we get that

$$
\begin{aligned}
(Q^{-\dot{1}}(t))Q(t) + Q^{-1}(t)\dot{Q}(t) &= 0, \\
(Q^{-\dot{1}}(t)) &= -Q^{-1}(t)\dot{Q}(t)Q^{-1}(t).
\end{aligned}
$$

□

Since in this chapter we work with multiple fuzzy summations, we make use of Lemma 2.4 to computationally implement them.

In order to illustrate the assumptions and notations presented in this section, we present an example system.

Example 6.1. Consider a system with four rules, with

$$
A_1 = \begin{bmatrix} -3 & 2 \\ 0 & -0.9 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.8 & 3 \\ 0 & -0.9 \end{bmatrix}
$$
$$
A_3 = \begin{bmatrix} -1.9 & 2 \\ -0.5 & 0.1 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0.1 & 3 \\ -0.5 & -2 \end{bmatrix}
$$

in a polytopic region given by vertices

$$
\mathbf{x}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \qquad \mathbf{x}_4 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}
$$

and membership functions

$$
h_1(\mathbf{x}) = \frac{\left(4 - x_1^2\right)\left(4 - x_2^2\right)}{16}, \qquad h_2(\mathbf{x}) = \frac{\left(4 - x_1^2\right) x_2^2}{16},
$$
$$
h_3(\mathbf{x}) = \frac{x_1^2 \left(4 - x_2^2\right)}{16}, \qquad h_4(\mathbf{x}) = \frac{x_1^2 x_2^2}{16},
$$

The Jacobian of the membership functions' is given by

$$
\frac{\partial \mathbf{h}}{\partial \mathbf{x}} = \frac{1}{8} \begin{bmatrix} -x_1(4 - x_2^2) & -x_2(4 - x_1^2) \\ -x_1 x_2^2 & x_2(4 - x_1^2) \\ x_1(4 - x_2^2) & -x_2 x_1^2 \\ x_1 x_2^2 & x_2 x_1^2 \end{bmatrix},
$$

and we can write a 16 rule TS model for it with

$$J_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 2 & 0 \\ -1 & 0 \\ -2 & 1 \\ 1 & -1 \end{bmatrix}, \quad J_4 = \begin{bmatrix} 2 & 2 \\ -1 & -2 \\ -2 & -1 \\ 1 & 1 \end{bmatrix},$$

$$J_5 = \begin{bmatrix} 0 & -2 \\ 1 & 2 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad J_6 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}, \quad J_7 = \begin{bmatrix} 2 & -2 \\ -1 & 2 \\ -2 & 1 \\ 1 & -1 \end{bmatrix}, \quad J_8 = \begin{bmatrix} 2 & 0 \\ -1 & 0 \\ -2 & -1 \\ 1 & 1 \end{bmatrix},$$

$$J_9 = \begin{bmatrix} -2 & 0 \\ 1 & 0 \\ 2 & 1 \\ -1 & -1 \end{bmatrix}, \quad J_{10} = \begin{bmatrix} -2 & 2 \\ 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad J_{11} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad J_{12} = \begin{bmatrix} 0 & 2 \\ -1 & -2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix},$$

$$J_{13} = \begin{bmatrix} -2 & -2 \\ 1 & 2 \\ 2 & 1 \\ -1 & -1 \end{bmatrix}, \quad J_{14} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \\ 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad J_{15} = \begin{bmatrix} 0 & -2 \\ -1 & 2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad J_{16} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

and

$$\begin{aligned}
g_1(\mathbf{x}) &= \omega_{11}\omega_{21}\omega_{31}\omega_{41}, & g_2(\mathbf{x}) &= \omega_{11}\omega_{21}\omega_{31}\omega_{42}, \\
g_3(\mathbf{x}) &= \omega_{11}\omega_{21}\omega_{32}\omega_{41}, & g_4(\mathbf{x}) &= \omega_{11}\omega_{21}\omega_{32}\omega_{42}, \\
g_5(\mathbf{x}) &= \omega_{11}\omega_{22}\omega_{31}\omega_{41}, & g_6(\mathbf{x}) &= \omega_{11}\omega_{22}\omega_{31}\omega_{42}, \\
g_7(\mathbf{x}) &= \omega_{11}\omega_{22}\omega_{32}\omega_{41}, & g_8(\mathbf{x}) &= \omega_{11}\omega_{22}\omega_{32}\omega_{42}, \\
g_9(\mathbf{x}) &= \omega_{12}\omega_{21}\omega_{31}\omega_{41}, & g_{10}(\mathbf{x}) &= \omega_{12}\omega_{21}\omega_{31}\omega_{42}, \\
g_{11}(\mathbf{x}) &= \omega_{12}\omega_{21}\omega_{32}\omega_{41}, & g_{12}(\mathbf{x}) &= \omega_{12}\omega_{21}\omega_{32}\omega_{42}, \\
g_{13}(\mathbf{x}) &= \omega_{12}\omega_{22}\omega_{31}\omega_{41}, & g_{14}(\mathbf{x}) &= \omega_{12}\omega_{22}\omega_{31}\omega_{42}, \\
g_{15}(\mathbf{x}) &= \omega_{12}\omega_{22}\omega_{32}\omega_{41}, & g_{16}(\mathbf{x}) &= \omega_{12}\omega_{22}\omega_{32}\omega_{42},
\end{aligned}$$

with

$$\begin{aligned}
\omega_{11}(\mathbf{x}) &= \frac{2 - x_1}{4}, & \omega_{12}(\mathbf{x}) &= \frac{2 + x_1}{4}, \\
\omega_{21}(\mathbf{x}) &= \frac{2 - x_2}{4}, & \omega_{22}(\mathbf{x}) &= \frac{2 + x_2}{4}, \\
\omega_{31}(\mathbf{x}) &= \frac{8 - x_1 x_2^2}{16}, & \omega_{32}(\mathbf{x}) &= \frac{8 + x_1 x_2^2}{16}, \\
\omega_{41}(\mathbf{x}) &= \frac{8 - x_2 x_1^2}{16}, & \omega_{42}(\mathbf{x}) &= \frac{8 + x_2 x_1^2}{16}.
\end{aligned}$$

6.3 Stability Conditions

The main idea of the stability conditions proposed in this section is to avoid using bounds over the membership functions time derivative by explicitly including the state vector into the stability conditions. By using a polytopic representation of a local region in state space, we are then able to get LMI conditions.

Consider a Lyapunov function of the form

$$V(\mathbf{x}) = \mathbf{x}^T P_h \mathbf{x},$$

its time derivative is given by

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^T P_h \mathbf{x} + \mathbf{x}^T P_h \dot{\mathbf{x}} + \mathbf{x}^T \dot{P}_h \mathbf{x}.$$

However, note that the last term can be written as

$$\begin{aligned}
\mathbf{x}^T \dot{P}_h \mathbf{x} =& \sum_{k}^{r} \dot{h}_k \mathbf{x}^T P_k \mathbf{x}, \\
=& \mathbf{x}^T \begin{bmatrix} P_1 \mathbf{x} & \dots & P_r \mathbf{x} \end{bmatrix} \begin{bmatrix} \dot{h}_1 \\ \vdots \\ \dot{h}_r \end{bmatrix}, \\
=& \mathbf{x}^T \begin{bmatrix} P_1 \mathbf{x} & \dots & P_r \mathbf{x} \end{bmatrix} \dot{\mathbf{h}},
\end{aligned}$$

and, by considering a polytopic region in the state space given by (6.4), we get that

$$\begin{aligned}
\mathbf{x}^T \dot{P}_h \mathbf{x} &= \mathbf{x}^T \begin{bmatrix} P_1 \mathbf{x}_\alpha & \dots P_r \mathbf{x}_\alpha \end{bmatrix} \dot{\mathbf{h}}, \\
&= \mathbf{x}^T Q_\alpha \dot{\mathbf{h}}, \\
&= \frac{1}{2} \left(\mathbf{x}^T Q_\alpha \dot{\mathbf{h}} + \dot{\mathbf{h}}^T Q_\alpha^T \mathbf{x} \right),
\end{aligned}$$

in which we used $\mathbf{x}_\alpha$ to indicate that we replace the state vector by its polytopic representation.

From this point forward, we only need the relations

$$\begin{aligned}
\dot{\mathbf{x}} &= A_h \mathbf{x}, \\
\dot{\mathbf{h}} &= J_g \dot{\mathbf{x}}, \\
\sum_{i=1}^{r} \dot{h}_i &= 0,
\end{aligned}$$

in order to get suitable LMI conditions for the stability problem.

In the following, we present a generalization of this idea to the case in which the Lyapunov function may be described by multiple fuzzy summations. In addition, we apply the conditions to the system in Example 6.1 to illustrate its use.

Theorem 6.1: Nonquadratic stability conditions

Given a system described by (6.1) *in a polytopic region given by* (6.4) *whose membership functions' Jacobian matrix can be described by* (6.3), *and a desired number q of fuzzy summations in the Lyapunov function*

$$V(\mathbf{x}) = \mathbf{x}^T P_{h^q} \mathbf{x}. \tag{6.6}$$

If there exists P_{h^q} and $N_{h^{q-1}\alpha}$ such that

$$\begin{aligned} &P_{h^q} > 0, \\ &\Lambda_{h^q\alpha} + N_{h^{q-1}\alpha}^T \tilde{A}_{hg} + \tilde{A}_{hg}^T N_{h^{q-1}\alpha} < 0, \end{aligned}$$

with

$$\begin{aligned} \Lambda_{h^q\alpha} &= \begin{bmatrix} 0 & P_{h^q} & \frac{1}{2} Q_{h^{q-1}\alpha} \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}, \\ \tilde{A}_{hg} &= \begin{bmatrix} A_h & -I & 0 \\ 0 & J_g & -I \\ 0 & 0 & \mathbf{1}^T \end{bmatrix}, \end{aligned}$$

and

$$Q_{h^{q-1}\alpha} = \begin{bmatrix} P_{h^{q-1}}^{(1)} \mathbf{x}_\alpha & \dots & P_{h^{q-1}}^{(r)} \mathbf{x}_\alpha \end{bmatrix}, \tag{6.7}$$

$$P_{h^{q-1}}^{(k)} = \sum_{i_1=1}^{r} \cdots \sum_{i_{q-1}=1}^{r} \left(\prod_{\ell=1}^{q-1} h_{i_\ell}(\mathbf{x}) \right) \left(P_{k i_1 \dots i_{q-1}} + P_{i_1 k \dots i_{q-1}} + \dots + P_{i_1 \dots k i_{q-1}} + P_{i_1 \dots i_{q-1} k} \right), \tag{6.8}$$

then the system is locally asymptotically stable with the Lyapunov function (6.6) *and an estimate of its domain of attraction is given by the largest sublevel set of the Lyapunov function that is inside the polytopic region described by* (6.4).

Proof. Consider that we have a fuzzy Lyapunov function of the form (6.6).

In order to assure that it is positive definite, it suffices that

$$P_{h^q} > 0.$$

Its time derivative is given by

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^T P_{h^q} \mathbf{x} + \mathbf{x}^T P_{h^q} \dot{\mathbf{x}} + \mathbf{x}^T \dot{P}_{h^q} \mathbf{x}.$$

However, note that we can write

$$
\begin{aligned}
\dot{P}_{h^q} &= \sum_{i_1=1}^{r} \cdots \sum_{i_q=1}^{r} \left(\dot{h}_{i_1} \prod_{\substack{\ell=1 \\ \ell\neq 1}}^{q} h_{i_\ell}(\mathbf{x}) + \cdots + \dot{h}_{i_q} \prod_{\substack{\ell=1 \\ \ell\neq q}}^{q} h_{i_\ell}(\mathbf{x}) \right) P_{i_1\ldots i_q}, \\
&= \sum_{i_1=1}^{r} \cdots \sum_{i_q=1}^{r} \left(\sum_{\kappa=1}^{q} \dot{h}_{i_\kappa} \prod_{\substack{\ell=1 \\ \ell\neq \kappa}}^{q} h_{i_\ell}(\mathbf{x}) \right) P_{i_1\ldots i_q}, \\
&= \sum_{k}^{r} \dot{h}_k \sum_{i_1=1}^{r} \cdots \sum_{i_{q-1}=1}^{r} \left(\prod_{\ell=1}^{q-1} h_{i_\ell}(\mathbf{x}) \right) \\
&\quad \left(P_{k i_1\ldots i_{q-1}} + P_{i_1 k\ldots i_{q-1}} + \cdots + P_{i_1\ldots k i_{q-1}} + P_{i_1\ldots i_{q-1} k} \right).
\end{aligned}
$$

By using $P_{h^{q-1}}^{(k)}$ defined in (6.8) we get that

$$
\begin{aligned}
\mathbf{x}^T \dot{P}_{h^q} \mathbf{x} &= \sum_{k}^{r} \dot{h}_k \mathbf{x}^T P_{h^{q-1}}^{(k)} \mathbf{x}, \\
&= \mathbf{x}^T \begin{bmatrix} P_{h^{q-1}}^{(1)} \mathbf{x} & \ldots & P_{h^{q-1}}^{(r)} \mathbf{x} \end{bmatrix} \begin{bmatrix} \dot{h}_1 \\ \vdots \\ \dot{h}_r \end{bmatrix}, \\
&= \mathbf{x}^T \begin{bmatrix} P_{h^{q-1}}^{(1)} \mathbf{x} & \ldots & P_{h^{q-1}}^{(r)} \mathbf{x} \end{bmatrix} \dot{\mathbf{h}}.
\end{aligned}
$$

Inside the region described by (6.4), we can write

$$
\begin{aligned}
\mathbf{x}^T \dot{P}_{h^q} \mathbf{x} &= \mathbf{x}^T \begin{bmatrix} P_{h^{q-1}}^{(1)} \mathbf{x}_\alpha & \ldots & P_{h^{q-1}}^{(r)} \mathbf{x}_\alpha \end{bmatrix} \dot{\mathbf{h}}, \\
&= \mathbf{x}^T Q_{h^{q-1}\alpha} \dot{\mathbf{h}}, \\
&= \frac{1}{2} \mathbf{x}^T Q_{h^{q-1}\alpha} \dot{\mathbf{h}} + \frac{1}{2} \dot{\mathbf{h}}^T Q_{h^{q-1}\alpha}^T \mathbf{x},
\end{aligned}
$$

with $Q_{h^{q-1}\alpha}$ given by (6.7).

Therefore, the time derivative of the Lyapunov function can be written as

$$
\begin{aligned}
\dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T P_{h^q} \mathbf{x} + \mathbf{x}^T P_{h^q} \dot{\mathbf{x}} + \frac{1}{2} \mathbf{x}^T Q_{h^{q-1}\alpha} \dot{\mathbf{h}} + \frac{1}{2} \dot{\mathbf{h}}^T Q_{h^{q-1}\alpha}^T \mathbf{x}, \\
&= \begin{bmatrix} \mathbf{x}^T \dot{\mathbf{x}}^T \dot{\mathbf{h}}^T \end{bmatrix} \begin{bmatrix} 0 & P_{h^q} & \frac{1}{2} Q_{h^{q-1}\alpha} \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \\ \dot{\mathbf{h}} \end{bmatrix}, \\
&= \boldsymbol{\zeta}^T \Lambda_{h^q\alpha} \boldsymbol{\zeta}.
\end{aligned}
$$

So to check if $\dot{V}(\mathbf{x}) < 0$, we need only to check that $\boldsymbol{\zeta}^T \Lambda_{h^q\alpha} \boldsymbol{\zeta} < 0$ under the equality constraints

$$
\begin{aligned}
\begin{bmatrix} A_h & -I & 0 \\ 0 & J_g & -I \\ 0 & 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \\ \dot{\mathbf{h}} \end{bmatrix} &= 0, \\
\tilde{A}_{hg} \boldsymbol{\zeta} &= 0,
\end{aligned}
$$

in which $\mathbf{1}^T$ represents a row vector of ones.

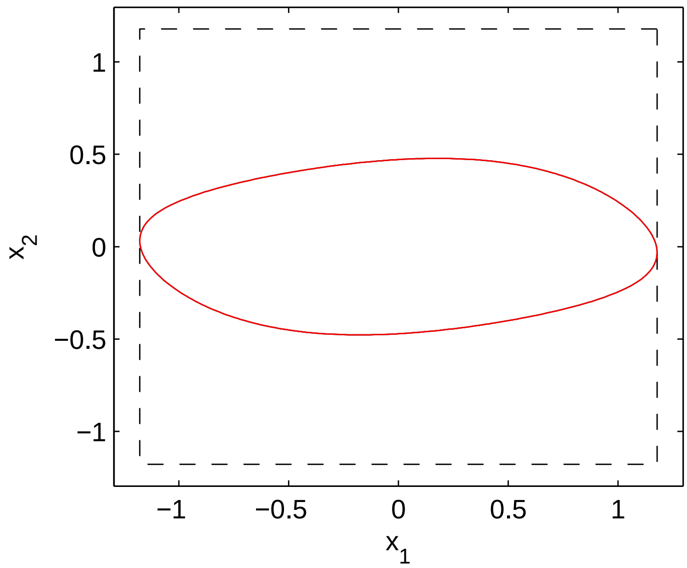

Figure 6.1: Largest domain of attraction found in Example 6.2 using Theorem 6.1 with $q = 1$.

So applying Finsler's Lemma (3 implies 1), and choosing the slack matrices in a way that we do not increase the number of fuzzy summations, we get that sufficient conditions for $\dot{V}(\mathbf{x}) < 0$ are given by

$$\Lambda_{h^q\alpha} + N^T_{h^{q-1}\alpha}\tilde{A}_{hg} + \tilde{A}^T_{hg}N_{h^{q-1}\alpha} < 0.$$

□

Remark 6.1

By making use of Lemma 2.4 to implement Theorem 6.1, we have $\dfrac{(r+q-1)!}{r!(q-1)!}(n+(2n+r)n_g n_\alpha)$ rows of LMIs and $\dfrac{r^q(n^2+n)}{2} + n_\alpha r^{q-1}(n+r+1)(2n+r)$ scalar decision variables, with n the number of states, r the number of rules of the TS model of the system, n_α the number of vertices used to described the local region, q the number of fuzzy summations in the Lyapunov function, and n_g the number of vertices used to describe the Jacobian of the membership functions vector.

Example 6.2. Consider the system in Example 6.1. If we apply the conditions from Theorem 6.1 with $q = 1$, they are not feasible for the region presented in Example 6.1. However, by doing a bisection search, we find that the conditions are feasible in the region $\|\mathbf{x}\|_\infty \leq 1.178$. Figure 6.1 presents the largest Lyapunov sublevel set that fits inside of the polytopic region which is the largest domain of attraction we are able to estimate using $q = 1$ in Theorem 6.1.

Using Theorem 6.1 with $q = 2$, the conditions work inside of the desired region, *i.e.* $\|\mathbf{x}\|_\infty \leq 2$. Figure 6.2 presents the largest Lyapunov sublevel set that fits inside of the polytopic region which is the largest domain of attraction we are able to estimate using $q = 2$ in Theorem 6.1. Figure 6.3 compares this domain of attraction with the ones found using (Lee, Joo, and Tak 2014, Theorem 1) and (Lee and Kim 2014, Theorem 1).

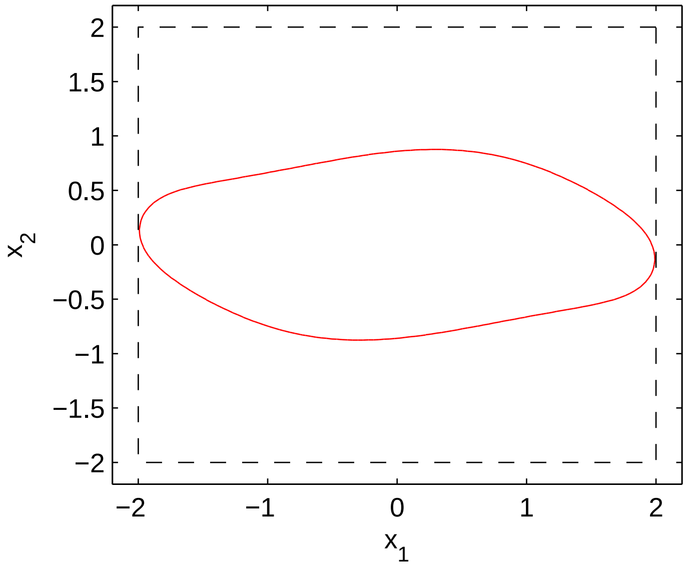

Figure 6.2: Largest domain of attraction found in Example 6.2 using Theorem 6.1 with $q = 2$.

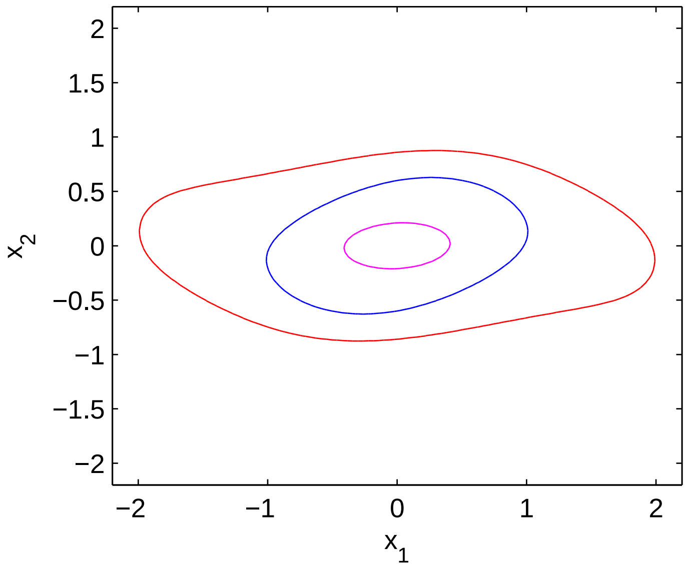

Figure 6.3: Different domains of attraction found in Example 6.2. The outermost region is the one found by Theorem 6.1 with $q = 2$ and $\|\mathbf{x}\|_\infty \leq 2$. The middle one is the one found by using the V-s iteration algorithm of (Lee, Joo, and Tak 2014) with $q = 2$ and 20 iterations. The innermost region is the one found by Theorem 1 of (Lee and Kim 2014) with $q = 2$ and $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 2.59$.