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Cohomology of Finite Groups

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Definition 1. Let H be a group. An abelian group N on which H acts (on the left) is called a *H-module.*

Definition 2. If H is a finite group and N is a H-module, define $C^0(H,N) = N$ and for $n \geq 1$ define $C^n(H,N)$ to be the collection of all maps from $H^n = H \times \ldots \times H$ (n times) to N. The elements of $C^n(H,N)$ are called *n-cochains (of H with values in N).*

Definition 3. For $n \geq 0$, define the n^{th} *coboundary* homomorphism from $C^n(H,N)$ to $C^{n+1}(H,N)$ by

$$d_n(f)(h_1,\ldots,h_{n+1}) = h_1 \cdot f(h_2,\ldots h_{n+1}) + \sum_{i=1}^{n}(-1)^i f(h_1,\ldots,h_{i-1},h_ih_{i+1},h_{i+2},\ldots,h_{n+1}) + (-1)^{n+1}f(h_1,\ldots,h_n)$$

where the product h_ih_{i+1} occupying the i^{th} position of f is taken in the group H.

Definition 4. Let $Z^n(H,N) = \ker d_n$ for $n \geq 0$. The elements of $Z^n(H,N)$ are called *n-cocycles.* Let $B^n(H,N) =$ image d_{n-1} for $n \geq 1$ and let $B^0(H,N) = 1$. The elements of $B^n(H,N)$ are called *n-coboundaries.*

Definition 5. For any H-module A and any $n \geq 0$ the quotient group $Z^n(H,N)/B^n(H,N)$ is called the n^{th} *cohomology group of H with coefficients in* N and is often denoted by $H^n(H,N)$.

Definition 6. Suppose N is a H-module and N' is a H'-module. Group homomorphisms $\varphi : H' \to H$ and $\psi : N \to N'$ are said to be compatible if ψ is a H'-module homomorphism when N is made into a H'-module by means of φ, i.e., if $\psi(\varphi(h')n) = h'\psi(n)$ for all $h' \in H'$ and $n \in N$.

Remark 7. Compatible maps induce a homomorphism λ_n between cohomology groups by way of

$$\lambda_n : C^n(H,N) \to C^n(H',N') \text{ defined by } f \mapsto \psi \circ f \circ \varphi^n.$$

Definition 8. Let K be a subgroup of a group H and let N be a H-module. The inclusion map $\varphi : K \to H$ and the identity $\psi : N \to N$ are compatible homomorphisms. We call the induced map the *restriction homomorphism*:

$$\text{Res} : H^n(H,N) \to H^n(K,N), \qquad n \geq 0.$$

Definition 9. The *corestriction homomorphism* is a map induced by the map Cor : $C^n(K,N) \to C^n(H,N)$, $n \geq 0$ defined by $\text{Cor}(f)(p) = \sum_{i=1}^m h_i f(h_i^{-1}p)$ for $p \in P_n = \mathbb{Z}H \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} \mathbb{Z}H$ ($n+1$-times) and $f \in \text{Hom}_{\mathbb{Z}K}(P_n,N)$ that is a cocycle.

Proposition 10. Suppose K is a subgroup of H of index m. Then Cor $\circ$ Res $= m$, i.e., if c is a cohomology class in $H^n(H,N)$ for some H-module N, then $\text{Cor}(\text{Res}(c)) = mc \in H^n(H,N)$ for all $n \geq 0$.

Corollary 11. Suppose the finite group H has order m. Then $mH^n(H,N) = 0$ for all $n \geq 1$ and any H-module N.

Definition 12. Let N,G,H be groups. Then G is an *extension of H by N* if there exists a short exact sequence $1 \to N \xrightarrow{\psi} G \xrightarrow{\varphi} H \to 1$. Two extensions are said to be equivalent if there exists an isomorphism $\alpha : G \to G'$ such that the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "\psi"] \arrow[d, "id"] & G \arrow[r, "\varphi"] \arrow[d, "\alpha"] & H \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & N \arrow[r, "\psi'"] & G' \arrow[r, "\varphi'"] & H \arrow[r] & 1
\end{tikzcd}$$

Definition 13. Let N and H be groups and let $\varphi : H \to \text{Aut}(N)$. Let $\cdot$ denote the (left) action of N on K determined by φ. Then the *semidirect product of N and H* is a group with set $N \times H$ and binary operation defined by $(n_1,h_1)(n_2,h_2) = (n_1h_1 \cdot n_2, h_1h_2)$.

Definition 14. The extension $1 \to N \xrightarrow{\psi} G \xrightarrow{\varphi} H \to 1$ is said to *split* if G can be written as a semidirect product of N and H.

Theorem 15. Let N be a H-module. Then there is a bijection between the equivalence classes of extensions H by N and the cohomology classes in $H^2(H,N)$. Under the bijection split extensions correspond to the trivial cohomology class.

Corollary 16. Every extension of H by the abelian group N splits if and only if $H^2(H,N) = 0$.

Corollary 17. If N is a finite abelian group and $(|N|,|H|) = 1$ then every extension of H by N splits.