PHYS20672 Complex Variables and Vector Spaces: Examples 12

1. The problems below illustrate the effect of using different trig functions as multipliers when summing series by the contour-integral method. They also give practice in finding residues at poles of high order.

   Use an appropriate contour integral of the functions suggested to obtain the following sums of series:

   (a) $f(z) = \dfrac{\cot z}{z^4}, \quad \dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \dfrac{1}{4^4} + \ldots = \dfrac{\pi^4}{90};$

   (b) $f(z) = \dfrac{\tan z}{z^4}, \quad \dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \dfrac{1}{7^4} + \ldots = \dfrac{\pi^4}{96};$

   (c) $f(z) = \dfrac{1}{z^4 \sin z}, \quad \dfrac{1}{1^4} - \dfrac{1}{2^4} + \dfrac{1}{3^4} - \dfrac{1}{4^4} + \ldots = \dfrac{7\pi^4}{720};$

   (d) $f(z) = \dfrac{1}{z^5 \cos z}, \quad \dfrac{1}{1^5} - \dfrac{1}{3^5} + \dfrac{1}{5^5} - \dfrac{1}{7^5} + \ldots = \dfrac{5\pi^5}{1536}.$

2. (a) Determine the poles and residues of the function

$$f(z) = \frac{\cot \pi z}{(z-a)(z-b)},$$

   where a and b are distinct complex numbers that are not integers.
   Use an appropriate contour integral of $f(z)$ to show that

$$S(a,b) \equiv \sum_{n=-\infty}^{\infty} \frac{1}{(n-a)(n-b)} = -\pi \left\{ \frac{\cot \pi a - \cot \pi b}{a-b} \right\}.$$

   This might seem like a random exercise, but sums of rational functions of n often appear in applications of quantum field theory to problems in statistical physics.

   The remaining parts of this question are optional, but may be of interest. No further contour integrations are involved.

   (b) By considering $S(a,-a)$, show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - a^2} = -\frac{\pi \cot \pi a}{a};$$

   then, for $|a| < 1$, expand $(n^2 - a^2)^{-1}$ in powers of a^2 and rearrange your result to show that

$$\sum_{k=1}^{\infty} \zeta(2k)\, a^{2k} = \tfrac{1}{2}(1 - \pi a \cot \pi a),$$

   where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function, which you have met in your course on statistical mechanics. Thus, the Riemann zeta function for positive even integer arguments can be read off from the coefficients in the series expansion of cot.

   (c) By choosing suitable arguments a and b for the function $S(a,b)$, evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + c^2},$$

   where $c \neq 0$ is real. Verify your result by considering the limiting behaviour (not the limiting *value*) for small and large values of c. Note that for large c, the sum can be approximated by an integral.