Homework 3

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Part 1 : Matching Markets and Exchange Networks

(1).

> Suppose we have a set of two sellers labeled a and b, and a set of two buyers labeled x andy. Each seller is offering a distinct house for sale; buyer x has a value of 2 for a's house and 4 for b's house, while buyer y has a value of 3 for a's house and 6 for b's house.
>
> Now suppose that a charges a price of 0 for his house, and b charges a price of 1 for his house.Is this set of prices market-clearing? Give a brief (1-3 sentence) explanation; as part of your answer, say what the preferred choice graph is with this given set of prices, and use this inyour explanation.

This set of prices is not market clearing with the following preferred choice graph since there is no perfect matching:

```
graph LR
    subgraph SELLERS
    A["A : 0"]
    B["B : 1"]
    end
    subgraph BUYERS
    X["X : (2,4)"]-->B
    Y["Y : (3,6)"]-->B
    end
```

In the preferred choice graph above, here are the utilities for each player:

$$\text{Utility for Buyer X} = [2,4] - [0,1] = [2,3]$$
$$\text{Utility for Buyer Y} = [3,6] - [0,1] = [3,5]$$

Which means that both buyer X and buyer Y will prefer house B since they both receive the highest (positive utility) with this choice.

(2).

> Suppose we have a set of three sellers labeled a, b, and c, and a set of three buyers labeled x,y, and z. Each seller is offering a distinct house for sale, and the valuations of the buyers forthe houses are given in the following table. Suppose that sellers a and b each charge 2, and seller c charges 1. Is this set of prices market-clearing? Give a brief explanation; as part of your answer, say what the preferred choicegraph is with this given set of prices, and use this in your explanation.

This set of prices is not market clearing with the following preferred choice graph since there is no perfect matching:

```
graph LR
    subgraph SELLERS
    A["A : 2"]
    B["B : 2"]
    C["C : 1"]
    end
    subgraph BUYERS
    X["X : (5, 7, 1)"]-->B
    Y["Y : (2, 3, 1)"]-->B
    Z["Z : (5, 4, 4)"]-->C
    end
```

The table below represents the utilities for all three buyers:

UTILITIES	Seller A's House	Seller B's House	Seller C's House	COMMENTS
Buyer X	$5-2=3$	$7-2=5$	$1-2=0$	X prefers B
Buyer Y	$2-2=0$	$3-2=1$	$1-1=0$	Y prefers B
Buyer Z	$5-2=3$	$4-2=2$	$4-1=3$	Z is indifferent to A and C

(Bonus Question 1).

(a).

> We observed (Claim 8.5) that an exchange network consisting of a cyclic graph of 3 nodes has no stable outcome. Does this generalize to every cyclic graph? If not, can we characterize for which values of n the n-node cyclic graph has a stable outcome? Briefly justify your answers.

No, it doesn't.

Claim: For a cyclic graph with even number of edges, there will be a stable outcome. This isn't expected to be the case with cyclic graphs with odd number of edges.

Proof: Because in a graph with odd number of edges, we see that there will always be one node that is not paired with any other node, after rest of the nodes are paired with each other.

Consider the following set-up: for every edge e, set the value of the edge, $v(e)$, to some integer $k \geq 2$. Assume by contradiction that we have a stable outcome (M, a) in G. Since G is a graph with odd count of nodes, any matching in G can contains one node that is unpaired.

If M contains no edges, $a(x)=0$ for every node x, and thus for any pair of edges (x, y) we have $a(x)+a(y)=0<v(x, y)=k$, so (M, a) is not stable.

If M contains$#edges < \frac{n-1}{2}$ then $a(x)=0$ for multiple nodes x, and thus for any of these edges (x, y) we have $a(x)+a(y)=0 < v(x, y) = k$, so (M, a) is not stable.

Consider next the case when M contains edges (x, y) where each paired node, say x, must have $a(x) \leq \frac{k}{2}$, and the other node z must have $a(z)=0$ (since it is unmatched), thus $a(x)+a(z) \leq \frac{k}{2} < k$, which means that the outcome is not stable.

(b).

> Show by constructing an example that an exchange network that contains a 3-node cycle (but doesn't necessarily entirely consist of one) can still have a stable outcome.

```
graph LR
    A((A))---D((D))
    A---B((B))
    A---C((C))
    B---C
```

We see that this graph has a 3-node cycle and yet can have a stable outcome. We can see that A and D can enter into a partnership, as well as B and C.

Part 2 : Auctions and Mechanism Design

(3).

> In this question we will consider an auction in which there is one seller who wants to sell one unit of a good and a group of bidders who are each interested in purchasing the good. The seller will run a sealed-bid, second-price auction. Your firm will bid in the auction, but it does not know for sure how many other bidders will participate in the auction. There will be either two or three other bidders in addition to your firm. All bidders have independent, private values for the good. Your firm's value for the good is c. What bid should your firm submit, and how does it depend on the number of other bidders who show up? Give a brief (1-3 sentence) explanation for your answer.

Since this is a 2nd price auction, the players should submit a bid of c. This is regardless of what the number of other players show up, since bidding truthfully always provides a non-negative utility.

(4).

> In this problem we will ask how the number of bidders in a second-price, sealed-bid auction affects how much the seller can expect to receive for his object. Assume that there are two bidders who have independent, private values v_i which are either 1 or 3. For each bidder,the probabilities of v_i being 1 and 3 are independent (from other bidders) and both 1/2. (If there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x.)

(a).

> What is the seller's expected revenue from the auction? Justify your answer.

We see the probabilities of all the possible cases arising from this game:

$$P(v_i=1, v_2=1) = P(v_i=1, v_2=3) = P(v_i=3, v_2=1) = P(v_i=3, v_2=3) = \frac{1}{4}$$

Therefore, the Seller's expected revenue is $R=\frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 3 = \frac{6}{4} = \frac{3}{2}$.

(b).

> Assume that we add a third bidder that behaves identically to the first two (whose v_i is also chosen independently of those for the first two). What is the seller's expected revenue? Justify your answer.

We see the probabilities of all the possible cases arising from this game:

$$P(v_1, v_2, v_3) = \frac{1}{8}, \forall v_1, v_2, v_3 \in \{1, 3\}$$

Therefore, the Seller's expected revenue is:

$$R = \frac{1}{8} \cdot (1+1+1+1+3+3+3+3) = \frac{16}{8} = 2$$

(c).

> Explain the trend you noticed between parts (a) and (b)—that is, why changing the number of bidders affects (or doesn't affect) the seller's expected revenue.

In this setup, it looks like, the more bidders, the higher the seller's expected revenue.

As a generalization for n players,

-

$R=\frac{1}{2^n} \cdot [1 + n \cdot 1 + 3 \cdot (2^n - n - 1)]$ and

(5).

> A seller will run a second-price, sealed-bid auction for an object. There are two bidders, a and b, who have independent, private values vi which are either 0 or 1. For both bidders theprobabilities of vi being 0 or 1 are independent (from the other bidder) and both 1/2. Both bidders understand the auction, but bidder b sometimes makes a mistake about his value for the object. Half of the time his value is 1 and he is aware that it is 1; the other half of the time his value is 0 but occasionally he mistakenly believes that his value is 1. Let's suppose that when b's value is 0 he acts as if it is 1 with probability ½ and as if it is 0 with probability ½. So in effect bidder b sees value 0 with probability ¼ and value 1 with probability ¾.
>
> Bidder a never makes mistakes about his value for the object, but he is aware of the mistakes that bidder b makes. Both bidders bid optimally given their perceptions of the value of the object. Assume that if there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x.

We see that $P(v_i =0)=P(v_i = 1) = 0.5$ for both players.

However, for player 2, given that he understands bidder b's biases, following is the game he expects (and bids accordingly):

$$P(v_i = 0) = 0.25$$
$$P(v_i = 1) = 0.75$$

(a).

> Is bidding his true value still a dominant strategy for bidder a? Explain briefly.

Yes it is. We see the probabilities of all the possible cases arising from this game:

$$P(v_1(a=0), v_2(b=1)) = 0.5 \cdot 0.75 = \frac{3}{8}$$
$$P(v_1(a=0), v_2(b=0)) = 0.5 \cdot 0.25 = \frac{1}{8}$$
$$P(v_1(a=1), v_2(b=1)) = 0.5 \cdot 0.75 = \frac{3}{8}$$
$$P(v_1(a=0), v_2(b=0)) = 0.5 \cdot 0.25 = \frac{1}{8}$$

We see that, bidding truthfully always results in non-negative utility (since the player can never end up with an item for a price higher than his valuation).

(b).

> What is the seller’s expected revenue? Explain briefly.

Seller's expected revenue, equals the weighted sum of the bids made by players:

$$R = \frac{3}{8} \cdot 1 + \frac{1}{8} \cdot 0 + \frac{3}{8} \cdot 1 + \frac{1}{8} \cdot 1$$

(Bonus Question 2).

> Let’s say we define a “third-price auction” in the same manner that we defined the first-price and second-price auctions. Is this mechanism DST, NT, or neither? Does it DST-implement, NT-implement, Nash-implement, or fail to implement social welfare maximization? Justify your answers, by proof or by counterexample.

A 3rd price auction is neither DST, nor NT.

Proof: Consider player i with valuation v, and 2 other players with valuations $v '$ and v'' such that $v<v''<v'$. The ith player can benefit from bidding above his true valuation in the following manner (assuming all other players bid their true valuation):

- **Bid 1:** Bid of the i th player (which is not equal to its true value)
- **Bid 2:**
  - v' (highest valuation amongst all players)
  - v : true valuation of the i th player
- **Bid 3:** v'' (2nd highest valuation amongst all players)

As long as i bids below v', his utility remains 0 (since he/she loses). In contrast, if he bids above v', he/she wins the auction and receives the item for a price v'' (the 3rd highest bid) which is less than his valuation v.

Since we see that his player has an incentive to lie, this auction is neither DST nor NT.

Part 3 : Sponsored Search

(6).

> Suppose a search engine has two ad slots that it can sell. Slot a has a clickthrough rate of 10 and slot b has a clickthrough rate of 5. There are three advertisers who are interested in these slots. Advertiser x values clicks at 3 per click, advertiser y values clicks at 2 per click, and advertiser z values clicks at 1 per click.
>
> Compute the socially optimal allocation and the VCG prices for it. Give a brief explanation for your answer.

We see that the following allocation would be socially optimal:

```
graph LR
    subgraph SLOT
    A["A : CTR 10"]
    B["B : CTR 5"]
    end
    subgraph ADVERTISER
    X[" X : 3/click"]-->A
    Y[" Y : 2/click"]-->B
    Z[" Z : 1/click"]
    end
```

This is because the total social utility is maximized when the advertisers with the highest value per click get paired with the slots with the highest CTR.

"Plain" VCG prices:

$$p_1 = -(2 \cdot 10 + 1 \cdot 5) = -25$$
$$p_2 = -(3 \cdot 10 + 1 \cdot 5) = -35$$
$$p_3 = -(3 \cdot 10 + 2 \cdot 5) = -40$$

(7).

> Suppose a search engine has three ad slots that it can sell. Slot a has a clickthrough rate of 6, slot b has a clickthrough rate of 5, and slot c has a clickthrough rate of 1. There are three advertisers who are interested in these slots. Advertiser x values clicks at 4 per click,advertiser y values clicks at 2 per click, and advertiser z values clicks at 1 per click.
>
> Compute the socially optimal allocation and the VCG prices for it. Give a brief explanationfor your answer.

We see that the following allocation would be socially optimal:

```
graph LR
    subgraph SLOT
    A["A : CTR 6"]
    B["B : CTR 5"]
    C["C : CTR 1"]
    end
    subgraph ADVERTISER
    X[" X : 4/click"]-->A
    Y[" Y : 2/click"]-->B
    Z[" Z : 1/click"]-->C
    end
```

This is because the total social utility is maximized when the advertisers with the highest value per click get paired with the slots with the highest CTR.

"Plain" VCG prices:

$$\begin{aligned} p_1 &= -(2 \cdot 5 + 1 \cdot 1) = -11 \\ p_2 &= -(4 \cdot 6 + 1 \cdot 1) = -25 \\ p_3 &= -(4 \cdot 6 + 2 \cdot 5) = -34 \end{aligned}$$

Part 4 : Richard Thaler

(8).

> Read about the contributions to behavioral economics made by Richard Thaler, the winner of the 2017 Nobel Prize in Economic Sciences. Write a short (roughly ½ page) essay summarizing some of his work and connecting it to some of the topics we have presented in this class. (You should write this individually, though you may discuss ideas with your group. Furthermore, we may award bonus credit for outstanding essays.)

Most modern economic theories assume that actors are rational. In other words, that they will make decisions based on which ever option carries the highest payoff. But the reality is much more complex than a simple payoff matrix and it's hard to model the irrational behavior of economic actors. Professor Richard Thaler, who teaches at the University of Chicago Booth School of Business, has done pioneering work in trying to predict irrational behavior and thus improving the understanding of human behavior. As a result of this work, he was awarded the 2017 Nobel Prize in Economic Sciences.

Part 5 : Implementing Matching Market Pricing

> The goal of this exercise is to implement an algorithm for finding market-clearing and VCG prices in a bipartite matching market. For each part of questions 9 and 10 (and bonus question 3, should you attempt it) that asks you to implement an algorithm, unless otherwise stated, ==submit your code along with its execution on a few (3-4) small (10-20 node) test graphs==, one of which must be the example in figure 7.3 from the notes.

(9).

> Recall the procedure constructed in Theorem 7.8 of the notes to find a market equilibrium in a matching market.

(a).

> The first step of this procedure involves either finding a perfect matching or a constricted set. Recall that this can be done using maximum flow. Now, either using your maximum-flow implementation from assignment 2 or a new implementation, implement an algorithm that finds either a perfect matching M or a constricted set S in a bipartite graph.

(b).

> Now, given a bipartite matching frame with n players, n items, and values of each playerfor each item, implement the full procedure to find a market equilibrium.

(10).

> Now, given a matching market frame, we wish to implement VCG pricing in this frameaccording to the results of Theorem 9.8 in the notes.

(a).

> In order to do this, we must find the socially optimal outcome. Briefly justify that theoutcome that your algorithm from the previous question finds is socially optimal (thiscan be done by simply stating 1-2 theorems from the notes).

(b).

> Finally, implement the Clark pivot rule to construct an algorithm that produces a posi- tive set of VCG prices in any matching market frame.

(11).

> Now simulate your VCG pricing algorithm in two different contexts.

(a).

> First, construct a graph of 20 buyers and 20 items. First assume that item i is a bundleof i identical goods. Now assign each player a random value per good (say, from 1 to 50;ties can be allowed). You shouldn't need to do any additional work on your algorithmto do this, instead just set each buyer's value per bundle appropriately. How should weset these values?

(b).

> Having set the values accordingly, run your VCG pricing algorithm and turn in theresults. Explain why the results you obtained make sense in the context above.

(c).

> Now construct a different context where, rather than assuming each item is a bundle ofidentical goods, we take each buyer's values for the 20 items from the previous contextand randomly permute them over the 20 items. (For instance, if we had 3 items, anda buyer had values (2, 4, 6) for these items, we randomly permute the values so

> thatthey could now have values (4, 2, 6), (6, 2, 4), etc.) Run your VCG pricing algorithmagain for this new context. How do the prices change? Intuitively explain the differencesbetween this and the previous context that might cause this change.

(Bonus Question 3).

> Bonus Question 3. (Please note that each part of this question is intended to be harder than the previous parts; each part will be graded separately, and you are not required to do the whole thing.)

(a).

> Implement an algorithm for GSP pricing in a matching-market context. Compare yourVCG prices from the previous question to the GSP prices when run in the same contexts(with the same randomness). Also, try to find and characterize some contexts whereVCG and GSP prices are similar, and where they are wildly different.

(b).

> GSP isn't truthful, so interesting things might happen if we run BRD on a GSP matchingmarket auction (where a player's "strategy" is their valuation report). Implement analgorithm that picks a random starting point and runs BRD to attempt to find a GSPequilibrium state. What happens? Does BRD converge; if so, how quickly?

(c).

> (Very difficult.) Either prove that BRD will converge in a GSP context, or disprove it by counterexample.