Chapter 1 Homework

1. *Write a formula expressing* $z = \langle\langle x,y\rangle, \langle v,w\rangle\rangle$ *using just* ϵ *and* $=$.

SOLUTION.

$<x, y> = \{\{x\}, \{x, y\}\}$

$<v, w> = \{\{v\}, \{v, w\}\}$

$z = << x, y>, <v, w> > = \{\{<x, y>\}, \{<x, y>, <v, w>\}\} = \{\{\{\{x\}, \{x, y\}\}\}, \{\{\{x\}, \{x, y\}\}, \{\{v\}, \{v, w\}\}\}\}$

2. *(a) Show that* $\alpha<\beta$ *implies that* $\gamma+\alpha<\gamma+\beta$ *and* $\alpha+\gamma\leqslant\beta+\gamma$. *(b)Give an example to show that the "*$\leq$*" cannot be replaced by "*$<$*". (c) Also show:* $\alpha\leq\beta\rightarrow\exists!\delta(\alpha+\delta=\beta)$.

SOLUTION.

(a) Suppose $\alpha<\beta$.

(i) The element $<\alpha,1>\in\beta\times\{1\}$, but $<\alpha,1>\notin\alpha\times\{1\}$, which implies that $\gamma\times\{0\}\cup\alpha\times\{1\}<\gamma\times\{0\}\cup\beta\times\{1\}$ with the ordering from the definition of "+".

(ii) Towards a contradiction, suppose $\alpha+\gamma>\beta+\gamma$. Then there is some element in $c\in\alpha\times\{0\}\cup\gamma\times\{1\}$ such that $c\notin\beta\times\{0\}\cup\gamma\times\{1\}$. This implies $\beta>\alpha$, a contradiction.

(b) Let $\gamma=\omega$, $\alpha=0$, and $\beta=1$. Then $0+\omega=\omega=1+\omega$, and hence there is no strict inequality.

(c) If $\alpha=\beta$, then existence is trivial ($\delta=0$) and uniqueness is clear since, for $\delta>0$, $\beta+\delta>\beta$. So suppose $\alpha<\beta$.

Existence: Transfinite induction on β.

Base: $\beta=0$. Then this is trivial with $\alpha=0, \gamma=0$.

Successor: $\beta=S(\zeta)=\zeta+1$. Then by IH $\exists\delta(\alpha+\delta=\zeta)$. But given associativity of addition, $\alpha+(\delta+1)=(\alpha+\delta)+1=\zeta+1=\beta$. (We've already shown associativity.)

Limit: β is a limit ordinal. By IH for all $\zeta<\beta, \exists\delta(\alpha+\delta=\zeta)$.

[I AM HERE]

Consider $\delta'=\sup\{\delta\colon\alpha+\delta=\zeta, \zeta<\beta\}$. If δ' is a limit ordinal, then consider $\sup\{\alpha+\delta\colon\delta<\delta'\}$

Consider $\sup\{\alpha+\delta\colon\delta<\beta\}=\alpha+\beta$

Want $\beta=\sup(\lambda\colon\lambda<\beta)=\alpha+\delta'$

Uniqueness: Suppose $\alpha+\delta_1=\alpha+\delta_2=\beta$. Then by (a), $\delta_1\not<\delta_2$ and $\delta_2\not<\delta_1$, hence $\delta_1=\delta_2$.

3. *(a) Show that if* $\gamma>0$, *then* $\alpha<\beta$ *implies that* $\gamma\cdot\alpha<\gamma\cdot\beta$ *and* $\alpha\cdot\gamma\leqslant\beta\cdot\gamma$. *(b) Give an example to show that the "*$\leqslant$*" cannot be replaced by "*$<$*". (c) Also show* $(\alpha\leqslant\beta\wedge\alpha>0)\rightarrow\exists!\delta,\zeta(\zeta<\alpha\wedge\alpha\cdot\delta+\zeta=\beta)$.