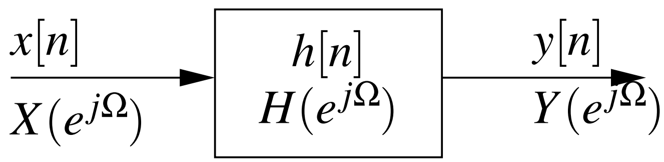
6 FIR Filter Design

6.1 Introduction

LTI-System:



Time domain

impulse response:

$$h[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot \delta[n-k]$$

I/O:

$$y[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

Frequency domain

frequency response:

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-jn\Omega}$$

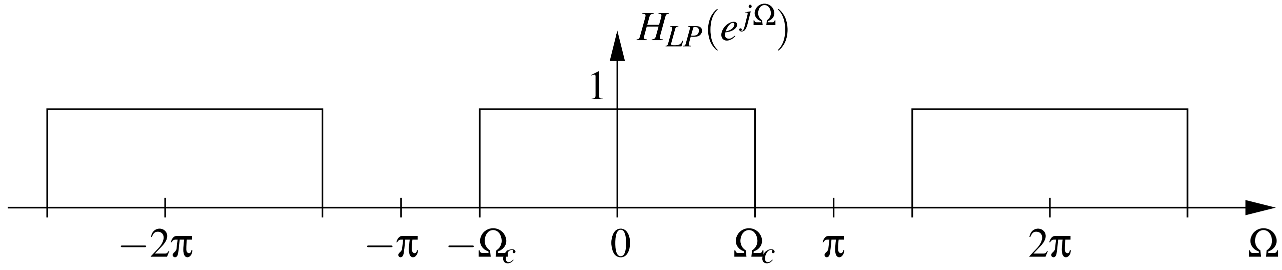
I/O:

$$Y(e^{j\Omega}) = H(e^{j\Omega}) \cdot X(e^{j\Omega})$$

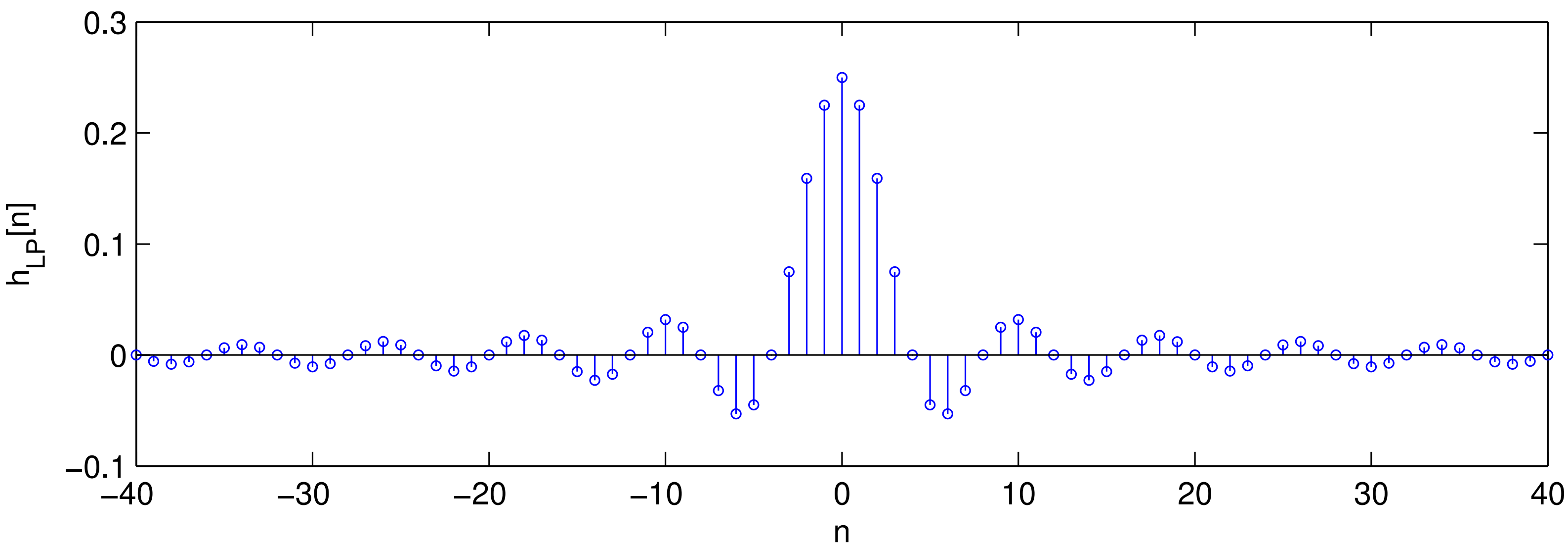
6.2 Filter Design using Window Functions

Example: Ideal lowpass filter

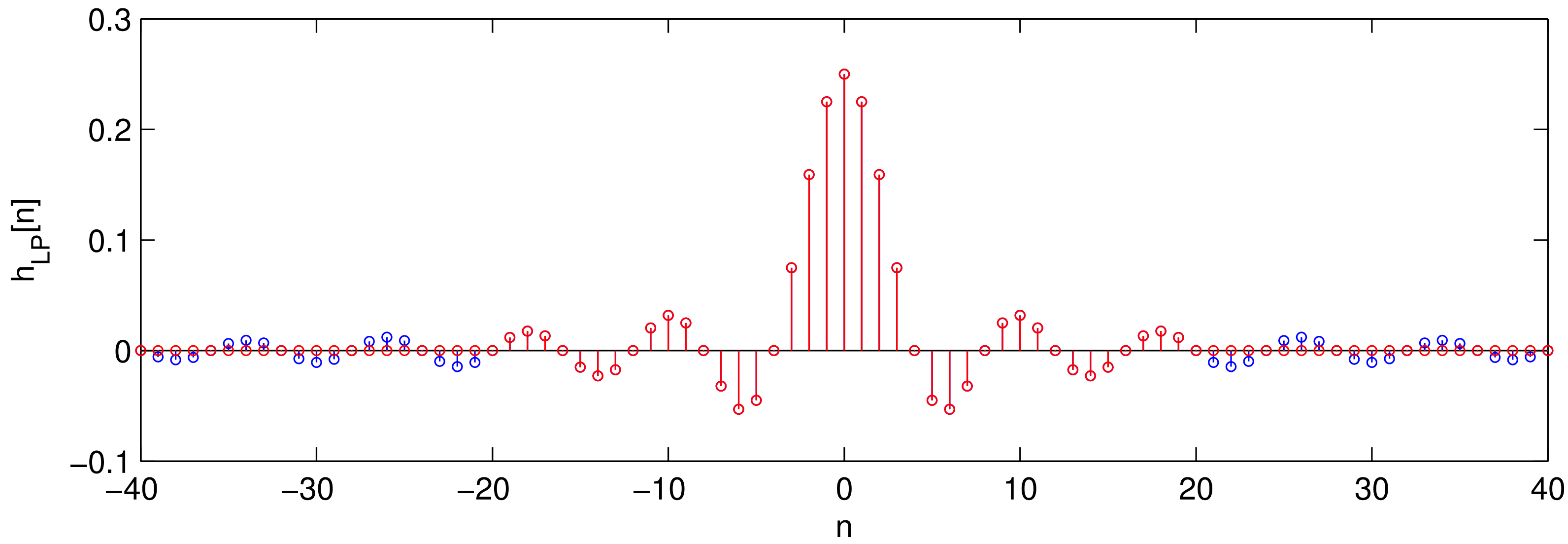
$$H_{LP}(e^{j\Omega}) = \begin{cases} 1 & \text{for} \quad |\Omega| < \Omega_c \\ 0 & \text{for} \quad \Omega_c < |\Omega| < \pi \end{cases}$$



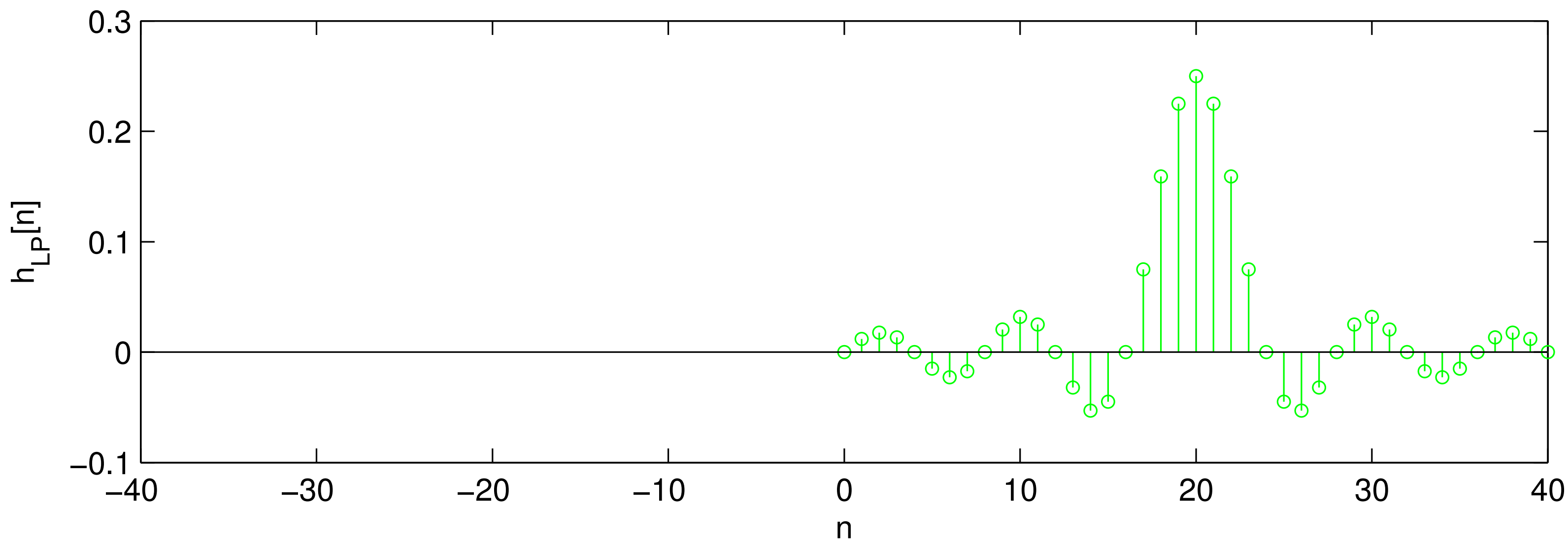
$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\Omega})e^{jn\Omega}d\Omega = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} 1 \cdot e^{jn\Omega}d\Omega = \frac{\Omega_c}{\pi} \cdot \frac{\sin(\Omega_c n)}{\Omega_c n}$$



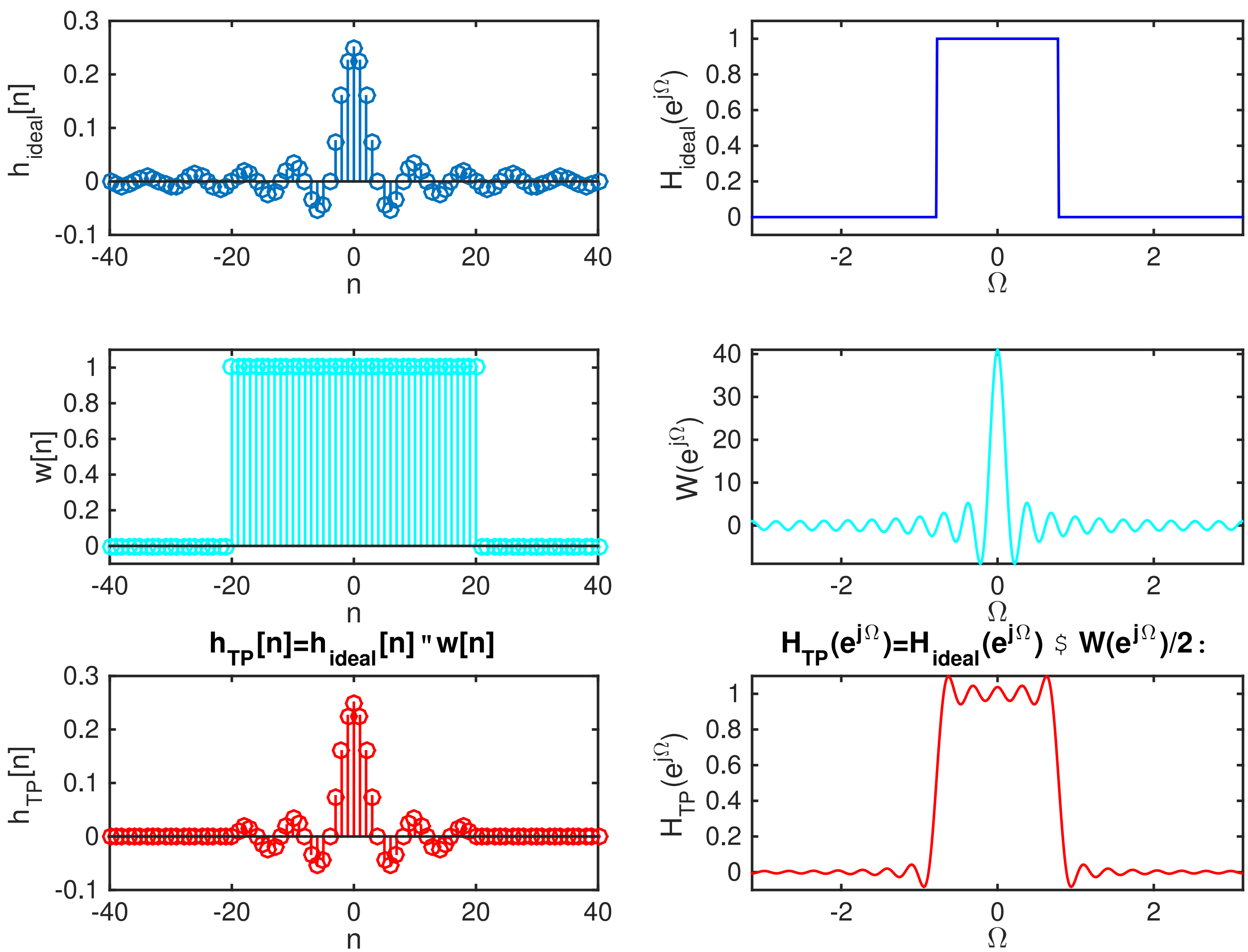
Windowing:



Time Delay:



6.2.1 Influence of Window Function in the Frequency Domain



6.2.2 Typical Window Functions (Causal Representation, $0 \leq n \leq N$)

Rectangular window: $$w_{rec}[n] = 1$$

Hanning window: $$w_{Han}[n] = 0.5 - 0.5\cos\left(\frac{2\pi n}{N}\right)$$

Hamming window: $$w_{Ham}[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N}\right)$$

Blackman window: $$w_{Bla}[n] = 0.42 - 0.5\cos\left(\frac{2\pi n}{N}\right) + 0.08\cos\left(\frac{4\pi n}{N}\right)$$

Kaiser window: $$w_{Kai}[n] = \frac{I_0\left(\beta\sqrt{1-\left(\frac{2n}{N}-1\right)^2}\right)}{I_0(\beta)}$$

$$\beta = \begin{cases} 0.1102(\alpha_S - 8.7), & \alpha_S > 50 \text{ dB} \\ 0.5842(\alpha_S - 21), & 21 \text{ dB} \leq \alpha_S \leq 50 \text{ dB} \\ 0, & \alpha_S < 21 \text{ dB} \end{cases}$$

I_0: modified Bessel function of order zero

β determines minimal stopband attenuation α_S in dB

6.3 Least Squares Error Design

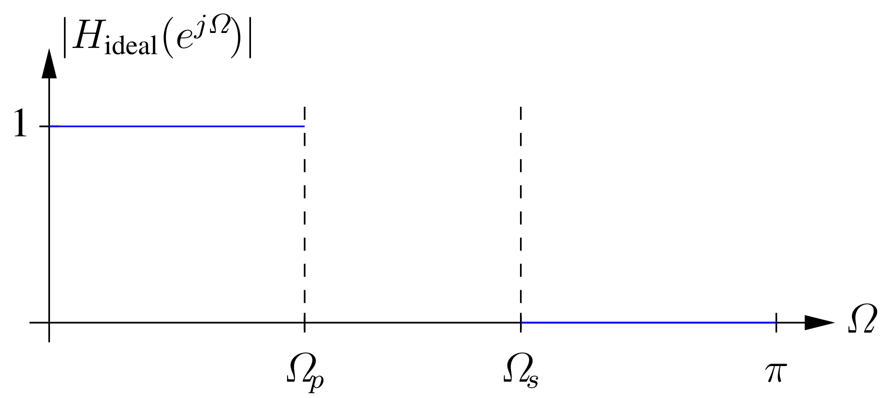
Error criterium:

$$e = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\Omega}) - H_{\text{ideal}}(e^{j\Omega})|^2 d\Omega = \sum_{n=-\infty}^{\infty} |h[n] - h_{\text{ideal}}[n]|^2$$

$$= \sum_{n=-\infty}^{-1} |h_{\text{ideal}}[n]|^2 + \sum_{n=0}^{N} |h[n] - h_{\text{ideal}}[n]|^2 + \sum_{n=N+1}^{\infty} |h_{\text{ideal}}[n]|^2$$

$$e_{\min} = \sum_{n=-\infty}^{-1} |h_{\text{ideal}}[n]|^2 + \sum_{n=N+1}^{\infty} |h_{\text{ideal}}[n]|^2$$

Introduction of a "Don't Care" transition band:



Matlab: h=firls(N, $[0, \Omega_p/\pi, \Omega_s/\pi, 1]$, [1,1,0,0])

6.4 Parks-McClellan (Equiripple) Filter Design

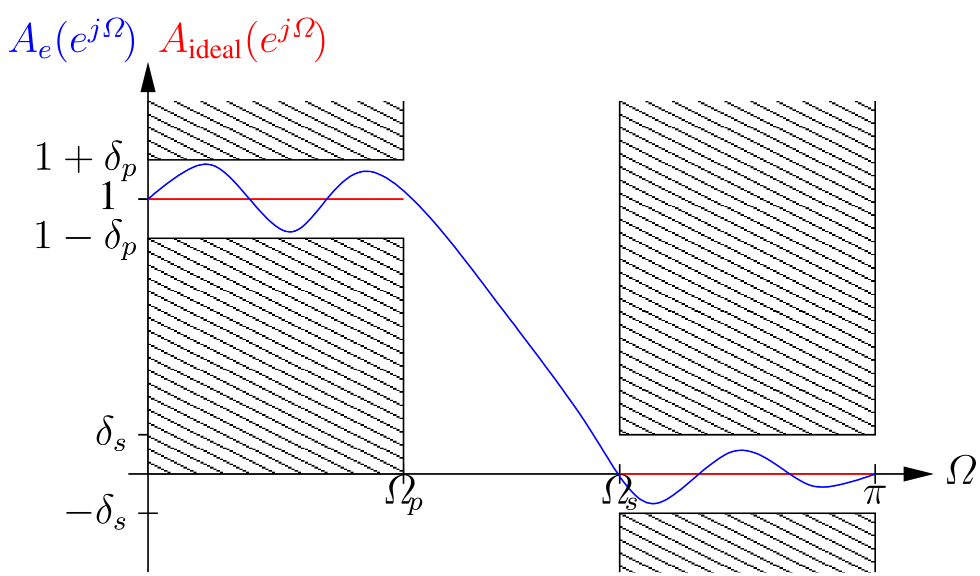
6.4.1 Design Steps for a Linear Phase Lowpass Filter

Zero phase, symmetric FIR filter:

$$a_e[-n] = a_e[n], \qquad A_e(e^{j\omega}) = \sum_{-N/2}^{N/2} a_e[n] e^{-jn\omega}$$

Causal filter: $H(e^{j\Omega}) = A_e(e^{j\Omega}) e^{-j\Omega N/2}$

Tolerance scheme:



Design criterium:

given: N, Ω_p, Ω_s, δ_p/δ_s

error function: $E(\Omega) =$

$W(\Omega)[A_{\text{ideal}}(e^{j\Omega}) - A_e(e^{j\Omega})]$

$W(\Omega)$: weighting function

$$\min_{\{a_e[n]:0\le n\le N/2\}} (\max_{\Omega\in F} |E(\Omega)|)$$

$F : \Omega \in [0, \Omega_p],\ \Omega \in [\Omega_s, \pi]$

6.4.2 Estimation of Filter Order

Kaiser's approximation:

$$N \approx \frac{-10\log_{10}(\delta_p\,\delta_s) - 13}{14.6(\Omega_s - \Omega_p)/2\pi}$$

Bellanger's Approximation:

$$N \approx -\frac{2\log_{10}(10\delta_p\,\delta_s)}{3(\Omega_s - \Omega_p)/2\pi} - 1$$

6.5 Special FIR Filters

6.5.1 Linear Phase FIR Filters

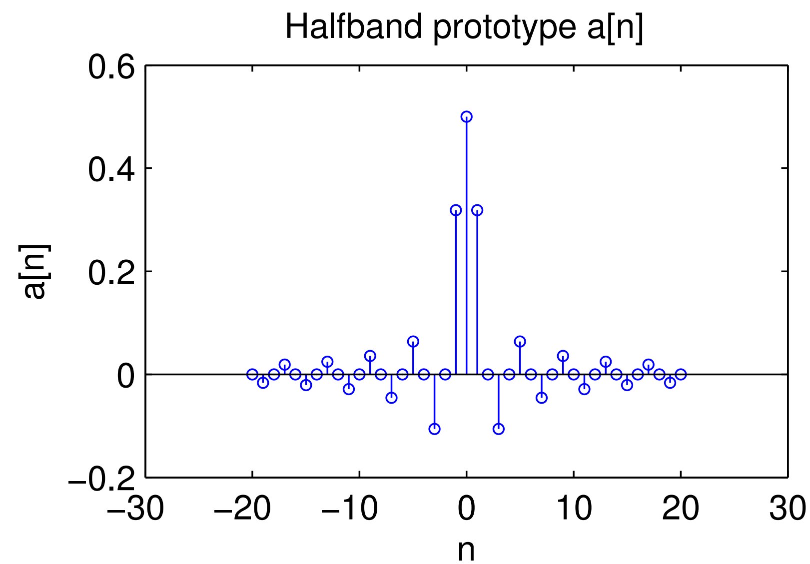
Linear phase filters have impulse responses with the following symmetry properties:

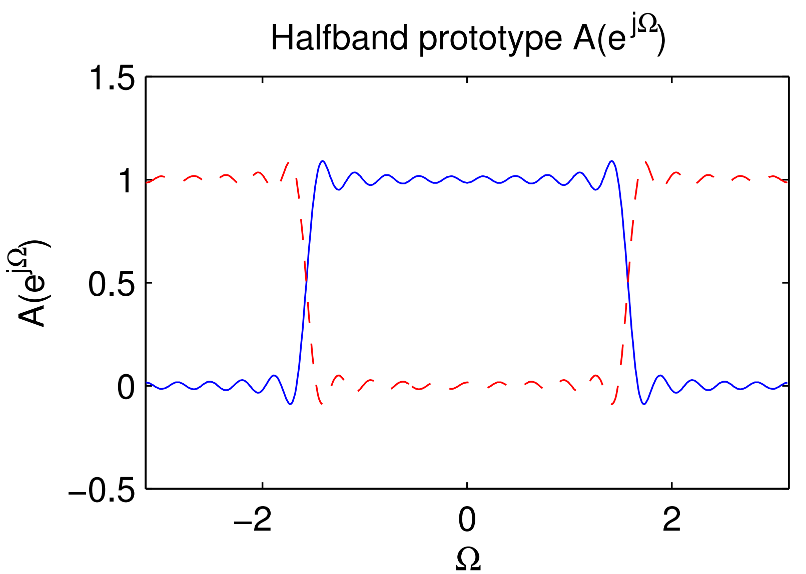
$$h[n] = h[N-n], \qquad h[n] = -h[N-n]$$

filter type	filter order N	symmetry	$H(e^{j\Omega})\big\vert_{\Omega=0}$	$H(e^{j\Omega})\big\vert_{\Omega=\pi}$
type 1	even	even	no restrictions	no restrictions
type 2	odd	even	no restrictions	$H(e^{j\pi}) = 0$
type 3	even	odd	$H(e^{j0}) = 0$	$H(e^{j\pi}) = 0$
type 4	odd	odd	$H(e^{j0}) = 0$	no restrictions

Linear phase filters have a constant group delay.

6.5.2 Halfband Filters (Nyquist Filters)





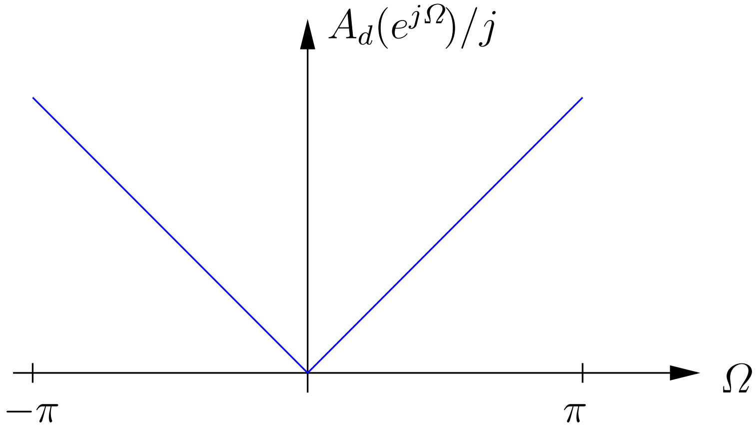
Zero-phase halfband lowpass filter:

- 6 dB cutoff frequency: $\Omega_{gr} = \pi/2$
- $\Omega_p = \pi - \Omega_s$
- $\delta_p = \delta_s$
- $A(e^{j\Omega}) - 0.5 = 0.5 - A(e^{j(\pi-\Omega)})$
- $A(e^{j\Omega}) + A(e^{j(\Omega-\pi)}) = 1$
- $a(2n) = 0$ for $n \neq 0$, $a(0) = 1/2$

Design:

- Windowing of an ideal lowpass filter with $\Omega_c = \pi/2$
- Parks-McClellan design with "trick"
- Cosine roll-off halfband filter

6.5.3 Differentiator



Ideal differentiator:

$$A_d(e^{j\Omega}) = j\Omega, \quad 0 \leq \omega \leq \pi$$

$$a_d[n] = \frac{\cos(\pi n)}{n}, \quad n \neq 0, \quad a_d(0) = 0$$

non causal, infinite length impulse response with odd symmetry

Approximation:

- Type 3 or type 4 FIR filter
- limitation of bandwidth
- Design with Parks-McClellan Algorithm
- Weighting of error $\sim W/f$