Computational Fluid Dynamics With a Paper Airplane

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Abstract

In this paper, we investigate the relationship between an airplane's shape and its performance. Our results show that ...

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1 Introduction

This thesis covers the simulation of the aerodynamics of an airplane, using our own Computational Fluid Dynamics model (CFD).

The goal is to simulate the airflow around an airplane's body. CFD is a branch of fluid mechanics that uses numerical methods and algorithms to solve and analyze problems that involve fluid flows. It is used in many fields, including aerospace engineering, automotive engineering, and meteorology.

The application of CFD to an airplane is important because it allows testing of a model's aerodynamic performance without building a physical model, or have to set up a wind tunnel. The practical alternative is much more expensive and time-consuming.

The final aim is to determine how the shape of an airplane's body affects its performance. Part of the project is to dynamically generate 3D models of airplanes, and

simulate the airflow around them. The intention is to use machine learning to optimize the shape of the airplane's body to maximize performance.

CFD is challenging in the sense that it requires a good understanding of fluid dynamics, as well as a good understanding of the math involved. CFD is also very expensive from a computational perspective, so code optimization is important.

The thesis questions are the following:

- How do the different aspects of fluid dynamics work and how do we implement it in a computer program?
- How do we dynamically generate 3D models?
- How does an airplane's wing shape influence its performance?

2 Execution

2.1 Preliminary: Lagrangian Fluid Simulation

One method of simulating fluid dynamics is the Lagrangian method. This method models the fluid as a particle collision system, where the air is represented by particles that interact with each other to emulate a fluid. Our implementation is based on Rigid Body Collision Resolution (Hakenberg, 2005). We used this paper as a guide for all the math involved.

The math relies on the momentum, inertia, and velocity of the particles to calculate the collision normal and point of contact. The collision normal is the direction in which the particles are moving away from each other, and the point of contact is the point at which the particles collide.

The following variables are necessary to perform the calculations:

Angular momentum L $(kg \cdot m^2/s)$:	$L = mvr;$
Inertia tensor I $(kg \cdot m^2)$:	$I = \frac{L}{\omega};$
Angular velocity ω (rad/s) :	$\omega = \frac{\Delta\theta}{\Delta t};$

Collision normal($n \in \mathbb{R}^3$) in world coordinates away from body;
Point of contact ($r_i \in \mathbb{R}^3$) in world coordinates with respect to p_i;
Orientation ($R_i \in SO(3)$) transforming from object to world coordinates;

Where i represents one of two particles in a given collision:

Velocity after collision	$\tilde{v}_i,$
Angular velocity after collision	$\tilde{\omega}_i,$
Constant	$\lambda,$

The following formulas represent the relation between particles:

$$\tilde{v}_1 = v_1 - \frac{\lambda}{m_1}n;$$
$$\tilde{v}_2 = v_2 + \frac{\lambda}{m_2}n;$$
$$\tilde{\omega}_1 = \omega_1 - \Delta q_1;$$
$$\tilde{\omega}_2 = \omega_2 + \Delta q_2;$$
$$\text{where } q_i := I_i^{-1} \cdot R_i^{-1} \cdot (r_i \times n),$$
$$\text{and } \lambda = 2\frac{nv_1 - nv_2 + \omega_1 I_1 q_1 - \omega_2 I_2 q_2}{(\frac{1}{m_1} + \frac{1}{m_2})n^2 + q_1 I_1 q_1 + q_2 I_2}$$

This was implemented using Go and raylib, and the code is available at github.com/dtasada/paper at the `lagrangian-go` branch.

When dealing with a lot of particles, the simulation stops performing as well, because the number of calculations and iterations of the formulas listed above has a time complexity of $O(n^2)$, where n is the number of particles. This is because every particle handles collisions with every other particlee every single frame. This can be elegantly mitigated by implementing a three-dimensional grid system, where every particle is assigned to a cell in the grid. This way, particles only need to check for collisions with other particles in the same cell or neighboring cells. If the particles are evenly distributed within the container, each particle only needs to handle collisions with particles in its own cell, and its 26 neighboring cells. This reduces the time complexity to $O(n)$.

The simulation was unstable at higher densities, where particles would phase into each other instead of cleanly bouncing. The bug is likely quite simple to fix, but we got distracted by another method. Despite the optimizations we made, Lagrangian simulation is still quite computationally expensive, and Go probably wasn't doing it any favors (Go isn't very memory efficient). In the end it was put aside, and the Lagrangian implementation was never completed.

2.2 Eulerian Fluid Simulation

The other method is the Eulerian method. This method models the fluid as a cellular automata. The fluid is represented by a grid of cells, each of which has its own properties and interacts with its neighbor cells, creating a mesh.

Our fluid simulation involves a three-dimensional grid of cells, each of which have velocity and density fields. Each frame, the cells interact with each other according to the Navier-Stokes equations.

The specific math involved in the base equations for the fluid simulation is quite complex, and we couldn't have derived it on our own, so we used Mike Ash's implementation Fluid Simulation for Dummies (Ash, 2006) as the bedrock of our program. Mike Ash's which is in turn based on Jos Stam's brilliant work on Real-Time Fluid Dynamics for Games (Stam, 2003).

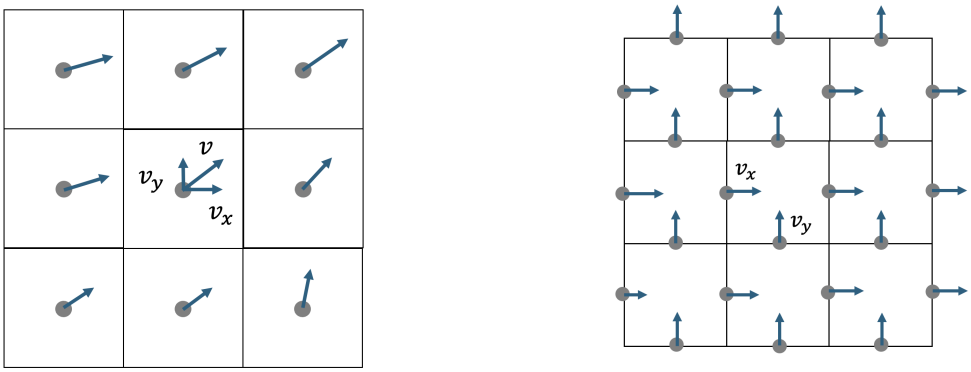
2.2.1 The Math

In Eulerian fluid dynamics we represent fluids with a velocity vector field. This means we assign a velocity vector to every point in space. The Navier-Stokes equations show us how these velocity vectors evolve over time with an infinitely small timestep.

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u + v\nabla^2 + f$$

$$\frac{\partial \rho}{\partial t} = -(u \cdot \nabla)\rho + k\nabla^2\rho + S$$

The first equation returns the change in velocity in a vector notation. Unlike in Lagrangian fluid simulation, in Eularian fluid simulation the fluid is not represented by individual particles. Thats why, instead, fluid density is used, which tells us the amount of particles present in a point in space. The second equation shows us the change of this density. Although to understand the math behind this simulation it isn't needed to fully understand these equations, it should be noted that the two equations above look a lot like each other, as this was helpful in developing the simulation. (Stam, 2003)



(a) Collocated grid (b) Staggered grid

Figure 1: Computational grids

There are two different types of velocity vector fields we can use: collocated and staggered. In the collocated grid all the variables are stored in the center of the cell. Whereas in the staggered grid the scalar variables are stored at the center but the velocity variable is stored at the cell face. Usually the use of a staggered grid is preferred, because its easier to see how much fluid flows from one cell to another. (Müller, 2022) However because Mike Ash and Jos Stam used a collocated grid in their works we decided to use the same grid for simpliciy sake.

2.2.2 Density Solver

First, lets look at the density solver. The density solver follows three steps: first adding forces, secondly diffusion and lastly moving the densities. If we look at the density equation in figure 1, we see that the change in density in a single time steps is influenced by three terms. The first term states that the density should follow the velocity field, the second term represents the density's rate of diffusion and the third term says that the density can increase due to sources. The density solver will apply these terms in reverse order. So, as stated before, first the forces are added, then the diffusion is applied and finally the densities are moved.

Adding Forces

The first step is adding forces from a source. In our case forces will be added if the user clicks on the screen. If that happens we will simply add a constant amount of density to the intial density. So:

$$\rho_{t+dt} = \rho_t + S \cdot dt$$

Diffusion

The second step is diffusion. Diffusion is the process where spread the fluid from high density cells to low density cells. The diffusion happens at a constant rate that we will

call D. Now we can simply calculate the diffusion rate a by multiplying it with the time step:

$$a = dt \cdot D$$

If we take a look at a single cell, we'll see that it exchanges densities with its four neighbors. The total difference of the densities between the cell and its neighbours can be calculated by the following formula, where $\rho_{x,y}$ is the current density of the cell at position (x, y):

$$\rho_{x+1,y} + \rho_{x-1,y} + \rho_{x,y+1} + \rho_{x,y-1} - 4 \cdot \rho_{x,y}$$

By multiplying the difference in densities with the diffusion rate a and adding it to the current density, we calculate new density of the cell. This can be seen in the following formula, where $P_{x,y}$ is equal to the density of cell at position (x, y) after the time step:

$$P_{x,y} = \rho_{x,y} + a \cdot (\rho_{x+1,y} + \rho_{x-1,y} + \rho_{x,y+1} + \rho_{x,y-1} - 4 \cdot \rho_{x,y})$$

But here we run into a problem. If the diffusion rate is big enough the density in a cell might end up becoming negative. Imagine, for example, a situation wehere the diffusion rate is equal to 1, the density of a cell is 1 and the densities of its neighbors are all 0. This means that the density of the cell will become -3. This of course is not possible. The solution to this problem is to find the density of the cell when being diffused backwards in time. Which results in the following equation:

$$\rho_{x,y} = P_{x,y} + a \cdot (P_{x+1,y} + P_{x-1,y} + P_{x,y+1} + P_{x,y-1} - 4 \cdot P_{x,y})$$

To solve this system of lineair equations for the unknowns $P_{x,y}$ the Gauss-Seidel method is used. (Stam, 2003)

Advection

The final step of the density solver is moving the densities following the velocity field. Because you can't really move grid cells, we will use a technique where we represent a density as a set of particles. An obvious way to move the densities is to act like the center of a cell is a particle and move it through the velocity field. But because we have to convert the particles back to a grid, we will use a different method. This method is called semi-lagrangian advection. We will first look at the particles that end up at the center of a cell and use the velocity of the cell to trace the particles one time step back in time. Then we will look at where the particles end up and calculate the weighted average

of the densities of the four closest cells from a particle and set this as the new density of the current cell. To do this we need to use two different grids: one that contains all the density values from the previous time step and on that contains the new density values. (Stam, 2003)

2.2.3 Velocity Solver

Now we will look at the velocity solver. Just like with the density solver if we look at the equations in section **??** we see that the change in velocity in a single time step is influenced by three factors. These are: the addition of forces, viscous diffusion and self-advection. Because the velocity equation looks so much like the density equation we can use the same steps that we're shown in section 2.2.2 to solve it. The only difference is that the velocity solver has an extra step called projection.

Projection

This step ensures that mass is conserved. The law of conservation of mass is an important property of real-life fluids. The projection step will be the last step that is executed because the previous steps might not always conserve mass. To make the fluid mass conserving we will use a mathematical concept called Hodge decomposition. Hodge composition states that every vector field is the sum of a mass conserving field and a gradient field. So all we have to do is compute a gradient field and subtract it from the velocity field. The gradient field Computing the gradient field involves the solution of a lineair system called a Poisson equation. To solve this system we will we again use the Gauss-Seidel method that was also mentioned in section 2.2.2 in the diffusion step. (Stam, 2003)

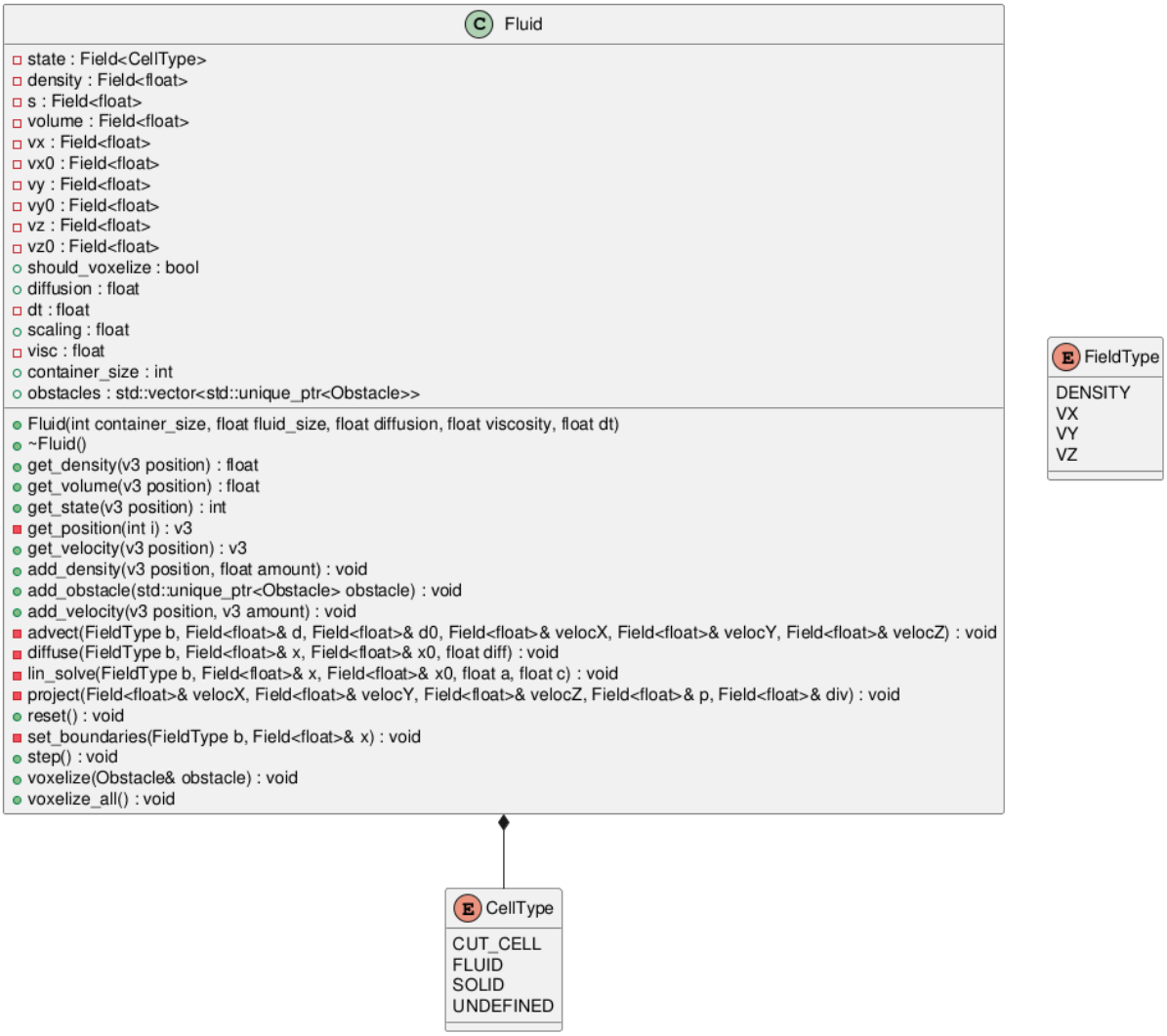
2.2.4 The Implementation

Our code is structured as follows:

```
paper
├── neural ..... Contains ML engine
└── simulation/ ..... Contains the physics simulation code
    ├── include/ ..... Our own engine headers
    ├── resources/ ..... Resources like images, fonts, shaders
    ├── src/ ..... Actual engine source
    ├── config.toml ..... Configuration file
    └── Makefile
```

This codebase is written in C++, and uses the raylib because it's a really simple and powerful library that allows us to focus on the simulation itself, as it provides a bunch of tools to easily render 3D graphics. The reason this codebase is in C++ was initially because we never got the Eulerian model to work in Go, likely due to a mistake of our own. Nevertheless we think C++ was the right choice, as it's more appropriate for the vector physics and low overhead we need for the simulation.

The main class we use for the simulation is the `Fluid` class. The implementation is available at `/simulation/src/Fluid.cpp`. The interface is as follows:



The class contains the cell property fields as well as all the functions related to the physics simulation, including the advection, diffusion and projection procedures. With this we have a basic working fluid simulator.

The next step is to add geometry. We did this by adding a boolean field to the Fluid properties and some getters and setters.

```
class Fluid {
   private:
    // rest of private members
    bool *solid;

   public:
    // rest of public members
    bool is_solid(v3 position);
    void set_solid(v3 position, bool set);
}
```

Then we change the advection and boundary handling functions to account for solid boundaries. This is done by checking whether each cell is solid, and if so, the cell properties aren't advected.

```
// in advection function, for each cell:
```

```
if (solid[IX(i, j, k)]) {
    if (b != FieldType::DENSITY) { // For velocity components
        d[IX(i, j, k)] = 0;
    }
    continue;
}

// in boundary function, for each cell:
if (solid[IX(x, y, z)]) {
    // For velocity components, enforce no-slip condition
    if (b == FieldType::VX) f[IX(x, y, z)] = 0; // x velocity
    if (b == FieldType::VY) f[IX(x, y, z)] = 0; // y velocity
    if (b == FieldType::VZ) f[IX(x, y, z)] = 0; // z velocity

    // For density and pressure, use average of neighboring
    // non-solid cells
    if (b == FieldType::DENSITY) {
        float sum = 0;
        int count = 0;

        if (!solid[IX(x-1, y, z)]) { sum += f[IX(x-1, y, z)]; count++; }
        if (!solid[IX(x+1, y, z)]) { sum += f[IX(x+1, y, z)]; count++; }
        if (!solid[IX(x, y-1, z)]) { sum += f[IX(x, y-1, z)]; count++; }
        if (!solid[IX(x, y+1, z)]) { sum += f[IX(x, y+1, z)]; count++; }
        if (!solid[IX(x, y, z-1)]) { sum += f[IX(x, y, z-1)]; count++; }
        if (!solid[IX(x, y, z+1)]) { sum += f[IX(x, y, z+1)]; count++; }

        f[IX(x, y, z)] = count > 0 ? sum / count : f[IX(x, y, z)];
    }
}
```

This allows us to add a cube of a given size at a given position, which the fluid will treat as a solid object. Unfortunately, a plane is more complex than a cube. There are a few ways to handle this. The most obvious and least efficient is to raise the resolution of the grid. Currently we've been experimenting with a 24x24x24 or a 32x32x32 grid, and that's already not as performant as we'd like. A 24x24x24 grid has to iterate $24^3 = 13824$ times per frame. And to simulate complex shapes, the higher resolution the better. But simply increasing the resolution to something like 128x128x128 or higher just isn't viable.

We're still implementing this at the time of writing the concept, but the goal is to use the method explained above in combination with the Cut-Cell method, which will dynamically create more of these cells along the borders of the model, in this case a plane. This would create a high-resolution grid near the important and complex areas of the simulation, while preserving resources by keeping the unimportant areas (like large patches of just air) from being computed.

3 Results

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3.1 Conclusion

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