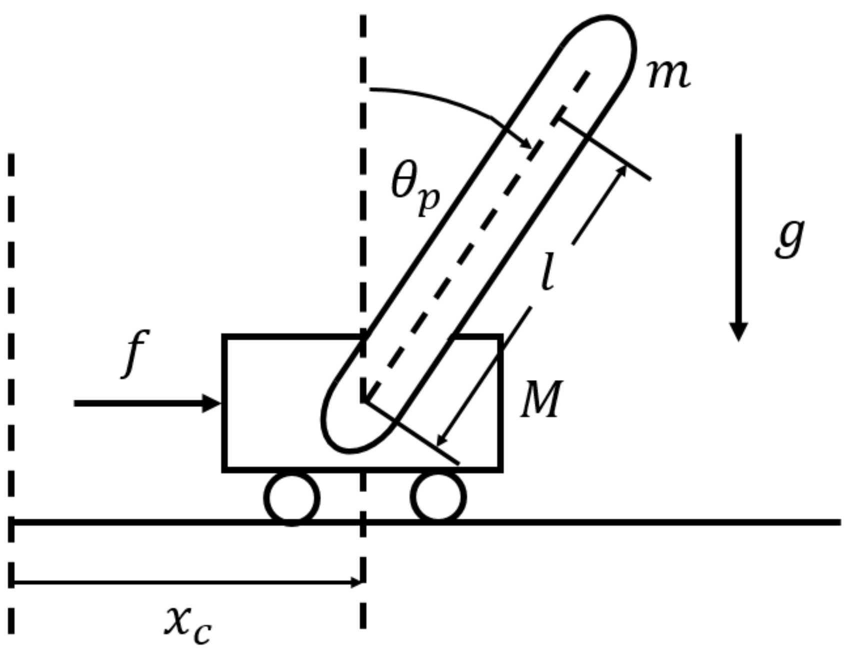
FRA 331 : Basic Control Theory
Homework Assignment 2: State-space

Written

1 : Segway (Inverted Pendulum)

A segway can be modelled as an inverted pendulum, which consists of a cart, and a pole. Let x_c and θ_p be the absolute position of the cart and the angular position of the pole relative to the upright position as seen in the figure. Let f be the only



Problem 1

input force acting on the cart. And let M, m, l, and g be parameters, which are constant.

The dynamic of an inverted pendulum can be written as the followings.

$$(M+m)\ddot{x}_c + ml\cos(\theta_p)\ddot{\theta}_p - ml\sin(\theta_p)\dot{\theta}_p^2 = f$$

$$ml\cos(\theta_p)\ddot{x}_c + ml^2\ddot{\theta}_p - mgl\sin(\theta_p) = 0$$

Let the output of the system be the position of the cart and the angular position of the pole.

a). Determine the order of the system, state variable, input, and output of the system

b.) Rewrite the dynamics of the inverted pendulum in form of state-space (both state equation and output equation). Clearly assign the state variables $\vec{x}$

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2 : Quadrotor

Th equation of linear motion of a quadrotor can be written as the following set of equations.

$$m \begin{bmatrix} \ddot{x}_q \\ \ddot{y}_q \\ \ddot{z}_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R(\phi, \theta, \psi)\vec{T}$$

where

$$R(\phi, \theta, \psi) = \begin{bmatrix} c_\phi c_\psi - s_\phi c_\theta s_\psi & -c_\phi s_\psi - c_\phi c_\theta s_\psi & s_\theta s_\psi \\ c_\phi s_\psi - s_\phi c_\theta c_\psi & -s_\phi s_\psi + c_\phi c_\theta c_\psi & -s_\theta c_\psi \\ s_\phi s_\theta & c_\phi s_\theta & c_\theta \end{bmatrix}$$

$$\vec{T} = \begin{bmatrix} 0 \\ 0 \\ u_1^2 + u_2^2 + u_3^2 + u_4^2 \end{bmatrix}$$

- m : mass of the quadrotor $[kg]$
- x_q, y_q, z_q : position of the quadrotor in Cartesian space $[m]$
- g : gravitational acceleration $\left[\frac{m}{s^2}\right]$
- ϕ, θ, ψ : roll, pitch, yaw of the quadrotor $[rad]$
- u_i : input rotor speed or rotor i $[rad/s]$

Note: $s_\alpha = \sin(\alpha)$ and $c_\beta = \cos(\beta)$

The equation of angular motion of the quadrotor can be written as the following set of equations.

$$\dot{\vec{\omega}} = \begin{bmatrix} \dot{\omega_x} \\ \dot{\omega_y} \\ \dot{\omega_z} \end{bmatrix} = \begin{bmatrix} \frac{1}{I_{xx}}\tau_\phi \\ \frac{1}{I_{yy}}\tau_\theta \\ \frac{1}{I_{zz}}\tau_\psi \end{bmatrix} - \begin{bmatrix} \frac{I_{yy}-I_{zz}}{I_{xx}}\omega_y\omega_z \\ \frac{I_{zz}-I_{xx}}{I_{yy}}\omega_x\omega_z \\ \frac{I_{xx}-I_{yy}}{I_{zz}}\omega_x\omega_y \end{bmatrix}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

where

$$\begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} Lk(u_1^2 - u_3^2) \\ Lk(u_2^2 - u_4^2) \\ b(u_1^2 - u_2^2 + u_3^2 - u_4^2) \end{bmatrix}$$

- ω_x, ω_y, ω_z : angular velocity of the quadrotor around each axis $\left[\frac{rad}{s}\right]$
- I_{xx}, I_{yy}, I_{zz} : inertia of the quadrotor $[kg \cdot m^2]$
- L : location of the rotor relative to the center of mass $[m]$
- k, b :aerodynamic constants

Let the Cartesian position x_q, y_q, and z_q, and the yaw angle ψ be the output of the system.

a). Determine the order of the system, state variable, input, and output of the system

b.) Rewrite the dynamics of the quadrotor in form of state-space (both state equation and output equation). Clearly assign the state variables $\vec{x}$. You may leave your answer in term of matrix multiplication, but you must specify those matrices in term of state variables x_i, and input variables u_i

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3 : RLC Circuit

Given the same RLC circuits introduced in the class, we can also write the dynamics of the circuit as the following second-order differential equation.

$$R_2C_1\frac{dv_{in}}{dt} + v_{in} = R_2LC_1\frac{d^2i_1}{dt^2} + (R_1R_2C_1 + L)\frac{di_1}{dt} + (R_1 + R_2)i_1$$

$$v_{out} = -R_1i_1 - L\frac{di_1}{dt} + v_{in}$$

a). Determine the order of the system, state variable, input, and output of the system

b.) Rewrite the dynamics of the RLC in form of state-space (both state equation and output equation). Clearly assign the state variables $\vec{x}$.

c.) Since this is a LTI system, determine state matrix A, input matrix B, Output matrix C, and feedforward matrix D

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

$$\vec{y} = C\vec{x} + D\vec{u}$$

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Programming

MATLAB Programming

Write a MATLAB function called "doublePendulum" that computes the time-derivative of the state variables of a double pendulum. The state-space of the double pendulum can be written as follows.

$$\vec{x} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ M^{-1}\Big[-V \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} - G + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Big] \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 l_{c_1}^2 + m_2(l_1^2 + l_{c_2}^2) + I_{zz_1} + I_{zz_2} + 2m_2 l_1 l_{c_2}\cos(x_2) & m_2 l_{c_2}^2 + I_{zz_2} + m_2 l_1 l_{c_2}\cos(x_2) \\ m_2 l_{c_2}^2 + I_{zz_2} + m_2 l_1 l_{c_2}\cos(x_2) & m_2 l_{c_2}^2 + I_{zz_2} \end{bmatrix}$$

$$V = \begin{bmatrix} -m_2 l_1 l_{c_2}\sin(x_2)x_4 + b_1 & -m_2 l_1 l_{c_2}\sin(x_2)(x_3 + x_4) \\ m_2 l_1 l_{c_2}\sin(x_2)x_3 & b_2 \end{bmatrix}$$

$$G = \begin{bmatrix} m_1 g l_{c_1}\sin(x_1) + m_2 g(l_1\sin(x_1) + l_{c_2}\sin(x_1 + x_2)) \\ m_2 g l_{c_2}\sin(x_1 + x_2) \end{bmatrix}$$

The function "doublePendulum" must take input time t, state variables x, and control inputs u. (Hint : $\dot{x} = doublePendulum(t, x, u)$) The value of all parameters are provided in the template. Run the $testScript$ to see the simulation.