PHYS 10672 Advanced Dynamics : 2024

Section 3 : Central forces and gravitation

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Central forces and gravitation - see FS chapter 9

3.1 Conservative forces - see FS9.1

The work done by a force $\mathbf{F}$ on a particle moving over an infinitesimal interval $d\mathbf{r}$ is

$$dW = \mathbf{F}\cdot d\mathbf{r} = |\mathbf{F}||d\mathbf{r}|\cos\theta\,. \tag{3.1}$$

The important things that you learnt in the Dynamics course were : (i) you need to integrate this expression if the force varies along the trajectory - don't just multiply by the distance; (ii) it is the component of the force tangential to the component which does work - that is there is the "$\cos\theta$". Using this we can calculate the work done along a trajectory from point A to point B as

$$W_{AB} = \int_A^B \mathbf{F}\cdot d\mathbf{r}\,. \tag{3.2}$$

Now N2 implies that $\mathbf{F} = m\frac{d\mathbf{v}}{dt}$ and so

$$W_{AB} = m\int_A^B \frac{d\mathbf{v}}{dt}\cdot d\mathbf{r} = m\int_A^B \frac{d\mathbf{v}}{dt}\cdot \mathbf{v}dt = \frac{1}{2}m\int_A^B \frac{d}{dt}\left(|\mathbf{v}|^2\right)dt = \frac{1}{2}\left(|\mathbf{v}_B|^2 - |\mathbf{v}_A|^2\right)\,, \tag{3.3}$$

that is, the work done is the change in the kinetic energy, ΔKE.

For a conservative force the the work done, W_{AB}, is independent of the path between the two points. and it implies the existence of a potential, $U(\mathbf{r})$ such that $dU = -\mathbf{F}\cdot d\mathbf{r}$. Therefore, we find that

$$\Delta\mathrm{KE} = \int_A^B \mathbf{F}\cdot d\mathbf{r} = -\int_A^B dU = U(\mathbf{r}_A) - U(\mathbf{r}_B) = -\Delta\mathrm{PE}\,, \tag{3.4}$$

and $\frac{1}{2}m|\mathbf{v}|^2 + U(\mathbf{r}) \equiv$ const. Hopefully, most of this is familiar to you from your previous studies, even if it has not been formulated in this formal way.

Now

$$dU = dU(x,y,z) = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz = \frac{\partial U}{\partial r_i}dr_i\,, \tag{3.5}$$

with $\mathbf{r} = (x,y,z) = (r_1,r_2,r_3)$. If we define the "gradient" operator to be

$$\boldsymbol{\nabla} = \mathbf{e}_i\frac{\partial}{\partial r_i}\,, \tag{3.6}$$

then

$$\boldsymbol{\nabla} U = \mathbf{e}_i \frac{\partial U}{\partial r_i}\,, \tag{3.7}$$

and $dU = \boldsymbol{\nabla} U \cdot d\mathbf{r}$.

Remembering that

$$\begin{aligned} \mathbf{A}\times\mathbf{B} &= \mathbf{e}_i \epsilon_{ijk} A_j B_k\,, \\ \mathbf{A}\cdot\mathbf{B} &= A_i B_i\,, \end{aligned} \tag{3.8}$$

we can set $A_j = \frac{\partial}{\partial r_j}$ and $B_k = v_k$ in order to define the "curl" of the vector $\mathbf{v}$ to be

$$\boldsymbol{\nabla}\times\mathbf{v} = \mathbf{e}_i \epsilon_{ijk} \frac{\partial v_k}{\partial r_j}\,, \tag{3.9}$$

and the "divergence" to be

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \delta_{ij}\frac{\partial v_i}{\partial r_j} = \frac{\partial v_i}{\partial r_i}\,. \tag{3.10}$$

These derivatives of vectors are studied in the Maths II course and are also important in the Electricity of Magnetism course. You should try to make every attempt to become familiar with them. Note that they can be thought of in a compact way in terms of the summation convention and the isotropic tensors δ_{ij} and ϵ_{ijk}.

Now consider the curl of the force of a conservative potential

$$\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{e}_i \epsilon_{ijk} \frac{\partial}{\partial r_j}\left(-\frac{\partial U}{\partial r_k}\right)\,. \tag{3.11}$$

One can write

$$\epsilon_{ijk}\frac{\partial^2 U}{\partial r_j \partial r_k} = \epsilon_{ikj}\frac{\partial^2 U}{\partial r_k \partial r_j} = -\epsilon_{ijk}\frac{\partial^2 U}{\partial r_j \partial r_k} = 0\,, \tag{3.12}$$

where the first equality is achieved by renaming the summed indices $j \leftrightarrow k$ and the second uses $\epsilon_{ijk} = -\epsilon_{ikj}$ and

$$\frac{\partial^2 U}{\partial r_j \partial r_k} = \frac{\partial^2 U}{\partial r_k \partial r_j}\,, \tag{3.13}$$

that is, the partial derivatives commute. The expression is zero since we have shown that $A_i = -A_i$ for some vector. Note that this is a special case of the more general results that $A_{ij}B_{ij} = 0$ if A_{ij} is symmetric and the B_{ij} is antisymmetric. Hence, we deduce that the condition for a force $\mathbf{F}$ to be conservative, that is, it can be created from a potential is that

$$\boldsymbol{\nabla}\times\mathbf{F} \equiv 0\,. \tag{3.14}$$

3.2 Central forces - see FS9.1

A force law is called a "Central Force" if it only depends on $r = |\mathbf{r}|$ and is in the direction $\hat{\mathbf{r}}$, that is,

$$\mathbf{F}(\mathbf{r}) = f(r)\hat{\mathbf{r}} = f(r)\frac{\mathbf{r}}{r}\,. \tag{3.15}$$

Now if the force is also conservative then $dU = -\mathbf{F}\cdot d\mathbf{r}$ and so

$$U(\mathbf{r}) - U(\mathbf{r}_0) = -\int_{\mathbf{r}_0}^{\mathbf{r}} f(r)\hat{\mathbf{r}}\cdot d\mathbf{r} = -\int_{r_0}^{r} dr' f(r')\,. \tag{3.16}$$

Note that $U(r_0)$ is arbitrary and does not effect the physics, that is, the force $\mathbf{F}$ is not changed by the choice of $U(r_0)$. A common choice is $U(\infty) = 0$ in which case

$$U(\mathbf{r}) = -\int_{\infty}^{r} dr' f(r')\,, \tag{3.17}$$

and $-U(r)$ is the kinetic energy of a particle at r which was initially at rest at ∞.

3.3 Newtonian Gravity - see FS9.1

Newtonian gravity is an example of a conservative central force law where the force between two particle of masses m and M is

$$\mathbf{F} = -G_{\mathrm{N}}\frac{Mm}{r^2}\hat{\mathbf{r}}\,, \tag{3.18}$$

where Newton's constant is $G_{\mathrm{N}} = 6.67\times 10^{-11}\,\mathrm{m}^2\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ and we can read off $f(r) = -G_{\mathrm{N}}Mm/r^2$. From this we can calculate the potential

$$U(r) = G_{\mathrm{N}}Mm\int_{\infty}^{r}\frac{dr'}{(r')^2} = G_{\mathrm{N}}Mm\left[-\frac{1}{r'}\right]_{\infty}^{r} = -G_{\mathrm{N}}\frac{Mm}{r}\,. \tag{3.19}$$

Now consider the other way around can calculate the force:

$$\mathbf{F} = -\boldsymbol{\nabla}U = -\begin{pmatrix}\frac{\partial U}{\partial x}\\ \frac{\partial U}{\partial y}\\ \frac{\partial U}{\partial z}\end{pmatrix} = G_{\mathrm{N}}Mm\boldsymbol{\nabla}\left(\frac{1}{r}\right)\,. \tag{3.20}$$

One can can calculate a general expression for $\boldsymbol{\nabla}g(r)$

$$\boldsymbol{\nabla}g = \mathbf{e}_i\frac{\partial g}{\partial r_i} = \mathbf{e}_i\frac{\partial g}{\partial r}\frac{\partial r}{\partial r_i}\,. \tag{3.21}$$

But $r^2 = r_j r_j = \sum_j r_j r_j$ and therefore, by differentiation with respect to r_i, we find that

$$2r\frac{\partial r}{\partial r_i} = \frac{\partial r_j}{\partial r_i}r_j + r_j\frac{\partial r_j}{\partial r_i} = \delta_{ij}r_j + r_j\delta_{ij} = 2r_i\,, \tag{3.22}$$

since $\frac{\partial r_j}{\partial r_i} = \delta_{ij}$. We have that

$$\frac{\partial r}{\partial r_i} = \frac{r_i}{r} = \hat{r}_i \tag{3.23}$$

and so

$$\boldsymbol{\nabla} g = \frac{\partial g}{\partial r} \hat{\mathbf{r}} \,. \tag{3.24}$$

In the present context with $g = 1/r$ we find that $\boldsymbol{\nabla}(1/r) = -\hat{\mathbf{r}}/r^2$ and so

$$\mathbf{F} = -G_{\mathrm{N}} \frac{Mm}{r^2} \hat{\mathbf{r}} \,, \tag{3.25}$$

as expected.

Now define the gravitational field strength and the gravitational potential

$$\begin{aligned} \mathbf{g}(\mathbf{r}) &= \frac{1}{m}\mathbf{F} = -\frac{G_{\mathrm{N}} M}{r^2}\hat{\mathbf{r}} = -\boldsymbol{\nabla}\Phi \,, \\ \Phi(\mathbf{r}) &= \frac{1}{m} U = -\frac{G_{\mathrm{N}} M}{r} \,. \end{aligned} \tag{3.26}$$

END OF 6TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 6 Exercises

1. Show that $\nabla \cdot \mathbf{r} = 3$ and $\nabla \times \mathbf{r} = 0$.
2. Show that $\nabla \times \mathbf{F} = 0$ for any central force.
3. The electrostatic potential due to a particle of charge Q on a particle of charge q is

$$\Phi = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \,.$$

Calculate the electric field strength $\mathbf{E} = -\nabla\Phi$ and the force on the particle with charge q, $\mathbf{F} = q\mathbf{E}$.

START OF 7TH LECTURE

A distance h above the surface of the Earth the gravitational potential is

$$\Phi = -\frac{G_{\mathrm{N}} M_{\mathrm{E}}}{R_{\mathrm{E}} + h} = -\frac{G_{\mathrm{N}} M_{\mathrm{E}}}{R_{\mathrm{E}}} \left(1 + \frac{h}{R_{\mathrm{E}}}\right)^{-1} \,. \tag{3.27}$$

Now using Taylor series we can can approximate $(1 + x)^{-1} \approx 1 - x$ for small x and so

$$\Phi(R_{\mathrm{E}} + h) \approx -\frac{G_{\mathrm{N}} M}{R_{\mathrm{E}}} \left(1 - \frac{h}{R_{\mathrm{E}}}\right) \,, \tag{3.28}$$

for $h \ll R_{\mathrm{E}}$. We can define the change in the gravitational potential to be

$$\Delta\Phi(h) = \Phi(R_{\mathrm{E}} + h) - \Phi(R_{\mathrm{E}}) \approx gh\,, \tag{3.29}$$

and we have derived the result you are very familiar with that $\Delta U(h) \approx mgh$ which we now see as an approximation which will be accurate when the height the body is above the Earth's surface is much smaller than the radius of the Earth, $\mathrm{R_E} \approx 6400\,\mathrm{km}$. We can calculate the value of g to be

$$g = \frac{G_{\mathrm{N}} M_{\mathrm{E}}}{R_{\mathrm{E}}^2} \approx \frac{6.67\times\ \mathrm{kg^{-1}\,m^3\,s^{-2}} \times 6\times 10^{24}\,\mathrm{kg}}{(6400\,\mathrm{km})^2} \approx 9.8\,\mathrm{m\,s^{-2}}\,. \tag{3.30}$$

Note that it is a good idea to put numbers into formulae in the way that I have done above for two reasons. The first is that the units will cancel out to give, in this case, $\mathrm{m\,s^{-2}}$ providing an important check that you are using the correct formula, and secondly if the units are not expressed in SI units then you will, hopefully, remember to make the appropriate unit conversions correctly as, for example, I did above in writing $1\,\mathrm{km} = 10^3\,\mathrm{m}$.

3.4 Calculating gravitational potentials - see FS9.2

The objective of this section is to calculate the gravitational potential from continuous mass distributions. Consider an infinitesimal mass element of mass

$$dM = \rho(\mathbf{r}')dV'\,, \tag{3.31}$$

for a volume density ρ, or equally $\sigma(\mathbf{r}')dA'$ and $\mu(\mathbf{r}')d\ell'$ for surface density (mass per unit area) σ and line density (mass per unit length) μ.

We want to calculate the gravitational potential at some point $\mathbf{r}$ due to the infinitesimal mass $dM(\mathbf{r}')$ at position $\mathbf{r}'$ as shown in the lecture slides. Using the results from the previous section this is given by

$$d\Phi(\mathbf{r}) = -G_{\mathrm{N}} \frac{dM(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,, \tag{3.32}$$

where $|\mathbf{r}-\mathbf{r}'|$ is the distance between the two points and $|\mathbf{r}-\mathbf{r}'|^2 = r^2 + (r')^2 - 2\mathbf{r}\cdot\mathbf{r}' = r^2 + (r')^2 - 2rr'\cos\theta$. The angle θ is shown in the slides and this is the cosine rule for the triangle shown between O, P and the position of the infinitesimal volume element. So we have that for a volume density

$$\Phi(\mathbf{r}) = -G_{\mathrm{N}} \int_V \frac{\rho(\mathbf{r}')dV'}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{3.33}$$

As an aside note that Φ satisfies the Poisson equation $\nabla^2\Phi = 4\pi G_{\mathrm{N}}\rho$ where $\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

In what follows we will calculate the gravitational potential for a few examples. A specific integral will come up and so to decouple its calculation from the physics, we will compute it here. Let us define

$$I(a,b) = \int_0^\pi d\theta \frac{\sin\theta}{(a^2+b^2-2ab\cos\theta)^{1/2}}\,, \tag{3.34}$$

where $a, b > 0$. The numerator of the integrand can be related to the derivative of the integrand so it is easy to see that

$$\begin{aligned} I(a,b) &= \left[\frac{1}{ab}\left(a^2+b^2-2ab\cos\theta\right)^{1/2}\right]_0^\pi \\ &= \frac{1}{ab}\left\{\left[(a+b)^2\right]^{1/2} - \left[(a-b)^2\right]^{1/2}\right\}\,. \end{aligned} \tag{3.35}$$

At this stage one has to be careful. The first term inside the curly brackets is always $a+b$ and one might presume that the second is $a-b$. However, it is necessary to take into account the fact that the square root is always positive and there when $a > b$ it is $a-b$, whereas if $b > a$ it is $b-a$. Using this we can deduce that $I(a,b) = 2/a$ when $a > b$ and $I(a,b) = 2/b$ when $b > a$.

3.4.1 Newton's shell theorem

Now let us prove a result due to Newton, known as Newton's Shell Theorem. Consider a spherical shell of mass M and radius R which is infinitesimally thin. In this case it is necessary to use the results we have derived earlier for a surface density $\sigma = M/(4\pi R^2)$ and $dA' = R^2\sin\theta d\theta d\phi$. We find that

$$\Phi(\mathbf{r}) = -\frac{G_\mathrm{N}M}{4\pi R^2}\int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{R^2\sin\theta}{(r^2+R^2-2rR\cos\theta)^{1/2}}\,, \tag{3.36}$$

and, because there is no ϕ dependence in the integrand, we have that

$$\Phi(\mathbf{r}) = -\frac{G_\mathrm{N}M}{2} I(r,R)\,. \tag{3.37}$$

Therefore, we find that for $r > R$,

$$\Phi(\mathbf{r}) = -\frac{G_\mathrm{N}M}{r}\,, \tag{3.38}$$

which is that of a point particle of mass M, whereas for $r < R$

$$\Phi(\mathbf{r}) = -\frac{G_\mathrm{N}M}{R}\,, \tag{3.39}$$

which is a constant. Hence, we find that outside the shell the system behaves as though there is just a point mass at the centre of the shell and there is no force inside the shell since it is the derivative of constant. This means that inside the shell the gravitational forces balance out due to the spherical symmetry. Perhaps you might have worked this out intuitively, but we have managed to prove it mathematically.

3.4.2 Spherically symmetric mass distribution

Now consider a spherically symmetric mass distribution where $\rho(\mathbf{r}) = \rho(r)$ for $r < R$ and $\rho(\mathbf{r}) = 0$ for $r > R$. In this case we have that

$$d\Phi(\mathbf{r}) = -G_{\rm N}\frac{\rho(\mathbf{r}')dV'}{(r^2 + (r')^2 - 2rr'\cos\theta)^{1/2}}\,, \quad dV' = (r')^2 dr' \sin\theta d\theta d\phi\,, \tag{3.40}$$

and, therefore,

$$\begin{aligned} \Phi(\mathbf{r}) &= -G_{\rm N}\int_0^R \rho(r')(r')^2 dr' \int_0^{2\pi} d\phi \int_0^{\pi} \frac{\sin\theta d\theta}{(r^2 + (r')^2 - 2rr'\cos\theta)^{1/2}} \\ &= -2\pi G_{\rm N}\int_0^R \rho(r')(r')^2 dr' I(r, r')\,. \end{aligned} \tag{3.41}$$

One has to be careful in evaluating this integral. If $r > R$ then for the whole of the range of the integration we have that $r' < r$ and therefore one finds that

$$\Phi(\mathbf{r}) = -\frac{4\pi G_{\rm N}}{r}\int_0^R \rho(r')(r')^2 dr' = -\frac{G_{\rm N}M_{\rm tot}}{r}\,, \tag{3.42}$$

and so the gravitational potential outside the mass distribution is again that of a point particle with mass $M_{\rm tot}$. In the case where $r < R$ we have to split the range of integration into the range 0 to r where $r' < r$ and r to R where $r' > r$. In this case we have that

$$\Phi(\mathbf{r}) = -4\pi G_{\rm N}\left\{\frac{1}{r}\int_0^r \rho(r')(r')^2 dr' + \int_r^R \rho(r')r'dr'\right\}\,. \tag{3.43}$$

The best way to understand what is going on in this case to calculate the force via the derivative of the gravitational potential

$$\begin{aligned} \frac{d\Phi}{dr} &= -4\pi G_{\rm N}\left\{-\frac{1}{r^2}\int_0^r \rho(r')(r')^2 dr' + \frac{1}{r}\rho(r)r^2 - \rho(r)r\right\} \\ &= \frac{G_{\rm N}M(<r)}{r^2}\,, \end{aligned} \tag{3.44}$$

where $M(<r)$ is the mass inside radius r given by

$$M(<r) = 4\pi\int_0^r (r')^2\rho(r')dr'\,, \tag{3.45}$$

and so

$$\frac{1}{m}\mathbf{F} = -\frac{d\Phi}{dr}\hat{\mathbf{r}} = -\frac{G_{\rm N}M(<r)}{r^2}\hat{\mathbf{r}}\,. \tag{3.46}$$

The expression for the force is that of a force due to a point particle with mass equal to the mass inside the radius r which is compatible with what we found in the context of the spherical shell in Newton's shell theorem.

END OF 7TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 7 Exercises

1. The mass of the Moon is $M_{\text{moon}} \approx 7.3 \times 10^{22}$ kg and its radius is ≈ 1700 km. Calculate the escape velocity and the gravitational field strength for the Moon.

2. A particle falls towards the Earth starting from rest at distance $R \gg R_{\text{E}}$ from the centre of the Earth. Derive an expression for the time taken to reach a distance $R/2$ from the centre.

3. Calculate the gravitational potential for a sphere of radius R with constant density ρ_0. Be careful to delinate the cases where $r < R$ and $r > R$.

START OF 8TH LECTURE

3.5 Motion in a central potential - FS 9.4 and 9.5

You will have come across Kepler's Laws of planetary motion in the course on Introductory Astronomy and Cosmology. They are:

- planetary orbits around the Sun are elliptical with the Sun at one of the two foci (K1);
- lines between the planet and the Sun sweep out equal areas in equal intervals (K2);
- the orbital period of a planet squared is proportional to the cube of the semi-major axis (K3).

In this section we will prove that these laws are a consequence of Newtonian gravity and discuss how to solve for the trajectory of a particle in a central potential.

We will work within the framework of the 2-body interacting system discussed in section 1.5 and in particular in the Centre of Mass (CoM) frame and only concentrate on the relative motion. If you remember we showed that the velocity of the CoM frame was constant for a central potential and we will set it to be zero, that is, $\dot{\mathbf{R}} \equiv 0$.

The equation of motion for the relative motion is $\mu\ddot{\mathbf{r}} = \mathbf{F} = -\frac{dU}{dr}\hat{\mathbf{r}}$. we can take the dot product with $\dot{\mathbf{r}}$ yielding

$$\mu\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -\frac{dU}{dr}\hat{\mathbf{r}} \cdot \dot{\mathbf{r}}\,. \tag{3.47}$$

Now $r^2 = \mathbf{r}\cdot\mathbf{r}$ which implies that $2\dot{r}r = 2\mathbf{r}\cdot\dot{\mathbf{r}}$ and so $\dot{r} = \hat{\mathbf{r}}\cdot\dot{\mathbf{r}}$ and, therefore,

$$\mu\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = -\dot{r}\frac{dU}{dr} = -\frac{dU}{dt}\,, \tag{3.48}$$

which implies that

$$E = \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 + U(\mathbf{r})\,, \tag{3.49}$$

is a constant which we define as the energy of the system.

In the CoM frame $\mathbf{L} = \mu\mathbf{r}\times\dot{\mathbf{r}}$ and we can calculate

$$\frac{d\mathbf{L}}{dt} = \mu\left(\dot{\mathbf{r}}\times\dot{\mathbf{r}} + \mathbf{r}\times\ddot{\mathbf{r}}\right) = -\mathbf{r}\times\frac{dU}{dr}\hat{\mathbf{r}} = \mathbf{0}\,, \tag{3.50}$$

so we see that the angular momentum is constant. Moreover, $\mathbf{L}\cdot\dot{\mathbf{r}} = \mu\mathbf{r}\times\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = 0$ and, therefore, the $\mathbf{L}$ is perpendicular to $\dot{\mathbf{r}}$ and hence the morion must be in a plane - which is part of K1.

Since the motion is in a plane we can use 2D polar coordinates to describe it: $\mathbf{r} = r\hat{\mathbf{r}}$ and $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$ with $|\dot{\mathbf{r}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2$ so that

$$E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r)\,, \quad \mathbf{L} = \mu\mathbf{r}\times\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right) = \mu r^2\dot{\theta}\hat{\mathbf{e}}_z\,, \tag{3.51}$$

and therefore $L = \mu r^2\dot{\theta}$ is constant since $\frac{d\mathbf{L}}{dt} = 0$. Replacing $\dot{\theta}$ in the expression for the energy gives

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + U(r) = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r)\,, \tag{3.52}$$

where we have defined the potential for effective 1D motion $U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + U(r)$ in the region $r > 0$.

Now consider the segment $d\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ with $d\theta$ the angle between them. The area of this element is $dA = \frac{1}{2}r^2 d\theta$ and so

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2\mu}\,, \tag{3.53}$$

which is constant. Therefore, we find that the conservation of angular momentum that we have proven leads to K2.

In order to solve for $r(\theta)$ we will define a change of variables $u = 1/r$ (note that this should not be confused with U which is the potential!). With this definition $\dot{u} = -\dot{r}/r^2 = -\dot{r}u^2$ and $\dot{\theta} = Lu^2/\mu$. From this we can derive the equation

$$\left(\frac{du}{d\theta}\right)^2 = \left(\frac{\dot{u}}{\dot{\theta}}\right)^2 = \left(\frac{\mu}{L}\right)^2\dot{r}^2 = \frac{2\mu}{L^2}\left[E - U\left(\frac{1}{u}\right)\right] - u^2\,. \tag{3.54}$$

In order to make further progress, we will use $U = -\alpha/r = -\alpha u$ where $\alpha = -Gm_1m_2$ in the case of gravity and $\alpha = -q_1q_2/(4\pi\varepsilon_0)$ for electrostatics. In this case we find that

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{L^2}\left(E + \alpha u\right) - u^2 = \varepsilon^2 u_0^2 - \left(u - u_0\right)^2 , \tag{3.55}$$

where $u_0 = 1/r_0 = \alpha\mu/L^2$ and $\varepsilon^2 = 1 + 2L^2E/(\alpha^2\mu)$. Now define $\bar{u} = (u - u_0)/(\varepsilon u_0)$ in which case

$$\left(\frac{d\bar{u}}{d\theta}\right)^2 = 1 - \bar{u}^2 . \tag{3.56}$$

We must take the square root, but $du/d\theta \propto -\dot{r}$ so it is negative for positive $\dot{r}$. Moreover we need to chose a boundary condition and it is conventional to choose $\bar{u} = 0$ - corresponding to $u = u_0$, $r = r_0$ - at $\theta = \pi/2$ which just says where the trajectory crosses the axis. This choice yields $\bar{u} = \cos\theta$ and so

$$u = u_0(1 + \varepsilon\cos\theta) , \quad r = \frac{r_0}{1 + \varepsilon\cos\theta} . \tag{3.57}$$

This equation for the trajectory in polar coordinates can be converted to Cartesian coordinates using $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$. In particular, we find that $r + \varepsilon x = r_0$ and so

$$r^2 = (r_0 - \varepsilon x)^2 \quad \rightarrow \quad x^2 + y^2 = r_0^2 - 2\varepsilon r_0 x + \varepsilon^2 x^2 . \tag{3.58}$$

This can be rearranged when $\varepsilon \neq 1$ to give

$$x^2 + \frac{2\varepsilon r_0}{1-\varepsilon^2}x + \frac{y^2}{1-\varepsilon^2} = \frac{r_0^2}{1-\varepsilon^2} \quad \rightarrow \quad \left(x + \frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{r_0^2}{(1-\varepsilon^2)^2} . \tag{3.59}$$

Now if we assume that $0 \leq \varepsilon < 1$ then

$$\frac{\left(x + \frac{\varepsilon r_0}{1-\varepsilon}\right)^2}{\left(\frac{r_0}{1-\varepsilon^2}\right)^2} + \frac{y^2}{\left(\frac{r_0}{\sqrt{1-\varepsilon^2}}\right)^2} = 1 . \tag{3.60}$$

If we define $X = x + \frac{\varepsilon r_0}{1-\varepsilon^2}$, $Y = y$ then

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 , \tag{3.61}$$

where $a = r_0/(1-\varepsilon^2)$ and $b = r_0/\sqrt{1-\varepsilon^2}$. This is the equation of an ellipse with semi-major axis a and semi-minor axis b. We find that $b = \sqrt{1-\varepsilon^2}\,a$ and $\varepsilon^2 = 1 - (b/a)^2$. The parameter ε is known as the eccentricity with $\varepsilon = 0$ corresponding to a circle with $a = b$. Therefore, if $0 \leq \varepsilon < 1$ we have an elliptical or circular orbit which implies $K1$.

Note that $1-\varepsilon^2 = -2L^2E/(\alpha^2\mu) = -2Er_0/\alpha$ and therefore $E<0$ and $0 \le \varepsilon < 1$ implies

$$-\frac{\alpha^2\mu}{2L^2} = -\frac{\alpha}{2r_0} \le E < 0\,. \tag{3.62}$$

We will consider the cases where $\varepsilon \ge 0$ in section 3.7

Now K2 - which we have already shown is a consequence of angular momentum conservation - implies that

$$\frac{dA}{dt} = \frac{L}{2\mu}\,, \tag{3.63}$$

which is constant. Hence, by integrating (3.63) the time period of rotation, T, is given by

$$\frac{LT}{2\mu} = A = \pi ab = \pi a^2\sqrt{1-\varepsilon^2}\,. \tag{3.64}$$

Now we have that $r_0 = L^2/(\alpha\mu)$ which implies that

$$\frac{L}{\mu} = \left(\frac{\alpha}{\mu}\right)^{1/2} r_0^{1/2} = \left(\frac{\alpha}{\mu}\right)^{1/2} a^{1/2}\sqrt{1-\varepsilon^2}\,, \tag{3.65}$$

and so we can deduce that

$$T = 2\pi\frac{\mu}{L}a^2\sqrt{1-\varepsilon^2} = 2\pi a^{3/2}\left(\frac{\mu}{\alpha}\right)^{1/2}\,. \tag{3.66}$$

Hence, we find that K2 and K1 lead naturally to K3.

END OF 8TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 8 Exercises

1. Deduce the effective potential for motion for a system with reduced mass μ in a central potential $\mathbf{F} = -\frac{dU}{dr}\hat{\mathbf{r}}$ assuming that it is in a plane using the formula for acceleration in 2D polar coordinates derived in lecture2 and show that this is the same as that which was derived in the lecture.

2. A particle of mass m is in circular motion with position vector $\mathbf{r} = r_0\hat{\mathbf{r}}$ and constant angular speed ω under the influence of a central potential $U(\mathbf{r})$. Calculate $\ddot{\mathbf{r}}$ and show that the

$$\omega^2 = \frac{1}{mr_0}\frac{dU}{dr}(r_0)\,.$$

Show that, if $U \propto -r^{-n}$ then the rotational period is $T \propto r_0^{\frac{n+2}{2}}$ and is compatible with Kepler's third law for $n=1$.

START OF 9TH LECTURE

3.6 Motion in the effective potential

In the previous lecture we have proven Kepler's laws of planetary motion by solving the equations of motion for the trajectory $r(\theta)$ from Newton's theory of gravity with a potential $U = -\alpha/r$. The calculation is pretty heavy going in terms of the all the substitutions (although you don't have to come up with them yourself since Newton did it for you!). This calculation is much more difficult for a general potential. In this section we will discuss a simpler way to understand what is going on in terms of the effective 1D motion in the region $r > 0$. Remember that

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r)\,, \tag{3.67}$$

where

$$U_{\text{eff}}(r) = U(r) + \frac{L^2}{2\mu r^2}\,. \tag{3.68}$$

As an example we will consider the potential $U = -\alpha/r$ and the cases $\alpha > 0$ and $\alpha < 0$ separately as an illustration of the methodology before trying to piece things together in generality. In all cases the process involves sketching the curve $U_{\text{eff}}(r)$ by identifying and classifying the stationary points, where it crosses the axis and the asymptotes. From this we will be able to piece together the dynamical behaviour by thinking about how a particle of a given energy would behave in that "potential landscape".

3.6.1 $\alpha > 0$

The derivative of the effective potential is

$$\frac{dU_{\text{eff}}}{dr} = -\frac{L^2}{\mu r^3} + \frac{\alpha}{r^2}\,, \tag{3.69}$$

and so we identify a stationary point at $r = L^2/(\alpha\mu)$ which is $r = r_0$ with the same definition for r_0 used in the previous lecture. The second derivative is given by

$$\frac{d^2U_{\text{eff}}}{dr^2} = \frac{1}{r^4}\left(\frac{3L^2}{\mu} - 2\alpha r\right)\,, \tag{3.70}$$

and therefore $\frac{d^2U_{\text{eff}}}{dr^2}(r_0) = \frac{L^2}{\mu r_0^4} > 0$ implying that the stationary point is a minimum. Moreover, we can calculate

$$U_{\text{eff}}(r_0) = -\frac{\mu\alpha^2}{2L^2} = -\frac{\alpha}{2r_0} = \frac{1}{2}U(r_0)\,. \tag{3.71}$$

At small r we have that $U_{\text{eff}} \sim L^2/(2\mu r^2) \to \infty$ as $r \to 0$ and for large r we find that $U_{\text{eff}} \sim -\alpha/r \to 0$ from below as $r \to \infty$. On the basis of the calculations we can piece together that the potential has the form shown in the lecture slides.

In order to understand the dynamics we need to consider various values of E.

(a) $E = -\mu\alpha^2/(2L^2)$, $\varepsilon = 0$: The particle is at the minimum of the potential and in this case there is stable circular orbit with $r = r_0$. There will be small oscillations about the circular orbit with effective spring constant $k = L^2/(\mu r_0^4)$.

(b) $-\mu\alpha^2/(2L^2) < E < 0$, $0 < \varepsilon < 1$; The particle will roll around in the bottom of the potential but does not have sufficient energy to escape. One can calculate the minimum and maximum values of r by setting $\dot{r} = 0$ (one of the exercises). These are the distance of closest approach. r_{min}, (pericentre) and distance of furthest approach, r_{max}, (apocentre) on an ellipse. We have not proved that it is an ellipse as we did in the previous section, but what else could a bound orbit with a variable radius be?

(c) $E = 0$, $\varepsilon = 1$: We did not consider this case in the previous section but it is in fact a perfectly reasonable solution of the equations of motion. Looking at the sketch of the potential on the slides, a ball with this energy would roll down to the minimum of the potential before getting to the point where $U_{\text{eff}} = 0$ with zero velocity before rolling back up the potential back to infinity just getting there with zero velocity. This corresponds to a an unbound parabolic orbit as we will see in the next lecture.

(d) $E > 0$, $\varepsilon > 1$: This case is similar to the parabolic orbit, but the particle has more energy, gets closer into the centre and returns to infinity with easily enough energy to get there - asymptotically it has non-zero velocity. This is an unbound hyperbolic orbit as we will see in the next lecture.

3.6.2 $\alpha < 0$

This case cannot happen in Newtonian gravity, but can take place in electrostatics for the scattering of particles with the same sign for the charge. As such it is a classical model for Rutherford scattering of α particles off a nucleus.

There are no stationary points - refer back to the expression for $\frac{dU_{\text{eff}}}{dr}$ when $\alpha < 0$ and the effetcive potential now $\to 0$ from above as $r \to \infty$. The sketch on the lecture slides shows a monotonic curve which goes from zero at $r = \infty$ to ∞ at $r = 0$. This will lead to unbound hyperbolic orbits for any value of E; the particle comes in, hits the barrier and bounces back.

3.6.3 Stability of circular orbits

Let us consider the effective potential expanded around a stationary point $r = r_0$. Using Taylor series we find that

$$U_{\rm eff}(r_0 + \delta r) = U_{\rm eff}(r_0) + \delta r \frac{dU_{\rm eff}}{dr}(r_0) + \frac{1}{2}\delta r^2 \frac{d^2 U_{\rm eff}}{dr^2}(r_0) + .. \,, \tag{3.72}$$

where the first term is constant and unimportant for the dynamics. More importantly the second term is zero since it is a stationary point, so we see the key result (which you should have already come across a few times) that for sufficiently small perturbations around a stationary point it is "harmonic", that is, is a simple harmonic oscillator and the effective spring constant is given by

$$k = \frac{d^2 U_{\rm eff}}{dr^2}(r_0)\,, \tag{3.73}$$

This has implications for the stability of the stationary point. If $\frac{d^2U_{\rm eff}}{dr^2}(r_0) > 0$ then local to the stationary point the potential is like a $y = x^2$ parabola as shown in the lecture slides and the point is stable, but is it is < 0 then it is like $y = -x^2$ and the point is unstable.

As an example of this let us consider $U = -\alpha r^5$ (note that the α here has different dimensions to that in the case of Newtonian gravity) and we will consider $\alpha > 0$. One can calculate

$$\begin{aligned} U_{\rm eff} &= -\frac{\alpha}{r^5} + \frac{L^2}{2\mu r^2}\,, \\ \frac{dU_{\rm eff}}{dr} &= \frac{5\alpha}{r^6} - \frac{L^2}{\mu r^3}\,, \\ \frac{d^2U_{\rm eff}}{dr^2} &= \frac{1}{r^4}\left(\frac{3L^2}{\mu} - \frac{30\alpha}{r^3}\right)\,. \end{aligned} \tag{3.74}$$

The stationary point is $r = r_0 = \left(\frac{5\mu\alpha}{L^2}\right)^{1/3}$ and at this point the second derivative is < 0 which implies that it is unstable. Considering the behaviour small and large r we find that $U_{\rm eff} \sim -\alpha/r^5$ for $r \to 0$ and $U_{\rm eff} \sim L^2/(2\mu r^2)$ as $r \to \infty$, so we can sketch the effective potential as shown in the lecture slides. A particle with $E > U_{\rm eff}(r_0)$ will always fall into $r = 0$ whereas if $E < U_{\rm eff}(r_0)$ a particle incoming from infinity will bounce of the potential barrier back to infinity but one from $r < r_0$ will ultimately fall into the centre.

END OF 9TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 9 Exercises

1. For the system with a central potential $U = -\alpha/r$ studied in the lectures with an energy $-\frac{\mu\alpha^2}{2L^2} < E < 0$ (i.e. an ellipse), calculate the apsides (minimum and maximum values of r) in terms of the semi-major axis and the eccentricity of the ellipse

2. A particle of mass m and angular momentum, L orbits in a central potential $U = \frac{1}{2}kr^2$. Calculate the radius of the circular orbit and decide whether it is stable or not.

START OF 10TH LECTURE

3.7 Conic sections and the classification of orbits

We have shown that the general solution to the equations of motion in Newtonian gravity leads to a trajectory

$$r = \frac{r_0}{1 + \varepsilon\cos\theta}\,, \tag{3.75}$$

where we have traded the physical parameters E and L for r_0 and ε which described the form of the trajectory more naturally. We presented expressions to make this conversion in section 3.5. This equation for the trajectory in polar coordinates can be converted in Cartesian coordinates

$$(1 - \varepsilon^2)x^2 + y^2 + 2\varepsilon r_0 x = r_0^2\,. \tag{3.76}$$

We showed that when $0 \leq \varepsilon < 1$ which can be transformed in the the standard form an ellipse. If you remember we found that a transformation $(X, Y) = \left(x + \frac{\varepsilon r_0}{1-\varepsilon^2}\right) = (x + \sqrt{a^2 - b^2}, y)$ allowed us to write the equation for the trajectory as

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1\,, \tag{3.77}$$

where $a = r_0/(1 - \varepsilon^2)$ and $b = r_0/\sqrt{1 - \varepsilon^2}$. The position shift in the position of the centre of the ellipse is to the point known as the focus - see below.

We will now consider the cases where $\varepsilon \geq 1$.

$\varepsilon = 1$: In this special case $y^2 = r_0^2 - 2r_0 x$ which can be transformed into $Y^2 = 4aX$ if $r_0 = 2a$ and $(X, Y) = (\frac{1}{2}r_0 - x, y)$. This is a parabola. The specific choice of a and the orientation (in the $y^2 = x$ direction as opposed to $y = x^2$) is just the convention used in many books etc.

$\varepsilon > 1$: We can complete the square as we did with $\varepsilon < 1$ and obtain

$$\left(x + \frac{\varepsilon r_0}{1 - \varepsilon^2}\right)^2 + \frac{y^2}{1 - \epsilon^2} = \frac{r_0^2}{(1 - \varepsilon^2)^2}\,. \tag{3.78}$$

This is slightly different now since the coefficient of y^2 is negative, whereas it was positive in the case of the ellipse. We make the transformation $(X,Y) = \left(x - \frac{\varepsilon r_0}{\varepsilon^2-1}, y\right) = \left(x - \sqrt{a^2+b^2}, y\right)$ to obtain

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1\,, \tag{3.79}$$

where $a = r_0/(\varepsilon^2 - 1)$, $b = r_0/\sqrt{\varepsilon^2 - 1}$ and $\varepsilon^2 = 1 + (b/a)^2$. This is the standard equation for a hyperbola with asymptotes $Y = \pm\left(\frac{b}{a}\right)X$. Notice that the shift in the position has a somewhat different dependence on a and b to the case of an ellipse.

So we see that the equation (3.75) can describe a range of well known curves (circle, ellipse, parabola, hyperbola) depending on the value of ε. This family of curves are known as conic sections since they can be created by slicing through a cone in various ways as shown in the lecture slides. This is not particularly important, but it is the name that they are given. But the fact that the orbits can be classified in terms of these four possibilities is a powerful theorem of dynamics - and we have proved it. It goes beyond Kepler's laws which only apply when $0 \le \epsilon < 1$.

The conic sections can be generated in a number of ways. One is by specification of a *directrix*. If we define the focus to be the point $(f, 0)$ denoted F and the directrix to the the vertical line generated by the locus of points (d, Y) for $-\infty < Y < \infty$ denoted D. The conic section with eccentricity ε is generated by demanding that the point (X, Y) denoted P satisfies $|PF| = \varepsilon|PD|$ which implies that

$$(X-f)^2 + Y^2 = \varepsilon^2(X-d)^2 \quad \to \quad (1-\varepsilon^2)X^2 + Y^2 - 2(f - \varepsilon^2 d^2)X = \varepsilon^2 d^2 - f^2\,, \tag{3.80}$$

which has some the same form as (3.76). There are two interesting cases.

First choose $f = \varepsilon^2 d$ and $\varepsilon \neq 1$ which allows us to write

$$\frac{X^2}{(\varepsilon d)^2} + \frac{Y^2}{(\varepsilon d)^2(1-\varepsilon^2)} = 1\,. \tag{3.81}$$

if we choose $d = \pm a/\varepsilon$ and $b^2 = \pm a^2(1-\varepsilon^2)$ from which we deduce that $f = \pm\varepsilon a$ and so we can generate ellipses (+) and hyperbolae (-). The second case has $\epsilon = 1$ and $f = -d$ (the case $f = d$ yields the trivial case $Y = 0$) which yields $Y^2 = 4fX$ which is a parabola. So we see that we can generate the conic sections by making various choices for the directrix.

In the lecture slides I have provided some sketches of the conic sections showing the various features we have derived here and a table of the properties. You should study these figures and the table to familiarise yourself with them. One additional feature to point out is what is called the *semi-latus rectum* which is the value of Y when $X = f$ and is given by $r_0 = \ell = b^2/a$ for the ellipses and hyperbolae and is $2a$ for the parabola. Note that there are other ways of generating the ellipses and hyperbola - see the exercises and problem sheets - which are possibly more intuitive or at least more familiar, but they are not the same and cannot generate the parabola.

3.8 Solar system

Now let us consider the solar system, that is, the case of gravity with $m_1 = M_\odot$ - the mass of the sun - and $m_2 = M_{\rm P}$ - the mass of a planet. The reduced mass is

$$\mu = \frac{M_{\rm P}}{1 + \frac{M_{\rm P}}{M_\odot}} \approx M_{\rm P}\,, \tag{3.82}$$

to within 0.1% or better since $M_{\rm JUP}/M_\odot \approx 10^{-3}$. We will ignore the gravitational interactions that the planets exert on each other.

For the solar system we have that $\alpha/\mu \approx 39\,{\rm Au}^3\,{\rm yr}^{-2}$ which implies that

$$T \approx 1\,{\rm yr}\left(\frac{a}{\rm Au}\right)^{3/2}. \tag{3.83}$$

The planets move on elliptical orbits with $r_{\rm min} = a(1-\varepsilon)$ and $r_{\rm max} = a(1+\epsilon)$ and if we define $\mathcal{R} = r_{\rm min}/r_{\rm max}$ then

$$\varepsilon = \frac{1-\mathcal{R}}{1+\mathcal{R}}. \tag{3.84}$$

Interestingly most of the planetary orbit, have relatively low eccentricities. It is thought that if they were very eccentric many more collisions would take place - which could have been the case during the early life of the solar system - but that this is self regulating in the long term limit toward low eccentricity.

Now consider the speed, v tangential to the curve at the minimum and maximum values of r. The angular momentum is $L = \mu r v$ at these points. Hence, we find that

$$(1-\varepsilon^2)a = r_0 = \frac{L^2}{\alpha\mu} = \frac{\mu}{\alpha}(rv)^2\,. \tag{3.85}$$

Since rv is constant we have that

$$\frac{v_{\rm max}}{v_{\rm min}} = \frac{r_{\rm min}}{r_{\rm max}} = \mathcal{R}\,, \tag{3.86}$$

where $v_{\rm max}$ is the velocity is in velocity at $r = r_{\rm max}$, and similarly $v_{\rm min}$ is that at $r = r_{\rm min}$.

Moreover, we can deduce an expression for this characteristic minimum velocity

$$v_{\rm min} \approx \frac{6.2\,{\rm Au}\,{\rm yr}^{-1}}{\mathcal{R}^{1/2}}\left(\frac{\rm Au}{a}\right)^{1/2}. \tag{3.87}$$

END OF 10TH LECTURE. NOW MAKE SURE YOU HAVE A GO AT THE EXERCISES BEFORE THE ANSWERS ARE RELEASED ON MONDAY NEXT WEEK.

Lecture 10 Exercises

1. The two focal points, F_1 and F_2, of a hyperbola are at $(\pm\sqrt{a^2+b^2}, 0)$, respectively. Show that the hyperbola can be generated as the locus of points P which satisfy

$$||PF_2| - |PF_1|| = 2a\,.$$

2. Halley's comet moves around the Sun in a highly elliptical orbit with an eccentrcity of 0.967 and a period of 76 years. Calculate its maximum and minimum distances from the Sun.

Section 3 in a Nutshell - what to remember

- $\boldsymbol{\nabla} = \mathbf{e}_i \frac{\partial}{\partial r_i}$.
- Div, grad and curl : $\boldsymbol{\nabla}\cdot\mathbf{v}$ - a scalar, $\boldsymbol{\nabla} U$ - a vector, $\boldsymbol{\nabla}\times\mathbf{v}$ - a vector - see the Maths II and E& M courses.
- Conservative force fields can be generated by a potential
- Condition for the conservative force field $\boldsymbol{\nabla}\times\mathbf{F} = 0$.
- Central force fields $\mathbf{F} = f(r)\hat{\mathbf{r}}$.
- $\boldsymbol{\nabla} g(r) = \frac{\partial g}{\partial r}\hat{\mathbf{r}}$.
- Gravitational field strength $\mathbf{g}(\mathbf{r}) = \frac{1}{m}\mathbf{F}$ and gravitational potential $\Phi = \frac{1}{m}U$.
- Calculating gravitational potentials

$$d\Phi(\mathbf{r}) = -G_{\mathrm{N}}\frac{dM(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{3.88}$$

- Two constants of motion for central potentials - energy E and L - with

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{\mathrm{eff}}(r) = \frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{L^2}{2\mu r^2}\,. \tag{3.89}$$

  Motion is in a plane.
- General solution for $U = -\alpha/r$ is

$$r = \frac{r_0}{1+\epsilon\cos\theta}\,, \tag{3.90}$$

  where r_0 and ε are related to E and L. This solution leads to circles when $\epsilon = 0$, ellipses when $0 < \varepsilon < 1$, a parabola when $\varepsilon = 1$ and hyperbolae when $\varepsilon > 1$. These curves are known as conic sections and can be generated from a directrix.
- In more general cases sketch the effective potential $U_{\mathrm{eff}}(r)$ in the region $r > 0$ by identifying and classifying the stationary points, crossing points and asymptotes.
- Stationary points are stable if $\frac{d^2U_{\mathrm{eff}}}{dr^2} > 0$ at the stationary point and unstable if it is < 0.