<u>ONE HOUR THIRTY MINUTES</u>

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Complex Variables and Vector Spaces

24 May 2019, 09:45 - 11:15

Answer **<u>ALL</u>** parts of question 1 and **<u>TWO</u>** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. (a) For a function $f(z) = u(x, y)+iv(x, y)$, where $z = x+iy$, state the Cauchy-Riemann conditions for $f(z)$ to be differentiable. Show that they are satisfied for $f(z) = z^3$.

[6 marks]

(b) Show by explicit integration that $\oint_C \frac{1}{z-1}\,\mathrm{d}z = 2\pi i$, where C is a circle of radius r centred on $z = 1$.

[6 marks]

(c) Sketch the following three regions of the complex plane:

$$|z| > |z - 1 - i|\,; \qquad -\pi/4 < \mathrm{Arg}\,[z - 1] < \pi/4\,; \qquad 1 < |z - 1 + i| < 2\,.$$

[6 marks]

(d) The real functions $u_n(x)$ have the form $u_n(x) = e^{-x^2/2}\,H_n(x)$, where $H_n(x)$ is a polynomial of degree n in the real variable x. The scalar product is given by

$$\langle u_m|u_n\rangle = \int_{-\infty}^{\infty} u_m(x)u_n(x)\,\mathrm{d}x\,.$$

Find $H_2(x)$, given that the functions u_0, u_1 and u_2 are mutually orthogonal. (Your answer for H_2 does not need to be normalized.)

You may use without proof the standard integrals

$$\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi} \qquad \text{and} \qquad \int_{-\infty}^{\infty} x^2 e^{-x^2}\mathrm{d}x = \sqrt{\pi}/2.$$

[7 marks]

2. (a) An analytic function $f(z) = u + iv$, where $z = x + iy$, has real part $u(x, y) = x^2 - y^2 + 3y$.

(i) Show that $u(x, y)$ is a harmonic function.

[2 marks]

(ii) Find $v(x, y)$, the imaginary part of $f(z)$, given that $v(1, 1) = 1$.

[6 marks]

(iii) Express $f(z)$ in terms of z.

[3 marks]

(b) In the remainder of this question we consider the mapping $w = u + iv = 1/z$, where $z = x + iy$.

(i) Show that the line $x = 0$, $y \neq 0$ is mapped into a straight line in the w plane.

[2 marks]

(ii) By considering the definition $u = \text{Re}(1/z)$, show that lines of constant $u \neq 0$ are circles in the z plane: $(x - x_0)^2 + y^2 = R^2$. Determine x_0 and R in terms of u.

[4 marks]

(iii) Draw lines of constant u in the xy plane for the cases $u = 0$, ± 1 and ± 2.

[5 marks]

(iv) Suggest **one** electrostatics problem for which $u(x, y)$ would be the potential. Specify carefully any conductors and their potentials.

[3 marks]

3. (a) Use Cauchy's residue theorem and a suitable choice of contour (which you should sketch) to evaluate **one** of the following integrals:

either (i) the principal value integral $\displaystyle ⨍_{-\infty}^{\infty} \frac{\cos(kx)}{x-1}\,\mathrm{d}x$ for real $k > 0$;

or (ii) $\displaystyle \int_0^{\infty} \frac{\sqrt{x}}{x^2+9}\,\mathrm{d}x.$

If Jordan's lemma is used, show that the conditions for its validity are satisfied.

[12 marks]

(b) (i) Find the poles and residues of the function $f(z) = \dfrac{1}{z^3\cos(\pi z)}$.

[4 marks]

(ii) Use your results from (i) and a suitable contour integral to show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

[9 marks]

4. (a) Give short definitions of (i) the Hermitian conjugate (or adjoint) of a linear operator, (ii) an Hermitian operator, and (iii) a unitary operator.

[4 marks]

(b) For each of the following matrices, in which $\theta \neq 0$ is real, state whether the matrix is Hermitian, unitary, both, or neither:

$$\text{(i)}\ \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}; \quad \text{(ii)}\ \begin{pmatrix} 0 & 1 \\ e^{i\theta} & 0 \end{pmatrix}; \quad \text{(iii)}\ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

[3 marks]

(c) A linear operator $\widehat{A}$ is defined as follows by its action on square-integrable functions of a real variable x:

$$\langle x|\widehat{A}|f\rangle = f(x) - f(-x).$$

The scalar product of functions f and g is defined as

$$\langle f|g\rangle = \int_{-\infty}^{\infty} \overline{f}(x) g(x)\, \mathrm{d}x.$$

(i) By considering the action of $\widehat{A}^2$ on a function $f(x)$, find an algebraic equation relating $\widehat{A}^2$ to $\widehat{A}$.

[5 marks]

(ii) Determine whether the operator $\widehat{A}$ is Hermitian. Is it also unitary? Explain your answer.

[6 marks]

(iii) Calculate the eigenvalues of $\widehat{A}$. Give **one** explicit example of a normalizable eigenfunction for **each** eigenvalue. (The functions you give need not be normalized.)

[7 marks]

END OF EXAMINATION PAPER