EE-556 Formula Sheet

Some useful Mathematical formulas and identities:

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$, $\cos(\theta) = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$, $\sin(\theta) = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, $\cos(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\cos(\alpha)\cos(\beta) = \dfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$

$\sin(\alpha)\sin(\beta) = \dfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$\sin(\alpha)\cos(\beta) = \dfrac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$

$\mathrm{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

Geometric series: $\displaystyle\sum_{k=m}^{n} a^k = \frac{a^m - a^{n+1}}{1-a}$

Fourier series:

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right] = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} ,$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt , \qquad a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)dt , \qquad b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)dt , \qquad c_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt$$

Fourier transform and inverse Fourier transform:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt , \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}dt$$

Some useful Fourier transform pairs:

$1 \leftrightarrow 2\pi\delta(\omega)$

$u(t) \leftrightarrow \pi\delta(\omega) + \dfrac{1}{j\omega}$

$\delta(t) \leftrightarrow 1$

$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$

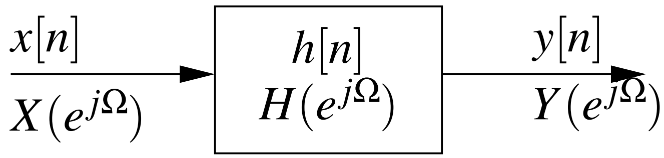
$\mathrm{rect}\left(\dfrac{t}{T}\right) = T\mathrm{sinc}\left(\dfrac{\omega T}{2\pi}\right)$

$\cos(\omega_0 t) \leftrightarrow \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$

$\sin(\omega_0 t) \leftrightarrow j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$

6.1 Introduction

LTI-System:



Time domain

impulse response:

$$h[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot \delta[n-k]$$

I/O:

$$y[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

Frequency domain

frequency response:

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-jn\Omega}$$

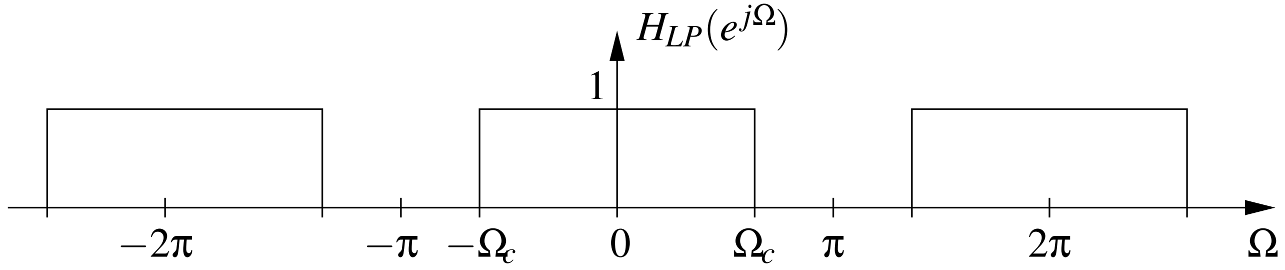
I/O:

$$Y(e^{j\Omega}) = H(e^{j\Omega}) \cdot X(e^{j\Omega})$$

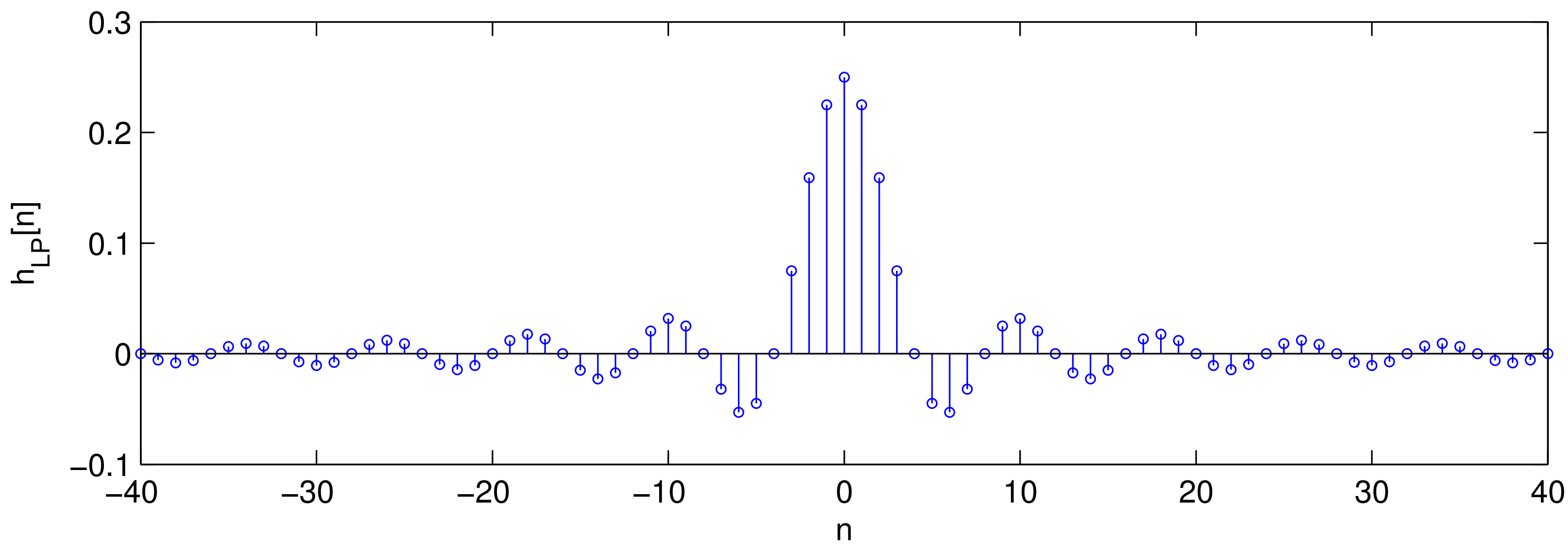
6.2 Filter Design using Window Functions

Example: Ideal lowpass filter

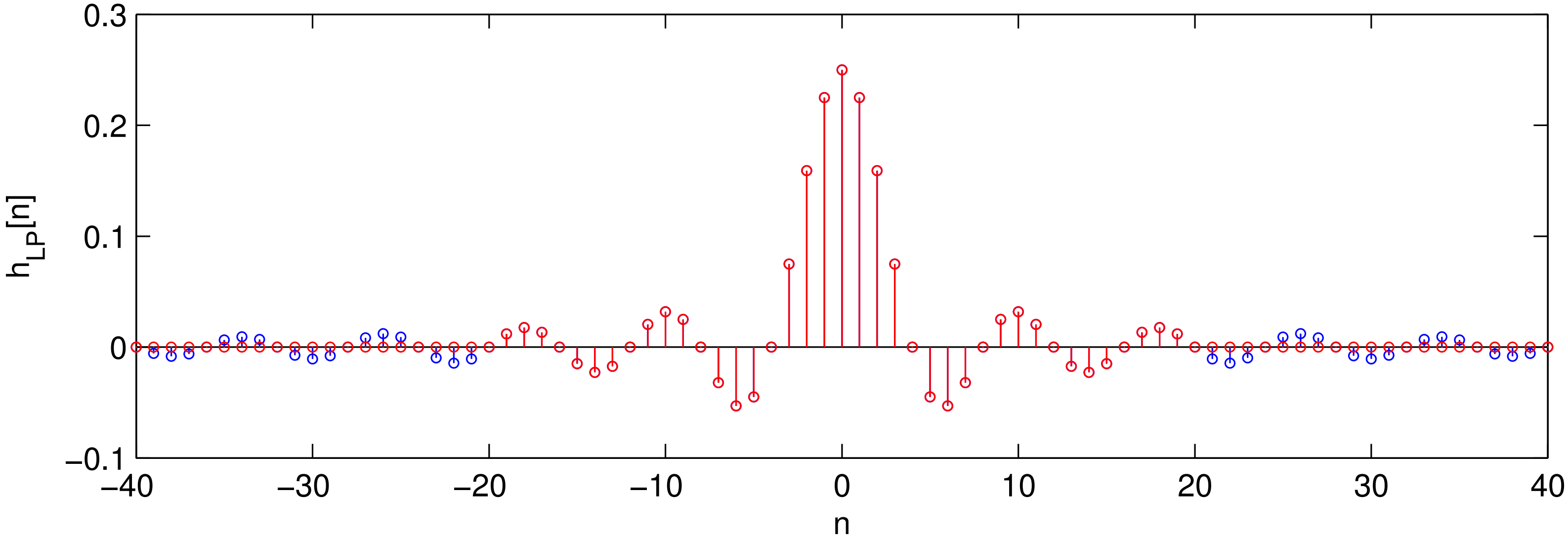
$$H_{LP}(e^{j\Omega}) = \begin{cases} 1 & \text{for} \quad |\Omega| < \Omega_c \\ 0 & \text{for} \quad \Omega_c < |\Omega| < \pi \end{cases}$$



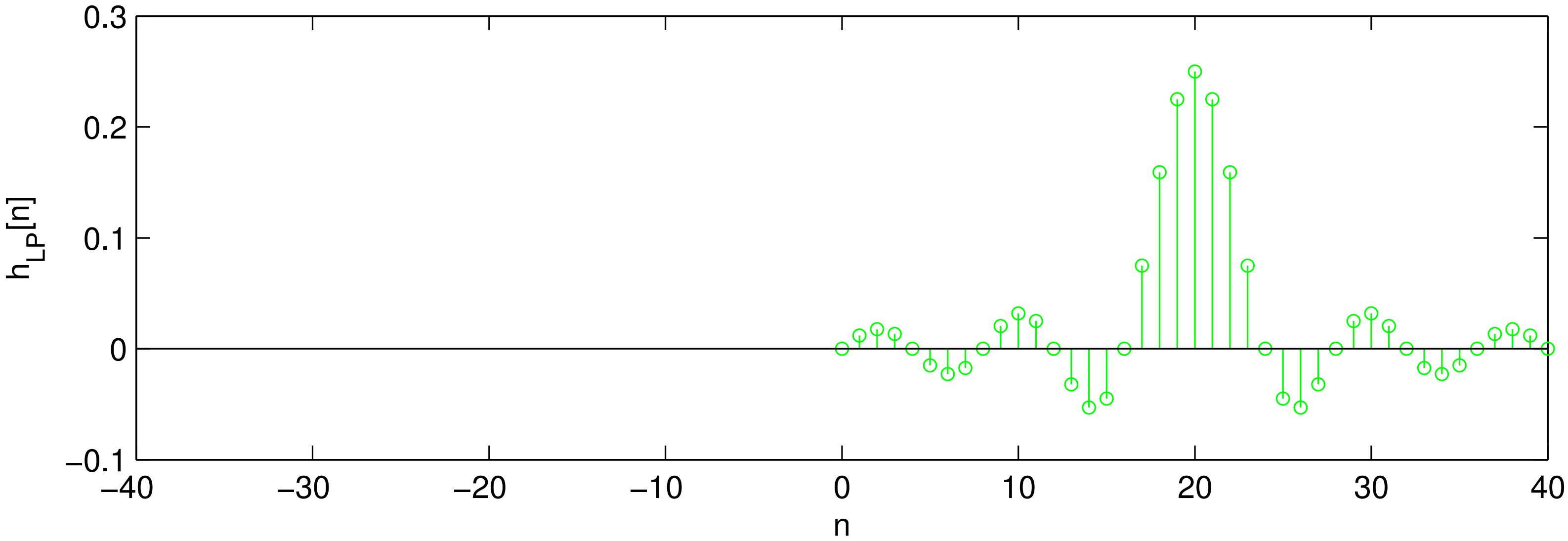
$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\Omega})e^{jn\Omega}d\Omega = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} 1\cdot e^{jn\Omega}d\Omega = \frac{\Omega_c}{\pi}\cdot\frac{\sin(\Omega_c n)}{\Omega_c n}$$



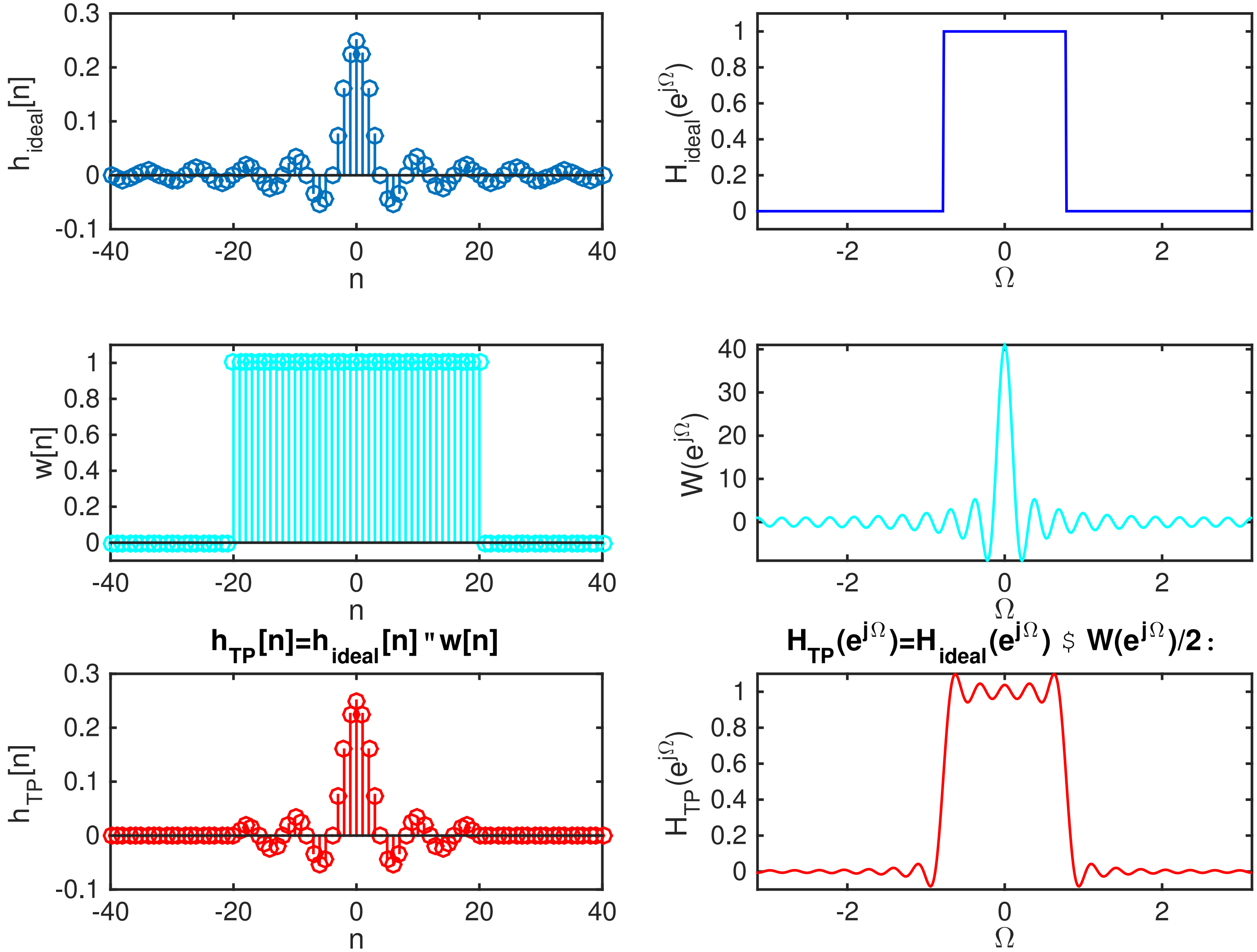
Windowing:



Time Delay:



6.2.1 Influence of Window Function in the Frequency Domain



6.2.2 Typical Window Functions (Causal Representation, $0 \leq n \leq N$)

Rectangular window: $$w_{rec}[n] = 1$$

Hanning window: $$w_{Han}[n] = 0.5 - 0.5\cos\left(\frac{2\pi n}{N}\right)$$

Hamming window: $$w_{Ham}[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N}\right)$$

Blackman window: $$w_{Bla}[n] = 0.42 - 0.5\cos\left(\frac{2\pi n}{N}\right) + 0.08\cos\left(\frac{4\pi n}{N}\right)$$

Kaiser window: $$w_{Kai}[n] = \frac{I_0\left(\beta\sqrt{1-\left(\frac{2n}{N}-1\right)^2}\right)}{I_0(\beta)}$$

$$\beta = \begin{cases} 0.1102(\alpha_S - 8.7), & \alpha_S > 50 \text{ dB} \\ 0.5842(\alpha_S - 21), & 21 \text{ dB} \leq \alpha_S \leq 50 \text{ dB} \\ 0, & \alpha_S < 21 \text{ dB} \end{cases}$$

I_0: modified Bessel function of order zero

β determines minimal stopband attenuation α_S in dB

6.3 Least Squares Error Design

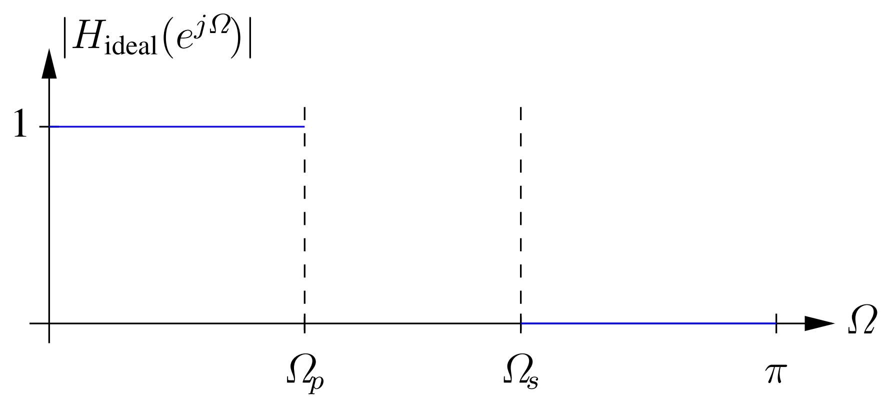
Error criterium:

$$e = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\Omega}) - H_{\text{ideal}}(e^{j\Omega})|^2 d\Omega = \sum_{n=-\infty}^{\infty} |h[n] - h_{\text{ideal}}[n]|^2$$

$$= \sum_{n=-\infty}^{-1} |h_{\text{ideal}}[n]|^2 + \sum_{n=0}^{N} |h[n] - h_{\text{ideal}}[n]|^2 + \sum_{n=N+1}^{\infty} |h_{\text{ideal}}[n]|^2$$

$$e_{\min} = \sum_{n=-\infty}^{-1} |h_{\text{ideal}}[n]|^2 + \sum_{n=N+1}^{\infty} |h_{\text{ideal}}[n]|^2$$

Introduction of a "Don't Care" transition band:



Matlab: $\boxed{\text{h=firls}(N,\ [0,\ \Omega_p/\pi,\ \Omega_s/\pi,\ 1],\ [1,1,0,0])}$

6.4 Parks-McClellan (Equiripple) Filter Design

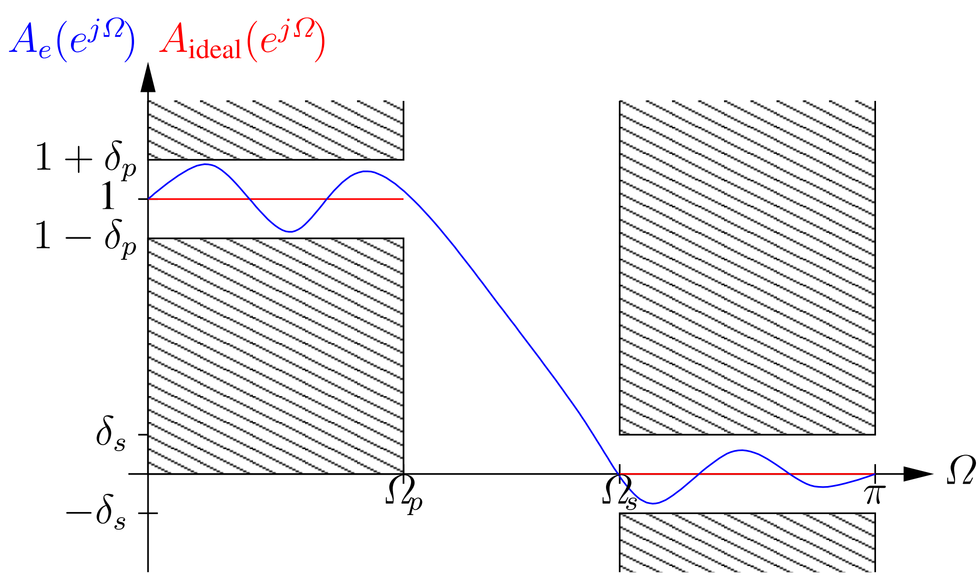
6.4.1 Design Steps for a Linear Phase Lowpass Filter

Zero phase, symmetric FIR filter:

$$a_e[-n] = a_e[n], \qquad A_e(e^{j\omega}) = \sum_{-N/2}^{N/2} a_e[n]e^{-jn\omega}$$

Causal filter: $H(e^{j\Omega}) = A_e(e^{j\Omega})e^{-j\Omega N/2}$

Tolerance scheme:



Design criterium:

given: N, Ω_p, Ω_s, δ_p/δ_s

error function: $E(\Omega) =$

$$W(\Omega)[A_{\text{ideal}}(e^{j\Omega}) - A_e(e^{j\Omega})]$$

$W(\Omega)$: weighting function

$$\min_{\{a_e[n]:0\le n\le N/2\}} (\max_{\Omega\in F} |E(\Omega)|)$$

$$F : \Omega \in [0, \Omega_p], \ \Omega \in [\Omega_s, \pi]$$

6.4.2 Estimation of Filter Order

Kaiser's approximation:

$$N \approx \frac{-10\log_{10}(\delta_p\,\delta_s) - 13}{14.6(\Omega_s - \Omega_p)/2\pi}$$

Bellanger's Approximation:

$$N \approx -\frac{2\log_{10}(10\delta_p\,\delta_s)}{3(\Omega_s - \Omega_p)/2\pi} - 1$$

6.5 Special FIR Filters

6.5.1 Linear Phase FIR Filters

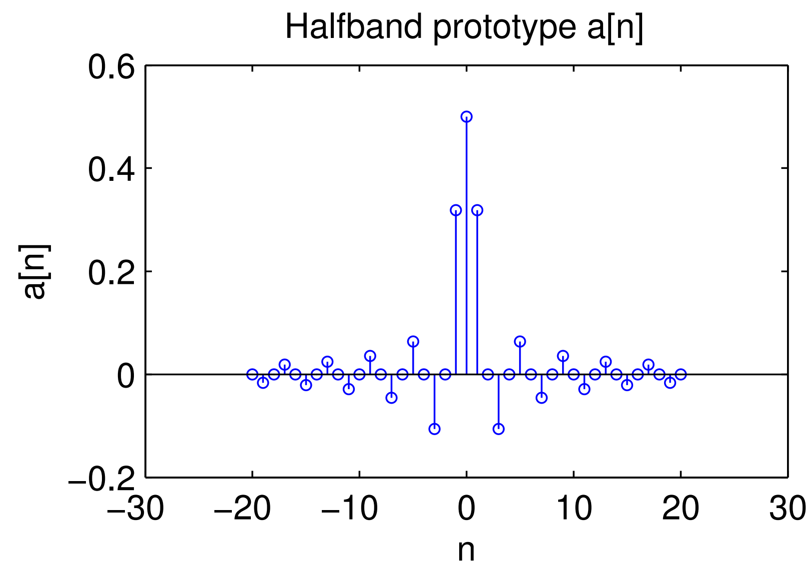
Linear phase filters have impulse responses with the following symmetry properties:

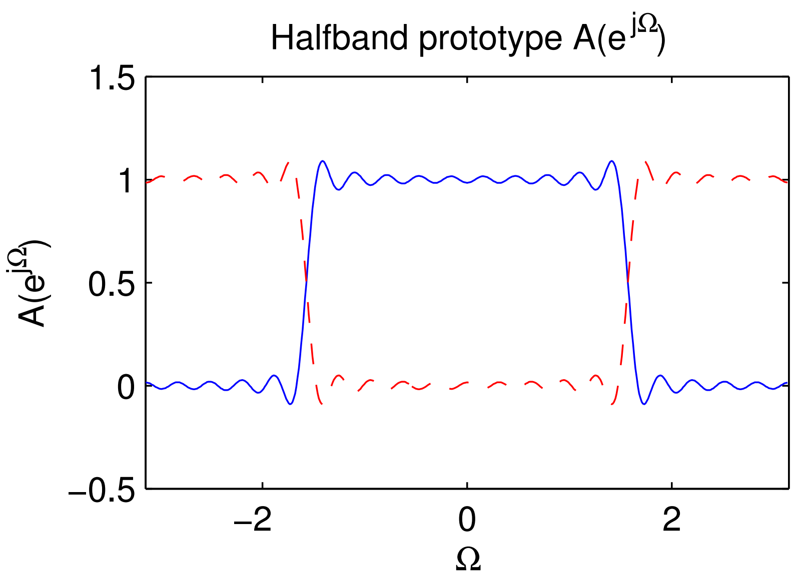
$$h[n] = h[N-n], \qquad h[n] = -h[N-n]$$

filter type	filter order N	symmetry	$H(e^{j\Omega})\big\|_{\Omega=0}$	$H(e^{j\Omega})\big\|_{\Omega=\pi}$
type 1	even	even	no restrictions	no restrictions
type 2	odd	even	no restrictions	$H(e^{j\pi}) = 0$
type 3	even	odd	$H(e^{j0}) = 0$	$H(e^{j\pi}) = 0$
type 4	odd	odd	$H(e^{j0}) = 0$	no restrictions

Linear phase filters have a constant group delay.

6.5.2 Halfband Filters (Nyquist Filters)





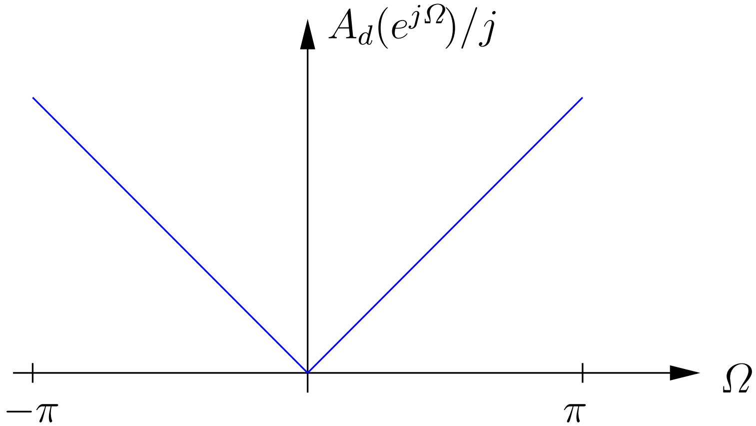
Zero-phase halfband lowpass filter:

- 6 dB cutoff frequency: $\Omega_{gr} = \pi/2$
- $\Omega_p = \pi - \Omega_s$
- $\delta_p = \delta_s$
- $A(e^{j\Omega}) - 0.5 = 0.5 - A(e^{j(\pi-\Omega)})$
- $A(e^{j\Omega}) + A(e^{j(\Omega-\pi)}) = 1$
- $a(2n) = 0$ for $n \neq 0$, $a(0) = 1/2$

Design:

- Windowing of an ideal lowpass filter with $\Omega_c = \pi/2$
- Parks-McClellan design with "trick"
- Cosine roll-off halfband filter

6.5.3 Differentiator



Ideal differentiator:

$$A_d(e^{j\Omega}) = j\Omega, \quad 0 \leq \omega \leq \pi$$

$$a_d[n] = \frac{\cos(\pi n)}{n}, \quad n \neq 0, \quad a_d(0) = 0$$

non causal, infinite length impulse response with odd symmetry

Approximation:

- Type 3 or type 4 FIR filter
- limitation of bandwidth
- Design with Parks-McClellan Algorithm
- Weighting of error $\sim W/f$

Energy and power of discrete-time signals:

Energy: $E_x = \sum_{n=-\infty}^{+\infty} |x[n]|^2$, Power: non-periodic $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$, N-periodic: $P_x = \frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2$

Discrete-time linear convolution: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

Discrete-time Fourier Transform (DTFT): $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$

DTFT Properties:

Symmetry relations:

Sequence	DTFT
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$
$j\,\mathrm{Im}\{x[n]\}$	$X_{as}(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$
$x_{cs}[n]$	$X_{re}(e^{j\omega})$
$x_{as}[n]$	$jX_{im}(e^{j\omega})$

Note: Subscript "*cs*" and "*as*" mean conjugate symmetric and conjugate anti-symmetric signals, respectively.

DTFT of commonly used sequences:

Sequence	DTFT
$\delta[n]$	1
$\mu[n]$ (Unit Step Function)	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega + 2\pi k)$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
$a^n \mu[n]$, ($\lvert a\rvert < 1$)	$\frac{1}{1-ae^{-j\omega}}$
$(n+1)a^n \mu[n]$, ($\lvert a\rvert < 1$)	$\left(\frac{1}{1-ae^{-j\omega}}\right)^2$
$h[n] = \frac{\sin(\omega_c n)}{\pi n}$, ($-\infty < n < +\infty$)	$H(e^{j\omega}) = \begin{cases} 1 & 0 \le \lvert\omega\rvert \le \omega_c \\ 0 & \omega_c < \lvert\omega\rvert < \pi \end{cases}$

DTFT Theorems:

Theorem	Sequence	DTFT
	$g[n]$ $h[n]$	$G(e^{j\omega})$ $H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time Reversal	$g[-n]$	$G(e^{-j\omega})$
Time Shifting	$g[n - n_0]$	$e^{-j\omega n_0} G(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n} g[n]$	$G(e^{j(\omega-\omega_0)})$
Differentiation in Frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] * h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{-\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$
Parseval's Identity	$\sum_{n=-\infty}^{+\infty} g[n]h^*[n] = \dfrac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\omega})H^*(e^{j\omega})d\omega$	

Discrete Fourier Transform (DFT): $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$, $\quad 0 \le k \le N-1$

Inverse DFT: $x[n] = \dfrac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk}$, $\quad 0 \le n \le N-1$

Circular Convolution: $y_c[n] = \sum_{m=0}^{N-1} x[m]h[\langle n-m\rangle_N]$, $\quad 0 \le n \le N-1$

Symmetry properties of DFT:

Length-N Sequence	N-Point DFT
$x[n] = x_{re}[n] + jx_{im}[n]$	$X[k] = X_{re}[k] + jX_{im}[k]$
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$x_{re}[n]$	$X_{cs}[k] = \frac{1}{2}\left[X[k] + X^*[\langle -k\rangle_N]\right]$
$jx_{im}[n]$	$X_{as}[k] = \frac{1}{2}\left[X[k] - X^*[\langle -k\rangle_N]\right]$
$x_{cs}[n]$	$X_{re}[k]$
$x_{as}[n]$	$jX_{im}[k]$

DFT Theorems:

Theorem	Sequence	DTFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular Time Shifting	$g[\langle n - n_0 \rangle_N]$	$e^{-j\frac{2\pi}{N}kn_0} G[k]$
Circular Frequency Shifting	$e^{j\frac{2\pi}{N}nk_0} g[n]$	$G[\langle k - k_0 \rangle_N]$
Duality	$G[n]$	$Ng[\langle -k \rangle_N]$
N-Point Circular Convolution	$\sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle n-m \rangle_N]$
Parseval's Identity	$\sum_{n=0}^{N-1} \lvert g[n] \rvert^2 = \frac{1}{N}\sum_{n=0}^{N-1} \lvert G[k] \rvert^2$	

Z-Transform: $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$ **Inverse Z-Transform:** $x[n] = \frac{1}{2\pi j}\oint_C G(z)z^{n-1}dz$

Z-Transform Properties:

Property	Sequence	z-Transform	Region of Convergence
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	R_g R_h
Conjugation	$g^*[n]$	$G^*(z^*)$	R_g
Time Reversal	$g[-n]$	$G(1/z)$	$1/R_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $R_g \cap R_h$
Time Shifting	$g[n - n_0]$	$z^{-n_0} G(z)$	R_g except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert R_g$
Differentiation in the z-domain	$ng[n]$	$-z\frac{dG(z)}{dz}$	R_g except possibly the point $z = 0$ or ∞
Convolution	$g[n] * h[n]$	$G(z)H(z)$	Includes $R_g \cap R_h$

TABLE 5.1 Select (Unilateral) z-Transform Pairs

No.	$x[n]$	$X[z]$
1	$\delta[n-k]$	z^{-k}
2	$u[n]$	$\dfrac{z}{z-1}$
3	$nu[n]$	$\dfrac{z}{(z-1)^2}$
4	$n^2u[n]$	$\dfrac{z(z+1)}{(z-1)^3}$
5	$n^3u[n]$	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^n u[n]$	$\dfrac{z}{z-\gamma}$
7	$\gamma^{n-1}u[n-1]$	$\dfrac{1}{z-\gamma}$
8	$n\gamma^n u[n]$	$\dfrac{\gamma z}{(z-\gamma)^2}$
9	$n^2\gamma^n u[n]$	$\dfrac{\gamma z(z+\gamma)}{(z-\gamma)^3}$
10	$\dfrac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$	$\dfrac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^n \cos\beta n\, u[n]$	$\dfrac{z(z-\lvert\gamma\rvert\cos\beta)}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^n \sin\beta n\, u[n]$	$\dfrac{z\lvert\gamma\rvert\sin\beta}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\dfrac{rz[z\cos\theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n] \qquad \gamma=\lvert\gamma\rvert e^{j\beta}$	$\dfrac{(0.5re^{j\theta})z}{z-\gamma}+\dfrac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$ $r=\sqrt{\dfrac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}}$ $\beta=\cos^{-1}\dfrac{-a}{\lvert\gamma\rvert}$ $\theta=\tan^{-1}\dfrac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$	$\dfrac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$