Chapter 1 Homework

1. *Write a formula expressing* $z = <<x,y>, <v,w>>$ *using just* ϵ *and* $=$.

SOLUTION.

$<x, y> = \{\{x\}, \{x, y\}\}$

$<v, w> = \{\{v\}, \{v, w\}\}$

$z = <<x, y>, <v, w>> = \{\{<x, y>\}, \{<x, y>, <v, w>\}\} = \{\{\{\{x\}, \{x, y\}\}\}, \{\{\{x\}, \{x, y\}\}, \{\{v\}, \{v, w\}\}\}\}$

2. *(a) Show that* $\alpha < \beta$ *implies that* $\gamma + \alpha < \gamma + \beta$ *and* $\alpha + \gamma \leqslant \beta + \gamma$. *(b)Give an example to show that the "*$\leq$*" cannot be replaced by "*$<$*". (c) Also show:* $\alpha \leq \beta \rightarrow \exists! \delta(\alpha + \delta = \beta)$.

SOLUTION.

(a) Suppose $\alpha < \beta$.

(i) The element $<\alpha, 1> \in \beta \times \{1\}$, but $<\alpha, 1> \notin \alpha \times \{1\}$, which implies that $\gamma \times \{0\} \cup \alpha \times \{1\} < \gamma \times \{0\} \cup \beta \times \{1\}$ with the ordering from the definition of "+".

(ii) Towards a contradiction, suppose $\alpha + \gamma > \beta + \gamma$. Then there is some element in $c \in \alpha \times \{0\} \cup \gamma \times \{1\}$ such that $c \notin \beta \times \{0\} \cup \gamma \times \{1\}$. This implies $\beta > \alpha$, a contradiction.

(b) Let $\gamma = \omega$, $\alpha = 0$, and $\beta = 1$. Then $0 + \omega = \omega = 1 + \omega$, and hence there is no strict inequality.

(c) If $\alpha = \beta$, then existence is trivial ($\delta = 0$) and uniqueness is clear since, for $\delta > 0$, $\beta + \delta > \beta$. So suppose $\alpha < \beta$.

Existence: Transfinite induction on β.

Base: $\beta = 0$. Then this is trivial with $\alpha = 0, \gamma = 0$.

Successor: $\beta = S(\zeta) = \zeta + 1$. Then by IH $\exists \delta(\alpha + \delta = \zeta)$. But given associativity of addition, $\alpha + (\delta + 1) = (\alpha + \delta) + 1 = \zeta + 1 = \beta$.

Limit: β is a limit ordinal. By IH for all $\zeta < \beta, \exists \delta(\alpha + \delta = \zeta)$. For all such δs, consider $\delta' = \sup(\delta)$.

Then:

$\alpha + \delta' = \sup(\alpha + \delta)$ (note this is trivial is δ is not a limit ordinal; else it follows by def of addition, and δ as above)

$= \sup(\zeta : \zeta < \beta)$

$= \beta$

Uniqueness: Suppose $\alpha + \delta_1 = \alpha + \delta_2 = \beta$. Then by (a), $\delta_1 \nless \delta_2$ and $\delta_2 \nless \delta_1$, hence $\delta_1 = \delta_2$.

3. *(a) Show that if* $\gamma > 0$, *then* $\alpha < \beta$ *implies that* $\gamma \cdot \alpha < \gamma \cdot \beta$ *and* $\alpha \cdot \gamma \leqslant \beta \cdot \gamma$. *(b) Give an example to show that the "*$\leqslant$*" cannot be replaced by "*$<$*". (c) Also show* $(\alpha \leqslant \beta \wedge \alpha > 0) \rightarrow \exists! \delta, \zeta(\zeta < \alpha \wedge \alpha \cdot \delta + \zeta = \beta)$.

SOLUTION.

(a) (i)Suppose $\alpha < \beta$. Then $<\alpha, 0> \in \beta \times \gamma$, but this element is not in $\alpha \times \gamma$, from the definition of ordinal multiplication.

(ii) [TO BE SOLVED; SIMILAR TO (2)]

(b) $2 \cdot \omega = 3 \cdot \omega = \omega$, for example.

(c) [TO BE SOLVED]