**ONE HOUR THIRTY MINUTES**

A list of constants is enclosed.

# UNIVERSITY OF MANCHESTER

Advanced Quantum Mechanics

6th June 2017, 2.00 p.m. - 3.30 p.m.

Answer any **TWO** questions

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Electronic calculators may be used, provided that they cannot store text.

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The numbers are given as a guide to the relative weights of the different parts of each question.

You may assume the following formulae in any question *if proof is not explicitly requested*:

**Special integrals**

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2a)^n}\sqrt{\frac{\pi}{a}}, \qquad \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}, \quad a > 0.$$

**4-vectors**
Use notation $x^\mu = (x^0, \mathbf{x})$ for 4-position and $p^\mu = (E/c, \mathbf{p})$ for 4-momentum. In electromagnetism $A^\mu = (\Phi/c, \mathbf{A})$ for 4-potential.

**Klein-Gordon equation** in scalar $S$ and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[(\hat{\mathbf{p}} - \mathbf{V})^2 c^2 + (mc^2 + S)^2\right]\Psi = (i\hbar\partial_t - V_0)^2\Psi.$$

**Dirac equation** in scalar $S$ and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[\alpha\cdot(\hat{\mathbf{p}} - \mathbf{V})c + \beta(mc^2 + S)\right]\Psi = (i\hbar\partial_t - V_0)\Psi.$$

**Standard spinor solutions** to the free massive Dirac equation

$$u^{(s)}(E, \mathbf{p}) = \sqrt{E + mc^2}\begin{pmatrix} \phi^s \\ \frac{c\sigma\cdot\mathbf{p}}{(E+mc^2)}\phi^s \end{pmatrix}, \quad \phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$v^{(s)}(-|E|, -\mathbf{p},) = \sqrt{|E| + mc^2}\begin{pmatrix} \frac{c\sigma\cdot\mathbf{p}}{(|E|+mc^2)}\chi^s \\ \chi^s \end{pmatrix}, \quad \chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \chi^2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

**Standard matrices**

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix};$$

$$\gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}.$$

**Space-time metric** We use the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -, 1, -1)$.

**Laplacian operator** in terms of angular momentum operator $\hat{\mathbf{L}}$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{\hat{\mathbf{L}}^2}{\hbar^2 r^2}.$$

**Solution to Dirac equation** in spherical vector potential $(V_0(r)/c, \mathbf{0})$

$$\psi^\kappa_{jm}(\mathbf{r}) = \begin{pmatrix} f^\kappa_j(r)\mathcal{Y}^\kappa_{jm}(\theta,\phi) \\ ig^\kappa_j(r)\mathcal{Y}^{-\kappa}_{jm}(\theta,\phi) \end{pmatrix} \quad \text{with } F = rf \text{ and } G = rg,$$

$$\hbar c\left[\frac{d}{dr} - \frac{\kappa}{r}\right]G^\kappa_j(r) = -\left(E - V_0(r) - mc^2\right)F^\kappa_j(r),$$

$$\hbar c\left[\frac{d}{dr} + \frac{\kappa}{r}\right]F^\kappa_j(r) = \left(E - V_0(r) + mc^2\right)G^\kappa_j(r).$$

**1.** a) Show that the Hamiltonian $\hat{H}$ is the generator for time translation. Derive an expression in terms of $\hat{H}$ for the operator $\hat{U}_{t_0}$ representing a finite time $t_0$ translation for a time-independent Hamiltonian. Briefly state the reason for the assumption that the Hamiltonian is time-independent in your derivation. Show that the operator $\hat{U}_{t_0}$ is unitary.

[10 marks]

b) i) Write down the $(4 \times 4)$ Dirac vector matrix $\mathbf{\Sigma}$ in terms of the $(2 \times 2)$ Pauli vector matrix and show that $\mathbf{\Sigma}^2 = 3$. Prove the identity for the angular momentum operator $\hat{\mathbf{L}}$, $(\mathbf{\Sigma} \cdot \hat{\mathbf{L}})^2 = \hat{\mathbf{L}}^2 - \hbar \mathbf{\Sigma} \cdot \hat{\mathbf{L}}$.

[6 marks]

ii) The operator $\hat{K}$ is defined as $\hat{K} = \beta(\mathbf{\Sigma} \cdot \hat{\mathbf{L}} + \hbar)$ and commutes with the Dirac Hamiltonian of hydrogen. Use the identity in (i) to show that $\hat{K}^2 = \hat{\mathbf{J}}^2 + \frac{1}{4}\hbar^2$, where $\hat{\mathbf{J}}$ is the total angular momentum operator. Hence find the eigenvalues of $\hat{K}$.

[6 marks]

iii) Briefly discuss the physical meaning of $\hat{K}$. Determine its eigenvalues for the Dirac hydrogen energy levels $1S_{1/2}$, $2P_{1/2}$, and $2P_{3/2}$.

[3 marks]

**2.** a) State the selection rules for electric dipole transitions in light atoms. Which of the following transitions in carbon or nitrogen are NOT electric dipole transitions? Give your reasons.

i) $(2s)^2(2p)^3$, ${}^4S_{3/2} \rightarrow (2s)^2(2p)^2(3d)$, ${}^4P_{5/2}$
ii) $(2s)^2(2p)^2(3s)$, ${}^4D_{3/2} \rightarrow (2s)^2(2p)^2(3d)$, ${}^4P_{5/2}$
iii) $(2s)^2(2p)(3d)^2$, ${}^4D_{3/2} \rightarrow (2s)^2(2p)^3$, ${}^4S_{3/2}$
iv) $(2s)^2(2p)(3d)$, ${}^3D_0 \rightarrow (2s)^2(2p)^2$, ${}^3P_0$
v) $(2s)^2(2p)(3s)$, ${}^1P_1 \rightarrow (2s)^2(2p)^2$, ${}^1S_0$

[6 marks]

b) i) Find the energy spectrum of a nonrelativistic electron moving along a cylindrical surface of radius $R$ without any external field. Find the energy spectrum of the same electron (with $g = 0$) after a homogeneous magnetic field is applied parallel to the axis, described by the azimuthal vector potential $A_\varphi = Br/2$ for $r \leq R$. At what values of $B$ is the spectrum the same as for the zero-field case?

[9 marks]

ii) Now consider a relativistic electron moving along a cylindrical surface of radius $R$ with a homogeneous magnetic field applied parallel to the axis described by the azimuthal vector potential $A_\varphi = Br/2$ for $r \leq R$ as part (i). Given the identity for this potential,

$$[\sigma \cdot (\hat{\mathbf{p}} + e\mathbf{A})]^2 = (\hat{\mathbf{p}} + e\mathbf{A})^2 + e\hbar\sigma_z B,$$

express the Dirac equation of this electron in terms of the upper components only and find the energy eigenvalues. Consider the nonrelativistic limit and determine the electron's anomalous Zeeman energy.

[10 marks]

**3.** a) Briefly describe the three elementary rotations in terms of Euler angles $(\alpha, \beta, \gamma)$ for a general orientation. Show that the corresponding transformation operator is

$$\hat{U}(\alpha, \beta, \gamma) = e^{-i\alpha\hat{L}_z/\hbar} e^{-i\beta\hat{L}_x/\hbar} e^{-i\gamma\hat{L}_z/\hbar}.$$

You may use the transformation operator $\hat{U}_\phi = \exp(-i\phi\hat{L}_i/\hbar)$ for a rotation around the $i$-axis by angle $\phi$.

[8 marks]

b) A one-dimensional harmonic oscillator of mass $m$ and angular frequency $\omega$ is subject to a small constant force $F$ during the time period $0 < t < \tau$. Assuming that the oscillator is initially in the ground state, find the probability, in the first-order approximation, that the oscillator is in its first excited state after a sufficient time $(t > \tau)$. The ground and first excited state wavefunctions of the oscillator are

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right), \quad \phi_1(x) = \sqrt{\frac{2m\omega}{\hbar}}\, x\phi_0(x)$$

respectively. Determine the value of $\tau$ for the maximum transition probability.

[8 marks]

c) A $\pi^-$ meson is moving in the $xy$ plane in a perpendicular uniform magnetic field of magnitude $B$.

i) Show that such a magnetic field can be described by a vector potential $\mathbf{A} = (-yB, 0, 0)$.

ii) Derive the time-independent Klein-Gordon equation for the eigenfunctions of the meson. Obtain the meson's energy eigenvalues and eigenfunctions in terms of the known functions. Obtain the energy levels in the weak field limit.

iii) Show that the different vector potential $\mathbf{A} = (0, xB, 0)$ also describes the same magnetic field. Demonstrate the energy eigenvalues of the meson remain the same as obtained in part (ii).

[9 marks]

**END OF EXAMINATION PAPER**