Relaxed LMI Conditions for Local Stability and Local Stabilization of Continuous-Time Takagi–Sugeno Fuzzy Systems

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***Abstract*—This paper addresses the problems of local stability analysis, local stabilization, and computation of invariant subsets of the domain of attraction for continuous-time Takagi–Sugeno fuzzy systems. Improvements on the existing stability and stabilization conditions are achieved through the reduction of conservatism. To release the conservatism further, the so-called multidimensional fuzzy summation approach is adopted. The design and analysis conditions are expressed as one-parameter minimization problems which can be solved via a sequence of linear matrix inequality optimizations or treated as eigenvalue problems. Examples are given to show the validity of the proposed method.**

***Index Terms*—Convex optimization, domain of attraction (DA), local stability, nonlinear systems, Takagi–Sugeno (T–S) fuzzy systems.**

I. Introduction

OVER THE PAST few decades, several modeling formalisms based on the fuzzy logic have been developed to describe nonlinear systems. Among them are Takagi–Sugeno (T–S) fuzzy models [1], fuzzy models using fuzzy basis function approximation [2], and fuzzy hyperbolic models [3]. In particular, significant research has been focused on stability analysis and control design of T–S fuzzy models, described as convex combinations of r linear subsystems weighted by the normalized membership functions (NMFs) $h_1(z(t))$, $h_2(z(t)), \ldots, h_r(z(t))$, because they make it simple to take advantage of conventional linear system theory (see [5] and references therein). Most of them have been investigated based on Lyapunov stability theory; the simplest method is the use of the common quadratic Lyapunov functions [4]–[9]. However, this approach sometimes leads to conservative results due to the common Lyapunov matrix that should be found for all subsystems of the T–S fuzzy model.

A great deal of research has been done to overcome the conservatism of the quadratic stability approach, and recent efforts can be roughly classified into three categories. First, many researchers have focused on the use of information on NMFs' shape to reduce conservatism and analyze stability. For instance, Lian *et al.* [10] established the dependence of the quadratic stability upon NMFs using Kharitonov's theorem. Bounds on the products of the NMFs were used in [11]; Narimani and Lam [12] considered bounds on the multiplication of NMFs for the partitioned operating region of NMFs; and Kruszewski *et al.* [13] employed simplicial partitions of the standard simplex of the NMFs to form a finer mesh. The second direction is to construct more general classes of Lyapunov functions; for instance, piecewise Lyapunov functions that are piecewise quadratic with respect to some partitions of the state space [14]–[17], fuzzy Lyapunov functions (FLFs) linearly dependent on the NMFs for the continuous-time systems [18]–[21] and discrete-time systems [22]–[29], a class of Lyapunov functions using line integral [30], [31], and polynomial Lyapunov functions whose dependence on the state variables are expressed as polynomial forms [32]–[37]. Finally, Pólya's theorem can be exploited to provide a set of less conservative sufficient conditions for proving positivity of the double fuzzy summations and can be generalized to the generic multidimensional or q-dimensional fuzzy summations [38]. Until now, the approach has been applied to construct the multidimensional FLFs that are homogeneous polynomial in the NMFs [39]–[43]. In most of the approaches mentioned previously, conditions to search for control laws and Lyapunov functions were stated in terms of linear matrix inequalities (LMIs) which can be solved via openly or commercially available software [54]–[56]. Recently, the use of sum-of-squares (SOS) techniques [57] has become important in dealing with stability of polynomial T–S fuzzy systems [36] along with polynomial FLFs.

As for the FLF approach, although the FLF scheme has proven to be successful in dealing with the discrete-time systems, it practically has a drawback when the continuous-time systems are under consideration; in this case, the time derivative of NMFs $\dot{h}_\rho(z(t))$, $\forall \rho \in \{1, 2, \ldots, r\} =: \mathcal{I}_r$ arising from the Lyapunov inequality contains additional nonlinear terms, which do not have any polytopic bound as the T–S fuzzy systems. To circumvent this obstacle, approaches in [18]–[21] assumed that the time derivative of NMFs is bounded as follows:

$$\dot{h}_\delta(z(t)) \in [-\phi_\delta, \phi_\delta], \quad \forall \delta \in \mathcal{I}_r \tag{1}$$

where $\phi_\delta > 0$ are *a priori* given real numbers.

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On the other hand, whenever exploiting assumption (1), we should take into account the fact that the region satisfying (1) is not the whole state space but some local area, because in general the time derivative of the NMFs is nonlinear in the state variables and is linearly dependent on the time derivative of the state. Therefore, when employing the FLFs, the local stability concept should be taken into account. Local stability ensures only that there exists a neighborhood of the equilibrium point, called the domain of attraction (DA) [53], such that all trajectories of the system emanating from any initial point in the DA asymptotically converges to the equilibrium point. Recently, the problems of assessing the local stability and estimating the DA have been addressed by many researchers [44]–[50] for continuous-time T–S fuzzy systems. Specifically, an LMI condition was developed for the first time in the pioneering work [44] by employing the FLFs and considering more structured bounds on the time derivative of the NMFs. Afterward, it was generalized in [45] by applying the convergent LMI relaxation technique developed in [38] and employing the multidimensional FLFs, and was further extended in [46]–[49] to deal with control problems. There, absolute values of partial derivatives of the NMFs were bounded by some constants to cast the local stabilization problem into LMIs. Besides these, local stability of continuous-time polynomial T–S fuzzy systems was studied in [32] via polynomial FLFs together with SOS programming.

In our prior work [50], published around the same time as [46]–[49], based on the FLFs and assumption (1), sufficient conditions for the local stability and local stabilization were formulated as eigenvalue problems (EVPs), which are convex optimizations [52]. There, in order to cast assumption (1) into LMI constraints, initially motivated by Tanaka *et al.* [18], we considered a polytopic-type bound on the gradient of the NMFs: a polytope such that the gradient of the NMFs varies within it. The bounding technique allowed us to cast constraint (1) into LMIs and to estimate the DA from the solution to the proposed conditions in a numerically efficient fashion. It is also very important to note that the basically same idea was also pursued in [46]–[49] with a superficial difference: the polytopic bound on the gradient of the NMFs versus the constant bounds on the absolute values of the partial derivative of the NMFs with respect to the state variables.

It is worth pointing out that, although the previous research works on the local stability and local stabilization problems have proven to be effective in reducing conservatism, there still exists a source of conservatism in the sense that they only considered FLFs consisting of single or double fuzzy summations, except for [45] which studied local stability of T–S fuzzy systems using the the multidimensional fuzzy summations. In addition, despite the maturity of the recently developed convergent relaxation techniques such as the multidimensional fuzzy summation approaches [38], [40], [45] and the homogeneous polynomial parameter-dependent matrix approaches [39], [41], [43], up to the authors' best knowledge, the explicit consideration of local stabilization and the DA estimate problems within this framework has not been fully investigated yet.

Motivated by the aforementioned discussion, in this paper, we develop less conservative sufficient conditions to assess the local stability, design locally stabilizing control laws, and estimate invariant subsets of the DA for continuous-time T–S fuzzy systems. To release the conservatism further, we take benefits of the multidimensional fuzzy summations approach [38] and rely on the use of the multidimensional FLFs as in [39], [40], and [45]. In addition, the slack variable approach developed by Mozelli *et al.* [20], [21] is applied with a small modification. For the main theorems, more rigorous proofs are given through a systematic study on the inclusion relations between several sets of the state variables. The proposed conditions are expressed as one-parameter minimization problems, which can be efficiently solved via a sequence of LMI optimizations such as a bisection algorithm or a line search process, or can be treated as EVPs [52], which are tractable via LMI solvers [54]–[56]. Finally, examples demonstrate that the proposed strategy achieves the least conservative results in terms of the volume of the DA estimate.

Notation: A^T: transpose of matrix A; $A \succ 0$ ($A \prec 0$, $A \succeq 0$, and $A \preceq 0$, respectively): symmetric positive definite (negative definite, positive semidefinite, and negative semidefinite, respectively) matrix A; I_n and $0_{n\times m}$: $n \times n$ identity matrix and $n \times m$ zero matrix, respectively; 0_n: origin of $\mathbb{R}^n$; He$\{A\}$: shorthand notion for $A^T + A$; co$\{\cdot\}$: convex hull [52]; $\emptyset$: empty set; ∂S: boundary of set S; $\nabla f(x)$: gradient of function $f : \mathbb{R}^n \to \mathbb{R}$ at point $x \in \mathbb{R}^n$; e_i: $n \times 1$ unit vector with a 1 in the ith component and 0's elsewhere; $*$ inside a matrix: transpose of its symmetric term; $\mathcal{I}_r := \{1, 2, \ldots, r\}$. Throughout this paper, the following shorthand is used for ease of notation: $\Upsilon_z := \sum_{i=1}^r h_i(z(t))\Upsilon_i$, $\Upsilon_{zz} := \sum_{i=1}^r \sum_{j=1}^r h_i(z(t))h_j(z(t))\ \Upsilon_{ij}$; given symmetric positive-definite matrix P, $\Omega(P, \gamma)$ designates set $\{x \in \mathbb{R}^n : x^T P x \leq \gamma\}$.

II. Preliminaries

A. Problem Formulation

We consider continuous-time nonlinear system $\dot{x}(t) = f(x(t), u(t))$, where $x(t) := [\ x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)\ ]^T \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function such that $f(0_n, 0_m) = 0_n$, i.e., the origin is an equilibrium point of the system. By the sector nonlinearity approach [5], the nonlinear system can be represented by the T–S fuzzy system

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)), \quad \forall x(t) \in \mathcal{L} \tag{2}$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ are constant matrices, $z(t) := [\ z_1(t) \quad z_2(t) \quad \cdots \quad z_p(t)\ ]^T \in \mathbb{R}^p$ is the vector containing premise variables in the fuzzy inference rule, $i \in \mathcal{I}_r := \{1, 2, \ldots, r\}$ is the rule number, $h_i(z(t))$ is the NMF for ith rule fulfilling the properties $0 \leq h_i(z(t)) \leq 1$, $\sum_{i=1}^r h_i(z(t)) = 1$, and $\mathcal{L} \subseteq \mathbb{R}^n$ is a set of state variables satisfying $\mathcal{L} \subseteq \{x(t) \in \mathbb{R}^n : f(x(t), u(t)) = A_z x(t) + B_z u(t)\}$. In this paper, it is assumed that

$$z(t) = [\ x_{a_1}(t) \quad x_{a_2}(t) \quad \cdots \quad x_{a_p}(t)\ ]^T \in \mathbb{R}^p$$

where $\{a_1, a_2, \ldots, a_p\} \subseteq \{1, 2, \ldots, n\}$ is a set of indexes, and set $\mathcal{L}$ is described as the hyper rectangle

$$\mathcal{L} := \{x(t) \in \mathbb{R}^n : x_{a_l}(t) \in [-\bar{x}_{a_l}, \bar{x}_{a_l}], \ l \in \mathcal{I}_p\} \tag{3}$$

where $\bar{x}_{a_l} > 0$, $l \in \mathcal{I}_p$ are *a priori* given real numbers. Without loss of generality, we will use both $h_i(z(t))$ and $h_i(x(t))$ interchangeably whenever necessary. The problems addressed in this paper are summarized as follows.

1) *Local stability:* Establish if the zero equilibrium point of (2) with $u(t) = 0_m$ (open-loop system) is locally asymptotically stable and estimate an invariant subset of the DA [53].
2) *Local stabilization:* Determine a control law $u(t)$ such that the zero equilibrium point of (2) is locally asymptotically stable and estimate an invariant subset of the DA.

B. Basic Definitions and Assumptions

In this paper, to take into account the time derivative of the NMFs when deriving LMI conditions, we will adopt assumption (1). In other words, vector

$$\bar{h}(z(t)) := [\ \dot{h}_1(z(t)) \quad \dot{h}_2(z(t)) \quad \cdots \quad \dot{h}_r(z(t))\ ]^T \in \mathbb{R}^r \tag{4}$$

is assumed to belong to the hyper rectangle

$$\mathcal{R} := \{\xi \in \mathbb{R}^r : [-\phi_1, \phi_1] \times [-\phi_2, \phi_2] \times \cdots \times [-\phi_r, \phi_r]\}$$

where $\phi_\delta > 0$, $\delta \in \mathcal{I}_r$ are *a priori* given real numbers, which can be represented by the convex set $\mathcal{R} := \mathrm{co}\{\xi_1, \xi_2, \ldots, \xi_{2^r}\}$, where ξ_i, $i \in \mathcal{I}_{2^r}$ are defined as the vertices of hyper rectangle $\mathcal{R}$. To facilitate the development, following the ideas of [46]–[50], the set of state variables satisfying assumption (1) is defined as follows:

$$\mathcal{H} := \{x(t) \in \mathcal{L} : \dot{h}_\delta(z(t)) \in [-\phi_\delta, \phi_\delta], \ \delta \in \mathcal{I}_r\}. \tag{5}$$

An important step to develop numerically efficient way to determine inner estimates of the DA is to cast constraint (1) into LMIs. To this end, we apply the assumption introduced in [18] and used later in [50]. The approach is based on the observation that the time derivative of NMFs $h_\delta(z(t))$ is expressed as the gradient of $h_\delta(z(t))$ (with respect to $x(t)$) multiplied by $\dot{x}(t)$

$$\frac{dh_\delta(z(t))}{dt} = \nabla h_\delta(x(t))\dot{x}(t), \quad \forall x(t) \in \mathcal{L}$$

where $\nabla h_\delta(\eta)$ denotes the gradient of $h_\delta(z(t))$ with respect to the state $x(t)$ at point $\eta \in \mathbb{R}^n$. Then, a polytopic bound on the gradient of the NMFs is taken into account to cast constraint (1) into LMIs. More precisely, we construct $1 \times n$ vectors $g_{(\delta,\rho)} \in \mathbb{R}^n$, $(\delta, \rho) \in \mathcal{I}_r \times \mathcal{I}_w$ such that

$$\nabla h_\delta(\eta) \in \mathrm{co}\{g_{(\delta,1)}, g_{(\delta,2)}, \ldots, g_{(\delta,w)}\}, \quad \forall \eta \in \mathcal{L}. \tag{6}$$

It is worth pointing out here that, as mentioned earlier, the use of bounds on the partial derivative of the NMFs to cast constraint (1) into LMIs was also pursued in promising research works [44]–[49] for the local stability and local stabilization problems, where absolute values of partial derivatives of the NMFs were bounded by some constants.

C. Multiindex Notation

To handle the q-dimensional fuzzy summations of matrices $\Upsilon_{(i_1, i_2, \ldots, i_q)}$, $(i_1, i_2, \ldots, i_q) \in \mathcal{I}_r^q$

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_q=1}^{r} h_{i_1}(z(t))h_{i_2}(z(t))\cdots h_{i_q}(z(t))\Upsilon_{(i_1, i_2, \ldots, i_q)}$$

the following notation, from [38], will be used. $\mathbf{i}_q := (i_1, i_2, \ldots, i_q)$; $\mathbf{j}_q := (j_1, j_2, \ldots, j_q)$; $\mathbb{I}_q := \{1, 2, \ldots, r\}^q = \mathcal{I}_r^q$; $\mathbb{I}_q^+ := \{\mathbf{i} \in \mathbb{I}_q : i_1 \le i_2 \le \cdots \le i_q\}$; $h_{\mathbf{i}_q} := h_{i_1}(z(t))h_{i_2}(z(t))\cdots h_{i_q}(z(t))$; $\Upsilon_{z^q} := \sum_{\mathbf{i}_q \in \mathbb{I}_q} h_{\mathbf{i}_q}\Upsilon_{\mathbf{i}_q}$; $\mathcal{P}(\mathbf{i}_q)$: set of permutations of a given multiindex $\mathbf{i}_q \in \mathbb{I}_q$; $\Upsilon_{z^q}^{-1} := (\sum_{\mathbf{i}_q \in I_q} h_{\mathbf{i}_q}\Upsilon_{\mathbf{i}_q})^{-1}$. Finally, for $\mathbf{i}_{q-1} \in \mathbb{I}_{q-1}$, $\mathbf{i}_{q-1}(k, l) \in \mathbb{I}_q$, $(k, l) \in \mathcal{I}_q \times \mathcal{I}_r$ is defined as the q-dimensional index resulting from inserting l before kth index or after $(k-1)$th index of $\mathbf{i}_{q-1}$. For instance, for $r = 4$ and $\mathbf{i}_3 = \{1, 2, 3\}$, $\mathbf{i}_3(1, 4) = \{4, 1, 2, 3\}$, $\mathbf{i}_3(2, 4) = \{1, 4, 2, 3\}$, $\mathbf{i}_3(3, 4) = \{1, 2, 4, 3\}$, and $\mathbf{i}_3(4, 4) = \{1, 2, 3, 4\}$.

Throughout this paper, the following lemma will be used to prove the positivity of the q-dimensional fuzzy summations of symmetric matrices:

Lemma 1 ([38]): Given symmetric matrices $\Upsilon_{\mathbf{j}_q}$, $\mathbf{j}_q \in \mathbb{I}_q$, $\Upsilon_{z^q} \prec 0$ holds for all $x(t) \in \mathcal{L}$ if $\sum_{\mathbf{j}_q \in \mathcal{P}(\mathbf{i}_q)} \Upsilon_{\mathbf{j}_q} \prec 0$ is fulfilled for all $\mathbf{i}_q \in \mathbb{I}_q^+$.

III. Relaxed Stability Condition

In this section, we shall be looking for an inner estimate of the DA by finding Lyapunov functions whose sublevel sets are invariant subsets of the DA. To this end, let $V(x(t)) := x(t)^T P_{z^q} x(t)$, $P_{z^q} \succ 0$, $\forall x(t) \in \mathcal{L}$ be a candidate of the FLF and define γ-sublevel set of $V(x(t))$ as $\Omega(P_{z^q}, \gamma) := \{x(t) \in \mathcal{L} : V(x(t)) = x(t)^T P_{z^q} x(t) \le \gamma\}$. The problems addressed in this section can be stated as follows.

1) Establish if the zero equilibrium point of (2) with $u(t) = 0_m$ (open-loop system) is locally asymptotically stable.
2) Search for Lyapunov function $V(x(t))$ for the zero equilibrium point such that $\Omega(P_{z^q}, 1)$ is an invariant subset of the DA [53].
3) Enlarge sublevel set $\Omega(P_{z^q}, 1)$.

Remark 1: It is important to note that although $V(x(t)) \le \gamma$ holds for all $x(t) \in \Omega(P_{z^q}, \gamma)$ by definition, this does not imply that $V(x(t)) = \gamma$, $\forall x(t) \in \partial\Omega(P_{z^q}, \gamma)$ is fulfilled, where $\partial\Omega(P_{z^q}, \gamma)$ is the boundary of $\Omega(P_{z^q}, \gamma)$. Hence, in order to use $\Omega(P_{z^q}, \gamma)$ as an invariant subset of the DA, it is important to ensure that $\{x(t) \in \mathcal{L} : V(x(t)) = \gamma\}$ is topologically equivalent to a circle, that is, $V(x(t)) = \gamma$, $\forall x(t) \in \partial\Omega(P_{z^q}, \gamma)$. One way to do so is to impose the constraint $\partial\Omega(P_{z^q}, \gamma) \cap \partial\mathcal{L} = \emptyset$, because

$$\begin{aligned} & V(x(t)) = \gamma, \quad \forall x(t) \in \partial\Omega(P_{z^q}, \gamma) \\ & \Leftrightarrow \partial\Omega(P_{z^q}, \gamma) = \{x(t) \in \mathcal{L} : V(x(t)) = \gamma\} \end{aligned}$$

$$\Leftrightarrow \partial\Omega(P_{z^q},\ \gamma) \cap \partial\mathcal{L} \subseteq \{x(t) \in \mathcal{L} :\ V(x(t)) = \gamma\}$$
$$\Leftarrow \partial\Omega(P_{z^q},\ \gamma) \cap \partial\mathcal{L} = \emptyset.$$

In the sequel, we provide a sufficient local stability condition, from whose solution the DA is determined, and hence, additional numerical computations to estimate the DA are not required.

Theorem 1: If there exist symmetric matrices $P_{\mathbf{j}_q} \in \mathbb{R}^{n\times n}$, $X_{\mathbf{j}_q} \in \mathbb{R}^{n\times n}$, matrices $Y_{\mathbf{j}_q} \in \mathbb{R}^{2n\times n}$, and a real number $\beta > 0$ such that

$$\min_{P_{\mathbf{j}_q},\, X_{\mathbf{j}_q},\, Y_{\mathbf{j}_q},\, \beta} \beta \text{ subject to}$$
$$\sum\nolimits_{\mathbf{j}_q\in\mathcal{P}(\mathbf{i}_q)} \Upsilon^{(1)}_{(l,\,\mathbf{j}_q)} \prec 0, \quad \forall(\mathbf{i}_q,\ l) \in \mathbb{I}^+_q \times \mathcal{I}_p, \tag{7}$$
$$\sum\nolimits_{\mathbf{j}_{q+1}\in\mathcal{P}(\mathbf{i}_{q+1})} \Upsilon^{(2)}_{\mathbf{j}_{q+1}}(y) \prec 0$$
$$\forall(\mathbf{i}_{q+1},\ y) \in \mathbb{I}^+_{q+1} \times \{\xi_1,\ \xi_2, \ldots,\ \xi_{2^r}\} \tag{8}$$
$$\sum\nolimits_{\mathbf{j}_q\in\mathcal{P}(\mathbf{i}_q)} \Upsilon^{(3)}_{(\delta,\,\rho,\,\mathbf{j}_q)} \prec 0, \quad \forall(\delta,\ \rho,\ \mathbf{i}_q) \in \mathcal{I}_r \times \mathcal{I}_w \times \mathbb{I}^+_q \tag{9}$$
$$\sum\nolimits_{\mathbf{j}_q\in\mathcal{P}(\mathbf{i}_q)} \Upsilon^{(4)}_{\mathbf{j}_q} \prec 0, \quad \forall\mathbf{i}_q \in \mathbb{I}^+_q \tag{10}$$

where for any integer i, e_i denotes $n \times 1$ unit vector with a 1 in the ith component and 0's elsewhere

$$\Upsilon^{(1)}_{(l,\,\mathbf{j}_q)} := \bar{x}_{a_l}^{-2} e_{a_l} e_{a_l}^T - P_{\mathbf{j}_q} \in \mathbb{R}^{n\times n}$$
$$\Upsilon^{(2)}_{\mathbf{j}_{q+1}}(y) := \begin{bmatrix} \sum_{\delta=1}^r y_\delta (\sum_{k=1}^q P_{\mathbf{j}_{q-1}(k,\,\delta)} + X_{\mathbf{j}_q}) & * \\ P_{\mathbf{j}_q} & 0_{n\times n} \end{bmatrix}$$
$$+ \mathrm{He}\{Y_{\mathbf{j}_q} [\ A_{j_{q+1}} \quad -I_n\ ]\} \in \mathbb{R}^{2n\times 2n}$$
$$\Upsilon^{(3)}_{(\delta,\,\rho,\,\mathbf{j}_q)} := \begin{bmatrix} -P_{\mathbf{j}_q} & * \\ g_{(\delta,\,\rho)} A_{j_1} & -\phi_\delta^2 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$
$$\Upsilon^{(4)}_{\mathbf{j}_q} := P_{\mathbf{j}_q} - \beta I_n \in \mathbb{R}^{n\times n}$$

and $\bar{x}_{a_l} > 0$, $l \in \mathcal{I}_p$ are *a priori* given real numbers representing the vertices of set $\mathcal{L}$ defined in (3); then, (2) with $u(t) = 0_m$ is locally asymptotically stable. Moreover, an invariant subset of the DA for the system is given by $\Omega(P_{z^q},\ 1)$.

Proof: By Lemma 1, LMIs (7)–(10) guarantee

$$\Upsilon^{(1)}_{(l,\,z^q)} = \bar{x}_{a_l}^{-2}(e_{a_l} e_{a_l}^T) - P_{z^q} \prec 0, \quad \forall(x(t),\ l) \in \mathcal{L} \times \mathcal{I}_p \tag{11}$$
$$\Upsilon^{(2)}_{z^{q+1}}(y) = \begin{bmatrix} \sum_{\delta=1}^r y_\delta \bar{P}_{(z^q,\,\delta)} & * \\ P_{z^q} & 0_{n\times n} \end{bmatrix}$$
$$+ \mathrm{He}\{Y_{z^q} [\ A_z \quad -I_n\ ]\} \prec 0$$
$$\forall(x(t),\ y) \in \mathcal{L} \times \{\xi_1,\ \xi_2, \ldots,\ \xi_{2^r}\} \tag{12}$$
$$\Upsilon^{(3)}_{(\delta,\,\rho,\,z^q)} = \begin{bmatrix} -P_{z^q} & * \\ g_{(\delta,\,\rho)} A_z & -\phi_\delta^2 \end{bmatrix} \prec 0$$
$$\forall(x(t),\ \delta,\ \rho) \in \mathcal{L} \times \mathcal{I}_r \times \mathcal{I}_w \tag{13}$$
$$\Upsilon^{(4)}_{z^q} = P_{z^q} - \beta I_n \prec 0, \quad \forall x(t) \in \mathcal{L} \tag{14}$$

where $\bar{P}_{(z^q,\,\delta)} := \sum_{\mathbf{j}_{q-1}\in\mathbb{I}_{q-1}} h_{\mathbf{j}_{q-1}} (\sum_{k=1}^q P_{\mathbf{j}_{q-1}(k,\,\delta)}) + X_{z^q}$. First of all, we prove that $\partial\Omega(P_{z^q},\ 1) \cap \partial\mathcal{L} = \emptyset$. Multiplying (11) by $x(t)^T$ on the left and $x(t)$ on the right, one gets

$$\bar{x}_{a_l}^{-2} x_{a_l}(t)^2 < V(x(t)), \quad \forall(x(t),\ l) \in \mathcal{L}\backslash\{0_n\} \times \mathcal{I}_p$$

which implies $1 < V(x(t))$, $\forall x(t) \in \partial\mathcal{L}$, where $\partial\mathcal{L}$ is the boundary of $\mathcal{L}$. This means $\partial\Omega(P_{z^q},\ 1) \cap \partial\mathcal{L} = \emptyset$. Since $P_{z^q} \succ 0$, $\forall x(t) \in \mathcal{L}$ from (13), we have that $V(x(t)) = 1$, $\forall x(t) \in \partial\Omega(P_{z^q},\ 1)$.

On the other hand, (13) implies

$$\Psi := \begin{bmatrix} -P_{z^q} & * \\ \nabla h_\delta(x(t)) A_z & -\phi_\delta^2 \end{bmatrix} \prec 0, \quad \forall(x(t),\ \delta) \in \mathcal{L} \times \mathcal{I}_r. \tag{15}$$

With

$$\xi := \begin{bmatrix} I_n \\ \phi_\delta^{-2} \nabla h_\delta(x(t)) A_z \end{bmatrix} x(t) \in \mathbb{R}^{n+1}$$

it follows from (15) that

$$\begin{aligned} \xi^T \Psi \xi =& \phi_\delta^{-2} x(t)^T A_z^T \nabla h_\delta(x(t))^T \nabla h_\delta(x(t)) A_z x(t) - V(x(t)) \\ =& \phi_\delta^{-2} \dot{h}_\delta(z(t))^2 - V(x(t)) \\ <& 0, \quad \forall(x(t),\ \delta) \in \mathcal{L}\backslash\{0_n\} \times \mathcal{I}_r \\ \Leftrightarrow& \phi_\delta^{-2} \dot{h}_\delta(z(t))^2 - 1 < V(x(t)) - 1 \\ & \forall(x(t),\ \delta) \in \mathcal{L}\backslash\{0_n\} \times \mathcal{I}_r \\ \Leftrightarrow& \dot{h}_\delta(z(t))^2 - \phi_\delta^2 < \phi_\delta^2 (V(x(t)) - 1) \\ & \forall(x(t),\ \delta) \in \mathcal{L}\backslash\{0_n\} \times \mathcal{I}_r. \end{aligned}$$

Since $\Omega(P_{z^q},\ 1) \subset \mathcal{L}$, the aforementioned inequality implies

$$\begin{aligned} \Omega(P_{z^q},\ 1) \subset& \bigcap\nolimits_{\delta=1}^r \{x(t) \in \mathcal{L} :\ \dot{h}_\delta(z(t)) \in [-\phi_\delta,\ \phi_\delta]\} \\ \Leftrightarrow& \Omega(P_{z^q},\ 1) \subset \mathcal{H}. \end{aligned} \tag{16}$$

Next, we show that $\Omega(P_{z^q},\ 1)$ is an invariant subset of the DA. Inequality (11) implies $P_{z^q} \succ 0$, $\forall x(t) \in \mathcal{L}$. Thus, $V(x(t)) > 0$, $\forall x(t) \in \mathcal{L}\backslash\{0_n\}$ holds. On the other hand, using relation $\sum_{\delta=1}^r \dot{h}_\delta(z(t)) = 0$, it is easy to see by direct calculation that (12) implies

$$\begin{bmatrix} \sum\limits_{\delta=1}^r \dot{h}_\delta(z(t)) \sum\limits_{\mathbf{j}_{q-1}\in\mathbb{I}_{q-1}} h_{\mathbf{j}_{q-1}} \sum\limits_{k=1}^q P_{\mathbf{j}_{q-1}(k,\,\delta)} & * \\ P_{z^q} & 0_{n\times n} \end{bmatrix}$$
$$+ \mathrm{He}\{Y_{z^q} [\ A_z \quad -I_n\ ]\}$$
$$= \begin{bmatrix} \dot{P}_{z^q} & * \\ P_{z^q} & 0_{n\times n} \end{bmatrix} + \mathrm{He}\{Y_{z^q} [\ A_z \quad -I_n\ ]\}$$
$$\prec 0, \quad \forall x(t) \in \mathcal{H}.$$

Multiplying the aforementioned inequality by $x(t)^T [\ I_n \quad A_z^T\ ]$ on the left and its transpose on the right, we have

$$x(t)^T (A_z^T P_{z^q} + P_{z^q} A_z + \dot{P}_{z^q}) x(t) < 0$$
$$\forall x(k) \in \mathcal{H}\backslash\{0_n\}$$

from which one concludes $\mathcal{H}\backslash\{0_n\} \subseteq \{x(t) \in \mathcal{L} :\ \dot{V}(x(k)) < 0\}$. Together with (16), this means that $\Omega(P_{z^q},\ 1)\backslash\{0_n\} \subset \{x(t) \in \mathcal{L} :\ \dot{V}(x(t)) < 0\}$. By the Lyapunov argument, (2) with $u(t) = 0_m$ is locally asymptotically stable, and $\Omega(P_{z^q},\ 1)$ is an invariant subset of the DA [53]. Finally, (14) ensures

$$V(x(t)) < \beta x(t)^T x(t), \quad \forall x(t) \in \mathcal{L}\backslash\{0_n\}$$
$$\Leftrightarrow V(x(t)) - 1 < \beta(x(t)^T x(t) - 1/\beta), \quad \forall x(t) \in \mathcal{L}\backslash\{0_n\}$$

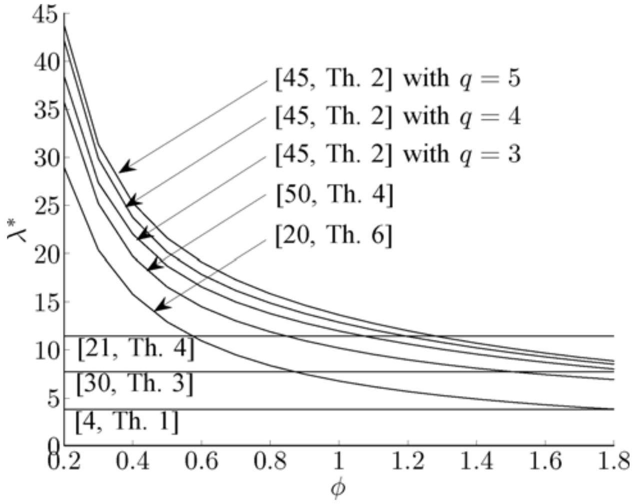

Fig. 1. Example 1. Stability bounds λ^* for different ϕ.

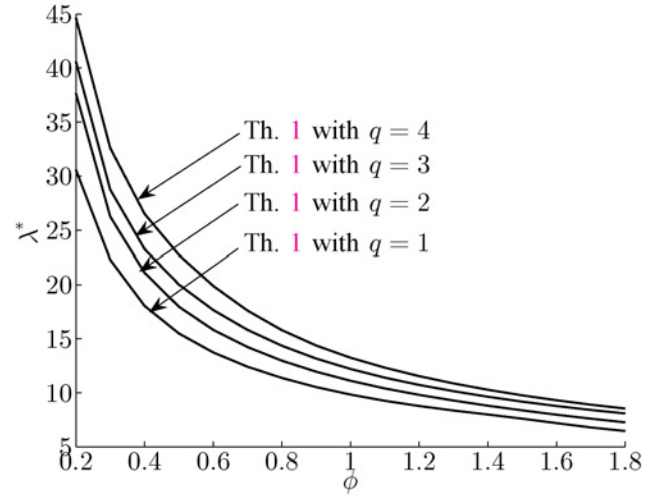

Fig. 2. Example 1. Stability bounds λ^* for different ϕ.

which implies $\{x(t) \in \mathcal{L} : x(t)^T x(t) \leq 1/\beta\} \subset \Omega(P_{z^q}, 1)$. Therefore, minimizing β while imposing constraint $\{x(t) \in \mathcal{L} : x(t)^T x(t) \leq 1/\beta\} \subset \Omega(P_{z^q}, 1)$ makes $\Omega(P_{z^q}, 1)$ to be enlarged. This completes the proof. ■

Remark 2: With a simple modification, we employ the slack variable technique developed in [20, Th. 6] to use relation $\sum_{\delta=1}^{r} \dot{h}_\delta(z(t)) = 0$ in Theorem 1. Specifically, LMIs $P_i + X \succeq 0,\ i \in \mathcal{I}_r$ in [20, Th. 6] is not considered in Theorem 1. Instead, we directly use the property $\sum_{\delta=1}^{r} \dot{h}_\delta(z(t)) X_{z^q} = 0_{n\times n}$ by checking the Lyapunov inequality including the time derivative of the NMFs for all vertices $\xi_i,\ i \in \mathcal{I}_{2^r}$ of $\bar{h}(z(t))$ defined in (4). It can be easily seen that this modified approach is not more conservative than that of [20, Th. 6].

Remark 3: The optimization problem of Theorem 1 is a single-parameter minimization problem, and for fixed β, conditions (7)–(10) are LMIs tractable via LMI solvers [54]–[56]. Thus, the optimization problem can be solved by means of a sequence of LMI optimizations, i.e., a line search or a bisection process over β. Moreover, the optimization problem belongs to the class of EVPs, which are convex optimizations [52], and hence, can be directly treated with the help of the LMI solvers.

Remark 4: The numerical complexity of optimization problems based on LMIs can be estimated from the total number N_D of scalar decision variables and the total row size N_L of the LMIs [52], [54]. For Theorem 1, $N_D = (n^2+n)r^q + 2n^2r^q + 1$ and $N_L = (np+(n+1)rw+n)(r+q-1)!/(q!(r-1)!) + 2^{r+1}n(r+q)!/((q+1)!(r-1)!)$.

Remark 5: One may easily conjecture that the smaller the bounds $\phi_\delta,\ \forall\delta \in \mathcal{I}_r$, the less conservative the condition of Theorem 1. Moreover, it can be easily verified that region $\mathcal{H}$ corresponding to $\phi_\delta = \hat{\phi}_\delta,\ \forall\delta \in \mathcal{I}_r$ always includes regions $\mathcal{H}$ corresponding to $\phi_\delta \leq \hat{\phi}_\delta,\ \delta \in \mathcal{I}_r$. In this sense, we can expect that the larger the bounds $\phi_\delta,\ \delta \in \mathcal{I}_r$ are, the larger the estimated DA is. This is because the estimation of the DA is confined to be within region $\mathcal{H}$ as shown in the proof of Theorem 1. Therefore, there is a tradeoff between conservative results with larger estimation of the DA and less conservative ones with the smaller estimation.

Remark 6: One remaining issue is how to choose bounds $\phi_\delta,\ \forall\delta \in \mathcal{I}_r$ in order to find a larger estimation of the DA using Theorem 1. In general, the task of choosing the bounds that lead to less conservative estimates of the DA is not trivial. At the present stage, there may be no formal and optimal algorithmic procedure to determine them, and they should be determined by a combination of previous expertise and trial and error. However, if we simplify this problem by setting $\phi_\delta = \phi,\ \forall\delta \in \mathcal{I}_r$, it can boil down to a single-parameter search problem. Since we know that the smaller bounds $\phi_\delta,\ \forall\delta \in \mathcal{I}_r$ lead to a less conservative stability condition with the DA estimation confined to a smaller region in the state space (as mentioned in the previous remark), we can set a sufficiently small positive number ϕ as an initial value of the search procedure, which once again boils down to a single-parameter maximization problem; with $\phi_\delta = \phi,\ \forall\delta \in \mathcal{I}_r$, we should compute the largest ϕ, denoted by ϕ^*, such that the condition of Theorem 1 admits a feasible solution for all $\phi \in [0, \phi^*]$. To compute ϕ^*, we can simply adopt a line search or bisection algorithm maximizing $\phi_\delta = \phi,\ \forall\delta \in \mathcal{I}_r$ subject to the condition of Theorem 1. Moreover, with some random initial values of $\phi_\delta,\ \forall\delta \in \mathcal{I}_r$, a locally convergent solution to the tuning problem can be obtained with the help of a numerical optimization algorithm, such as the program `fminsearch` in the optimization toolbox [51]. In this case, β can be selected as the objective of minimization.

All numerical experiments in the sequel were treated with the help of MATLAB 2006a running on a PC with dual Pentium 4 3.0 GHz CPU, 3 GB RAM. The LMI problems were solved with SeDuMi [55] and Yalmip [56].

Example 1: Let us consider (2) with $u(t) = 0_m$ (open-loop system) and

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -(2+\lambda) & -1 \end{bmatrix} \tag{17}$$

taken from [20] with $h_1(z(t)) = (1 + \sin x_1(t))/2$, $h_2(z(t)) = 1 - h_1(z(t))$ defined in $\mathcal{L} = \left\{x(t) \in \mathbb{R}^n : x_1(t) \in [-\pi/2, \pi/2]\right\}$, and premise variable $z(t) = z_1(t) = x_1(t)$. Since

TABLE I
EXAMPLE 1: NUMERICAL COMPLEXITY (N_D TOTAL NUMBER OF SCALAR DECISION VARIABLES; N_L TOTAL ROW SIZE OF CORRESPONDING LMI; TIME IN SECONDS)

Method	N_D	N_L	Time (s)
[4, Th. 1]	3	6	0.1391
[30, Th. 3]	7	12	0.0204
[20, Th. 6]	9	14	0.0171
[21, Th. 4]	12	12	0.0243
[50, Th. 4]	43	90	0.1686
[45, Th. 2] with $q = 3$	120	112	0.0376
[45, Th. 2] with $q = 4$	240	224	0.0805
[45, Th. 2] with $q = 5$	480	448	0.2125
Th. 1 with $q = 1$	29	80	0.1453
Th. 1 with $q = 2$	57	112	0.1899
Th. 1 with $q = 3$	113	144	0.2282
Th. 1 with $q = 4$	225	176	0.3296

$$\begin{aligned}\nabla h_1(x(t)) =& [\ 0.5\cos x_1(t) \quad 0\ ]\\ &\in \mathrm{co}\{[\ 0 \quad 0\ ], [\ 0.5 \quad 0\ ]\}\\ \nabla h_2(x(t)) =& -[\ 0.5\cos x_1(t) \quad 0\ ]\\ &\in \mathrm{co}\{[\ -0.5 \quad 0\ ], [\ 0 \quad 0\ ]\}\end{aligned}$$

for all $x(t) \in \mathcal{L}$, we have

$$\begin{cases} g_{(1,1)} = [\,0 \quad 0\,], & g_{(1,2)} = [\,0.5 \quad 0\,] \\ g_{(2,1)} = [\,-0.5 \quad 0\,], & g_{(2,2)} = [\,0 \quad 0\,]. \end{cases}$$

To compare Theorem 1 (Th. 1) with existing ones in terms of conservatism, the maximum values of λ, denoted by λ^*, such that asymptotic stability of the system is guaranteed for all $\lambda \in [0, \lambda^*]$ were computed for different $\phi_1 = \phi_2 = \phi$. Results of [4, Th. 1], [30, Th. 3], [21, Th. 4], [20, Th. 6], [50, Th. 4], [45, Th. 2] are depicted in Fig. 1, while the results of Theorem 1 are plotted separately in Fig. 2 to prevent confusion. In the figures, regions under each line are a set of pair (λ, ϕ) such that the system can be identified as stable by the corresponding approach. Note that assumption (1) with $\phi_1 = \phi_2 = \phi$ was used for [20, Th. 6], [21, Th. 4], [50, Th. 4], and Theorem 1, while assumption $|(\partial h_1(z(t))/\partial x_1(t))x_1(t)| \le \phi$ and $|(\partial h_1(z(t))/\partial x_1(t))x_2(t)| \le \phi$ was utilized for [45, Th. 4]. Fig. 2 shows that the larger the integer q or the smaller the real number ϕ, the less conservative the condition of Theorem 1. Moreover, by comparing Figs. 1 and 2, it can be observed that the condition of Theorem 1 outperforms the previous approaches, except for the condition of [45, Th. 2], which produces results similar to those of Theorem 1.

In addition, Table I lists the numerical complexity of several approaches in terms of N_D, N_L, and the average computational time (in seconds) spent by each test to provide a feasible solution with (17) and $\lambda = 1$. Here, $\phi = 0.3$ was used for [20, Th. 6], [21, Th. 4], [50, Th. 4], [45, Th. 2], and Theorem 1, and the average computational time for each test was obtained by taking the average of 20 measures. From the table, one concludes that Theorem 1 requires the highest computational burden among previous approaches except for those in [45] which also produce similar results in terms of the numerical complexity. Thus, one may say that the advantage of Theorem 1 comes at the price of a higher computational effort.

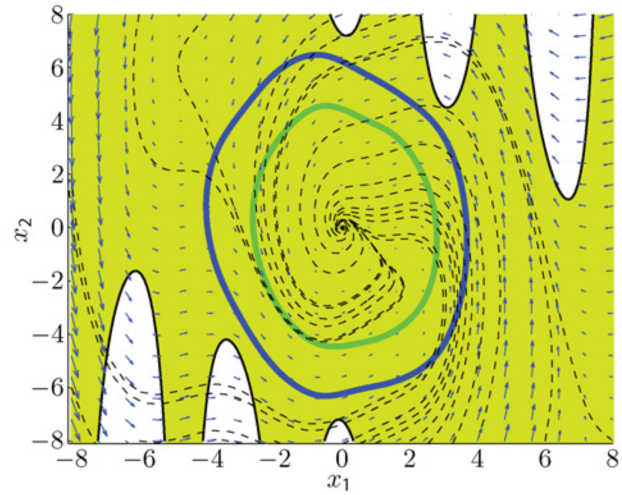

Fig. 3. Example 2. Trajectories (dashed lines), level set $\Omega(P_{z^q}, 1)$ (interior of the outer circular solid line) estimated by using Theorem 1 with $q = 3$ and $\phi = \phi^* = 14.4868$, the DA estimation (interior of the inner circular solid line) obtained by using [45, Th. 2] with $q = 5$, and the region of $\mathcal{H} := \{x(t) \in \mathcal{L} : \dot{h}_\delta(z(t)) \in [-14.4868, 14.4868], \delta \in \mathcal{I}_r\}$ (shaded area).

Example 2: Let us consider (2) with

$$\begin{aligned} &A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}\\ &h_1(z(t)) = (1 + \sin x_1(t))/2, \quad h_2(z(t)) = 1 - h_1(z(t))\\ &z(t) = z_1(t) = x_1(t)\end{aligned}$$

borrowed from [19]. Since the NMFs can be defined for any $x(t) \in \mathbb{R}^n$, $\mathcal{L}$ can be chosen to be arbitrarily large in $\mathbb{R}^n$. Thus, we can set $\mathcal{L} = \{x(t) \in \mathbb{R}^n : x_1(t) \in [-\bar{x}_1, \bar{x}_1]\}$ with $\bar{x}_1 = 8$. In this case, since

$$\begin{aligned}\nabla h_1(x(t)) =& [\,0.5\cos x_1(t) \quad 0\,]\\ &\in \mathrm{co}\{[\,0.5\min_{x(t)\in\mathcal{L}}(\cos x_1(t)) \quad 0\,],\\ &\qquad [\,0.5\max_{x(t)\in\mathcal{L}}(\cos x_1(t)) \quad 0\,]\}\\ \nabla h_2(x(t)) =& -[\,0.5\cos x_1(t) \quad 0\,]\\ &\in \mathrm{co}\{-[\,0.5\min_{x(t)\in\mathcal{L}}(\cos x_1(t)) \quad 0\,],\\ &\qquad -[\,0.5\max_{x(t)\in\mathcal{L}}(\cos x_1(t)) \quad 0\,]\}\end{aligned}$$

hold for all $x(t) \in \mathcal{L}$, vectors $g(\delta, \rho)$, $(\delta, \rho) \in \mathcal{I}_r \times \mathcal{I}_w$ defined in (6) can be calculated to be

$$\begin{aligned} g_{(1,1)} &= [\,-0.5 \quad 0\,], & g_{(1,2)} &= [\,0.5 \quad 0\,],\\ g_{(2,1)} &= [\,0.5 \quad 0\,], & g_{(2,2)} &= [\,-0.5 \quad 0\,].\end{aligned}$$

For the system, the maximum value of $\phi_1 = \phi_1 = \phi$, denoted by ϕ^*, such that the condition of Theorem 1 with $q = 3$ admits a feasible solution for all $\phi_1 = \phi_1 = \phi \in [0, \phi^*]$ was calculated to be $\phi^* = 14.4868$ by a bisection algorithm. Fig. 3 shows trajectories (dashed lines) and the level set $\Omega(P_{z^q}, 1)$ (interior of the outer circular solid line) estimated by using Theorem 1 with $q = 3$ and $\phi = \phi^* = 14.4868$. In addition, the interior of the inner circular solid line is the DA estimation computed by [45, Th. 2] with $q = 5$, $|(\partial h_1(z(t))/\partial x_1(t))x_1(t)| \le 2.23$, $|(\partial h_1(z(t))/\partial x_1(t))x_2(t)| \le$

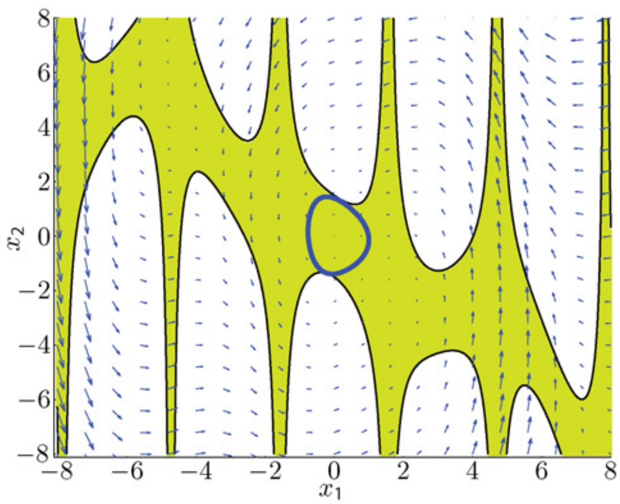

Fig. 4. Example 2. Lyapunov level set (interior of the circular solid line) estimated by using [41, Th. 1] with $(g, d) = (2, 2)$, $E_{\mathbf{i}}^{i} = 0_{2n\times 2n}$, $\phi_1 = \phi_2 = \phi = 2.9907$, and the region of $\mathcal{H} := \{x(t) \in \mathcal{L} : \dot{h}_\delta(z(t)) \in [-2.9907, 2.9907], \delta \in \mathcal{I}_r\}$ (shaded area).

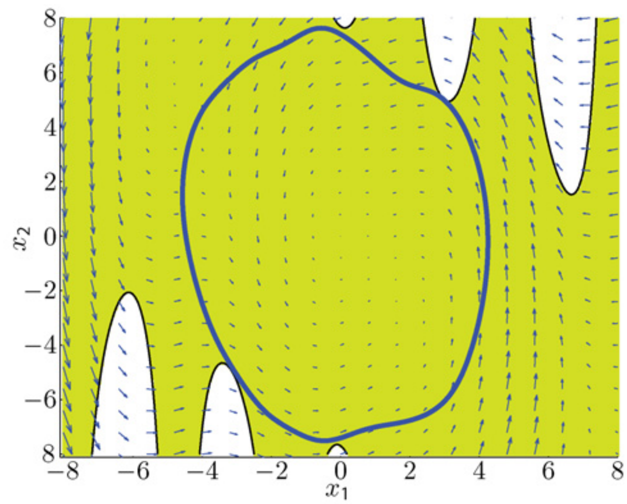

Fig. 5. Example 2. Lyapunov level set (interior of the circular solid line) estimated by using [43, Th. 1] with $g_1 = 4$, $\phi_1 = \phi_2 = \phi^* = 15.3687$, and the region of $\mathcal{H} := \{x(t) \in \mathcal{L} : \dot{h}_\delta(z(t)) \in [-15.3687, 15.3687], \delta \in \mathcal{I}_r\}$ (shaded area).

2.23, and the shaded area is the region of $\mathcal{H} := \{x(t) \in \mathcal{L} : \dot{h}_\delta(z(t)) \in [-14.4868, 14.4868], \delta \in \mathcal{I}_r\}$. From the results, we conclude that the DA estimated by the proposed approach includes that in [45, Th. 2]. For further comparison, we also applied the following approaches.

1) *Case 1:* [41, Th. 1] with $(g, d) = (2, 2)$, $E_{\mathbf{i}}^{i} = 0_{2n\times 2n}$, and $\phi_1 = \phi_2 = \phi$, where the maximum ϕ, denoted by ϕ^*, such that the condition admits a feasible solution for all $\phi \in [0, \phi^*]$ is searched via a bisection algorithm over the interval $[0, 100]$.
2) *Case 2:* [43, Th. 1] with $g_1 = 4$, $\phi_1 = \phi_2 = \phi$, where the maximum ϕ, denoted by ϕ^*, such that the condition admits a feasible solution for all $\phi \in [0, \phi^*]$ is searched via a bisection algorithm over the interval $[0, 100]$.

Case 1 with $\phi^* = 2.9907$ and Case 2 with $\phi^* = 15.3687$ admitted a feasible solution. However, these approaches do not deal with the problem of DA estimation, so a direct comparison of the proposed approach with them in terms of the volume of the DA estimation may be somewhat difficult. Nevertheless, since the system under consideration is a second-order system, one can numerically estimate a DA by calculating the largest Lyapunov level set included within $\mathcal{H} := \{x(t) \in \mathcal{L} : \dot{h}_\delta(z(t)) \in [-\phi^*, \phi^*], \delta \in \mathcal{I}_r\}$, namely $\Omega(P(t), \gamma)$, where $P(t)$ is the Lyapunov matrix corresponding to each method and $\gamma := \max\{c \in \mathbb{R} : \Omega(P(t), c) \subseteq \mathcal{H}\}$. Through an exhaustive search over a fine grid in the parameter space $(x_1(t), x_2(t)) \in [-8, 8] \times [-8, 8]$ and a line search procedure over $\{c \in \mathbb{R} : \Omega(P(t), c) \subseteq \mathcal{H}\}$, we could compute estimates of the DA (interiors of the circular solid lines) plotted in Figs. 4 and 5. From the figures, one can confirm that the DA estimation of the proposed approach shown in Fig. 3 includes that of Case 1 in Fig. 4, whereas the estimated DA of Case 2 in Fig. 5 outperforms our result in Fig. 3. However, it is worth mentioning that the stability analysis developed in [43] does not provide a systematic way to compute an invariant subset of the DA. Moreover, the gridding procedure adopted in this paper to evaluate the DA for Cases 1 and 2 cannot be easily applied to systems whose order is larger than two, because this extension requires computationally demanding searches over a parameter space of dimension larger than two. Most importantly, the proposed approach stands out as being better than previous approaches when we tackle the control design problem. An important difference of the stability analysis and stabilization is that in the former problem, the shape of region $\mathcal{H}$ is not changed, whereas in the latter problem, its shape can be easily altered, depending on the control gains. In this respect, the proposed strategy can establish superiority over other approaches, because it enables the control gains to be designed so that the estimated DA is enlarged. This will be shown in an example later. Finally, Table II summarizes the corresponding numerical complexities obtained in the same manner of Example 1. From the results, one concludes that the proposed method is computationally more demanding than the existing ones, except for [45, Th. 2] with $q = 5$, and its advantages come at the price of higher computational effort.

TABLE II

EXAMPLE 2: NUMERICAL COMPLEXITY (N_D TOTAL NUMBER OF SCALAR DECISION VARIABLES; N_L TOTAL ROW SIZE OF CORRESPONDING LMI; TIME IN SECONDS)

Method	N_D	N_L	Time (s)
Theorem 1 with $q = 3$	113	144	0.3782
[41, Th. 1] with $(g, d) = (2, 2)$ and $E_{\mathbf{i}}^{i} = 0_{2n\times 2n}$	48	49	0.0477
[43, Th. 1] with $g_1 = 4$	15	34	0.0297
[45, Th. 2] with $q = 5$	480	448	0.6852

IV. RELAXED STABILIZATION CONDITION

Let the control law and the Lyapunov function candidate be

$$u(t) = K_{z^q} P_{z^q}^{-1} x(t) \tag{18}$$

$$V_q(x(t)) = x(t)^T P_{z^q}^{-1} x(t) \tag{19}$$

respectively. Then, the resulting closed-loop system is obtained in the following form:

$$\dot{x}(t) = (A_z + B_z K_{z^q} P_{z^q}^{-1})x(t), \quad \forall x(t) \in \mathcal{L}. \tag{20}$$

In this section, the problem under study can be formulated as follows.

1) Determine control law (18) such that the zero equilibrium point of (20) is locally asymptotically stable.
2) Search for Lyapunov function (19) for the zero equilibrium point such that $\Omega(P_{z^q}^{-1}, 1)$ is an invariant subset of the DA [53].
3) Enlarge sublevel set $\Omega(P_{z^q}^{-1}, 1)$.

We obtain the following local stabilization condition.

Theorem 2: If there exist symmetric matrices $P_{\mathbf{j}_q} \in \mathbb{R}^{n\times n}$, $X_{\mathbf{j}_q} \in \mathbb{R}^{n\times n}$, matrices $K_{\mathbf{j}_q} \in \mathbb{R}^{m\times n}$, $Y_{\mathbf{j}_q} \in \mathbb{R}^{2n\times n}$, and a real number $\beta > 0$ such that the following optimization problem is satisfied:

$$\min_{P_{\mathbf{j}_q}, X_{\mathbf{j}_q}, K_{\mathbf{j}_q}, Y_{\mathbf{j}_q}, \beta} \beta \text{ subject to}$$

$$\sum\nolimits_{\mathbf{j}_q \in \mathcal{P}(\mathbf{i}_q)} \Upsilon^{(1)}_{(l,\mathbf{j}_q)} \prec 0, \quad \forall (\mathbf{i}_q, l) \in \mathbb{I}_q^+ \times \mathcal{I}_p \tag{21}$$

$$\sum\nolimits_{\mathbf{j}_{q+1} \in \mathcal{P}(\mathbf{i}_{q+1})} \Upsilon^{(2)}_{\mathbf{j}_{q+1}}(y) \prec 0$$
$$\forall (\mathbf{i}_{q+1}, y) \in \mathbb{I}_{q+1}^+ \times \{\xi_1, \xi_2, \dots, \xi_{2^r}\} \tag{22}$$

$$\sum\nolimits_{\mathbf{j}_{q+1} \in \mathcal{P}(\mathbf{i}_{q+1})} \Upsilon^{(3)}_{(\delta,\rho,\mathbf{j}_{q+1})} \prec 0$$
$$\forall (\delta, \rho, \mathbf{i}_{q+1}) \in \mathcal{I}_r \times \mathcal{I}_w \times \mathbb{I}_{q+1}^+ \tag{23}$$

$$\sum\nolimits_{\mathbf{j}_q \in \mathcal{P}(\mathbf{i}_q)} \Upsilon^{(4)}_{\mathbf{j}_q} \prec 0, \quad \forall \mathbf{i}_q \in \mathbb{I}_q^+ \tag{24}$$

where for any integer i, e_i denotes $n \times 1$ unit vector with a 1 in the ith component and 0's elsewhere

$$\Upsilon^{(1)}_{(l,\mathbf{j}_q)} := \begin{bmatrix} -P_{\mathbf{j}_q} & * \\ e_{a_l}^T P_{\mathbf{j}_q} & -\bar{x}_{a_l}^2 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

$$\Upsilon^{(2)}_{\mathbf{j}_{q+1}}(y) := \begin{bmatrix} \mathrm{He}\{B_{j_{q+1}} K_{\mathbf{j}_q}\} + \sum_{\delta=1}^r y_\delta (\sum_{k=1}^q P_{\mathbf{j}_{q-1}(k,\delta)} + X_{\mathbf{j}_q}) & * \\ P_{\mathbf{j}_q} & 0_{n\times n} \end{bmatrix} + \mathrm{He}\{Y_{\mathbf{j}_q} [\, A_{j_{q+1}}^T \quad -I_n \,]\} \in \mathbb{R}^{2n\times 2n},$$

$$\Upsilon^{(3)}_{(\delta,\rho,\mathbf{j}_{q+1})} := \begin{bmatrix} -P_{\mathbf{j}_q} & * \\ g_{(\delta,\rho)}(A_{j_{q+1}} P_{\mathbf{j}_q} + B_{j_{q+1}} K_{\mathbf{j}_q}) & -\phi_\delta^2 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

$$\Upsilon^{(4)}_{\mathbf{j}_q} := \begin{bmatrix} -\beta I_n & * \\ I_n & -P_{\mathbf{j}_q} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$$

and $\bar{x}_{a_l} > 0$, $l \in \mathcal{I}_p$ are *a priori* given real numbers representing the vertices of set $\mathcal{L}$ defined in (3), then (20) is locally asymptotically stable. Moreover, an invariant subset of the DA for the system is given by $\Omega(P_{z^q}^{-1}, 1)$.

Proof: Applying Lemma 1, we easily verify that LMIs (21)–(24) ensure

$$\Upsilon^{(1)}_{(l,z^q)} = \begin{bmatrix} -P_{z^q} & * \\ e_{a_l}^T P_{z^q} & -\bar{x}_{a_l}^2 \end{bmatrix} \prec 0$$
$$\forall (x(t), l) \in \mathcal{L} \times \mathcal{I}_p \tag{25}$$

$$\Upsilon^{(2)}_{z^{q+1}}(y) = \begin{bmatrix} \mathrm{He}\{B_z K_{z^q}\} + \sum_{\delta=1}^r y_\delta \bar{P}_{(z^q,\delta)} & * \\ P_{z^q} & 0_{n\times n} \end{bmatrix} + \mathrm{He}\left\{Y_{z^q} [\, A_z^T \quad -I_n \,]\right\} \prec 0$$
$$\forall (x(t), y) \in \mathcal{L} \times \{\xi_1, \xi_2, \dots, \xi_{2^r}\} \tag{26}$$

$$\Upsilon^{(3)}_{(\delta,\rho,z^{q+1})} = \begin{bmatrix} -P_{z^q} & * \\ g_{(\delta,\rho)}(A_z P_{z^q} + B_z K_{z^q}) & -\phi_\delta^2 \end{bmatrix} \prec 0$$
$$\forall (x(t), \delta, \rho) \in \mathcal{L} \times \mathcal{I}_r \times \mathcal{I}_w \tag{27}$$

$$\Upsilon^{(4)}_{z^q} = \begin{bmatrix} -\beta I_n & * \\ I_n & -P_{z^q} \end{bmatrix} \prec 0, \quad \forall x(t) \in \mathcal{L} \tag{28}$$

where $\bar{P}_{(z^q,\delta)} := \sum_{\mathbf{j}_{q-1} \in \mathbb{I}_{q-1}} h_{\mathbf{j}_{q-1}} (\sum_{k=1}^q P_{\mathbf{j}_{q-1}(k,\delta)}) + X_{z^q}$. Let us define vectors

$$\zeta_1 := \begin{bmatrix} P_{z^q}^{-1} \\ \bar{x}_{a_l}^{-2} e_{a_l}^T \end{bmatrix} x(t) \in \mathbb{R}^{n+1},$$

$$\zeta_2 := \begin{bmatrix} P_{z^q}^{-1} \\ \phi_\delta^{-2} \nabla h_\delta(x(t))(A_z + B_z K_{z^q} P_{z^q}^{-1}) \end{bmatrix} x(t) \in \mathbb{R}^{n+1},$$

$$\zeta_3 := \begin{bmatrix} P_{z^q}^{-1} \\ A_z^T P_{z^q}^{-1} \end{bmatrix} x(t) \in \mathbb{R}^{2n}, \quad \zeta_4 := \begin{bmatrix} I_n \\ P_{z^q}^{-1} \end{bmatrix} x(t) \in \mathbb{R}^{2n}.$$

Then, it can be seen by direct calculation that

$$\zeta_1^T \Upsilon^{(1)}_{(l,z^q)} \zeta_1 = \bar{x}_{a_l}^{-2} x_{a_l}(t)^2 - V_q(x(t)) < 0$$
$$\forall (x(t), l) \in \mathcal{L}\backslash\{0_n\} \times \mathcal{I}_p.$$

By resorting to the same reasoning adopted within the proof of Theorem 1, one concludes that $V_q(x(t)) = 1$, $\forall x(t) \in \partial\Omega(P_{z^q}^{-1}, 1)$. Moreover, using definition (6), it follows from (27) that

$$\begin{bmatrix} -P_{z^q} & * \\ \nabla h_\delta(x(t))(A_z P_{z^q} + B_z K_{z^q}) & -\phi_\delta^2 \end{bmatrix} \prec 0$$
$$\forall (x(t), \delta) \in \mathcal{L} \times \mathcal{I}_r.$$

Pre- and postmultiplying the aforementioned inequality by ξ_2^T and ξ_2 yields

$$\begin{aligned}
&-x(t)^T P_{z^q}^{-1} x(t) + \phi_\delta^{-2} x(t)^T (A_z + B_z K_{z^q} P_{z^q}^{-1})^T \nabla h_\delta(x(t))^T \\
&\quad \times \nabla h_\delta(x(t))(A_z + B_z K_{z^q} P_{z^q}^{-1}) x(t) \\
&\quad = -V_q(x(t)) + \phi_\delta^{-2} \dot{h}_\delta(z(t))^2 < 0 \\
&\quad\quad \forall (x(t), \delta) \in \mathcal{L}\backslash\{0_n\} \times \mathcal{I}_r \\
&\Leftrightarrow \phi_\delta^{-2} \dot{h}_\delta(z(t))^2 - 1 < V_q(x(t)) - 1 \\
&\quad\quad \forall (x(t), \delta) \in \mathcal{L}\backslash\{0_n\} \times \mathcal{I}_r \\
&\Leftrightarrow \dot{h}_\delta(z(t))^2 - \phi_\delta^2 < \phi_\delta^2 (V_q(x(t)) - 1) \\
&\quad\quad \forall (x(t), \delta) \in \mathcal{L}\backslash\{0_n\} \times \mathcal{I}_r.
\end{aligned} \tag{29}$$

Then, along the same lines to the proof of Theorem 1, it can be easily verified from (29) that $\Omega(P_{z^q}^{-1}, 1) \subset \mathcal{H}$ holds.

Next, we show that $\Omega(P_{z^q}^{-1}, 1)$ is an invariant subset of the DA. Inequality (25) ensures $P_{z^q} \succ 0$, $\forall x(t) \in \mathcal{L}$, which guarantees that P_{z^q} is invertible, $P_{z^q}^{-1} \succ 0$, $\forall x(t) \in \mathcal{L}$, and equivalently $V_q(x(t)) > 0$, $\forall x(t) \in \mathcal{L}\backslash\{0_n\}$. On the other hand, (26) ensures

$$\begin{bmatrix} \mathrm{He}\{B_z K_{z^q}\} - \sum_{\delta=1}^{r} \dot{h}_\delta(z(t))\bar{P}_{(z^q,\delta)} & * \\ P_{z^q} & 0_{n\times n} \end{bmatrix} + \mathrm{He}\{Y_{z^q}[\ A_z^T \quad -I_n\ ]\} \prec 0, \quad \forall x(t) \in \mathcal{H}. \tag{30}$$

Using relation $\sum_{\delta=1}^{r} \dot{h}_\delta(z(t)) = 0$, it is verified that (30) is equivalent to

$$\Pi := \begin{bmatrix} \mathrm{He}\{B_z K_{z^q}\} - \dot{P}_{z^q} & * \\ P_{z^q} & 0_{n\times n} \end{bmatrix} + \mathrm{He}\left\{Y_{z^q}[\ A_z^T \quad -I_n\ ]\right\} \prec 0, \quad \forall x(t) \in \mathcal{H}.$$

Then, we multiply the aforementioned inequality by ζ_3^T on the left and ζ_3 on the right and use relation $dP_{z^q}^{-1}/dt = -P_{z^q}^{-1}\dot{P}_{z^q}P_{z^q}^{-1}$ to show that

$$\begin{aligned} \xi_3^T \Pi \xi_3 &= x(t)^T(\mathrm{He}\{P_{z^q}^{-1}(A_z + B_z K_{z^q} P_{z^q}^{-1})\} + dP_{z^q}^{-1}/dt)x(t) \\ &= \dot{V}_q(x(t)) \\ &< 0, \quad \forall x(t) \in \mathcal{H}\backslash\{0_n\} \end{aligned}$$

which implies $\mathcal{H}\backslash\{0_n\} \subseteq \{x(t) \in \mathcal{L} : \dot{V}_q(x(t)) < 0\}$. Since $\Omega(P_{z^q}^{-1}, 1) \subset \mathcal{H}$, one has $\Omega(P_{z^q}^{-1}, 1)\backslash\{0_n\} \subset \{x(t) \in \mathcal{L} : \dot{V}_q(x(t)) < 0\}$. By the Lyapunov argument, (20) is locally asymptotically stable, and $\Omega(P_{z^q}^{-1}, 1)$ is an invariant subset of the DA [53]. Finally, it follows from (28) that

$$\begin{aligned} &\zeta_4^T \Upsilon_{z^q}^{(4)} \zeta_4 = V_q(x(t)) - \beta x(t)^T x(t) < 0, \quad \forall x(k) \in \mathcal{L}\backslash\{0_n\} \\ &\Leftrightarrow V_q(x(t)) - 1 < \beta(x(t)^T x(t) - 1/\beta), \quad \forall x(k) \in \mathcal{L}\backslash\{0_n\} \end{aligned}$$

from which the enlargement of $\Omega(P_{z^q}^{-1}, 1)$ is proved similarly to the proof of Theorem 1, and the proof is completed. ■

Remark 7: For Theorem 2, $N_D = (n^2+n)r^q + mnr^q + 2n^2r^q + 1$ and $N_L = ((n+1)+2n)p(r+q-1)!/(q!(r-1)!) + (2^{r+1}n + (n+1)rw)(r+q)!/((q+1)!(r-1)!)$.

Remark 8: As stated in Remark 6, bounds ϕ_δ, $\forall\delta \in \mathcal{I}_r$ can be determined by setting $\phi_\delta = \phi$, $\forall\delta \in \mathcal{I}_r$ and solving the single-parameter maximization problem via a line search or bisection algorithm maximizing $\phi_\delta = \phi$, $\forall\delta \in \mathcal{I}_r$ subject to the condition of Theorem 2. Moreover, with some random initial values of ϕ_δ, $\forall\delta \in \mathcal{I}_r$, a locally convergent solution to the tuning problem can be obtained with the program `fminsearch` in the optimization toolbox [51]. Note however that, unlike the local stability case, the volume of the DA estimate is not explicitly dependent on bounds ϕ_δ, $\forall\delta \in \mathcal{I}_r$, since the shape of region $\mathcal{H}$ can be easily altered by the gain matrix. Hence, in the control design case, developing more efficient algorithms to look for bounds ϕ_δ, $\forall\delta \in \mathcal{I}_r$ is left for future research.

Remark 9: The differences between the proposed approach and those in [44]–[49] can be summarized as follows.

1) The research works in [48] and [49] basically adopt the following sets of state variables:

$$\mathbf{R}_{\lambda_k} := \left\{ \begin{array}{c} x(t) \in \mathbb{R}^n : \left(\frac{\partial w_0^k(t)}{\partial \xi_k(t)}\right)^T \left(\frac{\partial w_0^k(t)}{\partial \xi_k(t)}\right) \le \lambda_k^2, \\ k \in \mathcal{I}_p \end{array} \right\}$$

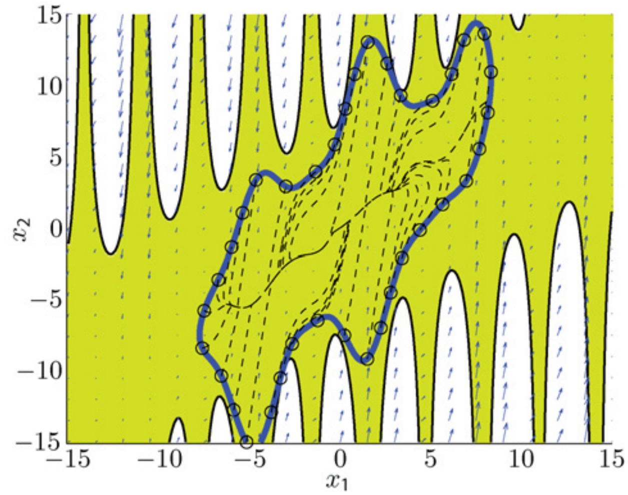

Fig. 6. Example 3. Trajectories (dashed lines) initialized at the "○" marks, level set $\Omega(P_{z^q}^{-1}, 1)$ (interior of the circular solid line) estimated by using Theorem 2 with $q = 3$, $\phi = \phi^* = 172.0994$, and the region of $\mathcal{H} := \{x(t) \in \mathcal{L} : \dot{h}_\delta(z(t)) \in [-172.0994, 172.0994], \delta \in \mathcal{I}_r\}$ (shaded area).

$$\mathbf{R}_{\lambda_x} := \left\{x(t) \in \mathbb{R}^n : x(t)^T x(t) \le \lambda_x^2\right\}$$

$$\mathbf{C} := \left\{x(t) \in \mathbb{R}^n : |x_i(t)| \le \bar{x}_i,\ i \in \mathcal{I}_r\right\}$$

$$\mathbf{R}_{\beta_k} := \left\{x(t) \in \mathbb{R}^n : \left|\frac{\partial w_0^k(t)}{\partial z_k(t)}\dot{z}_k(t)\right| \le \beta_k,\ k \in \mathcal{I}_p\right\}$$

where $\xi_k(t)$ is defined as a vector of the state variables such that the partial derivative of $z_k(t)$ with respect the state variables is not zero. The condition of [48, Th. 2] or [49, Th. 3] guarantees $\mathbf{R}_{\lambda_k} \cap \mathbf{R}_{\lambda_x} \cap \mathbf{C} \subseteq \mathbf{R}_{\beta_k}$ and $\dot{V}(x(t)) < 0$, $\forall x(t) \in (\mathbf{R}_{\beta_k} \cap \mathbf{C})\backslash\{0_n\} \Leftrightarrow (\mathbf{R}_{\beta_k} \cap \mathbf{C})\backslash\{0_n\} \subseteq \{x(t) \in \mathbb{R}^n : \dot{V}(x(t)) < 0\}$. This implies $(\mathbf{R}_{\lambda_k} \cap \mathbf{R}_{\lambda_x} \cap \mathbf{C})\backslash\{0_n\} \subseteq (\mathbf{R}_{\beta_k} \cap \mathbf{C})\backslash\{0_n\} \subseteq \{x(t) \in \mathbb{R}^n : \dot{V}(x(t)) < 0\}$. If we assume $\mathbf{C} \subseteq \mathbf{R}_{\lambda_k}$, then $(\mathbf{R}_{\lambda_x} \cap \mathbf{C})\backslash\{0_n\} \subseteq (\mathbf{R}_{\beta_k} \cap \mathbf{C})\backslash\{0_n\} \subseteq \{x(t) \in \mathbb{R}^n : \dot{V}(x(t)) < 0\}$ is guaranteed. Thus, the outmost Lyapunov level set contained within $\mathbf{R}_{\lambda_x} \cap \mathbf{C}$ is an inner estimate of the DA. More specifically, if $\{x(t) \in \mathbb{R}^n : V(x(t)) \le \gamma\}\backslash\{0_n\} \subseteq (\mathbf{R}_{\lambda_x} \cap \mathbf{C})\backslash\{0_n\}$, where γ is the maximum c such that $\{x(t) \in \mathbb{R}^n : V(x(t)) \le c\}\backslash\{0_n\} \subseteq (\mathbf{R}_{\lambda_x} \cap \mathbf{C})\backslash\{0_n\}$, then $\{x(t) \in \mathbb{R}^n : V(x(t)) \le \gamma\}\backslash\{0_n\}$ is an estimation of the DA. On the other hand, the condition of Theorem 2 ensures $\Omega(P_{z^q}^{-1}, 1) \subset \mathcal{H}$ and $\mathcal{H}\backslash\{0_n\} \subseteq \{x(t) \in \mathcal{L} : \dot{V}(x(t)) < 0\}$, so $\Omega(P_{z^q}^{-1}, 1)\backslash\{0_n\} \subset \mathcal{H}\backslash\{0_n\} \subseteq \{x(t) \in \mathcal{L} : \dot{V}(x(t)) < 0\}$. In [48] or [49], the boundary of $\mathbf{R}_{\lambda_x} \cap \mathbf{C}$ would lie between that of $\mathbf{R}_{\beta_k} \cap \mathbf{C}$ and the Lyapunov level set, while in our approach, the boundary of set $\Omega(P_{z^q}^{-1}, 1)$ can border on that of $\mathcal{H}$. In addition, if $\mathbf{C} \subseteq \mathbf{R}_{\lambda_k}$ in [48] or [49], $\mathbf{R}_{\beta_k}$ includes $\mathbf{C}$, whereas in our approach, $\mathcal{H}$ is defined within $\mathcal{L}$, albeit these points do not explicitly imply merits and demerits of both approaches.

2) The proposed approach ensures the enlargement of Lyapunov level set $\Omega(P_{z^q}, 1)$ or $\Omega(P_{z^q}^{-1}, 1)$ confined within $\mathcal{H}$ in (5). Therefore, the DA can be estimated via a one-

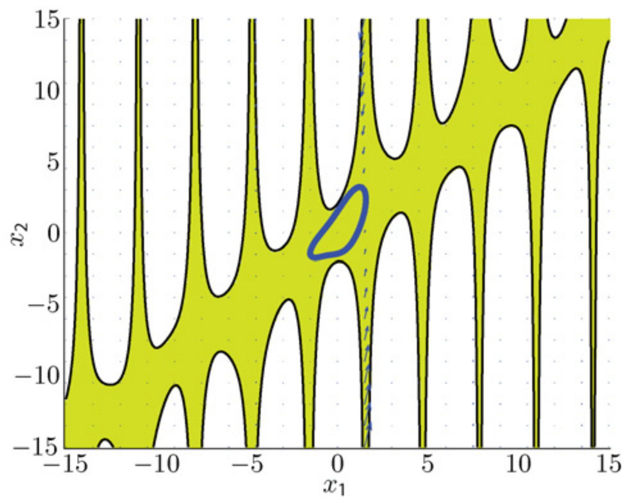

Fig. 7. Example 3. Lyapunov level set (interior of the circular solid line) obtained by using [50, Th. 9] with $\phi^* = 90.3662$, and the region of $\mathcal{H} := \{x(t) \in \mathcal{L} : \dot{h}_\delta(z(t)) \in [-90.3662, 90.3662], \delta \in \mathcal{I}_r\}$ (shaded area).

TABLE III

EXAMPLE 3: NUMERICAL COMPLEXITY (N_D TOTAL NUMBER OF SCALAR DECISION VARIABLES; N_L TOTAL ROW SIZE OF CORRESPONDING LMI; TIME IN SECONDS)

Method	N_D	N_L	Time (s)
Theorem 2 with $q = 1$	33	98	0.1382
Theorem 2 with $q = 2$	65	133	0.2219
Theorem 2 with $q = 3$	129	168	0.3132
[50, Th. 9]	57	129	0.2819
[49, Th. 3]	64	24	0.0470
[43, Th. 2] with $(g_1, s_1) = (3, 3)$	36	48	0.0578
[41, Th. 2] with $(g, d) = (3, 3)$	17	24	0.0282

step procedure of EVP, which is a convex optimization [52] and tractable via LMI solvers [54]–[56], or via a single-parameter minimization problem subject to LMIs. For this reason, the proposed method can be efficiently applied to deal with systems of order higher than two. On the other hand, approaches in [44]–[49] would require additional numerical procedures to calculate the outermost Lyapunov level set contained within $\mathbf{R}_{\lambda_x} \cap \mathbf{C}$. Furthermore, the enlargement of a certain Lyapunov level set at the stage of solving the proposed LMI condition allows the control gains to be designed in such a way that larger estimates of the DA are possible. The approach in [48] also offers a way to enlarge the DA by increasing the radius of circle $\mathbf{R}_{\lambda_x}$, but it would require an additional search procedure with LMI constraints for each step.

Example 3: Let us consider (2) with

$$\begin{cases} A_1 = \begin{bmatrix} 4 & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix} \\ B_1 = \begin{bmatrix} 1 \\ 10 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ h_1(z(t)) = (1 + \sin x_1(t))/2, \quad h_2(z(t)) = 1 - h_1(z(t)) \\ z(t) = z_1(t) = x_1(t) \end{cases}$$

taken from [50]. With $\mathcal{L} = \{x(t) \in \mathbb{R}^n : x_1(t) \in [-\bar{x}_1, \bar{x}_1]\}$ and $\bar{x}_1 = 15$, vectors $g(\delta, \rho)$, $(\delta, \rho) \in \mathcal{I}_r \times \mathcal{I}_w$ are calculated the

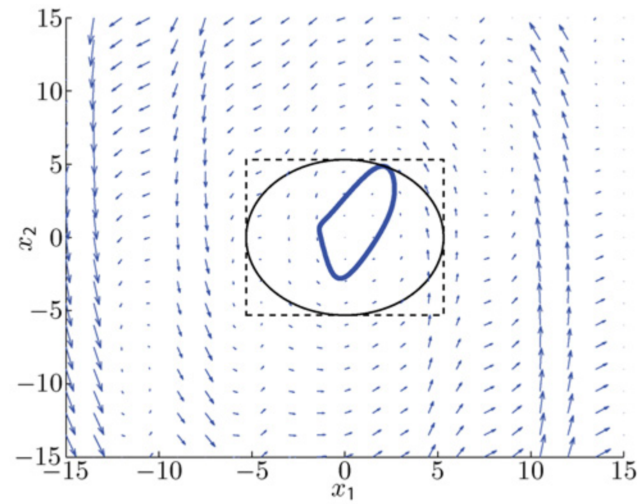

Fig. 8. Example 3. Lyapunov level set (interior of the inner solid line) obtained by using [49, Th. 3] with $\beta_1 = 370$, $\varepsilon = 1$, $\mathbf{C} = \{x(t) \in \mathbb{R}^n : x_1(t) \in [-\bar{x}_1, \bar{x}_1]\}$, $\bar{x}_1 = 5.3101$, $\lambda_x = 5.3101$, $\lambda_1 = \sqrt{0.25}$, the boundary of $\mathbf{C}$ (dashed line), and the boundary of $\{x(t) \in \mathbb{R}^n : ||x(t)|| \le \lambda_x = 5.3101\}$ (outer solid line (circle)).

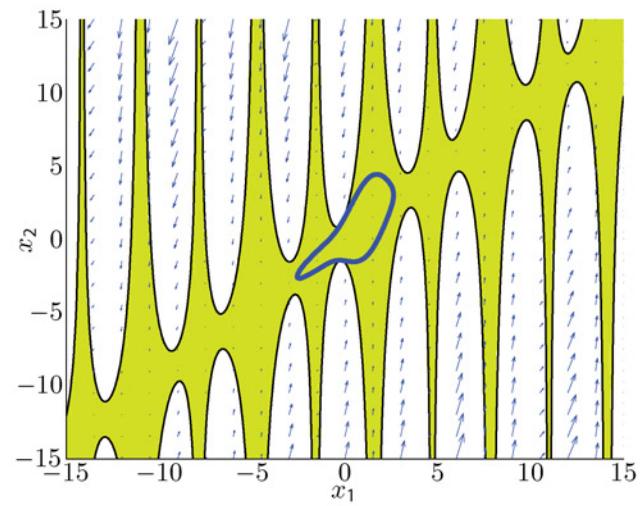

Fig. 9. Example 3. Lyapunov level set (interior of the circular solid line) obtained by using [43, Th. 3] with $(g, s) = (3, 3)$, $\beta = 10^{-3}$, $H_1 = \phi^*[\xi_1^T \ \xi_2^T]$, $\xi_1 = [-1 \ 1]$, $\xi_2 = [1, \ -1]$, and $\phi^* = 386.3884$, and the region of $\mathcal{H} := \{x(t) \in \mathcal{L} : \dot{h}_\delta(z(t)) \in [-386.3884, 386.3884], \delta \in \mathcal{I}_r\}$ (shaded area).

same as in Example 2. Via a bisection algorithm, the maximum value of $\phi_1 = \phi_2 = \phi$, denoted by ϕ^*, such that the LMI condition of Theorem 2 with $q = 3$ remains feasible was computed to be $\phi^* = 172.0994$. Fig. 6 shows conversing trajectories (dashed lines) initialized at the "○" marks, level set $\Omega(P_{z^q}^{-1}, 1)$ (interior of the circular solid line) estimated by using Theorem 2 with $q = 3$, $\phi = \phi^* = 172.0994$, and the region of $\mathcal{H} := \{x(t) \in \mathcal{L} : \dot{h}_\delta(z(t)) \in [-172.0994, 172.0994], \delta \in \mathcal{I}_r\}$ (shaded area). For comparison purpose, we conducted experiments with the following cases.

1) *Case 1:* [8, Th. 5] (quadratic approach).
2) *Case 2:* Iterative method developed in [17] with $N = 4$ and random initial values $\hat{\lambda}_{ijsk}(0) \ge 0$ uniformly distributed in the interval [0, 10].

3) *Case 3:* [46, Th. 3] with $\lambda_x = \sqrt{\pi^2/2}$, $\delta = 0.1$, $\lambda_1 = \sqrt{0.25}$, and β_1 searched over $\{10^{-6}, 10^{-5}, \ldots, 10^6\}$.
4) *Case 4:* [48, Th. 2] with $\lambda_x = \sqrt{\pi^2/2}$, $\lambda_1 = \sqrt{0.25}$, and β_1 searched over $\{10^{-6}, 10^{-5}, \ldots, 10^6\}$.
5) *Case 5:* [50, Th. 9] with $\phi_1 = \phi_2 = \phi$, where the maximum ϕ, denoted by ϕ^*, is searched over the interval [0, 1000].
6) *Case 6:* [49, Th. 3] with $\beta_1 = 370$, $\varepsilon = 1$, $\mathbf{C} = \{x(t) \in \mathbb{R}^n : x_1(t) \in [-\bar{x}_1, \bar{x}_1]\}$, $\lambda_x = \bar{x}_1$, and $\lambda_1 = \sqrt{0.25}$, where the maximum $\bar{x}_1$, denoted by $\bar{x}_1^*$, is searched over the interval $[\pi/2, 15]$.
7) *Case 7:* [43, Th. 2] with $(g_1, s_1) = (3, 3)$, β searched over $\{10^1, 10^0, \ldots, 10^{-9}\}$, $H_1 = \phi \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, and the maximum ϕ, denoted by ϕ^*, is searched over the interval [0, 100].
8) *Case 8:* [41, Th. 2] with $(g, d) = (3, 3)$, $E_{\mathbf{i}}^i = 0_{n\times n}$, and $\lambda_1 = \lambda_2 = \lambda$, where the minimum λ, denoted by λ^*, is searched over the interval $[-30, 0]$.

Cases 1–4 failed to find a feasible solution, while Case 5 with $\phi^* = 90.3662$, Case 6 with $\bar{x}_1^* = 5.3101$ and $\lambda_1 = \sqrt{0.25}$, Case 7 with $\beta = 10^{-3}$ and $\phi^* = 386.3884$, where $\beta = 10^{-3}$ was searched over $\{10^1, 10^0, \ldots, 10^{-9}\}$ with fixed $\phi = 221.0884$ and $\phi^* = 386.3884$ was searched over the interval [221.0884, 1000] with fixed $\beta = 10^{-3}$, and Case 8 with $\lambda^* = -8.8476$ were found feasible. The results of Cases 5–7 are plotted in Figs. 7–9. Note that, in order to find the maximum Lyapunov level set (interior of the circular solid line) confined within $\mathcal{H} := \{x(t) \in \mathcal{L} : \dot{h}_\delta(z(t)) \in [-386.3884, 386.3884], \delta \in \mathcal{I}_r\}$ (shaded area) in Fig. 9, an exhaustive search over a fine grid in the parameter space $(x_1(t), x_2(t)) \in [-15, 15] \times [-15, 15]$ and a line search procedure over $\{c \in \mathbb{R} : \Omega(P(t), c) \subseteq \mathcal{H}\}$ were used as stated in Example 2. Note also that, although the result of Case 8 is not presented here for space sake, from our own experiment, the corresponding Lyapunov level set confined within set $\{x(t) \in \mathcal{L} : h_\delta(z(t)) \geq \lambda_\delta, \delta \in \mathcal{I}_r\}$ was not larger than those presented in this paper. From the results, it is clear that the condition of Theorem 2 is better than the previous ones in terms of the volume of the DA estimation.

Finally, Table III provides the corresponding numerical complexities obtained in the same manner as Example 1. The results shown in the table suggest that, as q increases, the complexity also tends to increase, and the computational effort required by the condition of Theorem 2 is higher than those of the existing approaches. This suggests that the results proposed in this paper can be considered as a tradeoff between conservative results and less conservative ones with higher computational effort.

V. Conclusion

This paper proposed relaxed sufficient conditions for local stability and stabilization of continuous-time T–S fuzzy systems. The solution procedures have been formulated as optimization problems, which can be solved via a sequence of LMI optimizations or as EVPs. Finally, examples demonstrated the effectiveness of the proposed scheme.

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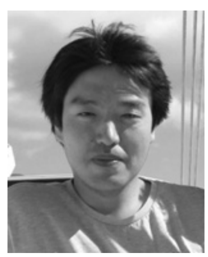

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