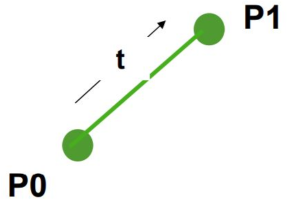
-The 3 reasons why "y=mx+b" doesn't cut it for us.

1. You cannot represent vertical lines (m = ∞)
2. You can only represent infinite lines, not finite line segments
3. You can only represent 2D lines, not 3D

-Parametric Line equation: two forms (shooting and blending), line segment intersection, parallel line intersection ("the math talks to you").

The “Shooting” Form



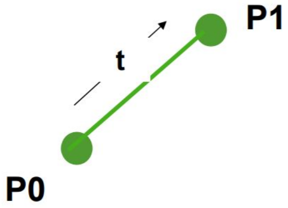
$$x = X_0 + t(X_1 - X_0)$$

$$y = Y_0 + t(Y_1 - Y_0)$$

$$z = Z_0 + t(Z_1 - Z_0)$$

$$0. \le t \le 1.$$

The “Blending” Form



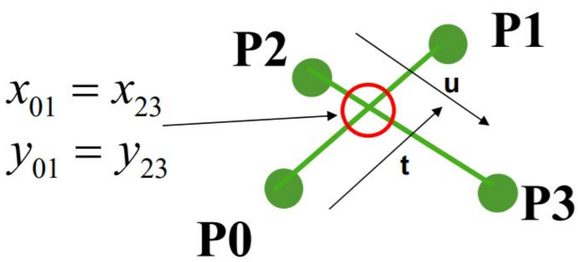
$$x = (1-t)X_0 + tX_1$$

$$y = (1-t)Y_0 + tY_1$$

$$z = (1-t)Z_0 + tZ_1$$

$$0. \le t \le 1.$$

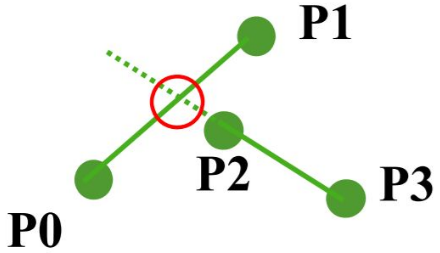
2D Line Segments



$$x_{01} = X_0 + t(X_1 - X_0) \qquad x_{23} = X_2 + u(X_3 - X_2)$$

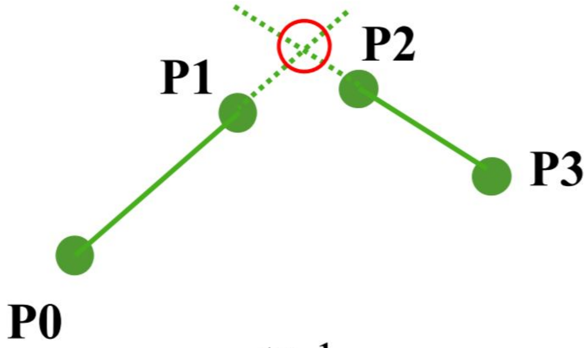
$$y_{01} = Y_0 + t(Y_1 - Y_0) \qquad y_{23} = Y_2 + u(Y_3 - Y_2)$$

Solve for t* and u*. If they are not each between 0. and 1., then the infinite lines intersect, but not the finite line segments.



$0. \le t \le 1.$

$u < 0.$



$t > 1.$

$u < 0.$

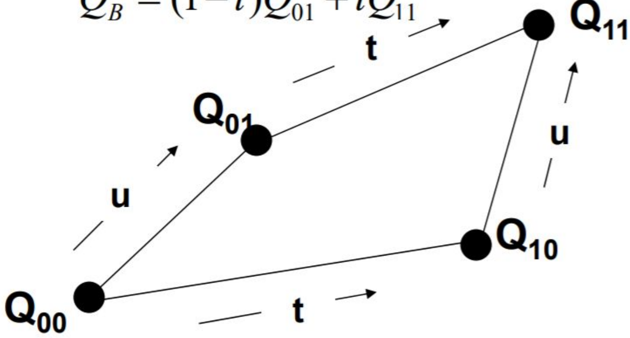
the Lines are Parallel
This shows up in the math by the expression for t* and u* becoming infinitely large, that is, there would be a divide by zero.

-Interpolation
linear

$$Q = (1-t)Q_0 + tQ_1$$

bilinear
writing the line blending equation twice and then blending the two lines

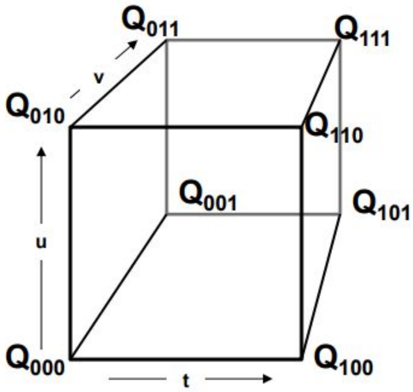
$$Q_B = (1-t)Q_{01} + tQ_{11}$$



$$Q_A = (1-t)Q_{00} + tQ_{10}$$

$$Q_{AB} = (1-u)Q_A + uQ_B = (1-t)(1-u)Q_{00} + t(1-u)Q_{10} + (1-t)uQ_{01} + tuQ_{11}$$

trilinear



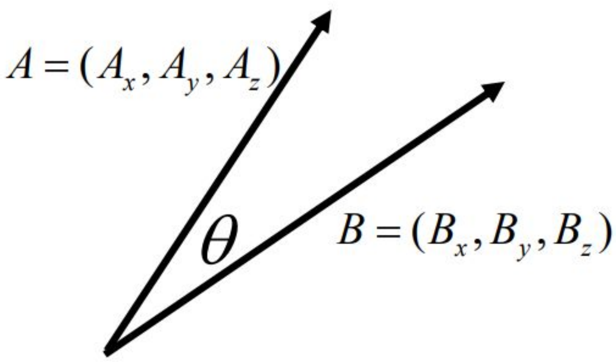
-Vectors: length, unit vector, dot product, physical meaning, cross product, physical meaning, right-hand rule, finding a surface normal, area of a 3D triangle, is a point in a 3D triangle, signed distance from a point to a plane, intersection of a line segment and a plane. (You are not responsible for the vector derivations of the Law of Sines and the Law of Cosines.)

$$\|V\| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

$$\hat{V} = \frac{V}{\|V\|}$$

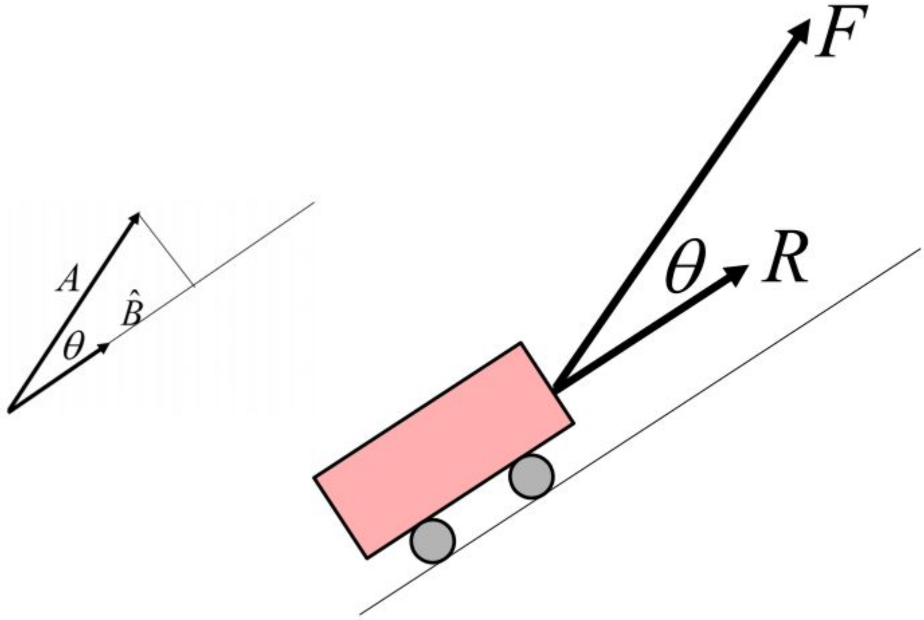
The circumflex (^) tells us this is a unit vector

Dot Product



$$A \bullet B = (A_x B_x + A_y B_y + A_z B_z) = \|A\| \|B\| \cos\theta$$

A Physical Interpretation of the Dot Product



$$F_{road} = F\cos\theta = F \bullet \hat{R}$$

vector lives on B

So, how much of A lives in the $\hat{B}$ direction is that magnitude times the B unit vector:

$$\hat{B}(A \cdot \hat{B})$$

That, plus the perpendicular vector equals A, so that how much of A is perpendicular to the $\hat{B}$ direction is:

$$\mathrm{A} - \hat{B}(A \cdot \hat{B})$$

Dot Products are Commutative

$$A \bullet B = B \bullet A$$

Dot Products are Distributive

$$A \bullet (B + C) = (A \bullet B) + (A \bullet C)$$

The Perpendicular to a 2D Vector

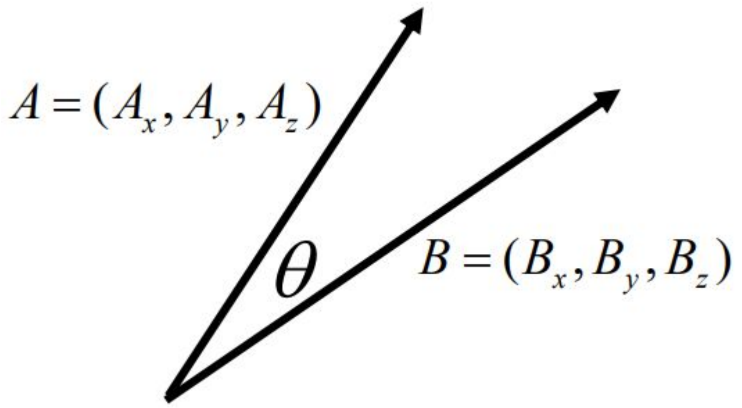
If $\mathrm{V} = (\mathrm{x}, \mathrm{y})$

then $\mathrm{V}_{\perp} = (-\mathrm{y}, \mathrm{x})$

You can tell that this is true because

$$\mathrm{V} \bullet \mathrm{V}_{\perp} = (\mathrm{x}, \mathrm{y}) \bullet (-\mathrm{y}, \mathrm{x}) = -\mathrm{xy} + \mathrm{xy} = 0 = \cos 90^{\circ}$$

Cross Product

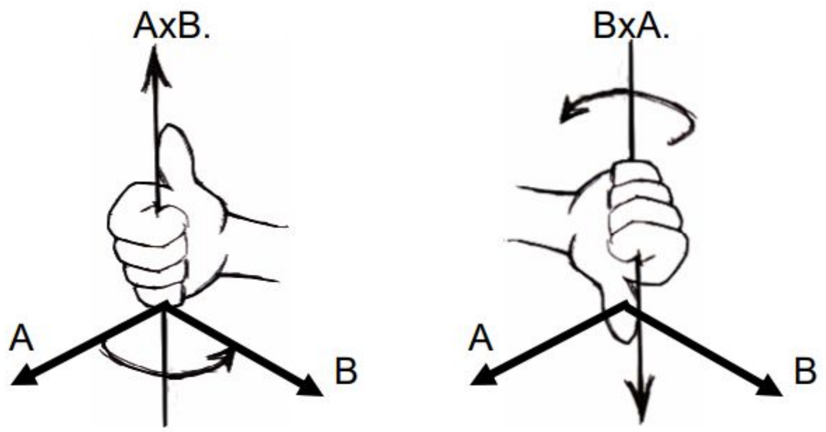


$$A \times B = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x)$$

$$\|A \times B\| = \|A\| \|B\| \sin\theta$$

Cross Products are *Not* Commutative

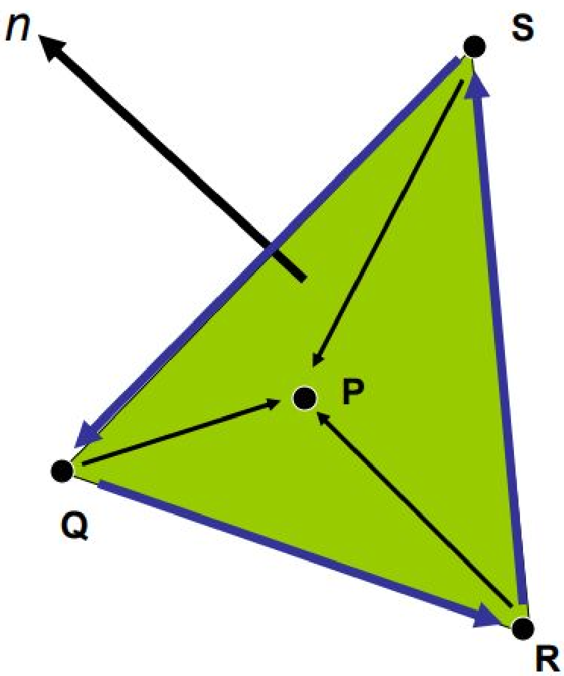
$$A \times B = -B \times A$$



Cross Products are Distributive

$$A \times (B + C) = (A \times B) + (A \times C)$$

A Use for the Cross and Dot Products :
Is a Point Inside a Triangle? – 3D (X-Y-Z) Version



Let:

$$n = (R - Q) \times (S - Q)$$

$$n_q = (R - Q) \times (P - Q)$$

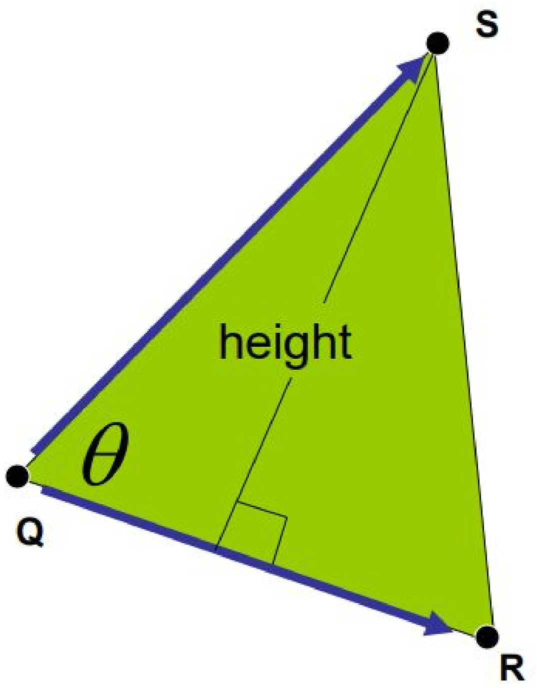
$$n_r = (S - R) \times (P - R)$$

$$n_s = (Q - S) \times (P - S)$$

If $(n \bullet n_q), (n \bullet n_r), and (n \bullet n_s)$

are all positive, then P is inside the triangle QRS

A Use for the Cross Product : Finding the Area of a 3D Triangle



$$Area = \frac{1}{2} \cdot Base \cdot Height$$

$$Base = \|QR\|$$

$$Height = \|QS\| \sin\theta$$

$$Area = \frac{1}{2} \cdot \|QR\| \cdot \|QS\| \cdot \sin\theta = \frac{1}{2} \cdot \|(R-Q) \times (S-Q)\|$$

Distance from a Point to a Plane

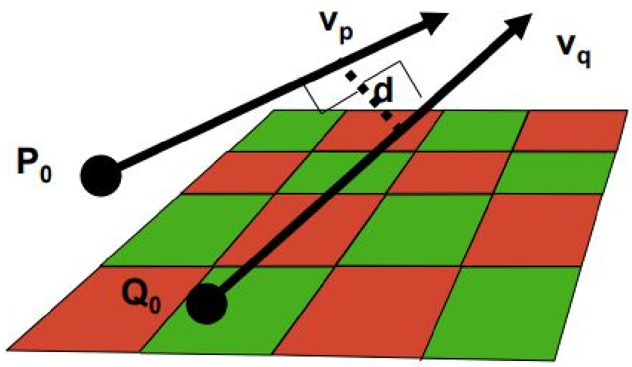
If you want the familiar equation of the plane, it is:

$$\left((x,y,z) - (Q_x,Q_y,Q_z)\right) \bullet (n_x,n_y,n_z) = 0$$

which expands out to become the more familiar Ax + By + Cz + D = 0

Note that this gives a signed distance. If d > 0., then P is on the same side of the plane as the normal points. This is very useful.

Minimal Distance Between Two 3D Lines



The equation of the lines are : $P = P_0 + t \cdot v_p \qquad Q = Q_0 + t \cdot v_q$

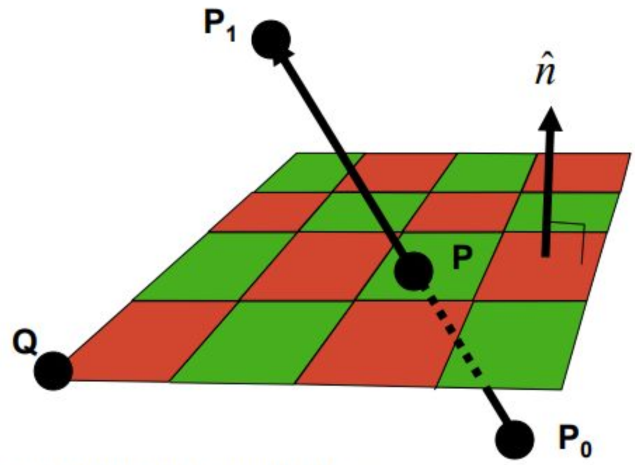
The minimal distance vector between the two lines must be perpendicular to both

A vector between them that is perpendicular to both is: $v_\perp = v_p \times v_q$

We need to answer the question “How much of (Q_0-P_0) is in the $v_\perp$ direction?”.
To do this, we once again use the dot product:

$$d = \left(P_0 - Q_0\right) \bullet \hat{v}_\perp$$

Where does a line segment intersect an infinite plane?



The equation of the line segment is:

$$P = (1-t)P_0 + tP_1$$

If point P is in the plane, then:

$$\left(\left(P_x, P_y, P_z\right) - \left(Q_x, Q_y, Q_z\right)\right) \bullet (n_x, n_y, n_z) = 0$$

If we substitute the parametric expression for P into the plane equation, then the only thing we don't know in that equation is t. Solve it for t*. Knowing t* will let us compute the (x,y,z) of the actual intersection using the line equation. If t* has a zero in the denominator, then that tells us that t*=∞, and the line must be parallel to the plane.

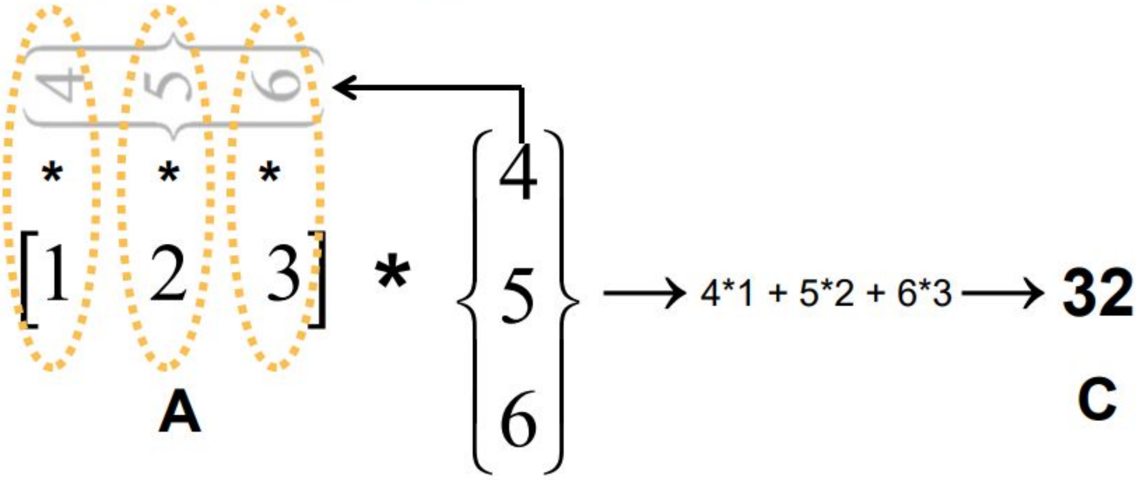
This gives us the point of intersection with the infinite plane. We could now use the method covered a few slides ago to see if P lies inside a particular triangle.

-GLM primer
http://web.engr.oregonstate.edu/~mjb/cs491/glmprimer.html

-Matrices: dimensions, multiplication, column matrix to hold a 3D point.

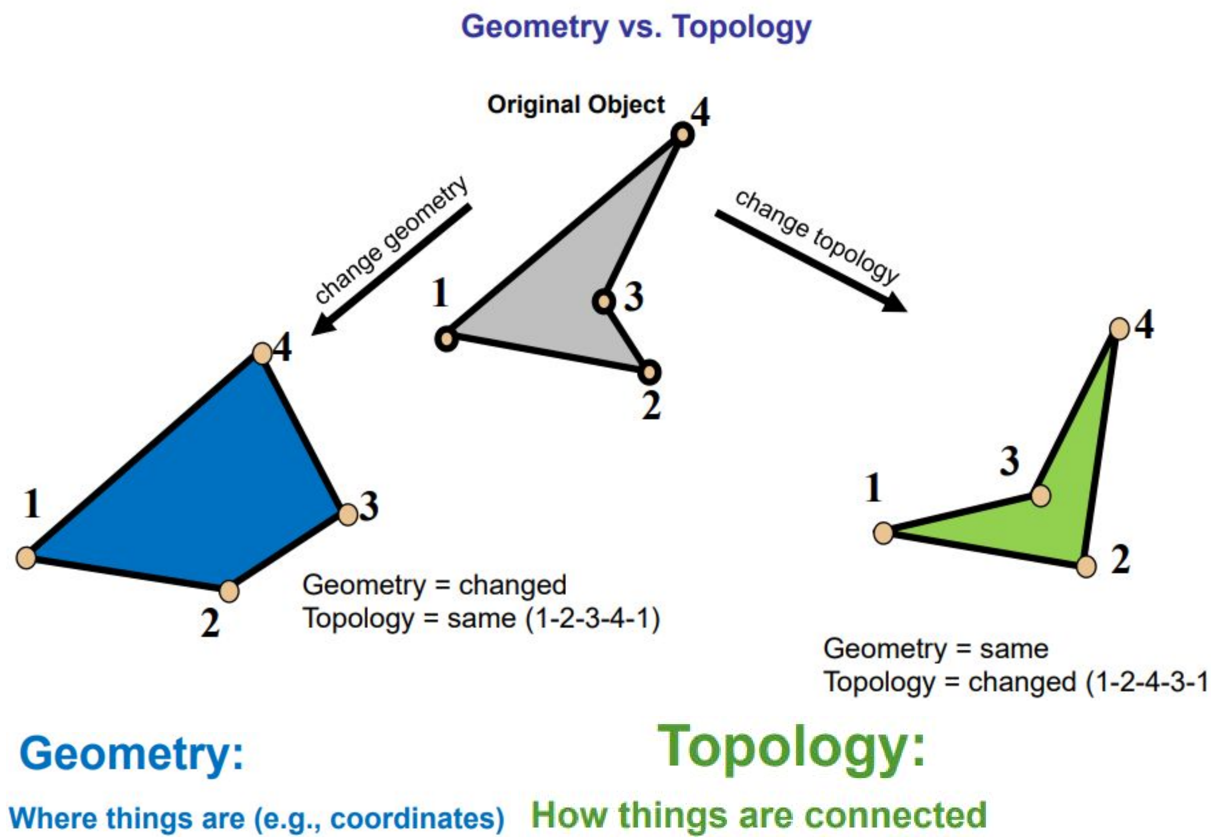
Matrix Multiplication

The basic operation of matrix multiplication is to pair-wise multiply a single row by a single column




```
for( int i = 0; i < numArows; i++ )
{
        for( int j = 0; j < numBcols; j++ )
        {
                C[ i ][ j ] = 0.;
                for( int k = 0; k < numAcols; k++ )
                {
                        C[ i ][ j ] += A[ i ][ k ] * B[ k ][ j ];
                }
        }
}
```

-Transformations: geometry vs. topology, RH vs. LH coordinate system, identity matrix, inverse matrix, transformation equations, transformation matrices, coordinate systems, positive rotation rule, compound transformations. (You *are* responsible for knowing how to derive the translation, scaling, and rotation matrices for *any coordinate system* and for *any positive rotation rule*.)



$$
\begin{array}{cccc}
\text{x consuming column} & \text{y consuming column} & \text{z consuming column} & \text{constant column}
\end{array}
$$

$$
= \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}
$$

Matrix Inverse

$$[M] \bullet [M]^{-1} = [I]$$

$$[M] \bullet [M]^{-1} = \text{"Nothing has changed"}$$

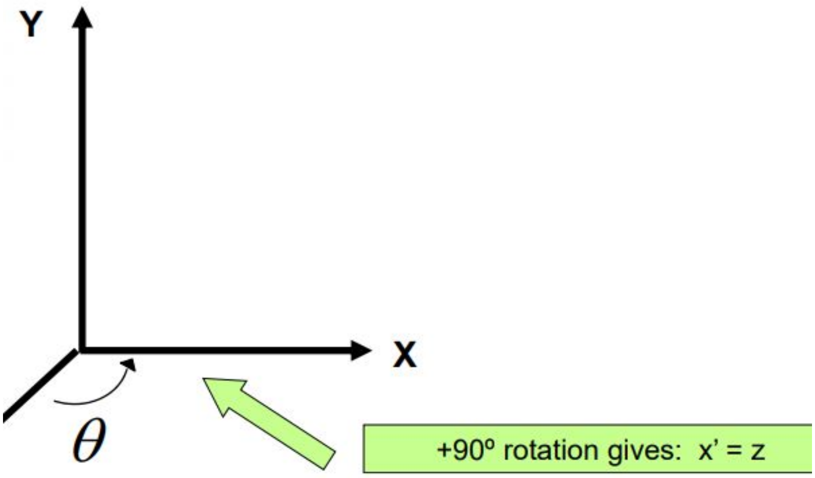
Translation Matrix

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

Scaling Matrix

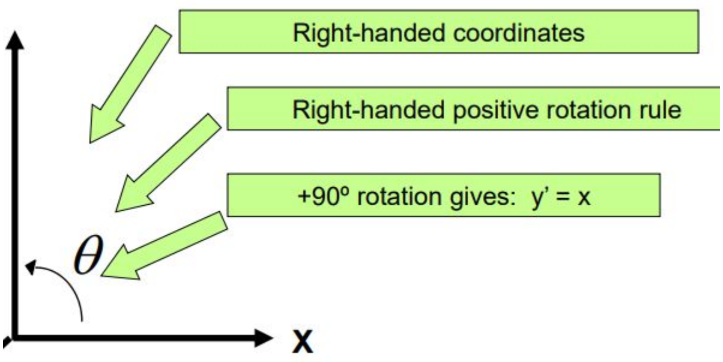
$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

3D Rotation Matrix About Y



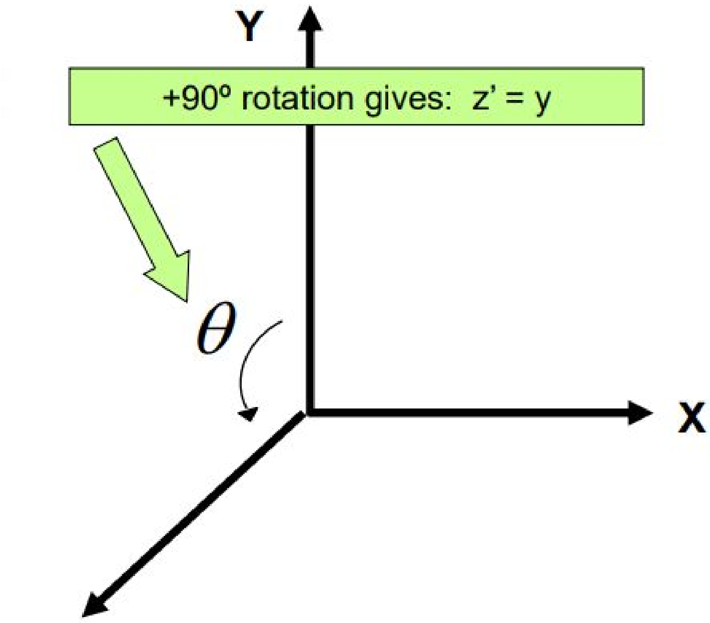
$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

3D Rotation Matrix About Z



$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

3D Rotation Matrix About X



$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

-Forward kinematics (hierarchical transformations): formulating the matrices in the correct order

Positioning Part #1 With Respect to Ground

Write it

$$[M_{1/G}] = [T_{1/G}] * [R_{\theta 1}]$$

Say it

Positioning Part #2 With Respect to Ground

1. Rotate by Θ2
2. Translate the length of part 1
3. Rotate by Θ 1
4. Translate by $T_{1/G}$

Write it

$$[M_{2/G}] = [T_{1/G}]*[R_{\theta 1}]*[T_{2/1}]*[R_{\theta 2}]$$

$$[M_{2/G}] = [M_{1/G}]*[M_{2/1}]$$

Positioning Part #3 With Respect to Ground

1. Rotate by Θ3
2. Translate the length of part 2
3. Rotate by Θ2
4. Translate the length of part 1
5. Rotate by Θ1
6. Translate by $T_{1/G}$

Write it

$$[M_{3/G}] = [T_{1/G}]*[R_{\theta 1}]*[T_{2/1}]*[R_{\theta 2}]*[T_{3/2}]*[R_{\theta 3}]$$

$$[M_{3/G}] = [M_{1/G}]*[M_{2/1}]*[M_{3/2}]$$

Say it

-Solving nonlinear equations using Newton's Method: how to do it, how it works, what can go wrong

Newton's Method for Solving a Nonlinear Equation

You can take the x you have, x_{have}, and plug it into the equation to produce y_{have} and thus see how close you are to y = 0. But now what?

From calculus, we know that:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} \quad \text{or} \quad \frac{dy}{dx}\Delta x = \Delta y$$

The required Δy to get from $y = y_{have}$ to y = 0. is:

$$\Delta y = y_{want} - y_{have} = 0. - y_{have}$$

So that: $$\frac{dy}{dx}\Delta x = \Delta y = y_{want} - y_{have} = 0. - y_{have}$$

which gives us: $$\Delta x = \frac{-y_{have}}{\frac{dy}{dx}}$$

We will use that to find the next value of x to try, and then repeat the process:

$$x_{have} = x_{have} + \Delta x = x_{have} + \frac{-y_{have}}{\frac{dy}{dx}}$$

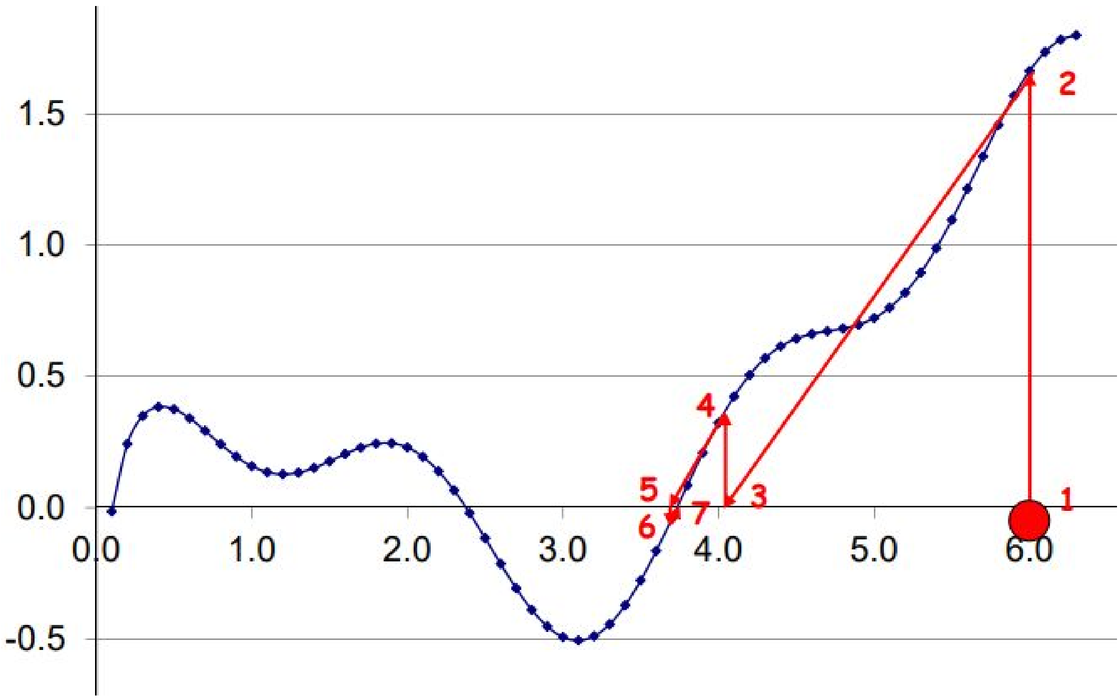
$$y_{have} = y(x_{have})$$

Watching Newton's Method Work

$$y = \cos^3 x + \log_{10} x = 0$$

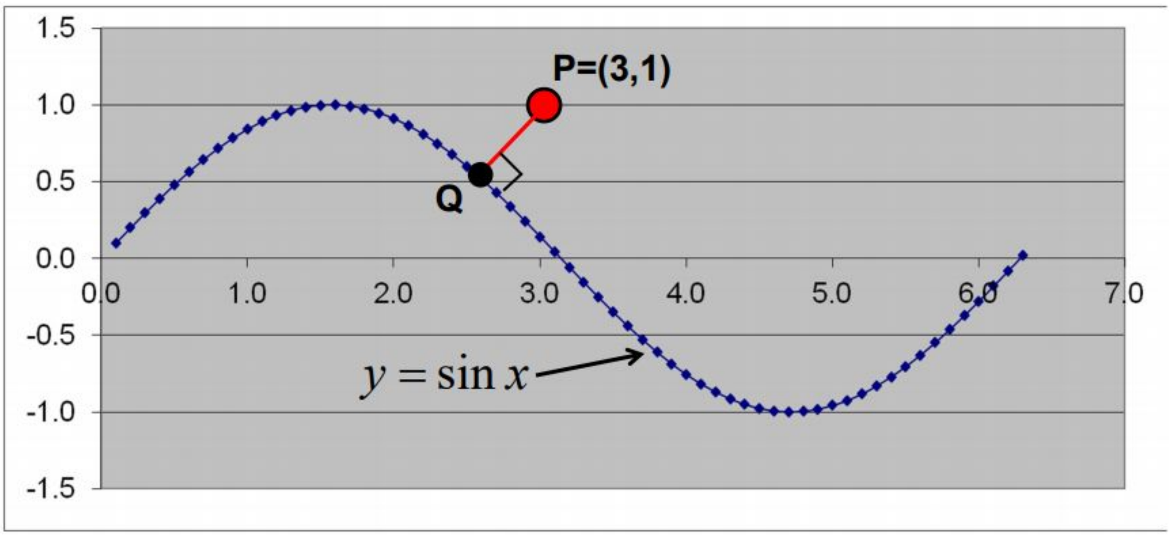
$$\frac{dy}{dx} = -3 * \sin x * \cos^2 x + \frac{1}{x \ln(10)}$$

x_{have}	y_{have}	dydx	x_{next}
6.00000	1.66336	0.84518	4.03196
4.03196	0.35651	1.03069	3.68607
3.68607	-0.05934	1.25483	3.73336
3.73336	0.00040	1.26907	3.73304
3.73304	0.00000	1.26903	3.73304
3.73304	0.00000	1.26903	3.73304
3.73304	0.00000	1.26903	3.73304
3.73304	0.00000	1.26903	3.73304
3.73304	0.00000	1.26903	3.73304
3.73304	0.00000	1.26903	3.73304
3.73304	0.00000	1.26903	3.73304



A Collision Detection Example

Let's say we have a nonlinear surface. How close is the point (3,1) to that surface?



Using our friend, the dot product:

$$(P_x - Q_x, P_y - Q_y) \bullet slope = 0$$

where the vector slope is:

$$slope = (dx, dy) = (1, \frac{dy}{dx}) = (1, \frac{d \sin x}{dx}) = (1, \cos x)$$

$$f(x) = (P_x - x, P_y - \sin x) \bullet (1, \cos x) = 0$$

$$f(x) = \left(P_x - x\right) + \cos x * \left(P_y - \sin x\right) = 0$$

A Collision Detection Problem Example

$$f(x) = \left(P_x - x\right) + \cos x * \left(P_y - \sin x\right) = 0$$

xhave	yhave	fhave	dfdx	xnext
1.00000	0.84147	2.08565	-1.42532	2.46328
2.46328	0.62748	0.24666	-1.84002	2.59733
2.59733	0.51778	-0.00988	-1.98158	2.59235
2.59235	0.52204	-0.00001	-1.97699	2.59234
2.59234	0.52205	0.00000	-1.97698	2.59234
2.59234	0.52205	0.00000	-1.97698	2.59234
2.59234	0.52205	0.00000	-1.97698	2.59234

$$dist = \sqrt{(3 - 2.59234)^2 + (1 - .52205)^2}$$

$$dist = .62819$$

-Physics

$$F_{12} = \frac{Gm_1m_2}{d_{12}^2} \quad \text{where: } G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$$

and d_{12} is the distance between body 1 and body 2

For an object, m, at or near the surface of the Earth (i.e., d_{12} is the radius of the Earth) this simplifies to:

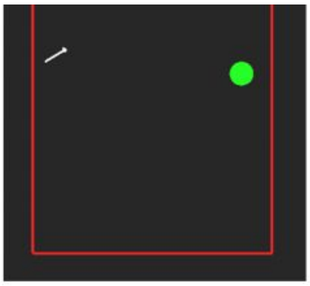
$$F = mg \quad \text{where: } g = -9.8 \frac{meters}{\sec^2} = -32.2 \frac{feet}{\sec^2}$$

g is known as the ***Acceleration Due to (Earth's) Gravity***

Constant-Acceleration Formulas
(these need to be memorized!)

$$v_1 = v_0 + at$$
$$d_1 = d_0 + v_0 t + \frac{1}{2} a t^2$$

A Ball Bouncing in a Box



Current Position = (x,y)
Current Velocity = (v_x,v_y)
How long until the *next* bounce?

$$x' = x + v_x t = x_{left} + radius$$

$$x' = x + v_x t = x_{right} - radius$$

$$y' = y + v_y t + \frac{1}{2} g t^2 = y_{floor} + radius \longrightarrow t^* = \frac{-v_y \pm \sqrt{v_y^2 - 2g\left(y - y_{floor} - radius\right)}}{g}$$

Spinning Motion:
Constant-Acceleration Formulas

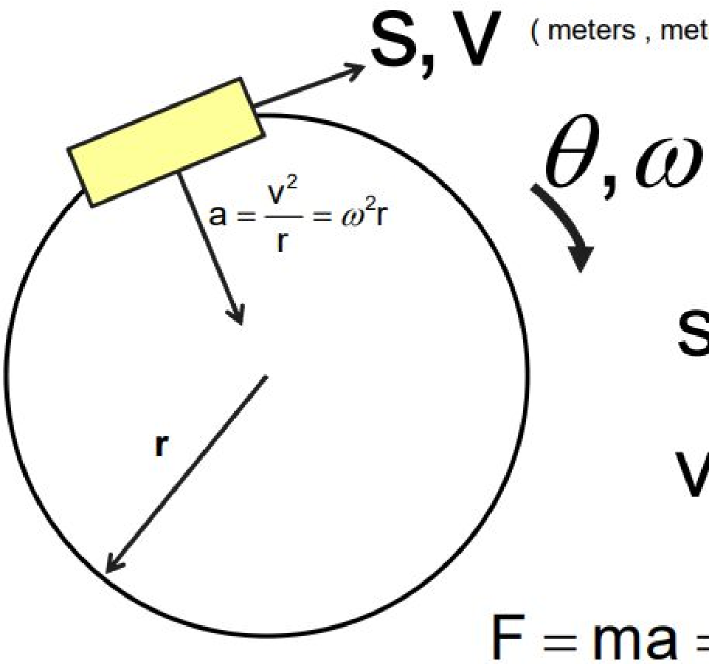
Angular acceleration (radians/sec^2)

$$\omega_1 = \omega_0 + \alpha t$$

Angular velocities (radians/sec)

$$\theta_1 = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega_1^2 = \omega_0^2 + 2\alpha(\theta_1 - \theta_0)$$



(meters , meters/sec)

(radians , radians/sec)

$$s = r\theta$$

$$v = r\omega$$

$$F = ma = m\frac{v^2}{r} = m\omega^2 r$$

This force points *towards* the instantaneous center of curvature