mathbf{A} = (0, xB, 0)$ and prove that the following identity holds for this potential:

$$[\boldsymbol{\sigma} \cdot (\hat{\mathbf{p}} + e\mathbf{A})]^2 = (\hat{\mathbf{p}} + e\mathbf{A})^2 + e\hbar\sigma_z B.$$

[4 marks]

ii) The Hamiltonian for a non-relativistic electron in the same potential is

$$\hat{H}_{\mathrm{NR}} = \frac{1}{2m}(\hat{\mathbf{p}} + e\mathbf{A})^2,$$

which has eigenvalues

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c.$$

Here, $n = 0, 1, 2, \ldots$ and $\omega_c = eB/m$ is the cyclotron frequency.

Express the Dirac equation for this electron in terms of the upper components only. Find the energy eigenvalues by treating the upper components as simultaneous eigenstates of $\hat{H}_{\mathrm{NR}}$ and of the spin operator $\hat{S}_z$.

[8 marks]

iii) For the weak-field case, $\hbar\omega_c \ll mc^2$, show that the energy levels of the relativistic electron are given by

$$E_n^{\uparrow} = mc^2 + (n+1)\hbar\omega_c, \quad E_n^{\downarrow} = mc^2 + n\hbar\omega_c, \quad n = 0, 1, 2\ldots$$

for spin-up and spin-down electrons, respectively.

[2 marks]

c) A beam of massless relativistic particles with charge $q = e$, energy E_p, and in the spin-up eigenstate of $\hat{S}_z$ (spinor ϕ_1) is moving in the positive z-direction and encounters an electrostatic step potential defined by

$$e\Phi(z) = \begin{cases} 0 & \text{for } z < 0, \\ V_0 & \text{for } z > 0. \end{cases}$$

Show that for $E_p > V_0$ all particles are transmitted.

[7 marks]

3. a) Consider a non-relativistic spinless particle of charge q and mass m in the 4-potential $A^\mu = (\Phi/c, \mathbf{A})$.

i) Write down the electromagnetic gauge transformation for the potentials Φ and $\mathbf{A}$ in terms of a differentiable function $\lambda(\mathbf{r}, t)$.

[2 marks]

ii) The particle is placed in a homogenous oscillatory electric field of frequency ω directed along the z-axis, with $E_z = E_0 \cos(\omega t)$. In two different gauges the electric field may be considered as arising purely from the potential Φ, or purely from the potential $\mathbf{A}$. Write down the corresponding potentials and find a function $\lambda(\mathbf{r}, t)$ that transforms from the first to the second. Given that $|\psi_1\rangle$ is a solution to the time-dependent Schrödinger equation in the first case, show that

$$|\psi_2\rangle = \exp(iq\lambda(\mathbf{r}, t)/\hbar)|\psi_1\rangle,$$

is a solution in the second case. [8 marks]

b) Now consider a relativistic spinless particle of charge e that is bound to a nucleus by a Coulomb-like *scalar* potential $S(r) = -Z\hbar c\alpha/r$, where α is the fine-structure constant.

i) Write down the time-independent Klein-Gordon equation for the eigenstates of the particle with energy E.

[2 marks]

ii) By considering an eigenstate of total angular momentum with quantum number l, show that the eigenvalue equation for the radial part $\psi(r)$ of the wave function of the particle can be written as

$$\left[2mE' + \hbar^2\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{l'(l'+1)}{r^2}\right) + \frac{2m\hbar cZ\alpha}{r}\right]\psi(r) = 0. \tag{1}$$

Express E' and $l'(l'+1)$ in terms of E and l.

[6 marks]

iii) The eigenvalues of equation (1) can be written in the form

$$E' = -\frac{1}{2}mc^2\left(\frac{Z\alpha}{n'}\right)^2,$$

where n' is related to l and the principal quantum number $n = 1, 2, 3\ldots$ by

$$n' = n - (l + 1/2) + \sqrt{(l + 1/2)^2 + (Z\alpha)^2}.$$

Find the energy levels $E_{n,l}$ of the particle and discuss whether or not there are any restrictions on the allowed values of $Z\alpha$. Considering the limit $Z\alpha \ll 1$ find the first *relativistic* energy correction for the case $l = 0$.

[7 marks]

END OF EXAMINATION PAPER