PHYS20672 Complex Variables and Vector Spaces: Examples 2

Some of these examples are copied from the exercises in the notes, but have been included here for completeness.

1. Confirm that the Euclidean dot product $\boldsymbol{a} \cdot \boldsymbol{b}$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ satisfies the definition of an inner product.
2. Show that if $\boldsymbol{w} = \lambda\boldsymbol{u} + \mu\boldsymbol{v}$, then we have: $\langle \boldsymbol{w}, \boldsymbol{a} \rangle = \bar{\lambda} \langle \boldsymbol{u}, \boldsymbol{a} \rangle + \bar{\mu} \langle \boldsymbol{v}, \boldsymbol{a} \rangle$. This property is called anti-linearity or conjugate linearity.

3. Regard $\boldsymbol{a} = (2, 3, -1)^T$, $\boldsymbol{b} = (0, 1, 2)^T$, $\boldsymbol{c} = (0, 0, -5)^T$ as vectors in $\mathbb{R}^3$. Apply the method of Gram-Schmidt orthonormalisation to the vectors $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ to obtain an orthonormal set. [In this problem the scalar product is the usual dot product, $\langle \boldsymbol{a}, \boldsymbol{b} \rangle \equiv \boldsymbol{a} \cdot \boldsymbol{b}$.]
4. Use the Schwarz inequality to prove the triangle inequality,
$$\|\boldsymbol{c}\| \leq \|\boldsymbol{a}\| + \|\boldsymbol{b}\| \quad \text{if} \quad \boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}.$$
[Both sides are nonnegative, so you can consider the squares of each side.]
Now use the triangle inequality to show that if $\boldsymbol{c}$ is defined as above, then
$$\Big| \|\boldsymbol{a}\| - \|\boldsymbol{b}\| \Big| \leq \|\boldsymbol{c}\|.$$
5. The inner product of two real functions $u(x)$ and $v(x)$, where $x \in [-1, 1]$, is defined by
$$\langle u, v \rangle = \int_{-1}^{1} u(x)v(x)\mathrm{d}x.$$
Let $p_n(x)$ be polynomials of degree n in x. Given that these polynomials form an orthonormal set of functions on the interval $[-1, 1]$, find p_0, p_1 and p_2. [The results are unique up to the choice of sign.] Where have you seen these polynomials before, perhaps with a different normalization?
6. For two vectors $\boldsymbol{\psi} = a\boldsymbol{e}_1 + b\boldsymbol{e}_2$ and $\boldsymbol{\phi} = c\boldsymbol{e}_1 + d\boldsymbol{e}_2$ where a, b, c and $d \in \mathbb{C}$, and $\boldsymbol{e}_i$ are orthonormal basis vectors, show that:
$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle = a\bar{c} + b\bar{d}. \tag{1}$$
Show that $\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \overline{\langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle}$ for this specific case, and generalise this to $\mathbb{C}^N$.
7. Write the column vectors,
$$\begin{pmatrix} 2+i \\ -i \end{pmatrix} \text{ and } \begin{pmatrix} 5+i \\ i \end{pmatrix} \tag{2}$$
in Dirac notation, and calculate their norms and the inner product assuming that the basis used is orthonormal.
8. A vector $\boldsymbol{a} \in V_N$ can be represented in the orthonormal basis $\{\boldsymbol{e}_j\}_{j=1}^N$ as $\boldsymbol{a} = \sum_{j=1}^N a_j \boldsymbol{e}_j$. Show that $a_j = \langle \boldsymbol{e}_j, \boldsymbol{a} \rangle$.
9. Let $\boldsymbol{a} = 3i\boldsymbol{e}_1 - 7i\boldsymbol{e}_2$ and $\boldsymbol{b} = \boldsymbol{e}_1 + 2\boldsymbol{e}_2$, where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are orthonormal. Show explicitly that $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy the triangle and Schwarz inequalities.
10. Show that if $\boldsymbol{a}$ is a nonzero vector in a vector space V_N, the set W of vectors orthogonal to $\boldsymbol{a}$ is a vector space of dimension $N-1$. [Since every vector in W is also a vector in V_N, we say that W is a subspace of V_N.]
11. Show that the set of all linear operators acting on a vector space is itself a vector space.
12. Suppose an inner product space V_N is spanned by the orthonormal basis $\{\boldsymbol{e}_j\}_{j=1}^N$, show that $\hat{P} = \boldsymbol{e}_1\boldsymbol{e}_1{}^\dagger + \boldsymbol{e}_2\boldsymbol{e}_2{}^\dagger$ is projection operator, and determine its action on an arbitrary state $\boldsymbol{b} = \sum_{k=1}^N b_k \boldsymbol{e}_k$.
13. Prove the following statements,
    (i) $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$.
    (ii) $(\lambda\hat{A})^\dagger = \bar{\lambda}\hat{A}^\dagger$ where λ is a scalar.
    (iii) If $\hat{Q} = \boldsymbol{c}\boldsymbol{a}^\dagger$ then $\hat{Q}^\dagger = \boldsymbol{a}\boldsymbol{c}^\dagger$.