TWO HOURS

UNIVERSITY OF MANCHESTER

Statistical Mechanics

2nd June 2021, 11 a.m. - 1 p.m.

Answer **TWO** questions

You MUST NOT confer with anyone in answering the questions on this assessment.

The numbers are given as a guide to the relative weights of the different parts of each question.

Solutions must be handwritten and scanned, or handwritten on a tablet, and uploaded to Blackboard **as a single pdf file**.

Order the pages so that the answers to different questions are sequential and make clear on every page which question part is being addressed. Number the pages and write your student ID on the first page.

Ensure the scan is clear. Do not use green or red pens.

One hour of the exam duration has been allowed for accessing the exam and uploading the answers. Multiple submissions are allowed and you should upload your first attempt 30 minutes before the deadline. Only the final submission will be marked.

Late penalties will apply to work submitted after the deadline.

If you are a DASS-registered student with extra time, write your own submission time, which will have been communicated to you in advance, on the first page of your solutions, and submit before that deadline.

Information which may be used in this paper:

The density of states for a spinless particle in two dimensions is $g(k) = \dfrac{Ak}{2\pi}$.

The symbol β is defined as $\beta = 1/(k_B T)$.

The symbol n_Q is defined as $n_Q = \left(\dfrac{m k_B T}{2\pi\hbar^2}\right)^{3/2}$

Stirling's approximation is $\ln N! \approx N \ln N - N$ for $N \gg 1$.

$$\sum_{n=0}^{\infty} (n+1)\, x^n = (1-x)^{-2} = \text{for } |x| < 1.$$

The following integrals may be useful (note $n!! \equiv n(n-2)(n-4)\ldots 1$ for odd n.)

$$\int_0^\infty \frac{x^{1/2}}{e^x+1} dx = 0.678094 \qquad \int_0^\infty \frac{x^{1/2}}{e^x-1} dx = 2.31516$$

$$\int_0^\infty \frac{x}{e^x+1} dx = \frac{\pi^2}{12} \qquad \int_0^\infty \frac{x}{e^x-1} dx = \frac{\pi^2}{6}$$

$$\int_0^\infty \frac{x^{3/2}}{e^x+1} dx = 1.15280 \qquad \int_0^\infty \frac{x^{3/2}}{e^x-1} dx = 1.78329$$

$$\int_0^\infty \frac{x^2}{e^x+1} dx = 1.80309 \qquad \int_0^\infty \frac{x^2}{e^x-1} dx = 2.40411$$

$$\int_0^\infty \frac{x^{5/2}}{e^x+1} dx = 3.08259 \qquad \int_0^\infty \frac{x^{5/2}}{e^x-1} dx = 3.74453$$

$$\int_0^\infty \frac{x^3}{e^x+1} dx = \frac{7\pi^4}{120} \qquad \int_0^\infty \frac{x^3}{e^x-1} dx = \frac{\pi^4}{15}$$

$$\int_0^\infty x^n e^{-x/a} dx = n!\, a^{n+1} \qquad \int_0^\infty x^n e^{-x^2/a^2} dx = \frac{(n-1)!!\sqrt{\pi} a^{n+1}}{2^{(n/2+1)}} \quad \text{for even } n.$$

1. a) The entropy of some isolated system, as a function of its energy E, volume V and particle number N, is

$$S = Nk_B\left(\ln\left(\frac{V}{\pi^2}\left(\frac{E}{3\hbar c}\right)^3\right)+4\right).$$

Find the temperature of the system in terms of these variables.

[5 marks]

b) An isolated system consists of N distinguishable atoms, each of which has three states available to it: one with energy 0 and two with energy ε.

i) Show that the number of microstates of the system with n particles in total in the upper two states is

$$\Omega = \frac{N!}{n!(N-n)!}2^n$$

[4 marks]

ii) Find an expression for the entropy of the system in terms of the energy, and hence find the temperature of the system.

[6 marks]

iii) From your expression for the temperature as a function of energy, find and sketch an expression for the average energy as a function of temperature, commenting on the high- and low-temperature limits. Discuss how this compares to the expression $\langle E\rangle = N\varepsilon(1+\mathrm{e}^{\varepsilon\beta})^{-1}$ which is obtained if there is only one excited state.

[5 marks]

iv) Two sets of these atoms, N_1 and N_2 respectively, are brought into thermal contact with one another, while still being isolated from the surroundings. Their combined energy is $E = n\varepsilon$. After reaching equilibrium, the average number of excited atoms in the first system is n_1. Find an expression for n_1.

[5 marks]

2. a) A system has 16 states. The energies are -4ε, -2ε, 0, 2ε and 4ε, and the degeneracies are 1, 4, 6, 4 and 1 respectively.

i) Find the partition function for this system and obtain expressions for the average energy and entropy at temperature T.

[8 marks]

ii) We can interpret this system as involving 4 independent subsystems, each with energies $\pm\varepsilon$ and partition function Z_1. What condition is required on the subsystems for $Z = (Z_1)^4$? Write down the partition function if that condition does not hold.

[4 marks]

b) Consider a system of N identical non-interacting spin-0 bosons moving ultra-relativistically in two dimensions on an area A; their mass may be neglected.

i) Find the general expression which links the chemical potential to the number density, and show that it cannot be satisfied for all temperatures and densities. Explain the resolution of this problem.

[7 marks]

ii) Hence derive the equation which fixes the condensation temperature T_C,

$$\frac{N}{A} = C\left(\frac{k_B T_C}{\hbar c}\right)^2,$$

and determine the value of the constant C.

[3 marks]

iii) At temperatures $T < T_C$ show that the number of particles in the condensate is

$$N_0 = \left(1 - \left(\frac{T}{T_c}\right)^2\right) N.$$

[2 marks]

iv) If the mass cannot be neglected at the lowest energies, what changes?

[1 marks]

3. a) i) Calculate the chemical potential of O_2 molecules in a box with a number density of 10^{26} m^{-3}, and a temperature of 27° C.

[5 marks]

ii) The walls of the box contain sites which can bind a single molecule of O_2 with a binding energy of 0.2 eV. What is the probability that a given site will be occupied?

[5 marks]

b) The grand potential for a system of non-interacting spin-$\frac{1}{2}$ fermions which can occupy single-particle states of energy ϵ_i, at chemical potential μ, is

$$\Phi_G = -k_B T \sum_i \ln\left(e^{(\mu-\epsilon_i)\beta} + 1\right).$$

i) For a 3D gas of massless fermions moving ultra-relativistically, show that we can write the grand potential in the continuum limit as:

$$\Phi_G = -\frac{k_B T V}{\pi^2(\hbar c)^3} \int_0^\infty \ln\left(e^{(\mu-\epsilon)\beta} + 1\right) \epsilon^2 d\epsilon.$$

[5 marks]

ii) By considering the limit of the integrand as $T \to 0$ separately for the two cases $\epsilon < \mu$ and $\epsilon > \mu$, show that at zero temperature,

$$\Phi_G = -\frac{V\mu^4}{12\pi^2(\hbar c)^3}.$$

[7 marks]

iii) Hence show that the internal energy per particle of the gas is $\frac{3}{4}\mu$.

[3 marks]

END OF EXAMINATION PAPER