PHYS20672 Complex Variables and Vector Spaces: Examples 3

Some of these examples are copied from the exercises in the notes, but have been included here for completeness.

1. Show that the columns of a unitary matrix are orthogonal.
2. Consider the two-dimensional inner-product space $\mathbb{C}^2$. This space has an orthonormal bases $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. We can use this basis to construct a new basis $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$, such that $\boldsymbol{e}_\pm = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_1 \pm \boldsymbol{e}_2\right)$.
   (a) Show that $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ is an orthonormal basis for $\mathbb{C}^2$.
   (b) Write the vector $\boldsymbol{\psi} = \alpha \boldsymbol{e}_1 + \beta \boldsymbol{e}_2$ in the $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ basis.
   (c) Consider another vector $\boldsymbol{\phi} = \delta \boldsymbol{e}_+ + \gamma \boldsymbol{e}_-$. Show that the inner product
   $$\langle \phi, \psi \rangle = \frac{1}{\sqrt{2}}\left(\bar{\delta}(\alpha+\beta) + \bar{\gamma}(\alpha-\beta)\right), \tag{1}$$
   is the same in both the $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ and $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ basis sets.
   (d) Show that any inner product is left invariant by a change of basis.
   (e) Write steps a-d above in braket notation.
3. The vector space $\mathbb{C}^2$ has an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. The Pauli operators $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ have the following matrix representations with respect to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$:
   $$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2}$$
   Find the eigenvalues and eigenvectors for each of the Pauli operators.
4. a) Consider the operator $\hat{A}$ in $\mathbb{C}^3$ with matrix representation in the Cartesian basis given by
   $$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$
   Find the eigenvalues and eigenvectors of this operator. Can you find a basis that diagonalises this operator?
   b) Repeat this for the operator $\hat{B}$ with matrix representation
   $$\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \tag{4}$$
5. A unitary operator $\hat{U}$ which satisfies $\hat{U}\hat{U}^\dagger = \hat{\mathbb{1}}$, has eigenvalue equation,
   $$\hat{U}\boldsymbol{u}_j = \lambda_j \boldsymbol{u}_j. \tag{5}$$
   Prove that these eigenvalues and eigenvectors satisfy the the following conditions,
   (i) $|\lambda_j| = 1$, with $\lambda_j = e^{i\theta_j}$ where $\theta_j \in \mathbb{R}$.
   (ii) If $\lambda_j \neq \lambda_k$ then $\langle \boldsymbol{u}_j, \boldsymbol{u}_k \rangle = 0$.
   (iii) The eigenvectors of the $\hat{U}$ can be chosen to be an orthonormal basis for V_N.