Chapter 1 Homework

1. *Write a formula expressing* $z = \,<<x,y>, <v,w>>$ *using just* ϵ *and* $=$.

SOLUTION.

$<x, y> = \{\{x\}, \{x, y\}\}$

$<v, w> = \{\{v\}, \{v, w\}\}$

$z = <<x, y>, <v, w>> = \{\{<x, y>\}, \{<x, y>, <v, w>\}\} = \{\{\{\{x\}, \{x, y\}\}\}, \{\{\{x\}, \{x, y\}\}, \{\{v\}, \{v, w\}\}\}\}$

2. *(a) Show that* $\alpha < \beta$ *implies that* $\gamma + \alpha < \gamma + \beta$ *and* $\alpha + \gamma \leqslant \beta + \gamma$*. (b)Give an example to show that the "*$\leq$*" cannot be replaced by "*$<$*". (c) Also show:* $\alpha \leq \beta \rightarrow \exists! \delta(\alpha + \delta = \beta)$.

SOLUTION.

(a) Suppose $\alpha < \beta$.

(i) The element $<\alpha, 1> \in \beta \times \{1\}$, but $<\alpha, 1> \notin \alpha \times \{1\}$, which implies that $\gamma \times \{0\} \cup \alpha \times \{1\} < \gamma \times \{0\} \cup \beta \times \{1\}$ with the ordering from the definition of "+".

(ii) Towards a contradiction, suppose $\alpha + \gamma > \beta + \gamma$. Then there is some element in $c \in \alpha \times \{0\} \cup \gamma \times \{1\}$ such that $c \notin \beta \times \{0\} \cup \gamma \times \{1\}$. This implies $\beta > \alpha$, a contradiction.

(b) Let $\gamma = \omega$, $\alpha = 0$, and $\beta = 1$. Then $0 + \omega = \omega = 1 + \omega$, and hence there is no strict inequality.

(c) If $\alpha = \beta$, then existence is trivial ($\delta = 0$) and uniqueness is clear since, for $\delta > 0$, $\beta + \delta > \beta$. So suppose $\alpha < \beta$.

Existence: Consider the set $\beta - \alpha$ (the complement) which exists by comprehension. By AC, this is well-orderable, hence isomorphic to some ordinal δ under that ordering R. This allows us to construct a well-ordering of the set β as $\alpha + \delta$, where the ordering on α is the ordinary $\in$ relation, and the ordering R for elements of the set $\beta - \alpha$. Thus as we've constructed it, the set $\alpha + \delta$ is well-ordered and has the same elements as β and hence must be isomorphic to β.

Uniqueness: Suppose $\alpha + \delta_1 = \alpha + \delta_2 = \beta$. Then by (a), $\delta_1 \not< \delta_2$ and $\delta_2 \not< \delta_1$, hence $\delta_1 = \delta_2$.