

Properties of Matter

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Chapter 1

Properties of Solids and Liquids

1.1 Introduction

The kinetic energy per particle is $\approx k_B T$. Solids form when,

$$k_B T \ll \epsilon \quad (1.1)$$

where ϵ is the binding energy. Further, liquids form when,

$$k_B T \approx \epsilon \quad (1.2)$$

and gasses when,

$$k_B T \gg \epsilon. \quad (1.3)$$

1.2 Elasticity of Solids

Solids exhibit elastic behaviour, i.e., when stress is applied they deform (strain) and return to their original shape once removed. The way solids resist strain is defined by elastic moduli, which are generally defined,

$$\text{Elastic Modulus} = \frac{\text{Stress}}{\text{Strain}}. \quad (1.4)$$

1.2.1 Young's Modulus

Definition. E - The ability of a solid to resist forces along a given axis.

Consider a setup like in fig. 1.1. Stress is given by, $\frac{F}{A}$ and strain by, $\frac{\Delta l}{l}$ where Δl is the deformation along the axis of the force. We then have,

$$E = \frac{F}{A} \left(\frac{\Delta l}{l} \right)^{-1}. \quad (1.5)$$

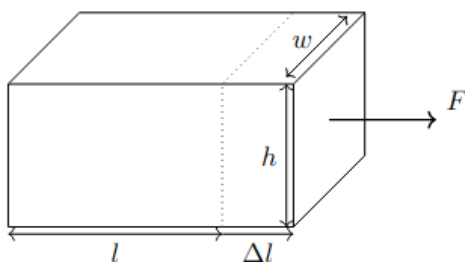


Figure 1.1: Solid undergoing deformation due to a force along its axis.

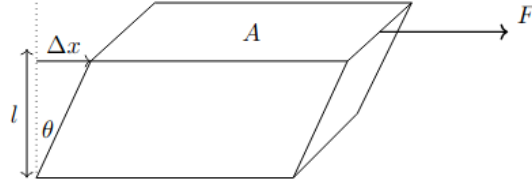


Figure 1.2: A solid undergoing shearing due to a force parallel to and along one of its faces.

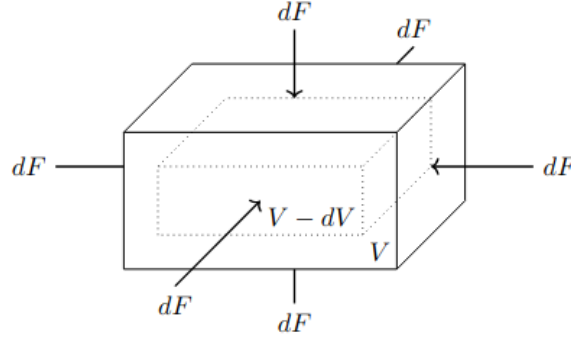


Figure 1.3: A solid of volume V being compressed by a force, dF , leading to an infinitesimal change in its volume dV .

1.2.2 Shear Modulus

Definition. G - The ability of a solid to resist shearing when a force is applied parallel to the surface of the solid.

Consider a setup like in fig. 1.2. If the solid shears by Δx due to a force F , then its strain is,

$$\frac{\Delta x}{l} = \tan \theta \cong \theta \quad (1.6)$$

for small θ . The shear modulus is then,

$$G = \frac{F}{A} \left(\frac{\Delta x}{l} \right)^{-1} = \frac{F}{A\theta}. \quad (1.7)$$

1.2.3 Bulk Modulus

Definition. K - Resistance of a material to compression.

The bulk modulus describes the stress due to a change in pressure dP . If we consider a setup like in fig. 1.3, the change in volume is $-dV$, the strain is dV/V , so the bulk modulus is,

$$K = -V \frac{dP}{dV}. \quad (1.8)$$

The minus sign indicates that an increase in pressure results in a decrease in volume.

1.3 Properties of Static Liquids

1.3.1 Hydrostatic Pressure

Consider a column of liquid of constant density ρ_l , a depth d , and cross-sectional area A as in fig. 1.4. Since the liquid is static (no net flow), we have $F_{\text{net}} = 0$. The force at d is then,

$$F = P_0 A + \rho_l g d A = P(d) A m \quad (1.9)$$

so the pressure at d is,

$$P(d) = P_0 + \rho_l g d. \quad (1.10)$$

\Rightarrow as d increases linearly, so does P .

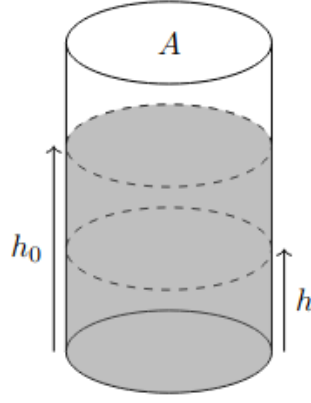


Figure 1.4: A column of liquid of constant density ρ , vertical height h_0 , and cross-sectional area A .

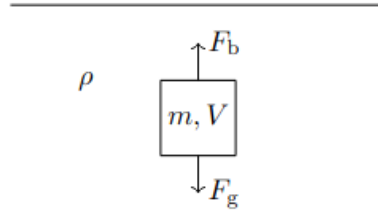


Figure 1.5: An arbitrary hollow shape of volume V filled with liquid of density ρ , immersed in a liquid of the same density.

1.3.2 Buoyancy and Archimedes Principle

Consider an arbitrary hollow shape of volume V filled with liquid of density ρ_l , immersed in a liquid of the same density, as in fig. 1.5. The weight of the object is given by,

$$F_w = \rho_l V g. \quad (1.11)$$

The pressure acting on the body must be equal and opposite to its weight. We call this the *buoyancy force*,

$$F_b = \rho_l V g. \quad (1.12)$$

The buoyancy force is *equal and opposite to the weight of the liquid displaced*. I.e., in the case that the mass has a density ρ_b such that $\rho_l > \rho_b$, the force it feels is still of the form eq. (1.12), so will float above the surface of the water.

1.4 Flow of Incompressible Liquid

The molecules in a liquid or gas can easily flow, with the same time and position dependent velocity, $\mathbf{v}(\mathbf{r}, t)$.

1.4.1 Conservation of mass and the continuity equation

The rate of mass flow through an area element of a surface $d\mathbf{S}$ is $\rho \mathbf{v} \cdot d\mathbf{S}$. The rate of mass flow into a volume bounded by a surface S is this expression evaluated over all space,

$$\frac{dm}{dt} = \rho \int_S \mathbf{v} \cdot d\mathbf{S}. \quad (1.13)$$

For a liquid of constant density, eq. (1.13) must equal 0,

$$\int_S \mathbf{v} \cdot d\mathbf{S} = 0 \quad (1.14)$$

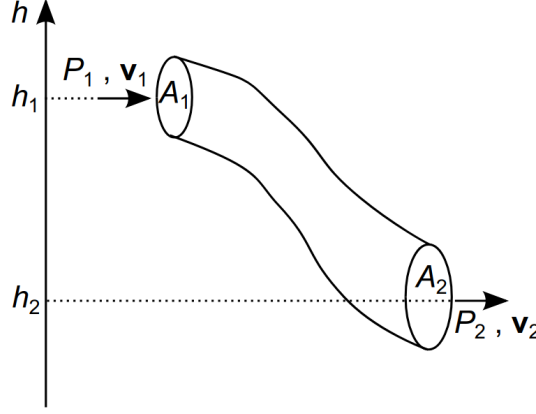


Figure 1.6: Frictionless pipe of arbitrary shape, with each end at a different height h_1 and h_2 such that $h_2 < h_1$.

and by the divergence theorem,

$$\boxed{\nabla \cdot \mathbf{v} = 0} \quad (1.15)$$

In the case of compressible liquids (i.e., where $\rho = \rho(\mathbf{r})$), conservation of mass is satisfied via the continuity equation,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}. \quad (1.16)$$

1.4.2 Bernoulli's Principle

We will investigate how pressure of a liquid depends on flow speed. We assume no surface or viscous forces, and we assume flow is time independent, i.e., $\mathbf{v} = \mathbf{v}(\mathbf{r})$.

Particles will flow in streamlines. Consider adjacent streamlines which form a tube of arbitrary shape, where each end is at a different vertical height, h_1 and h_2 such that $h_2 < h_1$, and there is no flow in or out of the tube, as in fig. 1.6. The liquid enters at height h_1 , with velocity v_1 , pressure P_1 , through an area A_1 . The liquid leaves at height h_2 , with velocity v_2 , pressure P_2 , through an area A_2 .

From the conservation of mass,

$$v_1 A_1 = v_2 A_2. \quad (1.17)$$

In a time dt , the work done by pressure P_1 on the liquid at entry is,

$$dW_1 = P_1 A_1 v_1 dt \quad (1.18)$$

and similarly for the exit,

$$dW_2 = P_2 A_2 v_2 dt. \quad (1.19)$$

The kinetic energy of the liquid is,

$$dK_1 = \frac{1}{2} (\rho v_1 A_1 dt) v_1^2, \quad (1.20)$$

and the potential energy,

$$dV = (\rho v_1 A_1 dt) gh_1, \quad (1.21)$$

and similarly for the exit. By conservation of energy,

$$\begin{aligned} dW_1 + dK_1 + dV_1 &= dW_2 + dK_2 + dV_2 \\ A_1 v_1 \left(P_1 + \frac{1}{2} \rho v_1^2 + \rho gh_1 \right) dt &= A_2 v_2 \left(P_2 + \frac{1}{2} \rho v_2^2 + \rho gh_2 \right) dt. \end{aligned} \quad (1.22)$$

By eq. (1.17), we can divide eq. (1.22) by $A_1 v_1 dt$,

$$P_1 + \frac{1}{2} \rho v_1^2 + \rho gh_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho gh_2 \quad (1.23)$$

which leads to Bernoulli's equation,

$$P(h, v) + \rho gh + \frac{1}{2}\rho v^2 = \text{constant} \quad (1.24)$$

\implies faster moving liquids have lower pressure.

1.4.3 Flow Properties and Viscosity

Laminar Flow

Definition. *Smooth, regular streamlines that are locally parallel to each other*

The drag force on a spherical object of radius R insterted into a liquid under laminar flow is caused by the liquid's resistance to shearing, and is given by Stoke's law,

$$\mathbf{F}_D = 6\pi\nu R\mathbf{v} \quad (1.25)$$

where ν is the viscosity of the liquid.

Turbulent Flow

Definition. *Streamlines are irregular and chaotic du to formation of eddies in the flow.*

The drag force on an object placed in the stream occurs due to the inertial force required to push the liquid out of the way of the object. The drag force is given by,

$$\mathbf{F}_D = \frac{1}{2}\rho v^2 C_D A \hat{\mathbf{v}} \quad (1.26)$$

where A is the cross-sectional area of the object and C_D is a dimensionless drag coefficient.

Laminar-Turbulent Transition and Reynold's Number

Whether a liquid experiences laminar or turbulent flow depends on the ratio of inertial force to viscous force. This ratio is given by Reynold's number,

$$Re = \frac{\rho L v}{\nu} \quad (1.27)$$

where L is the length of the object moving through the liquid.

1.5 Liquid Surfaces

Free Surface: The surface interface of a liquid is characterised by the surface free energy γ . This is the energy required to increase the surface area of the interface.

Liquid-Liquid Interface: At the interface between two liquids, it is only energetically favourable for them to mix if $\gamma < 0$.

Liquid-Solid Interface: For $\gamma < 0$, it is energetically favourable for the liquid to spread over the surface of the solid.

1.5.1 Liquids in a Container

Contact Angle

We are interested in the way a liquid-gas interface meets the solid wall of a container. The shape of the liquid near the wall is known as the meniscus, as in fig. 1.7. We can approximate the meniscus as linear, where the liquid meets the wall at the contact angle θ , at a height h , and horizontal distance l from the surface of the liquid.

Let us consider the following surface-free energies,

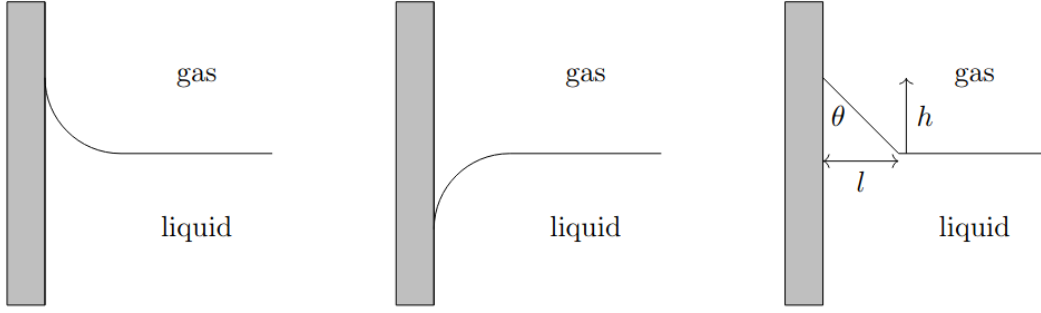


Figure 1.7

1. Solid-gas γ_{sg} ,
2. Solid-liquid γ_{sl} ,
3. Liquid-gas γ_{lg} .

The meniscus free energy (per unit length of the meniscus, thus having dimensions of force) is given by,

$$F_{\text{men}} = (\gamma_{sl} - \gamma_{sg}) + \gamma_{lg} \left(\sqrt{h^2 + l^2} - l \right). \quad (1.28)$$

The free energy is minimised when,

$$\frac{\partial F_{\text{men}}}{\partial h} = (\gamma_{sl} - \gamma_{sg}) + \gamma_{lg} \left(\frac{1}{2} \frac{2h}{\sqrt{h^2 + l^2}} \right) = 0. \quad (1.29)$$

We have that,

$$\frac{h}{\sqrt{h^2 + l^2}} = \cos \theta \quad (1.30)$$

which allows us to find Young's equation relating contact angle to surface free energy,

$$\boxed{\gamma_{lg} \cos \theta = \gamma_{sg} - \gamma_{sl}}. \quad (1.31)$$

Capillary Action

This occurs when a column of liquid is placed in a narrow tube. Consider a tube of radius R at height h relative to the surface of the bulk liquid. The surface free energy gain is,

$$\Delta F_s = 2\pi R h (\gamma_{sl} - \gamma_{sg}) \quad (1.32)$$

while the increase in gravitational potential energy is,

$$\Delta F_g = \pi R^2 h \rho g \frac{h}{2} \quad (1.33)$$

where $h/2$ is used as that is the location of the centre of mass.

The total free energy is,

$$F = \Delta F_s + \Delta F_g = \frac{\pi}{2} R^2 h^2 \rho g + 2\pi R h (\gamma_{sl} - \gamma_{sg}) \quad (1.34)$$

which is minimised when,

$$\frac{\partial F}{\partial h} = \pi R^2 h \rho g + 2\pi R (\gamma_{sl} - \gamma_{sg}) = 0, \quad (1.35)$$

which can be rearranged to give a function for h ,

$$\boxed{h = -\frac{2}{(\gamma_{sl} - \gamma_{sg})} \rho g R = \frac{2\gamma_{lg} \cos \theta}{\rho g R}} \quad (1.36)$$

1.5.2 Surface Tension and Surface Free Energy

Consider a wire frame, with one side of length l , free to move in the x direction. A thin film of liquid covers the space within the frame and shifts the wire a distance dx in order to minimise the overall free energy.

The work done on the wire to move a distance dx due to a force F applied by the thin film of liquid is,

$$dW = F dx. \quad (1.37)$$

The work also contributes to increasing the surface area of the liquid by,

$$dA = 2l dx. \quad (1.38)$$

The energy increase is given by,

$$dE_s = \gamma dA = 2\gamma l dx. \quad (1.39)$$

Equating eq. (1.37) and eq. (1.39) gives an expression for the *surface tension*,

$$\boxed{F = 2\gamma l}. \quad (1.40)$$

Chapter 2

Thermodynamics

In this chapter we deal with analysing the macroscopic properties of objects composed of many microscopic atoms and particles. We will first consider an idea gas, i.e., one in which,

- the particles have no volume,
- and the particles have no interatomic forces acting between them.

This allows us to write down the ideal gas equation in two forms,

$$\underbrace{P}_{\text{Pressure}} \underbrace{V}_{\text{Volume}} = \underbrace{n}_{\text{Moles}} \underbrace{R}_{\text{Molar gas constant}} \underbrace{T}_{\text{Temperature}} \quad (2.1)$$

$$PV = \underbrace{N}_{\text{Number of molecules}} \underbrace{k_b}_{\text{Boltzmann constant}} T. \quad (2.2)$$

1 mole of substance contains N_A (Avogadro's number) amount of molecules. We further have a relation between the molar gas constant and the Boltzman constant which states,

$$R = N_A k_b. \quad (2.3)$$

In this course, we will often discuss 2 different variables,

1. **Intensive Variables** - Value is not proportional to the amount of substance.
2. **Extensive Variables** - Value is linearly proportional to the amount of substance.

2.1 First law of thermodynamics

Law 1.

$$\underbrace{\Delta E}_{\text{Change in internal energy}} = \underbrace{W}_{\text{Work done on the system}} + \underbrace{Q}_{\text{Heat supplied to a system}} \quad (2.4)$$

We say that E is a function of state, meaning that the value of E is path independent. For an ideal gas, we have that $E = E(T)$ which is the energy due to the kinetic energy of the particles. If we take the limit as $\Delta E \rightarrow 0$,

$$\boxed{dE = dQ + dW}. \quad (2.5)$$

Furthermore, W and Q are not functions of state, meaning that their values are path dependent. This means that,

$$\Delta E = W + Q \equiv \text{Const} \quad (2.6)$$

however, W and Q will not be constant as they depend on the shape of the function they take on.

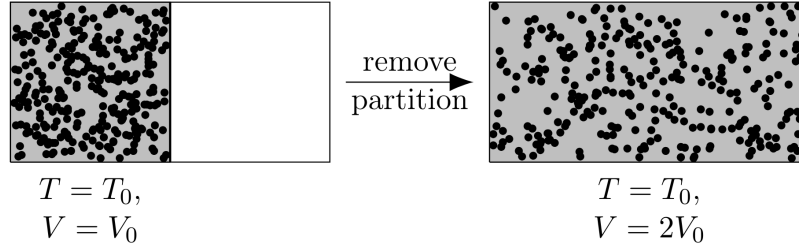


Figure 2.1

2.1.1 Cyclic Process

This is a process which starts and ends in the same equilibrium state, so there is no change in energy over the cycle,

$$\oint dE = E_{\text{final}} - E_{\text{initial}} = 0. \quad (2.7)$$

From the first law,

$$\oint dW + \oint dQ = 0 \quad (2.8)$$

and thus,

$$\boxed{\oint dW = - \oint dQ} \quad (2.9)$$

for a cyclic process.

2.1.2 Quasistatic Process

This is a process that occurs so slowly that the system is always instantaneously in thermal equilibrium. For a quasistatic process, the force applied is approximately equal to the pressure of the fluid itself,

$$\boxed{\frac{|\mathbf{F}|}{A} \simeq P}. \quad (2.10)$$

2.1.3 Reversible Processes

For a reversible process, we require,

1. The process must be *quasistatic*.
2. There is no external friction.
3. The process must not cause permanent change to the system.

Free expansion of an isolated gas

Let us consider a gas confined in a volume which is separated in half, with the gas initially in only one part of the volume as in fig. 2.1. If we remove the partition instantaneously, we have,

$$W = 0 \quad Q = 0 \quad \implies \quad \Delta E = 0. \quad (2.11)$$

However, this process is *non-reversible*, as we have caused a permanent change to the system.

Isothermal expansion of an ideal gas

Now consider a setup as in fig. 2.2, where we use a piston to change the volume of the container, and we use a source to keep the temperature of the gas constant. We have,

$$W < 0 \quad Q > 0 \quad \implies \quad (Q + W) = 0 \quad \implies \quad \Delta E = 0. \quad (2.12)$$

This process is *reversible* given that,

- The piston is frictionless,
- The piston moves slowly.

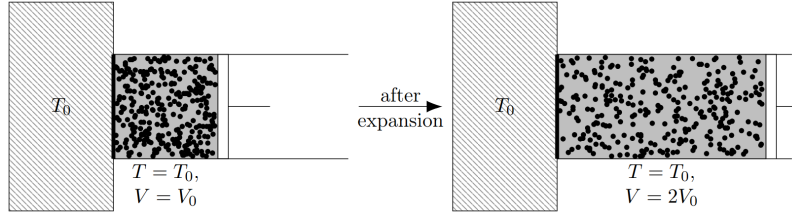


Figure 2.2

2.1.4 Work Done

Generally, the work done on a fluid is represented by the area under the curve on an *indicator diagram*, which plots pressure against volume. If we consider a piston of cross-sectional area A compressing a fluid, we can write the work done as,

$$\boxed{dW = \mathbf{F} \cdot d\mathbf{x} = \underbrace{\frac{|\mathbf{F}|}{A}}_P \underbrace{A dx}_{-dV} = -P dV.} \quad (2.13)$$

For an ideal gas at temperature, we can evaluate this as,

$$\begin{aligned} W &= - \int_{V_i}^{V_f} \frac{nRT}{V} dV \\ &= nRT \ln \left(\frac{V_i}{V_f} \right). \end{aligned} \quad (2.14)$$

For a cyclic process, the net work done is an enclosed area on the $P - V$ diagram.

Reversible Work

Stretching of a wire

If we have a string under tension Γ which undergoes an infinitesimal change in length $d\ell$, we have,

$$dW = \Gamma d\ell. \quad (2.15)$$

Increase in surface area

For a surface tension γ and a change in area dA ,

$$dW = \gamma dA. \quad (2.16)$$

2.1.5 Heat Capacity

Heat capacity C is the amount of energy required to raise the temperature by dT ,

$$dQ = C dT. \quad (2.17)$$

Heat capacity at constant volume

Rewriting the first law,

$$dQ = C_V dT = dE - dW = dE + \underbrace{P dV}_0 \quad (2.18)$$

where the subscript V indicates volume is being held constant. We can take $P dV = 0$ since V is constant, and $\Delta V \rightarrow 0, dV = 0$. Let us write $E = (V, T)$,

$$dE = \frac{\partial E}{\partial T} dT + \underbrace{\frac{\partial E}{\partial T}}_0 dV. \quad (2.19)$$

Equating eq. (2.18) to (2.19), we get,

$$\mathrm{d}Q = C_V \mathrm{d}T = \frac{\partial E}{\partial T} \mathrm{d}T \quad (2.20)$$

from which we obtain our final expression,

$$C_V = \frac{\partial E}{\partial T}. \quad (2.21)$$

Heat capacity at constant pressure

Rewriting the first law,

$$\mathrm{d}Q = C_P \mathrm{d}T = \mathrm{d}E + P \mathrm{d}V. \quad (2.22)$$

Writing $E = E(T, P)$,

$$\mathrm{d}E = \frac{\partial E}{\partial T} \mathrm{d}T + \underbrace{\frac{\partial E}{\partial P} \mathrm{d}P}_0. \quad (2.23)$$

We can then obtain our final relation,

$$\boxed{C_P = \frac{\partial E}{\partial T} + P \frac{\partial V}{\partial T}} \quad (2.24)$$

Ideal Gas

In an ideal gas, $E = E(T)$, i.e., energy has sole dependence on temperature. Thus,

$$\frac{\mathrm{d}E}{\mathrm{d}T} = \frac{\partial E}{\partial T} = C_V(T). \quad (2.25)$$

From the ideal gas law,

$$P \frac{\partial V}{\partial T} = nR. \quad (2.26)$$

From which we get the relation,

$$\boxed{C_P(T) = C_V(T) + nR}. \quad (2.27)$$

2.1.6 Adiabatic Processes

For an adiabatic process, there is *no heat transfer*, i.e., $Q = 0$. From the first law and eq. (2.25),

$$\mathrm{d}E - \mathrm{d}W = C_V \mathrm{d}T - \mathrm{d}W = 0. \quad (2.28)$$

From the ideal gas equation,

$$T = \frac{PV}{nR} \quad (2.29)$$

whose differential is,

$$\mathrm{d}T = \frac{1}{nR} (P \mathrm{d}V + V \mathrm{d}P) \quad (2.30)$$

from which the differential of E follows,

$$\mathrm{d}E = \frac{C_V}{nR} (P \mathrm{d}V + V \mathrm{d}P). \quad (2.31)$$

Substituting eq. (2.31) into eq. (2.25) and assuming a reversible process ($\mathrm{d}W = -P \mathrm{d}V$), we obtain,

$$\begin{aligned} C_V \mathrm{d}T + P \mathrm{d}V &= \frac{C_V}{nR} (P \mathrm{d}V + V \mathrm{d}P) + P \mathrm{d}V = 0 \\ &\quad C_V V \mathrm{d}P + \underbrace{(C_V nR)}_{C_P} P \mathrm{d}V = 0 \\ \implies \frac{1}{P} \mathrm{d}P + \frac{C_P}{C_V} \frac{1}{V} \mathrm{d}V &= 0. \end{aligned} \quad (2.32)$$

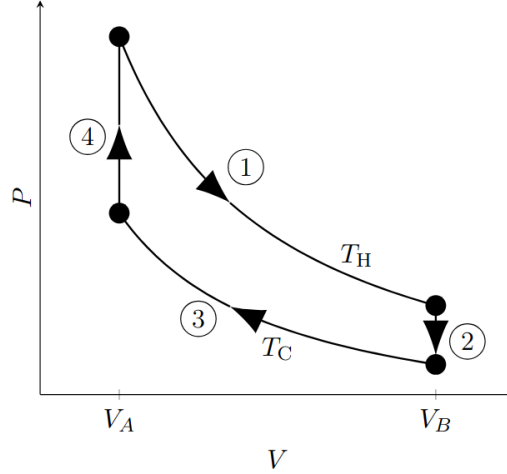


Figure 2.3

If we have that C_P and C_V are constant, let us define,

$$\gamma = \frac{C_P}{C_V} \quad (2.33)$$

which is also constant. We can integrate eq. (2.32) from $(P_1, V_1) \rightarrow (P_2, V_2)$,

$$\int_{P_1}^{P_2} \frac{1}{P} dP + \gamma \int_{V_1}^{V_2} \frac{1}{V} dV = 0 \implies \ln \left(\frac{P_2 V_2^\gamma}{P_1 V_1^\gamma} \right) = 0 \quad (2.34)$$

Removing the logarithm from eq. (2.34), we obtain an expression relating the initial and final pressure and volume for an adiabatic process,

$$\boxed{P_2 V_2^\gamma = P_1 V_1^\gamma} \quad (2.35)$$

and by using the ideal gas law, we can eliminate variables and find expressions involving temperature,

$$\boxed{T_2 V_2^{\gamma-1} = T_1 V_1^{\gamma-1}} \quad (2.36)$$

$$\boxed{T_2 P_2^{\frac{1}{\gamma}-1} = T_1 P_1^{\frac{1}{\gamma}-1}} \quad (2.37)$$

2.2 Second Law of Thermodynamics

Law 2. *The direction of heat flow approaches equilibrium, such that,*

- *Systems at an intermediate temperature reach equilibrium at an intermediate temperature.*
- *Heat naturally flows from hot bodies to cold bodies,*
- *We require an engine to produce work from heat.*

2.2.1 Heat engines

Heat engines produce work from heat. They operate in a cyclic process such that,

$$\Delta E = \oint dE = 0. \quad (2.38)$$

Sterling Engine

The Sterling engine is a piston operating between two heat reservoirs, with a hot reservoir at temperature T_H and a hot reservoir at temperature T_C where $T_C < T_H$ and contains an ideal gas. It works in a cycle such that,

1. *Isothermal expansion*

- Work done *by* gas,

$$W_1 = nRT_H \ln \left(\frac{V_B}{V_A} \right) \quad (2.39)$$

- Heat *absorbed*,

$$Q_1 = W_1 \quad (2.40)$$

2. *Contact cooling*

- No work done.
- Heat *ejected*,

$$Q_2 = E(T_H) - E(T_C) \quad (2.41)$$

3. *Isothermal compression*

- Work done *on* gas,

$$W_3 = nRT_C \ln \left(\frac{V_B}{V_A} \right) \quad (2.42)$$

- Heat *ejected*

$$Q_3 = W_3 \quad (2.43)$$

4. *Contact heating*

- No work done
- Heat *absorbed*,

$$Q_4 = E(T_H) - E(T_C) \quad (2.44)$$

This cycle can be visualised on an indicator diagram as in fig. 2.3. Furthermore, this cycle can all be derived from the first law and $dW = -P dV$.

Let us look at the cycle in more detail. The net work done *by* the gas is,

$$W = W_1 - W_3 = nR(T_H - T_C) \ln \left(\frac{V_B}{V_A} \right). \quad (2.45)$$

The heat absorbed from the reservoir is at T_H is,

$$Q_H = Q_1 + Q_4. \quad (2.46)$$

The heat ejected *to* the hot reservoir is,

$$Q_C = Q_2 + Q_3. \quad (2.47)$$

From the first law,

$$\Delta E = (-W) + (Q_H - Q_C) = 0 \quad (2.48)$$

and therefore,

$$W = Q_H - Q_C = nR \ln \left(\frac{V_B}{V_A} \right) T_H + C_V \Delta T - nR \ln \left(\frac{V_B}{V_A} \right) T_C = nR \ln \left(\frac{V_B}{V_A} \right) \Delta T. \quad (2.49)$$

where $\Delta T = T_H - T_C$. The efficiency of the engine is the ratio of desired output to required input, so,

$$\eta_E = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H} = \frac{nR \ln \left(\frac{V_B}{V_A} \right) \Delta T}{nR \ln \left(\frac{V_B}{V_A} \right) T_H + C_V \Delta T}. \quad (2.50)$$

A schematic of the heat flow of a Sterling engine is shown in fig. 2.4.

2.2.2 Heat pumps and refrigerators

These are the opposite of engines, they do work on a system to extract heat.

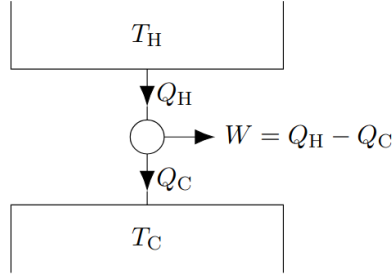


Figure 2.4

Refrigerator

Work is done to extract heat Q_C from a system to lower its temperature T_C . In this case, the efficiency of the system is given as,

$$\eta_R = \frac{Q_C}{Q_H - Q_C}. \quad (2.51)$$

Heat Pump

In a heat pump, we extract heat from a cooler reservoir at T_C to heat a system to T_H . The desired output is Q_H , so,

$$\eta_P = \frac{Q_H}{Q_H - Q_C}. \quad (2.52)$$

If an engine is reversible, then,

$$\eta_P = \frac{1}{\eta_E}. \quad (2.53)$$

2.2.3 Carnot's Cycle

Carnot's theorem states,

Theorem.

1. *A reversible engine is most efficient.*
2. *All reversible engines, operating between two heat baths, have the same efficiency, η_C depending only on T_H and T_C .*

This type of engine can be constructed using the Carnot cycle (whose P-V diagram is shown in fig. 2.5),

1. Isothermal Expansion at T_H ,

- $\Delta T = 0, \Delta E = 0$.
- $dW = nRT_H \ln\left(\frac{V_B}{V_A}\right)$.

2. Adiabatic Expansion

- $dQ \implies dE = dW$
- $PV^\gamma = \text{Const.} \implies P \propto V^{-\gamma}$
- By the ideal gas law, $P = \frac{nRT}{V} \implies \frac{nRT}{V} V^\gamma = \text{Const.} \implies T_H V_B^{\gamma-1} = T_C V_C^{\gamma-1}$.

3. Isothermal Compression

- $Q = nRT_C \ln\left(\frac{V_C}{V_D}\right)$

4. Adiabatic Compression

- $T_H V_A^{\gamma-1} = T_C V_D^{\gamma-1}$.

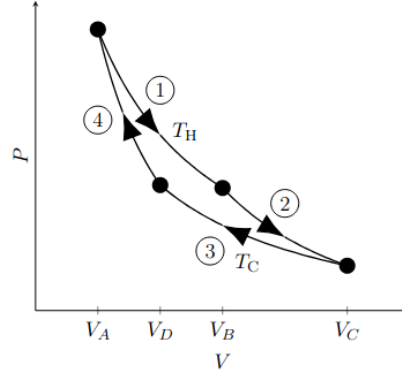


Figure 2.5

From the cycle above, we find that the ratio,

$$\frac{V_B}{V_A} = \frac{V_C}{V_D} \quad (2.54)$$

holds. From this, we can write down the efficiency of this engine,

$$\eta_C = \frac{W}{Q_H} = \frac{nRT_H \ln\left(\frac{V_B}{V_A}\right) - nRT_C \ln\left(\frac{V_C}{V_D}\right)}{nRT_H \ln\left(\frac{V_B}{V_A}\right)} = \frac{nRT_H \ln\left(\frac{V_B}{V_A}\right) - nRT_C \ln\left(\frac{V_B}{V_A}\right)}{nRT_H \ln\left(\frac{V_B}{V_A}\right)} \quad (2.55)$$

from which we can write down the final expression for the *Carnot efficiency*,

$$\boxed{\eta_C = 1 - \frac{T_C}{T_H}}. \quad (2.56)$$

Let us note that we find that the Stirling cycle approaches Carnot efficiency for small values of C_V .

2.2.4 Statements of the Second Law

It is difficult to state the second law in a concise, general way through onyl understanding heat. However, below are two common formulations.

Kelvin-Planck Statement

It is impossible to construct an engine which, operating in a cycle, produces no effect other than the extraction of heat from the reservoir and the performance of an equivalent amount of work.

Claussius Statement

It is impossible to construct a refrigerator which, operating in a cycle, produces no effect other than the transfer of heat from a cooler body to a hotter one.

2.3 Entropy

Entropy is encoded into Calusiu's inequality, which states that if a system is taken over a cycle, absorbing infinitesimal amounts of heat dQ , from various reservoirs at varing temperatures T , throughout, then

$$\oint \frac{dQ}{T} \leq 0. \quad (2.57)$$

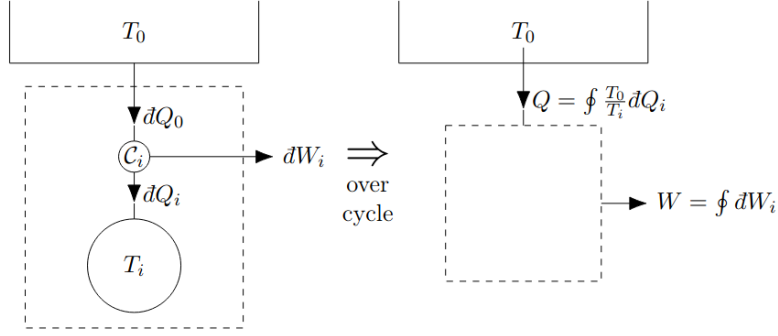


Figure 2.6

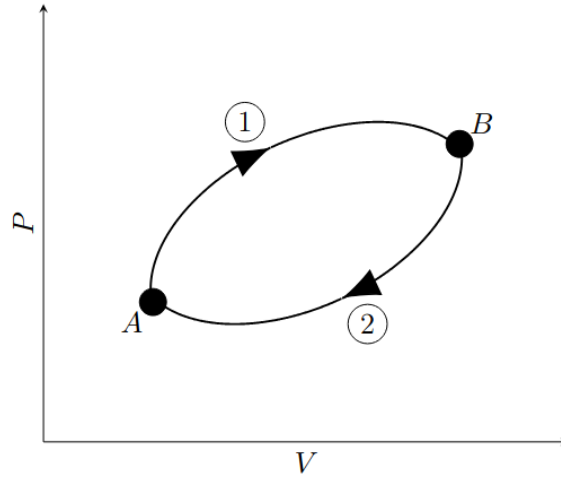


Figure 2.7

2.3.1 Proof

Let us consider a cycle as in fig. 2.6. We have that the heat input is,

$$Q_H = \oint dQ_H = T_H \oint \frac{dQ_i}{T_i} \quad (2.58)$$

and the total work done is,

$$W = \oint dW_i. \quad (2.59)$$

The only way for this cycle to be possible is if the work done is 0 or negative. Thus,

$$\oint dW_i \leq 0. \quad (2.60)$$

We can write $\oint dW_i = \oint dQ_i$ and,

$$\oint \frac{dQ_i}{T_i} \leq 0. \quad (2.61)$$

2.3.2 Mathematical definition of entropy

We define infinitesimal entropy of a system,

$$dS = \frac{dQ}{T}. \quad (2.62)$$

We only have constant entropy in equilibrium. Let us consider a process like in fig. 2.7. The total

entropy of this system is,

$$\oint \frac{dQ}{T} = \int_A^B \frac{dQ}{T} + \int_B^A \frac{dQ}{T} = 0$$

$$\Rightarrow \int_A^B \frac{dQ}{T} - \int_A^B \frac{dQ}{T} = 0$$
(2.63)

thus we have shown that entropy is path independent and thus is a function of state. Let us now define entropy further. Let us consider a reference state O as we go from A to B . We have,

$$\begin{aligned} \int_A^B \frac{dQ}{T} &= \int_A^O \frac{dQ}{T} + \int_O^B \frac{dQ}{T} \\ &= \int_A^O \frac{dQ}{T} - \int_O^A \frac{dQ}{T} \\ &= S(B) - S(A). \end{aligned}$$
(2.64)

We can then define the entropy,

$$\boxed{S(A) = \int_O^A \frac{dQ_{\text{rev}}}{T}}.$$
(2.65)

NOTE: Entropy is defined in terms of a reversible process.

2.3.3 The law of increase in entropy

Let us now consider a similar scenario to fig. 2.7, but we have an irreversible process as we go from A to B . Applying Clausius' inequality,

$$\begin{aligned} \int_A^B \frac{dQ_{\text{irrev}}}{T} + \int_B^A \frac{dQ_{\text{rev}}}{T} &\leq 0 \\ \int_A^B \frac{dQ_{\text{irrev}}}{T} &\geq \int_A^B \frac{dQ_{\text{rev}}}{T} \end{aligned}$$
(2.66)

which gives us our final statement,

$$S(B) - S(A) \geq \int_A^B \frac{dQ_{\text{irrev}}}{T}$$
(2.67)

Appendix A

Definitions

Isothermal Process: Happens at constant temperature, $\Delta E = 0$ for an ideal gas.

Isobaric Process: Happens at constant pressure.

Isochoric Process: Happens at constant volume $\implies dW = 0$.

Adiabatic Process: Happens with no heat flow $\implies dQ = 0$.

Heat Capacity: Relates heat flow to temperature change $dQ = C dT$.