Local stability analysis and estimation of domain of attraction of nonlinear speed droop system by T–S fuzzy modeling

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1 Problem definition

Let the nonlinear system

$$\dot{x}(t) = f(x(t)) \tag{1}$$

such that the origin is as equilibrium point, that is, $f(0) = 0$, $x(t) \in \mathbb{R}^n$. For a speed droop system one has $x_1(t) = P_f(t)$, $x_2(t) = Q_f(t)$ and $x_3(t) = \delta(t)$.

By the sector nonlinearity approach [1], the nonlinear system can be exactly represented by the T–S fuzzy system

$$\dot{x}(t) = A(\alpha)x(t) \tag{2}$$

$A(\alpha) \in \mathbb{R}^{n\times n}$, $\forall x(k) \in \mathcal{X}$, where $\mathcal{X}$ is a set of the state variables including the origin and

$$A(\alpha) = \sum_{i=1}^{N} \alpha_i(z)A_i,$$

$\alpha(z) = [\alpha_1 \cdots \alpha_N]' \in \Lambda_N$,

$$\Lambda_N = \left\{ \xi \in \mathbb{R}^p : \sum_{i=1}^{N} \alpha_i(z) = 1, \quad \alpha_i(z) \geq 0 \right\}$$

and $z(t)$ are the premise variables depending on the states, that is, $z(x(t))$.

The domain of validity of the T–S model (2) is given by the following polyhedral set [2] $\mathcal{X}$, with $0 \subset \mathcal{X}$,

$$\mathcal{X} = \{x \in \mathbb{R}^n : b_k' x \leq 1, \quad k = 1, \ldots, q \leq n\} \tag{3}$$

where $b_k \in \mathbb{R}^n$, $k = 1, \ldots, q$, are defined in the T–S fuzzy modeling approach.

Problem 1. *To verify the stability analysis of nonlinear system (1) and a estimation of its domain of attraction by means of the exact local representation (2) for all $x(k) \in \Omega \subset \mathcal{X}$.*

2 Stability Analysis

Let the Lyapunov function

$$V(x) = x(t)'P(\alpha)x(t), \qquad P(\alpha) = \sum_{i=1}^{N} \alpha_i(z)P_i, \qquad P_i \in \mathbb{R}_S^{n\times n}. \tag{4}$$

Then,[1]

$$\begin{aligned}\dot{V}(x) &= \dot{x}'P(\alpha)x + x'P(\alpha)\dot{x} + x'\dot{P}(\alpha)x \\ &= x'\left(A(\alpha)'P(\alpha) + P(\alpha)A(\alpha)\right)x + x'\dot{P}(\alpha)x.\end{aligned}$$

Observe that,

$$\dot{\alpha}(z) = J(\theta)\dot{x}$$

with

$$\dot{\alpha}(z) = \begin{bmatrix}\dot{\alpha}_1 \\ \vdots \\ \dot{\alpha}_N\end{bmatrix}, \qquad \dot{x} = \begin{bmatrix}\dot{x}_1 \\ \vdots \\ \dot{x}_n\end{bmatrix}$$

$$J(\theta) = \nabla_x \alpha(z) = \sum_{i=1}^{\vartheta} \theta_i(x)J_i.$$

where J_i are matrices obtained form the knowledge of $\alpha(z)$ and the set $\mathcal{X}$.

One has,

$$\begin{aligned}x'\dot{P}(\alpha)x &= x'\left(\sum_{i=1}^{N}\dot{\alpha}_i(z)P_i\right)x \\ &= x'\left(\dot{\alpha}_1P_1 + \ldots + \dot{\alpha}_NP_N\right)x \\ &= x'\begin{bmatrix}P_1x & \cdots & P_Nx\end{bmatrix}\begin{bmatrix}\dot{\alpha}_1 \\ \vdots \\ \dot{\alpha}_N\end{bmatrix} \\ &= x'\begin{bmatrix}P_1x & \cdots & P_Nx\end{bmatrix}\dot{\alpha}(z) \\ &= x'\begin{bmatrix}P_1x & \cdots & P_Nx\end{bmatrix}J(\theta)A(\alpha)x.\end{aligned}$$

The set $\mathcal{X}$ can be defined in terms of its κ vertices[2],

$$\mathcal{X} = co\{x^1, x^2, \ldots, x^\kappa\},$$

[1]The argument t is omitted hereafter.

[2]The Matlab toolbox *Multi-Parametric Toolbox* (MPT) can be used to convert the representations of the polytope, see Section *Polytope Library* therein.

then all $x \in \mathcal{X}$ can be written as

$$x(\gamma) = \sum_{k=1}^{\nu} \gamma_k(x) x^k.$$

One has,

$$\begin{aligned} & x' \begin{bmatrix} P_1 x & \cdots & P_N x \end{bmatrix} J(\theta) A(\alpha) x \\ & = x' \begin{bmatrix} P_1 x(\gamma) & \cdots & P_N x(\gamma) \end{bmatrix} J(\theta) A(\alpha) x \\ & = x' Q(\gamma) J(\theta) A(\alpha) x \end{aligned}$$

where

$$Q(\gamma) \triangleq \sum_{k=1}^{\nu} \gamma_k \begin{bmatrix} P_1 x^k & \cdots & P_N x^k \end{bmatrix}.$$

Therefore,

$$\begin{aligned} & \dot{V}(x) = x' \Big(A(\alpha)' P(\alpha) + P(\alpha) A(\alpha) + Q(\gamma) J(\theta) A(\alpha) \Big) x, \\ & \mathbf{1}' J(\theta) A(\alpha) x = 0 \end{aligned} \tag{5}$$

since $\sum_{i=1}^{N} \dot{\alpha}_i(z) = 0$ and

$$\begin{aligned} & \sum_{i=1}^{N} \dot{\alpha}_i(z) = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \vdots \\ \dot{\alpha}_N \end{bmatrix} \\ & = \mathbf{1}' \dot{\alpha} = \mathbf{1}' J(\theta) \dot{x} = \mathbf{1}' J(\theta) A(\alpha) x = 0, \qquad \mathbf{1} \triangleq \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}'. \end{aligned}$$

Lemma 1 (Finsler Lemma - short version)**.** *Consider $w \in \mathbb{R}^n$, $\mathcal{D} \in \mathbb{R}^{n \times n}$ and $\mathcal{B} \in \mathbb{R}^{m \times n}$ with rank$(\mathcal{B}) < n$. Then, the following are equivalents:*

i. $w' \mathcal{D} w < 0, \quad \forall w \neq 0, \quad \mathcal{B} w = 0$

ii. $\exists X \in \mathbb{R}^{n \times m} : \quad \mathcal{D} + X\mathcal{B} + \mathcal{B}' X < 0$

Applying Lemma 1 in (5), $\dot{V}(x) < 0$ holds if

$$A(\alpha)' P(\alpha) + P(\alpha) A(\alpha) + Q(\gamma) J(\theta) A(\alpha) + X(\alpha) \mathbf{1}' J(\theta) A(\alpha) + A(\alpha)' J(\theta)' \mathbf{1} X(\alpha)' < 0$$

or[3]

$$He \left\{ \left(P(\alpha) + X(\alpha) \mathbf{1}' J(\theta) \right) A(\alpha) \right\} + Q(\gamma) J(\theta) A(\alpha) < 0, \quad \forall \alpha \in \Lambda_N, \ \forall \theta \in \Lambda_\vartheta, \ \forall \gamma \in \Lambda_\nu. \tag{6}$$

[3] $He\{M\}$ means $He\{M\} = M + M'$

The largest invariant set contained in the polytope $\mathcal{X}$ is defined as

$$\Omega \triangleq \{x \in \mathbb{R}^n : \quad x'P(\alpha)x \leq 1\}.$$

The constraints $\Omega \subset \mathcal{X}$ holds if [2],

$$b_k'P(\alpha)^{-1}b_k \leq 1, \qquad k = 1, \ldots, q.$$

By applying Schur Complement,

$$\begin{bmatrix} 1 & b_k' \\ b_k & P(\alpha) \end{bmatrix} \geq 0, \qquad k = 1, \ldots, q, \quad \forall \alpha \in \Lambda_N. \tag{7}$$

The enlargement of Ω may be obtained by maximizing the radius $\beta > 0$ of a ball with center in origin in the state space contained in Ω, that is,

$$\min \beta \quad \text{such that} \quad P(\alpha) - \beta I_n < 0.$$

3 Main result

Theorem 1. *If there exist matrices $P(\alpha) = P(\alpha)' > 0$ and $X(\alpha)$ such that*

$$A(\alpha)'P(\alpha) + P(\alpha)A(\alpha) + Q(\gamma)J(\theta)A(\alpha) + X(\alpha)\mathbf{1}'J(\theta)A(\alpha) + A(\alpha)'J(\theta)'\mathbf{1}X(\alpha)' < 0 \tag{8}$$

$$\begin{bmatrix} 1 & b_k' \\ b_k & P(\alpha) \end{bmatrix} \geq 0, \quad k = 1, \ldots, q \tag{9}$$

for all $\alpha \in \Lambda_N$, $\theta \in \Lambda_\vartheta$ and $\gamma \in \Lambda_\nu$ then, the origin of the nonlinear system (1) is asymptotically stable and $\Omega \subset \mathcal{X}$ is an invariant set of the domain of attraction for (1).

Remark 1 (Tip)**.** *In Theorem 1, one has*

- *Variables: vertices of $P(\alpha), X(\alpha), Q(\gamma) = \sum_{k=1}^{\nu} \gamma_k \begin{bmatrix} P_1 x^k & \cdots & P_N x^k \end{bmatrix}$*
- *Known parameters: vertices of $A(\alpha), J(\theta)$ and b_k*
- *Constants: $\mathbf{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}'$*

References

[1] K. Tanaka and H. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. New York, NY: John Wiley & Sons, 2001.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM Studies in Applied Mathematics, 1994.