EE-556 Formula Sheet

Some useful Mathematical formulas and identities:

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$, $\cos(\theta) = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$, $\sin(\theta) = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, $\cos(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Geometric series: $\displaystyle\sum_{k=m}^{n} a^k = \frac{a^m - a^{n+1}}{1-a}$

Fourier series:

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right] = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} ,$$

$a_0 = \dfrac{1}{T_0}\displaystyle\int_{T_0} x(t)dt$, $a_n = \dfrac{2}{T_0}\displaystyle\int_{T_0} x(t)\cos(n\omega_0 t)dt$, $b_n = \dfrac{2}{T_0}\displaystyle\int_{T_0} x(t)\sin(n\omega_0 t)dt$, $c_n = \dfrac{1}{T_0}\displaystyle\int_{T_0} x(t)e^{-jn\omega_0 t}dt$

Fourier transform and inverse Fourier transform:

$X(\omega) = \displaystyle\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$, $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}dt$

Some useful Fourier transform pairs:

$1 \leftrightarrow 2\pi\delta(\omega)$

$u(t) \leftrightarrow \pi\delta(\omega) + \dfrac{1}{j\omega}$

$\delta(t) \leftrightarrow 1$

$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$

$\mathrm{rect}\left(\dfrac{t}{T}\right) = T\mathrm{sinc}\left(\dfrac{\omega T}{2\pi}\right)$

$\cos(\omega_0 t) \leftrightarrow \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$

$\sin(\omega_0 t) \leftrightarrow j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$

Energy and power of discrete-time signals:

Energy: $E_x = \sum_{n=-\infty}^{+\infty} |x[n]|^2$, Power: non-periodic $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$, N-periodic: $P_x = \frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2$

Discrete-time linear convolution: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

Discrete-time Fourier Transform (DTFT): $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$

DTFT Properties:

Symmetry relations:

Sequence	DTFT
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$
$j\,\mathrm{Im}\{x[n]\}$	$X_{as}(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$
$x_{cs}[n]$	$X_{re}(e^{j\omega})$
$x_{as}[n]$	$jX_{im}(e^{j\omega})$

Note: Subscript “*cs*” and “*as*” mean conjugate symmetric and conjugate anti-symmetric signals, respectively.

DTFT of commonly used sequences:

Sequence	DTFT
$\delta[n]$	1
$\mu[n]$ (Unit Step Function)	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega + 2\pi k)$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
$a^n\mu[n]$, ($\lvert a\rvert < 1$)	$\frac{1}{1-ae^{-j\omega}}$
$(n+1)a^n\mu[n]$, ($\lvert a\rvert < 1$)	$\left(\frac{1}{1-ae^{-j\omega}}\right)^2$
$h[n] = \frac{\sin(\omega_c n)}{\pi n}$, ($-\infty < n < +\infty$)	$H(e^{j\omega}) = \begin{cases} 1 & 0 \le \lvert\omega\rvert \le \omega_c \\ 0 & \omega_c < \lvert\omega\rvert < \pi \end{cases}$

DTFT Theorems:

Theorem	Sequence	DTFT
	$g[n]$ $h[n]$	$G(e^{j\omega})$ $H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time Reversal	$g[-n]$	$G(e^{-j\omega})$
Time Shifting	$g[n - n_0]$	$e^{-j\omega n_0} G(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n} g[n]$	$G(e^{j(\omega-\omega_0)})$
Differentiation in Frequency	$ng[n]$	$j \dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] * h[n]$	$G(e^{j\omega}) H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\dfrac{1}{2\pi} \int_{-\pi}^{-\pi} G(e^{j\theta}) H(e^{j(\omega-\theta)}) d\theta$
Parseval's Identity	$\sum_{n=-\infty}^{+\infty} g[n]h^*[n] = \dfrac{1}{2\pi} \int_{-\pi}^{+\pi} G(e^{j\omega}) H^*(e^{j\omega}) d\omega$	

Discrete Fourier Transform (DFT): $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$, $\quad 0 \le k \le N-1$

Inverse DFT: $x[n] = \dfrac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk}$, $\quad 0 \le n \le N-1$

Circular Convolution: $y_c[n] = \sum_{m=0}^{N-1} x[m] h[\langle n - m \rangle_N]$, $\quad 0 \le n \le N-1$

Symmetry properties of DFT:

Length-*N* Sequence	***N*-Point DFT**
$x[n] = x_{re}[n] + jx_{im}[n]$	$X[k] = X_{re}[k] + jX_{im}[k]$
$x^*[n]$	$X^*[\langle -k \rangle_N]$
$x^*[\langle -n \rangle_N]$	$X^*[k]$
$x_{re}[n]$	$X_{cs}[k] = \frac{1}{2}\left[X[k] + X^*[\langle -k \rangle_N]\right]$
$jx_{im}[n]$	$X_{as}[k] = \frac{1}{2}\left[X[k] - X^*[\langle -k \rangle_N]\right]$
$x_{cs}[n]$	$X_{re}[k]$
$x_{as}[n]$	$jX_{im}[k]$

DFT Theorems:

Theorem	Sequence	DTFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular Time Shifting	$g[\langle n - n_0 \rangle_N]$	$e^{-j\frac{2\pi}{N}kn_0} G[k]$
Circular Frequency Shifting	$e^{j\frac{2\pi}{N}nk_0} g[n]$	$G[\langle k - k_0 \rangle_N]$
Duality	$G[n]$	$Ng[\langle -k \rangle_N]$
N-Point Circular Convolution	$\sum_{m=0}^{N-1} g[m]h[\langle n - m \rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle n - m \rangle_N]$
Parseval's Identity	$\sum_{n=0}^{N-1} \lvert g[n] \rvert^2 = \frac{1}{N}\sum_{n=0}^{N-1} \lvert G[k] \rvert^2$	

Z-Transform: $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$ **Inverse Z-Transform:** $x[n] = \frac{1}{2\pi j}\oint_C G(z)z^{n-1}dz$

Z-Transform Properties:

Property	Sequence	z-Transform	Region of Convergence
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	R_g R_h
Conjugation	$g^*[n]$	$G^*(z^*)$	R_g
Time Reversal	$g[-n]$	$G(1/z)$	$1/R_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $R_g \cap R_h$
Time Shifting	$g[n - n_0]$	$z^{-n_0}G(z)$	R_g except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert R_g$
Differentiation in the z-domain	$ng[n]$	$-z\frac{dG(z)}{dz}$	R_g except possibly the point $z = 0$ or ∞
Convolution	$g[n] * h[n]$	$G(z)H(z)$	Includes $R_g \cap R_h$

TABLE 5.1 Select (Unilateral) z-Transform Pairs

No.	$x[n]$	$X[z]$
1	$\delta[n-k]$	z^{-k}
2	$u[n]$	$\frac{z}{z-1}$
3	$nu[n]$	$\frac{z}{(z-1)^2}$
4	$n^2u[n]$	$\frac{z(z+1)}{(z-1)^3}$
5	$n^3u[n]$	$\frac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^n u[n]$	$\frac{z}{z-\gamma}$
7	$\gamma^{n-1}u[n-1]$	$\frac{1}{z-\gamma}$
8	$n\gamma^n u[n]$	$\frac{\gamma z}{(z-\gamma)^2}$
9	$n^2\gamma^n u[n]$	$\frac{\gamma z(z+\gamma)}{(z-\gamma)^3}$
10	$\frac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$	$\frac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^n \cos\beta n\, u[n]$	$\frac{z(z-\lvert\gamma\rvert\cos\beta)}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^n \sin\beta n\, u[n]$	$\frac{z\lvert\gamma\rvert\sin\beta}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n]$	$\frac{rz[z\cos\theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n] \qquad \gamma=\lvert\gamma\rvert e^{j\beta}$	$\frac{(0.5re^{j\theta})z}{z-\gamma}+\frac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n]$ $r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}}$ $\beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert}$ $\theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$	$\frac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$