Computational Fluid Dynamics With a Paper Airplane

Tasada, Daniel Tse, Nathan

January 31, 2025

Abstract

In this paper, we investigate the relationship between an airplane's shape and its performance. Our results show that …

Contents

1 Introduction

This thesis covers the simulation of the aerodynamics of an airplane, using our own Computational Fluid Dynamics model (CFD).

The goal is to simulate the airflow around an airplane's body. CFD is a branch of fluid mechanics that uses numerical methods and algorithms to solve and analyze problems that involve fluid flows. It is used in many fields, including aerospace engineering, automotive engineering, and meteorology.

The application of CFD to an airplane is important because it allows testing of a model's aerodynamic performance without building a physical model, or have to set up a wind tunnel. The practical alternative is much more expensive and time-consuming.

The final aim is to determine how the shape of an airplane's body affects its performance. Part of the project is to dynamically generate 3D models of airplanes, and simulate the airflow around them. The intention is to use machine learning to optimize the shape of the airplane's body to maximize performance.

CFD is challenging in the sense that it requires a good understanding of fluid dynamics, as well as a good understanding of the math involved. CFD is also very expensive from a computational perspective, so code optimization is important.

The thesis questions are the following:

- How do the different aspects of fluid dynamics work and how do we implement it in a computer program?
- How do we dynamically generate 3D models?
- How does an airplane's wing shape influence its performance?

2 Preliminary: Lagrangian Fluid Simulation

One method of simulating fluid dynamics is the Lagrangian method. This method models the fluid as a particle collision system, where the air is represented by particles that interact with each other to emulate a fluid. Our implementation is based on Rigid Body Collision Resolution (Hakenberg, 2005). We used this paper as a guide for all the math involved.

The math relies on the momentum, inertia, and velocity of the particles to calculate the collision normal and point of contact. The collision normal is the direction in which the particles are moving away from each other, and the point of contact is the point at which the particles collide.

The following variables are necessary to perform the calculations:

Angular momentum L $(kg \cdot m^2/s)$:	$L = mvr;$
Inertia tensor I $(kg \cdot m^2)$:	$I = \frac{L}{\omega};$
Angular velocity ω (rad/s) :	$\omega = \frac{\Delta\theta}{\Delta t};$

Collision normal($n \in \mathbb{R}^3$) in world coordinates away from body;
Point of contact ($r_i \in \mathbb{R}^3$) in world coordinates with respect to p_i;
Orientation ($R_i \in SO(3)$) transforming from object to world coordinates;

Where i represents one of two particles in a given collision:

Velocity after collision	$\tilde{v}_i,$
Angular velocity after collision	$\tilde{\omega}_i,$
Constant	$\lambda,$

The following formulas represent the relation between particles:

$$\tilde{v}_1 = v_1 - \frac{\lambda}{m_1}n;$$
$$\tilde{v}_2 = v_2 + \frac{\lambda}{m_2}n;$$
$$\tilde{\omega}_1 = \omega_1 - \Delta q_1;$$
$$\tilde{\omega}_2 = \omega_2 + \Delta q_2;$$
$$\text{where } q_i := I_i^{-1} \cdot R_i^{-1} \cdot (r_i \times n),$$
$$\text{and } \lambda = 2\frac{nv_1 - nv_2 + \omega_1 I_1 q_1 - \omega_2 I_2 q_2}{(\frac{1}{m_1} + \frac{1}{m_2})n^2 + q_1 I_1 q_1 + q_2 I_2}$$

This was implemented using Go and raylib, and the code is available at github.com/dtasada/paper at the `lagrangian-go` branch.

When dealing with a lot of particles, the simulation stops performing as well, because the number of calculations and iterations of the formulas listed above has a time complexity of $O(n^2)$, where n is the number of particles. This is because every particle handles collisions with every other particlee every single frame. This can be elegantly mitigated by implementing a three-dimensional grid system, where every particle is assigned to a cell in the grid. This way, particles only need to check for collisions with other particles in the same cell or neighboring cells. If the particles are evenly distributed within the container, each particle only needs to handle collisions with particles in its own cell, and its 26 neighboring cells. This reduces the time complexity to $O(n)$.

The simulation was unstable at higher densities, where particles would phase into each other instead of cleanly bouncing. The bug is likely quite simple to fix, but we

got distracted by another method. Despite the optimizations we made, Lagrangian simulation is still quite computationally expensive, and Go probably wasn't doing it any favors (Go isn't very memory efficient). In the end it was put aside, and the Lagrangian implementation was never completed.

3 Execution: Eulerian Fluid Simulation

The other method is the Eulerian method. This method models the fluid as a cellular automata. The fluid is represented by a grid of cells, each of which has its own properties and interacts with its neighbor cells, creating a Cartesian mesh-style field.

Our fluid simulation involves a three-dimensional grid of cells, each of which have velocity and density fields. Each frame, the cells interact with each other according to the Navier-Stokes equations.

The specific math involved in the base equations for the fluid simulation is quite complex, and we couldn't have derived it on our own, so we used Mike Ash's implementation Fluid Simulation for Dummies (Ash, 2006) as the bedrock of our program. Mike Ash's code is in turn based on Jos Stam's brilliant work on Real-Time Fluid Dynamics for Games (Stam, 2003).

3.1 The Math

In Eulerian fluid dynamics we represent fluids with a velocity vector field. This means we assign a velocity vector to every point in space. The Navier-Stokes equations show us how these velocity vectors evolve over time with an infinitely small timestep.

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u + v\nabla^2 + f$$

$$\frac{\partial \rho}{\partial t} = -(u \cdot \nabla)\rho + k\nabla^2\rho + S$$

The first equation returns the change in velocity in a vector notation. Unlike in Lagrangian fluid simulation, in Eularian fluid simulation the fluid is not represented by individual particles. Thats why, instead, fluid density is used, which tells us the amount of particles present in a point in space. The second equation shows us the change of this density. Although to understand the math behind this simulation it isn't needed to fully understand these equations, it should be noted that the two equations above look a lot like each other, as this was helpful in developing the simulation. (Stam, 2003)

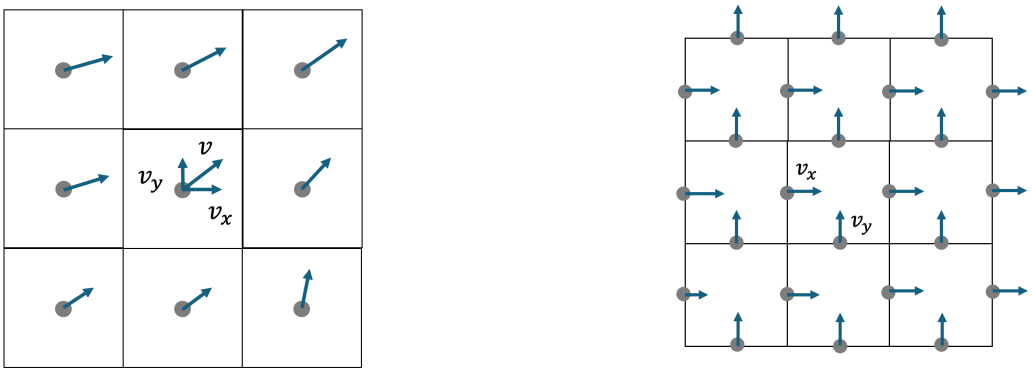


Figure 1: Computational grids

There are two different types of velocity vector fields we can use: collocated and staggered. In the collocated grid all the variables are stored in the center of the cell. Whereas in the staggered grid the scalar variables are stored at the center but the velocity variable is stored at the cell face. Usually the use of a staggered grid is preferred, because

its easier to see how much fluid flows from one cell to another. (Müller, 2022) However because Mike Ash and Jos Stam used a collocated grid in their works we decided to use the same grid for simpliciy sake.

3.2 Density Solver

First, lets look at the density solver. The density solver follows three steps: first adding forces, secondly diffusion and lastly moving the densities. If we look at the density equation in figure 1, we see that the change in density in a single time steps is influenced by three terms. The first term states that the density should follow the velocity field, the second term represents the density's rate of diffusion and the third term says that the density can increase due to sources. The density solver will apply these terms in reverse order. So, as stated before, first the forces are added, then the diffusion is applied and finally the densities are moved.

3.2.1 Adding forces

The first step is adding forces from a source. In our case forces will be added if the user clicks on the screen. If that happens we will simply add a constant amount of density to the intial density. So:

$$\rho_{t+dt} = \rho_t + S \cdot dt$$

3.2.2 Diffusion

The second step is diffusion. Diffusion is the process where spread the fluid from high density cells to low density cells. The diffusion happens at a constant rate that we will call D. Now we can simply calculate the diffusion rate a by multiplying it with the time step:

$$a = dt \cdot D$$

If we take a look at a single cell, we'll see that it exchanges densities with its four neighbors. The total difference of the densities between the cell and its neighbours can be calculated by the following formula, where $\rho_{x,y}$ is the current density of the cell at position (x, y):

$$\rho_{x+1,y} + \rho_{x-1,y} + \rho_{x,y+1} + \rho_{x,y-1} - 4 \cdot \rho_{x,y}$$

By multiplying the difference in densities with the diffusion rate a and adding it to the current density, we calculate new density of the cell. This can be seen in the following formula, where $P_{x,y}$ is equal to the density of cell at position (x, y) after the time step:

$$P_{x,y} = \rho_{x,y} + a \cdot (\rho_{x+1,y} + \rho_{x-1,y} + \rho_{x,y+1} + \rho_{x,y-1} - 4 \cdot \rho_{x,y})$$

But here we run into a problem. If the diffusion rate is big enough the density in a cell might end up becoming negative. Imagine, for example, a situation wehere the diffusion rate is equal to 1, the density of a cell is 1 and the densities of its neighbors are all 0. This means that the density of the cell will become -3. This of course is not possible. The solution to this problem is to find the density of the cell when being diffused backwards in time. Which results in the following equation:

$$\rho_{x,y} = P_{x,y} + a \cdot (P_{x+1,y} + P_{x-1,y} + P_{x,y+1} + P_{x,y-1} - 4 \cdot P_{x,y})$$

To solve this system of linear equations for the unknowns $P_{x,y}$ the Gauss-Seidel method is used. (Stam, 2003

3.2.3 Advection

The final step of the density solver is moving the densities following the velocity field. Because you can't really move grid cells, we will use a technique where we represent a density as a set of particles. An obvious way to move the densities is to act like the center of a cell is a particle and move it through the velocity field. But because we have to convert the particles back to a grid, we will use a different method. This method is called semi-lagrangian advection. We will first look at the particles that end up at the center of a cell and use the velocity of the cell to trace the particles one time step back in time. Then we will look at where the particles end up and calculate the weighted average of the densities of the four closest cells from a particle and set this as the new density of the current cell. To do this we need to use two different grids: one that contains all the density values from the previous time step and on that contains the new density values. (Stam, 2003)

3.3 Velocity Solver

Now we will look at the velocity solver. Just like with the density solver if we look at the equations in section 3.1 we see that the change in velocity in a single time step is influenced by three factors. These are: the addition of forces, viscous diffusion and self-advection. Because the velocity equation looks so much like the density equation we can use the same steps that we're shown in section 3.2 to solve it. The only difference is that the velocity solver has an extra step called projection.

3.3.1 Projection

This step ensures that mass is conserved. The law of conservation of mass is an important property of real-life fluids. The projection step will be the last step that is executed because the previous steps might not always conserve mass. To make the fluid mass conserving we will use a mathematical concept called Hodge decomposition. Hodge composition states that every vector field is the sum of a mass conserving field and a gradient field. So all we have to do is compute a gradient field and subtract it from the velocity field. The gradient field Computing the gradient field involves the solution of a linear system called a Poisson equation. To solve this system we will we again use the Gauss-Seidel method that was also mentioned in section 3.2 in the diffusion step. (Stam, 2003)

3.4 The Implementation

3.4.1 Introduction to the code

Our project is structured as follows:

```
paper
├── neural ..... Contains ML engine
└── simulation/ ..... Contains the physics simulation code
    ├── include/ ..... Our own engine headers
    ├── resources/ ..... Resources like images, fonts, shaders
    ├── src/ ..... Actual engine source
    └── config.toml ..... Configuration file
```

```
└── Makefile
```

This codebase is written in C++ and uses raylib again because it's a really simple and powerful library that allows us to focus on the simulation itself, as it provides a bunch of tools to easily render 3D graphics. The reason this codebase is in C++ was initially because we never got the Eulerian model to work in Go, likely due to a mistake of our own. Nevertheless we think C++ was the right choice, as it's more appropriate for the vector physics and low overhead we need for the simulation.

The main fluid simulation object we use for the simulation is the `Fluid` class. The implementation is is available at `/simulation/src/Fluid.cpp`. The interface is as follows:

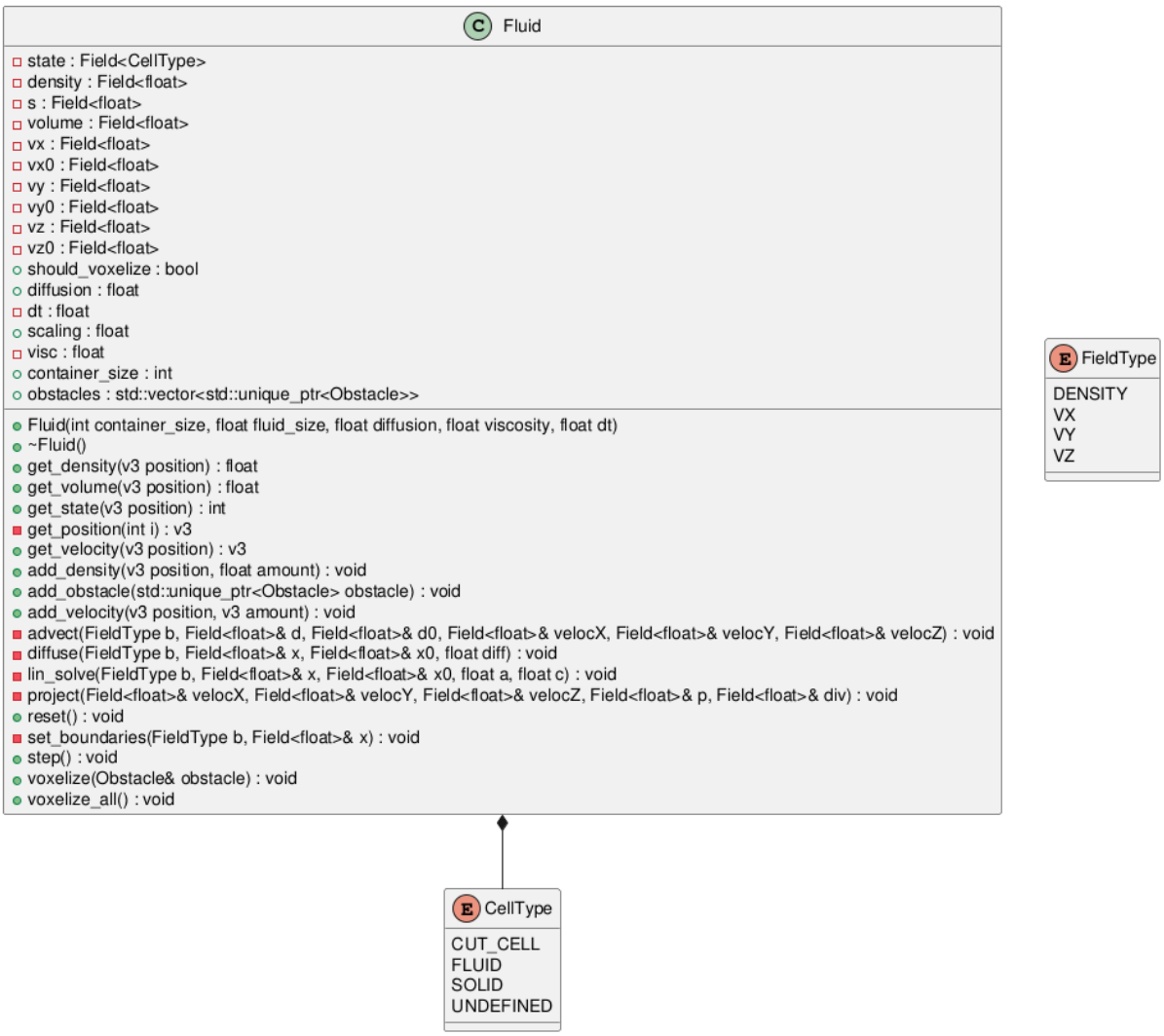


Figure 2: Fluid class diagram

This class contains the vector fields for the container as well as all the functions related to the physics simulation, including the advection, diffusion and projection procedures. These functions are the core of the CFD and this is the part that we borrowed from Mike Ash.

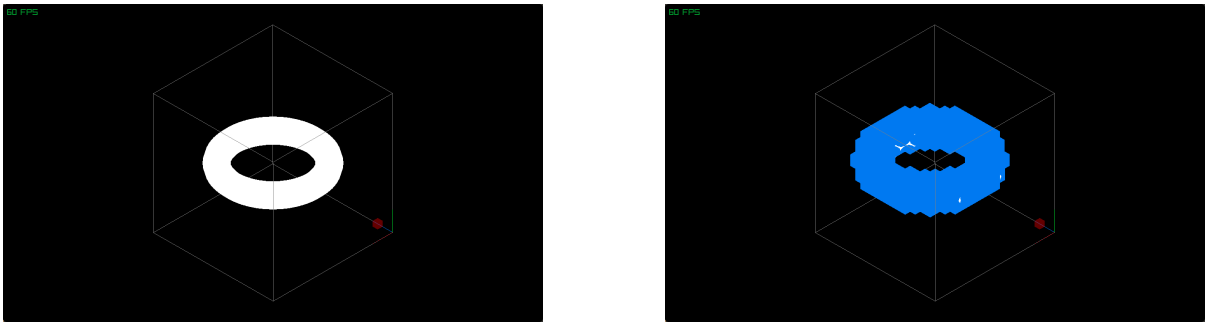
Figure 3: Basic fluid simulation

3.4.2 Voxelizing Geometry

The next step is to implement geometry collision. The graphics library we're using, Raylib, has a function to load geometry from object files. Loading in a torus is easy enough, but the real challenge lies in communicating to the CFD that there's a torus in a given position and it behaving correspondingly.

First, the 3D object that Raylib loads has to be converted to a BVH (Bounding Volume Hierarchy) object. This is a tree structure that contains the object's geometry in a way that makes it easy to check if a point is inside the object. This format is what FCL (Flexible Collision Library) uses to check for collisions. We are using FCL to easily process flexible collisions between 3D objects.

The next step is to write a voxelization function. Voxelization takes a 3D object's mesh data and checks the previously established 3D vector fields to see which if the object is in a given cell. FCL performs collision detection between two 3D objects. In this case those two objects are our obstacle (like a torus or a plane), and whichever cell is being iterated over. If the object is in a cell, we can pass a condition to the CFD. This is done for every cell in the grid, and the result is a 3D boolean array that represents the object. The source code is at `Fluid::voxelize`. Each obstacle is passed to this function, and as a result we get a 3D boolean field saved at `Fluid::state`. Below is a figure of the voxel map, with cells classified as solid highlighted in blue. Note that the edges of the torus are also marked as solid despite their cells obviously only being partial. This is because our implementation of the voxel mapping using FCL isn't working as intended.



(a) (b)

Figure 4: Voxel map

This is called the Cut-Cell method. The reason we even need this is because the more obvious option is to set up the grid to be high resolution so the voxels are small enough

to accurately approximate an arbitrarily complex object. The reason this isn't viable is purely due to performance. This might be reasonable with a supercomputer, but for our purposes we'd just like to optimize.

Another method would be to implement an AMR (Adaptive Mesh Refinement) system, which would adapt the mesh's resolution based on what's in a given region. So this would generate a high resolution grid around an obstacle's bounds, and a low resolution grid in any empty space. This is an interesting concept, but we would have to rewrite some of the fluid math code to make the irregular grid work, and it seems like a lot of work, so we stuck with the Cut-Cell method. TODO: finish this

3.4.3 Implementing obstacle collision

Each cell of the grid is now one of three states: `FLUID`, `SOLID`, or `CUT_CELL`. This method of classifying cells into the three states is called the Cut-Cell method.

1. Fluid cells
    - Cells that are fully inside a fluid region
    - Normal Navier-Stokes equations apply
2. Solid cells
    - Cells that are fully inside an obstacle
    - Applies no-slip condition: velocity is zero
3. Cut-cells
    - Cells partially inside an object
    - Modify fluid calculations

We'd like to preface this by saying that our implementation of Cut-Cell obstacle resolution is not completed or working, so some of the following is theoretical. The idea is that when a cell is classified as a Cut-Cell, we create a new field to track the amount of fluid that's inside the cell. For every cell, we calculate the fractional fluid volume:

$$\text{Fractional fluid volume } \alpha = \frac{\text{volume of fluid in cell}}{\text{total cell volume}}$$
$$\text{Fractional face area } \beta_d = \frac{\text{open d-face area}}{\text{total d-face area}} \text{ where d is one of X, Y, or Z}$$

The fractional fluid volume is the ratio of the volume of fluid in the cell to the total volume of the cell, where a solid cell has a value of 0 and an empty cell has a value of 1. The same goes for the fractional face area, which is the ratio of the open face area to the total area of a given cell.

With this information we can modify the advection step of the CFD to account for a cut-cell. We do this by scaling the advection by the fractional face area. Next we also adjust the projection step. // TODO POISSON EQUATION //

The last step is to make sure to apply the correct boundaries to the fluid, where any solid cells have a constant velocity of 0.

The reason we failed to implement this is twofold:

1. Incomplete voxelization process. Figure 4b is captured with a modified version of the program that only voxelizes to two states: solid and fluid, with no cut-cells. This is because we weren't capable of implementing the FCL collision detection correctly. We don't know what caused the issue, but unfortunately we didn't have time to debug it thoroughly.

2. We realized while writing this paper that we'd forgotten to modify the projection step to account for cut-cells, which explains a bug we experienced, where the fluid would leak through the cut-cells.

Improving on these two points might have yielded better results.

4 Results: Challenges & Failures

Due to the issues with the voxelization process and the object collisions, we weren't actually able to complete the model we wanted. We have a working fluid simulation, but most of the work we did on the obstacles was more theoretical than any tangible working prototype. Here's an outline of what went wrong:

4.1 Grid Collision

- **Issue**: air bleeds through obstacles
- **Causes**:
  - Cut-cell method not fully implemented
  - Failure in voxelization process
- **Solution**: it took a lot of trial and error, but no solution was found. We have some hypotheses, but we don't have any time to test, so it's possible that some of the solutions defined in this paper don't work in practice.
- **Result**: a dead end. Most of our project relied on this section working.

4.2 Performance

Despite being written in 2006, Mike Ash's isn't the fastest, even with today's hardware. The fluid simulation runs quite reasonably at a resolution, of say $32 \times 32 \times 32$, but bumping up the grid size yields worse and worse results. At that grid resolution of 32^3, we see about 20 FPS on my laptop. This isn't abysmal by any means, but 32^3 really isn't very high either. So we profiled the program to see what was the slowest.

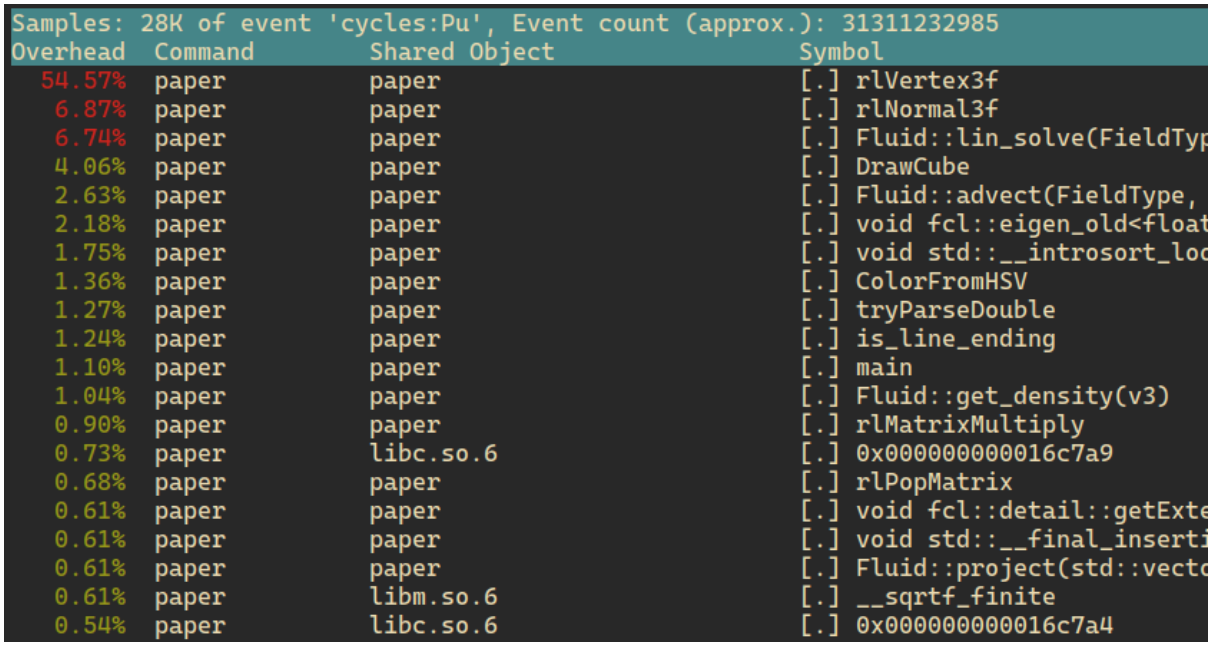


Figure 5: Profile of the fluid simulation program

Some of the steps involved in the Eulerian fluid simulation require iterating over every cell in the grid, and performing some arithmetic operations on that cell. Both the advection and projection steps require this, and the advection step alone runs four times every step. That's about $32^3 * 4 = 131\,072$ iterations every step, just for the advection. This is understating the amount of arithmetic operations that are executed, but it already gives an idea of the scale of the problem.

We haven't optimized the code that much, but the most important thing we did was parallelize the these large iterations. Anytime we have any meaningful iteration over the grid, we use OpenMP to parallelize it. This was a very quick and simple way to get a performance boost, and it works quite well. Still, there's some room for improvement.

We didn't actually go any further but we have some ideas for speed boosts. These are speed boosts and not real algorithmic optimizations because for that we'd have to really look into the Navier-Stokes equations and speed up Jos Stam's algorithms, but that's beyond the scope of what we'd want to do. Rather we're thinking that while we parallelized the long for loops, the code still isn't really asynchronous. OpenMP is a great tool for parallelization, which is what we're using it for, but we could go one step further and refactor our code to be truly asynchronous, and assign a thread to each task, for example.

Another factor is that while the client-side rendering is hardware accelerated, the actual fluid simulation all runs on the CPU. This is quite the bottleneck, as the CPU architecture isn't really designed for this kind of large scale parallel work for high resolution grids. Brute-forcing this would require a lot of compute cores, which CPUs don't have (modern CPUs have between 4 and 16 cores in the consumer market). The most obvious solution is to transfer the load to a dedicated GPU, which we can utilize to perform all these arithmetic operations for us would be a massive improvement. Migrating the CFD code to a compute shader or an OpenCL kernel would be a great way to speed up the simulation. This is something that would be very fun to explore in the future, but once again it's beyond the scope (and time limit) of our project.

5 Conclusion

Nam quis enim. Quisque ornare dui a tortor. Fusce consequat lacus pellentesque metus. Duis euismod. Duis non quam. Maecenas vitae dolor in ipsum auctor vehicula. Vivamus nec nibh eget wisi varius pulvinar. Cras a lacus. Etiam et massa. Donec in nisl sit amet dui imperdiet vestibulum. Duis porttitor nibh id eros.

References

Ash, M. (2006). Fluid simulation for dummies. www.mikeash.com/pyblog/fluid-simulation-for-dummies.html

contributors, W. (2024). Adaptive mesh refinement. https://en.wikipedia.org/wiki/Adaptive_mesh_refinement

d'Lélis, B. (2022). *Semi lagrangian scheme - a visual explanation*. www.youtu.be/kvBRFxRIJuY

Gonkee. (2021). *Coding a fluid simulation with my last 2 brain cells*. www.youtu.be/uG2mPez44eY

Gradience. (2024). *Teaching myself c so i can build a particle simulation*. www.youtu.be/NorXFOobehY

Hakenberg. (2005). Rigid body collision resolution. www.hakenberg.de/diffgeo/collision/rigid_body_collision_resolution.pdf

Ingram, D. M., & Causon, D. (2003). A cartesian cut cell method for incompressible viscous flow. www.researchgate.net/publication/222518000

Lague, S. (2023). *Coding adventure: Simulating fluids*. www.youtu.be/rSKMYc1CQHE

Müller, M. (2022). *17 - how to write an eulerian fluid simulator with 200 lines of code*. www.youtu.be/iKAVRgIrUOU

Shiffman, D. (2019). *Coding challenge #132 fluid simulation*. www.youtu.be/alhpH6ECFvQ

Stam, J. (2003). Real-time fluid dynamics for games. www.dgp.toronto.edu/public_user/stam/reality/Research/pdf/GDC03.pdf

Tucker, P. (2000). A cartesian cut cell method for incompressible viscous flow. www.sciencedirect.com/science/article/pii/S0307904X00000056

Unknown. (2012). Lattice-botlzmann fluid dynamics. www.physics.weber.edu/schroeder/javacourse/LatticeBoltzmann.pdf

Wikipedia. (2024a). Gauss–seidel method. https://en.wikipedia.org/wiki/Gauss%E2%80%93Seidel_method#References

Wikipedia. (2024b). Navier-stokes equations. https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations

Work, P. (2023). *Writing a physics engine from scratch - collision detection optimization*. www.youtu.be/9IULfQH7E90