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A fuzzy Lyapunov function approach to estimating the domain of attraction for continuous-time Takagi–Sugeno fuzzy systems

Dong Hwan Lee [a], Jin Bae Park [a,*], Young Hoon Joo [b]

[a] Department of Electrical and Electronic Engineering, Yonsei University, Seodaemun-gu, Seoul 120-749, Republic of Korea
[b] Department of Control and Robotics Engineering, Kunsan National University, Kunsan, Chonbuk 573-701, Republic of Korea

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ABSTRACT

This paper deals with stability analysis and control design problems for continuous-time Takagi–Sugeno (T–S) fuzzy systems. The first aim is to present less conservative linear matrix inequality (LMI) conditions to design controllers and assess the stability. The second relevant contribution is to present a new strategy to find an inner estimate of the domain of attraction (DA) via LMIs. The results are based on the fuzzy Lyapunov functions (FLFs) and non-parallel distributed compensation (non-PDC) approaches. Finally, examples illustrate the effectiveness and merits of the proposed methods.

1. Introduction

Let us consider a continuous-time Takagi–Sugeno (T–S) fuzzy system represented by

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}^m$ is the control input; $i \in \mathcal{I}_r := \{1, 2, \ldots, r\}$ is the rule number; $z(t) \in \mathbb{R}^p$ is the vector containing premise variables in the fuzzy inference rule; $h_i(z(t))$ are the membership functions that belong to class $\mathcal{C}^1$, i.e., they are continuously differentiable, and are subject to the following conditions:

$$0 \leqslant h_i(z(t)) \leqslant 1, \quad \sum_{i=1}^{r} h_i(z(t)) = 1. \tag{2}$$

The stability analysis and control design for (1) keep attracting researchers for decades [1,3–6,8–11,13–19,23–35]. The Lyapunov stability theory is the main approach for these kinds of problems. Among them, the simplest approach consists in looking for a common quadratic Lyapunov function (CQLF) [6,13,14,17,24,31,34,35]. However, the use of a CQLF often leads to overly conservative results because a common Lyapunov matrix should be found for all subsystems of (1).

* Corresponding author. Tel.: +82 2 2123 2773; fax: +82 2 362 4539.
E-mail addresses: hope2010@yonsei.ac.kr (D.H. Lee), jbpark@yonsei.ac.kr (J.B. Park), yhjoo@kunsan.ac.kr (Y.H. Joo).

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To get around this problem, fuzzy Lyapunov functions (FLFs), which depend on the same membership functions as T–S fuzzy system (1), have been investigated in both continuous-time systems [1,18,19,23,28–30,32] and discrete-time systems [5,8–10,15,16]. With respect to discrete-time fuzzy systems, extended FLFs and non-parallel distributed compensation (non-PDC) control laws were investigated in [5,8–10,15,16]. The resultant stability and stabilization conditions were shown to be more effective in reducing the conservativeness of previous results. However, unlike the discrete-time case, the use of FLFs for continuous-time fuzzy systems results in non-convex optimization problems because the time-derivatives of the membership functions appear in the Lyapunov inequality. In an attempt to overcome this obstacle, in [28–30], the stability analysis and control design problems were cast as linear matrix inequality (LMI) conditions, which are solvable through convex optimization techniques [2,7,20], under the assumption that the time-derivatives of the membership functions have the following upper bounds:

$$\left|\dot{h}_\rho(z(t))\right| \leqslant \phi_\rho, \quad \rho \in \mathcal{I}_r, \tag{3}$$

where ϕ_ρ are positive real numbers. In [28–30], a less conservative stability condition was presented by considering the property

$$\sum_{i=1}^{r} \dot{h}_i(z(t)) = 0, \tag{4}$$

and it was further developed in [18] by introducing an additional decision matrix variable. An alternative approach based on a new type of FLF using line-integral was developed in [23] to eliminate the terms involving time-derivatives of the membership functions. Efforts toward control design of continuous-time fuzzy systems using FLFs can be found in [4,19,32]. In [32], a descriptor system representation of the continuous-time fuzzy systems was considered to design a non-PDC control law. Results in [18,23] were extended in [19] to design PDC control laws and further investigated in [4] to design non-PDC control laws.

On the other hand, another important issue in stability analysis of nonlinear systems may be how to estimate the domain of attraction (DA). As in the stability problems, such estimates can be obtained based on the Lyapunov theory [12]. Specifically, for a Lyapunov function $V(x(t))$ which guarantees the local stability of the equilibrium, any sublevel set of the Lyapunov function is an inner estimate of the DA if the set belongs to the region where $V(x(t)) > 0$ and $\dot{V}(x(t)) < 0$ hold for all $x(t) \neq 0$ [12]. However, especially when dealing with continuous-time fuzzy systems along with FLFs, what makes the problem more challenging is that additional assumptions (2) and (3) should be considered. In [28,29],through assuming a polytopic bound on the derivatives of the membership functions, constraint (3) was turned into LMIs. Consequently, stability and stabilization conditions that depend on the initial states and guarantee (3) were derived in terms of LMIs. However, to date and to the best of our knowledge, systematic approaches to estimating the DA for continuous-time T–S fuzzy systems have not been fully investigated yet.

Motivated by the discussions above, this paper aims at establishing an effective and systematic framework to estimate the DA for stability analysis and control design of continuous-time T–S fuzzy systems. First, less conservative sufficient conditions for stability analysis and control design are derived by extending those in [18,19] and generalizing established FLFs and non-PDC control laws. Then, motivated by the work in [29], a strategy to compute inner estimates of the DA is presented. The proposed method is beneficial in that inner estimates of the DA can be determined by solving eigenvalue problems (EVPs) [2] which can be efficiently handled by means of convex optimization techniques [2,7,20]. Finally, examples illustrate the effectiveness and merits of the proposed methods.

2. Preliminaries

The following notation will be used throughout this paper: $\mathbb{R}$ denotes the set of real numbers. 0_n denotes the origin of $\mathbb{R}^n$. For a real matrix A, the transpose of A is denoted by A^T. The notion $A \succ 0$("$\succcurlyeq$", "$\prec$", and "$\preccurlyeq$", respectively) is used to denote positive definiteness (positive semi-definiteness, negative definiteness, and negative semi-definiteness, respectively), while $a > 0$("$\geqslant$", "<", and "$\leqslant$", respectively) means that scalar a is positive number (nonnegative, negative, and nonpositive number, respectively). An asterisk $(*)$ inside a matrix represents the transpose of its symmetric term. e_i denotes a vector with one at entry i and zeros elsewhere, i.e.

$$e_i := \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1}_{i-\text{th}} & 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{R}^n.$$

The following notations will be adopted for simplicity:

$$X_{z(t)} := \sum_{i=1}^{r} h_i(z(t))X_i, \quad X_{z(t)z(t)} := \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))X_{ij}, \quad X_{z(t)}^{-1} := \left(\sum_{i=1}^{r} h_i(z(t))X_i\right)^{-1}, \quad \dot{X}_{z(t)}^{-1} := \frac{dX_{z(t)}^{-1}}{dt},$$

$$\dot{X}_{z(t)z(t)} := \sum_{i=1}^{r}\sum_{j=1}^{r}\left\{\dot{h}_i(z(t))h_j(z(t))X_{ij} + h_i(z(t))\dot{h}_j(z(t))X_{ij}\right\}.$$

Let us consider the continuous-time nonlinear system

$$\dot{x}(t) = f(x(t))x(t) + g(x(t))u(t), \tag{5}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ are matrix-valued nonlinear functions such that $f(0_n) = 0_n$, i.e., the origin is an equilibrium point of (5) with $u(t) = 0_m$. A wide range of nonlinear systems in the form of (5) can be exactly represented by T–S fuzzy system (1) in the set $\mathbf{H}_1 \subseteq \mathbb{R}^n$ of state variables:

$$\mathbf{H}_1 := \left\{ x(t) \in \mathbb{R}^n \middle| f(x(t)) = \sum_{i=1}^{r} h_i(z(t))A_i, g(x(t)) = \sum_{i=1}^{r} h_i(z(t))B_i \right\},$$

which contains the origin. It is useful to introduce a subset $\mathbf{C}_1 \subseteq \mathbf{H}_1$ that can be described as

$$\mathbf{C}_1 := \left\{ x(t) \in \mathbb{R}^n \middle| |x_i(t)| \leqslant \bar{x}_{1,i}, i \in \mathbf{I} \right\}, \tag{6}$$

where $\mathbf{I} \subseteq \{1,2,\ldots,n\}$ is the set of indexes for state variables $x_i(t)$ that compose the premise variables. Note that, in many cases, one can set $\mathbf{H}_1 = \mathbf{C}_1$. For the membership functions that belong to class $\mathcal{C}^1$, define the set $\mathbf{R} \subset \mathbb{R}^n$ of state variables where constraint (3) holds:

$$\mathbf{R} := \left\{ x(t) \in \mathbb{R}^n \middle| \left| \dot{h}_\rho(z(t)) \right| \leqslant \phi_\rho, \ \rho \in \mathcal{I}_r \right\}, \tag{7}$$

which will be used through the development. The DA of the origin is the set of initial conditions for which the state asymptotically converges to the origin:

$$\mathbf{D} := \left\{ x(0) \in \mathbb{R}^n \middle| \lim_{t \to +\infty} x(t) = 0_n \right\}.$$

Suppose that $V(x(t)) = x^T(t)P(x(t))x(t)$ is a Lyapunov function for the origin in (1), i.e., for a domain $\mathbf{U} \subset \mathbb{R}^n$ containing the origin, $V : \mathbf{U} \to \mathbb{R}$ is a continuously differentiable function, $V(0_n) = 0$, and $V(x(t)) > 0$, $\forall x(t) \in \mathbf{U} - \{0_n\}$ such that the time derivative of $V(x(t))$ along the trajectories of (1) is locally negative definite, i.e. $\dot{V}(x(t)) < 0, \ \forall x(t) \in \mathbf{U} - \{0_n\}$. Then, for a real number $c > 0$, the sublevel set

$$\boldsymbol{\Omega}(P(x(t)), c) := \left\{ x(t) \in \mathbb{R}^n \middle| x^T(t)P(x(t))x(t) \leqslant c \right\}$$

is an inner estimate of the DA, i.e., $\boldsymbol{\Omega}(P(x(t)),c) \subseteq \mathbf{D}$ if $\boldsymbol{\Omega}(P(x(t)),c) \subseteq \mathbf{U}$ [12]. Moreover, the largest inner estimate of the DA whose shape is fixed by $V(x(t))$ is $\boldsymbol{\Omega}(P(x(t)),c^*)$, where $c^* = \max\{c \in \mathbb{R} | \boldsymbol{\Omega}(P(x(t)), c) \subseteq \mathbf{U}\}$. We end this section by introducing useful lemmas which play important roles in the development.

Lemma 1. ([22]) *The following two statements are equivalent:*

1) *Find $P = P^T \succ 0$, such that: $H + A^TP + PA \prec 0$.*
2) *Find $P = P^T \succ 0$, L, and R such that*

$$\begin{bmatrix} H + A^TL^T + LA & (*) \\ P - L^T + R^TA & -R - R^T \end{bmatrix} \prec 0.$$

Lemma 2. ([21]) *The following two statements are equivalent:*

1) *Find $P = P^T \succ 0$ such that $A^TPA - H \prec 0$.*
2) *Find $P = P^T \succ 0$ and G such that*

$$\begin{bmatrix} -H & (*) \\ GA & -G - G^T + P \end{bmatrix} \prec 0.$$

Lemma 3. ([31]) *Let the symmetric matrices $\Upsilon_{ij}, \ i, \ j \in \mathcal{I}_r$. Inequality $\Upsilon_{z(k)z(k)} \prec 0$ holds if the following condition is fulfilled:*

$$\Upsilon_{ij} + \Upsilon_{ji} \prec 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r.$$

Remark 1. For the purpose of fair comparison, we will apply Lemma 3 directly to all the results in this paper rather than the relaxation techniques presented in [13,17,31,33].

3. Stability

In this section, two stability conditions for the unforced system of (1) (by setting $u(t) = 0_m$) are presented by extending results in [18,19]. The key point for deriving the first stability condition is to introduce additional slack variables into the condition of Theorem 6 in [19].

Theorem 1. *Consider assumption (3). If there exist symmetric matrices P_i and M_{ij} such that*

$$P_i \succ 0, \quad i \in \mathcal{I}_r, \tag{8}$$

$$\Upsilon_{ij} + \Upsilon_{ji} \prec 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \tag{9}$$

$$\widetilde{\Upsilon}_{ij}^{\rho} + \widetilde{\Upsilon}_{ji}^{\rho} \succ 0, \quad i \leqslant j, \quad i,j,\rho \in \mathcal{I}_r \tag{10}$$

hold, where $\widetilde{\Upsilon}_{ij}^{\rho} := P_\rho + M_{ij}$ and $\Upsilon_{ij} := A_i^T P_j + P_j A_i + \sum_{\rho=1}^{r} \phi_\rho \widetilde{\Upsilon}_{ij}^{\rho}$, then the unforced system of (1) is asymptotically stable. Moreover, $\boldsymbol{\Omega}(P_{z(t)}, c^*)$ is an inner estimate of the DA, where

$$c^* = \max\left\{c \in \mathbb{R} \middle| \boldsymbol{\Omega}\left(P_{z(t)}, c\right) \subseteq \mathbf{R} \cap \mathbf{C}_1\right\}.$$

Proof. Let us consider $V(x(t)) = x^T(t)P_{z(t)}x(t)$ proposed in [10,28–30] a FLF candidate. Since the membership functions belong to class $\mathcal{C}^1$, $V(x(t))$ is continuously differentiable. From (8), $V(x(t)) > 0$, $\forall x(t) \in \mathbf{C}_1 - \{0_n\}$ is guaranteed. If (9) holds, then by Lemma 3, one has

$$A_{z(t)}^T P_{z(t)} + P_{z(t)} A_{z(t)} + \sum_{\rho=1}^{r} \phi_\rho \left(P_\rho + M_{z(t)z(t)}\right) \prec 0, \quad \forall x(t) \in \mathbf{C}_1.$$

The satisfaction of (10) ensures $P_\rho + M_{z(t)z(t)} \succ 0$, $\forall x(t) \in \mathbf{C}_1$, $\rho \in \mathcal{I}_r$. Then, from the definition (7), it follows that

$$\sum_{\rho=1}^{r} \dot{h}_\rho(z(t))\left(P_\rho + M_{z(t)z(t)}\right) \preccurlyeq \sum_{\rho=1}^{r} \phi_\rho \left(P_\rho + M_{z(t)z(t)}\right), \quad \forall x(t) \in \mathbf{R} \cap \mathbf{C}_1.$$

Also, according to (4), $\sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) M_{z(t)z(t)} = 0$, $\forall x \in \mathbf{C}_1$ is satisfied. Thus, one can prove

$$0 \succ A_{z(t)}^T P_{z(t)} + P_{z(t)} A_{z(t)} + \sum_{\rho=1}^{r} \phi_\rho \left(P_\rho + M_{z(t)z(t)}\right) \succcurlyeq A_{z(t)}^T P_{z(t)} + P_{z(t)} A_{z(t)} + \sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) P_\rho, \quad \forall x(t) \in \mathbf{R} \cap \mathbf{C}_1,$$

which implies $\dot{V}(x(t)) < 0$, $\forall x(t) \in (\mathbf{R} \cap \mathbf{C}_1) - \{0_n\}$, and hence, the unforced system of (1) is asymptotically stable. In summary, $V : \mathbf{R} \cap \mathbf{C}_1 \to \mathbb{R}$ is a continuously differentiable function, $V(0_n) = 0$ and $V(x(t)) > 0$, $\dot{V}(x(t)) < 0$, $\forall x(t) \in (\mathbf{R} \cap \mathbf{C}_1) - \{0_n\}$. Hence, if $\boldsymbol{\Omega}(P_{z(t)}, c) \subseteq \mathbf{R} \cap \mathbf{C}_1$, then $\boldsymbol{\Omega}(P_{z(t)}, c)$ is an inner estimate of the DA, and the largest inner estimate of the DA whose shape is fixed by $V(x(t)) = x^T(t)P_{z(t)}x(t)$ is $\boldsymbol{\Omega}(P_{z(t)}, c^*)$, where $c^* = \max\left\{c \in \mathbb{R} \middle| \boldsymbol{\Omega}\left(P_{z(t)}, c\right) \subseteq \mathbf{R} \cap \mathbf{C}_1\right\}$. □

Remark 2. The condition of Theorem 6 in [19] is recovered by setting $M_{ij} = M$ in that of Theorem 1. This means that the latter contains the former as a special case, and Theorem 1 always shows better results, or at least the same, than Theorem 6 in [19].

By generalizing Theorem 1 and Theorem 1 in [18], we have the next theorem.

Theorem 2. *Consider assumption (3). If there exist symmetric matrices P_{ij}, M_{ij}, matrices L_i and R_i such that*

$$P_{ij} + P_{ji} \succ 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \tag{11}$$

$$\Upsilon_{ij} + \Upsilon_{ji} \prec 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \tag{12}$$

$$\widetilde{\Upsilon}_{ij}^{\rho} + \widetilde{\Upsilon}_{ji}^{\rho} \succ 0, \quad i \leqslant j, \quad i,j,\rho \in \mathcal{I}_r \tag{13}$$

hold, where

$$\widetilde{\Upsilon}_{ij}^{\rho} := P_{\rho j} + P_{i\rho} + M_{ij}, \quad \Upsilon_{ij} := \begin{bmatrix} A_i^T L_j^T + L_j A_i + \sum_{\rho=1}^{r} \phi_\rho \widetilde{\Upsilon}_{ij}^{\rho} & (*) \\ P_{ij} - L_i^T + R_j^T A_i & -R_i - R_i^T \end{bmatrix},$$

then the unforced system of (1) is asymptotically stable. Moreover, $\boldsymbol{\Omega}(P_{z(t)z(t)}, c^*)$ is an inner estimate of the DA, where $c^* = \max\left\{c \in \mathbb{R} \middle| \boldsymbol{\Omega}\left(P_{z(t)z(t)}, c\right) \subseteq \mathbf{R} \cap \mathbf{C}_1\right\}$.

Proof. Let us consider an extended FLF candidate $V(x(t)) = x^T(t)P_{z(t)z(t)}x(t)$, which is continuously differentiable. According to (11) along with Lemma 3, $V(x(t)) > 0$, $\forall x(t) \in \mathbf{C}_1 - \{0_n\}$ is guaranteed. Also, it follows from (12) and Lemma 3 that

$$\begin{bmatrix} \begin{pmatrix} A_{z(t)}^T L_{z(t)}^T + L_{z(t)}A_{z(t)} \\ + \sum_{\rho=1}^{r} \phi_\rho \widetilde{\Upsilon}_{z(t)z(t)}^{\rho} \end{pmatrix} & (*) \\ P_{z(t)z(t)} - L_{z(t)}^T + R_{z(t)}^T A_{z(t)} & -R_{z(t)} - R_{z(t)}^T \end{bmatrix} \prec 0, \quad \forall x(t) \in \mathbf{C}_1.$$

Then, by Lemma 1, it follows that

$$A_{z(t)}^T P_{z(t)z(t)} + P_{z(t)z(t)}A_{z(t)} + \sum_{\rho=1}^{r} \phi_\rho \left\{ \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\big(P_{\rho j} + P_{i\rho} + M_{ij}\big) \right\} \prec 0, \quad \forall x(t) \in \mathbf{C}_1. \tag{14}$$

By using Lemma 3, the satisfaction of (13) ensures

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\big(P_{\rho j} + P_{i\rho} + M_{ij}\big) \succ 0, \quad \forall x(t) \in \mathbf{C}_1, \quad \rho \in \mathcal{I}_r.$$

Thus, from definition (7), we can prove that

$$\sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) \left\{ \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\big(P_{\rho j} + P_{i\rho} + M_{ij}\big) \right\} \preccurlyeq \sum_{\rho=1}^{r} \phi_\rho \left\{ \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\big(P_{\rho j} + P_{i\rho} + M_{ij}\big) \right\},$$
$$\forall x(t) \in \mathbf{R} \cap \mathbf{C}_1.$$

Since $\sum_{\rho=1}^{r} \dot{h}_\rho(z(t))M_{z(t)z(t)} = 0, \ \forall x \in \mathbf{C}_1$ is satisfied from property (4), it follows from (14) that

$$\begin{aligned} 0 &\succ A_{z(t)}^T P_{z(t)z(t)} + P_{z(t)z(t)}A_{z(t)} + \sum_{\rho=1}^{r} \phi_\rho \left\{ \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\big(P_{\rho j} + P_{i\rho} + M_{ij}\big) \right\} \succcurlyeq A_{z(t)}^T P_{z(t)z(t)} + P_{z(t)z(t)}A_{z(t)} \\ &\quad + \sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) \left\{ \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\big(P_{\rho j} + P_{i\rho} + M_{ij}\big) \right\} \\ &= A_{z(t)}^T P_{z(t)z(t)} + P_{z(t)z(t)}A_{z(t)} + \dot{P}_{z(t)z(t)}, \quad \forall x(t) \in \mathbf{R} \cap \mathbf{C}_1, \end{aligned}$$

which implies $\dot{V}(x(t)) < 0, \ \forall x(t) \in (\mathbf{R} \cap \mathbf{C}_1) - \{0_n\}$. Thus, the unforced system of (1) is asymptotically stable. Finally, similarly to Theorem 1, the largest inner estimate of the DA whose shape is fixed by $V(x(t))$ is $\boldsymbol{\Omega}(P_{z(t)z(t)}, c^*)$, where $c^* = \max\{c \in \mathbb{R} | \boldsymbol{\Omega}(P_{z(t)z(t)}, c) \subseteq \mathbf{R} \cap \mathbf{C}_1\}$. □

Remark 3.

1) The condition of Theorem 2 contains that of Theorem 1 as a special case because the latter is recovered by setting $P_{ij} = P_i$, $L_i = P_i$, $L_j = P_j$, $P_{kj} = P_{ik} = P_k$, and $R_i = R_j = \varepsilon I$ with sufficiently small $\varepsilon > 0$ in the former.
2) The condition of Theorem 1 in [18] is recovered by setting $P_{ii} = P_{ij} = P_{ji} = P_i$, $P_{kj} = P_{ik} = P_k$, $M_{ij} = M$, $R_i = R_j = R$, and $L_i = L_j = L$ in that of Theorem 2.

Example 1. Let us consider (1) taken from [19] with

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & a \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & -4 \\ \frac{1}{5}(3b-2) & \frac{1}{5}(3a-4) \end{bmatrix}, \quad A_3 = \begin{bmatrix} -3 & -4 \\ \frac{1}{5}(2b-3) & \frac{1}{5}(2a-6) \end{bmatrix}, \quad A_4 = \begin{bmatrix} -2 & -4 \\ b & -2 \end{bmatrix}.$$

The stability of this system is checked using Theorem 6 in [19], Theorem 1 in [18], Theorems 1, and 2 for several values of pairs (a,b), $a \in [-10,1]$, $b \in [0,200]$, and $\phi_\rho = 0.85, \ \rho \in \mathcal{I}_r$. The results are depicted in Fig. 1(a) and (b), which reveal that Theorems 1 and 2 provide less conservative results than Theorem 6 in [19] and Theorem 1 in [18], respectively.

Remark 4. Another kind of conservativeness can be induced from the bounds on the time-derivatives of the membership functions (i.e., constraint (3)). Although approaches based on FLFs can produce less conservative results in many cases, the region of the state variables that satisfies constraint (3) can be a tiny area around the origin especially when the parameters $\phi_\rho, \ \rho \in \mathcal{I}_r$ are small. As a result, the DA can be confined to narrow limits in the state-space. Hence, there is a trade-off between the degree of conservativeness with the bounds of the DA. The designer's task therefore is to find appropriate choices for $\phi_\rho, \ \rho \in \mathcal{I}_r$ to keep the benefit of the proposed approach. This point will be illustrated in Example 2.

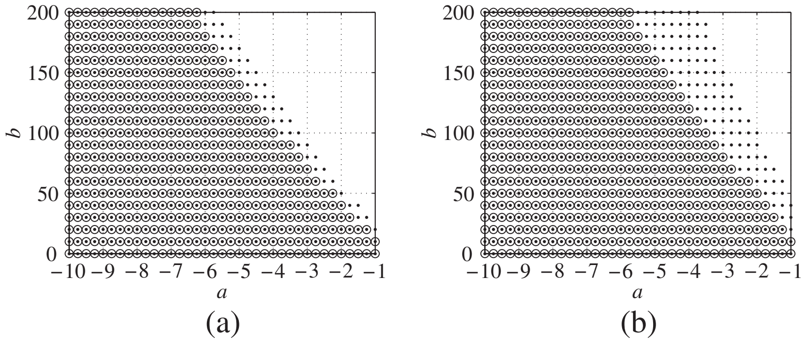

Fig. 1. Example 1. (a) Stability region based on Theorem 6 of [19] (∘) and Theorem 1 (·). (b) Stability region based on Theorem 1 of [18] (∘) and Theorem 2 (·).

Example 2. Let us consider (1) with

$$A_1 = \begin{bmatrix} -2 & 4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 4 \\ -(1+\lambda) & -2 \end{bmatrix}, \quad h_1(z(t)) = \frac{1+\sin x_1(t)}{2},$$
$$h_2(z(t)) = 1 - h_1(z(t)), \quad \mathbf{C}_1 = \{x(t) \in \mathbb{R}^n | |x_i(t)| \leqslant \pi/2, i \in \{1,2\}\}.$$

The maximum values of λ, denoted by λ^*, such that the stability is guaranteed were checked by using Theorems 1, and 2, Theorem 1 in [31] (CQLF approach), Theorem 6 in [19], and Theorem 1 in [18] for several values of $\phi_1 = \phi_2$. The results are illustrated in Fig. 2. As these can be seen, depending on the upper bound selection $\phi_1 = \phi_2$, the proposed approaches always show better results, or at least the same, than the previous ones. Also, as $\phi_\rho,\ \rho \in \mathcal{I}_2$ decrease, less conservative results are obtained. The boundaries of $\mathbf{R}$ (solid line), $\boldsymbol{\Omega}(P_{z(t)z(t)}, c^*)$ (dashdot line), $\mathbf{C}_1$ (dotted line), the region of $\mathbf{R} \cap \mathbf{C}_1$ (colored area), and a converging trajectory (dashed line) obtained by using Theorem 2 with $\lambda = 30,\ \phi_\rho = 1.4,\ \rho \in \mathcal{I}_r$ and $\lambda = 20,\ \phi_\rho = 3.4,\ \rho \in \mathcal{I}_r$ are depicted in Fig. 3(a) and (b), respectively, where the boundaries of the several regions were

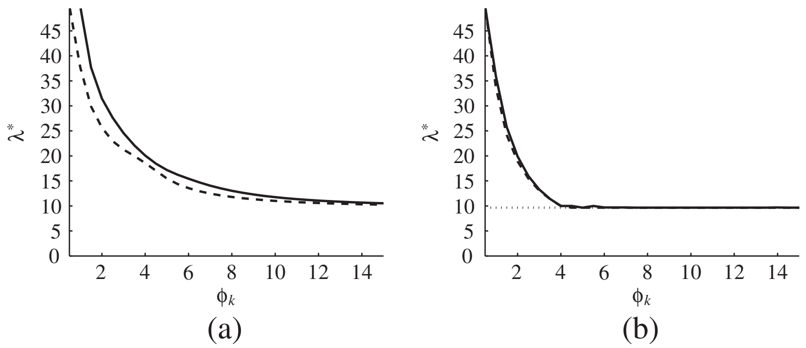

Fig. 2. Example 2. Allowable upper bounds of λ computed by using several approaches for different values of $\phi_1 = \phi_2$. (a) The results of Theorems 1 (dashed line) and 2 (solid line). (b) The results of Theorem 1 in [31] (dotted line), Theorem 6 in [19] (dashed line), and Theorem 1 in [18] (solid line).

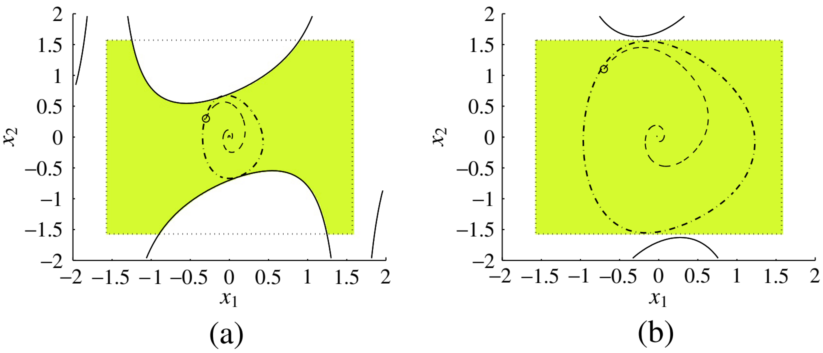

Fig. 3. Example 2. Boundaries of $\mathbf{R}$ (solid line), $\boldsymbol{\Omega}(P_{z(t)z(t)}, c^*)$ (dashdot line), and $\mathbf{C}_1$ (dotted line), the region $\mathbf{R} \cap \mathbf{C}_1$ (colored area), and a converging trajectory (dashed line) initialized at the "∘" mark. (a) The results of Theorem 2 with $\lambda = 30$ and $\phi_\rho = 1.4,\ \rho \in \mathcal{I}_r$. (b) The results of Theorem 2 with $\lambda = 20$ and $\phi_\rho = 3.4,\ \rho \in \mathcal{I}_r$.

computed using an exhaustive search over a fine grid in the parameter space, and c^* could be obtained by a visual evaluation or a simple unidimensional search procedure. Note that, for $\lambda = 30,\ \phi_\rho = 3.4,\ \rho \in \mathcal{I}_r$, the condition of Theorem 2 does not admit a feasible solution, which means that, as $\phi_\rho,\ \rho \in \mathcal{I}_r$ decrease, the conservativeness vanishes while the region of $\boldsymbol{\Omega}(P_{z(t)z(t)}, c^*)$ tends to shrink as it is shown in Fig. 3(a) and (b).

Note that, in Example 2, the upper bounds for the time-derivatives of the membership functions were chosen arbitrarily, and then, the boundaries of the several regions and c^* were computed via a gridding procedure. Conversely, the DA can be estimated in such a way that the values $\phi_\rho,\ \rho \in \mathcal{I}_r$ are computed in a prescribed region such as the following hyper-rectangle in the state-space:

$$\mathbf{S}(\delta) := \{x(t) \in \mathbb{R}^n | |x_i(t)| \leqslant \delta,\ i \in \{1, 2, \ldots, n\}\},$$

where δ is a positive real number, and then, the LMI problem of Theorem 1 or Theorem 2 is solved by using the computed values $\phi_\rho,\ \rho \in \mathcal{I}_r$. This method is summarized in the following procedure:

Step 1 Set $i = 1$. Initialize parameters $\delta_i > 0$ and $\Delta > 0$ small enough.

Step 2 Compute the values $\phi_\rho,\ \rho \in \mathcal{I}_r$ via a gridding procedure in the parameter space $\mathbf{S}(\delta_i)$:

$$\phi_\rho = \max_{x(t) \in \mathbf{S}(\delta_i)} \left| \dot{h}_\rho(z(t)) \right|, \quad \rho \in \mathcal{I}_r.$$

Step 3 With the values $\phi_\rho,\ \rho \in \mathcal{I}_r$ obtained in Step 2, solve the LMI problem in Theorem 1 or 2. If the LMI problem is feasible, set $i = i + 1$, $\delta_i = \delta_{i-1} + \Delta$, and return to Step 2. Iterate from Step 2 until the condition fails to find a feasible solution or the boundary of $\mathbf{S}(\delta_i)$ borders on that of $\mathbf{C}_1$. Consequently, obtain the maximum value of δ_i, denoted by δ^*, such that the system is identified as stable for $\phi_\rho,\ \rho \in \mathcal{I}_r$.

Step 4 Compute

$$c^* = \max\{c \in \mathbb{R} | \boldsymbol{\Omega}(\bullet, c) \subseteq \mathbf{S}(\delta^*)\}, \tag{15}$$

where $\bullet$ denotes any Lyapunov matrix. It can be numerically estimated via a gridding procedure by using the fact that (15) can be equivalently rewritten as $c^* = \min\{V(x(t)) \in \mathbb{R} | x(t) \in \partial\mathbf{S}(\delta^*)\}$, where $\partial\mathbf{S}(\delta^*)$ denotes the boundary of $\mathbf{S}(\delta^*)$. Note that $\mathbf{S}(\delta^*) \subseteq \mathbf{R} \cap \mathbf{C}_1$ is fulfilled, so $\boldsymbol{\Omega}(\bullet, c^*)$ is an inner estimate of the DA.

Example 3. To illustrate the aforementioned computational steps, let us consider the system used in Example 2 with $\lambda = 33$. With $\delta_1 = 0.1$ and $\Delta = 0.01$ and after 71 iterations, we obtained $c^* = 1.5494 \times 10^{-6}$ and $\phi_1 = \phi_2 = 1.8459$ by using Theorem 2. Fig. 4 shows the boundary of $\boldsymbol{\Omega}(\bullet, c^*)$ and this demonstrates the validity of the proposed method. Note that, for $\phi_1 = \phi_2 = 1.8459$, the LMI problem of Theorem 6 in [19] failed to find a feasible solution.

As shown in Examples 2 and 3, inner estimates of the DA can be computed throughout numerical procedures. However, these methods cannot offer a general way to determine the estimates, and numerically tedious computations are required. Furthermore, for higher order systems, to handle and evaluate the set $\boldsymbol{\Omega}(\bullet, c^*)$ may be a very difficult task. Hence, those possibilities are not further considered in this paper. Now, a question naturally arises: how can we obtain a sharp estimate of $\boldsymbol{\Omega}(\bullet, c^*)$ in a computationally efficient fashion? An answer to this question can be found in [29]. Following the concept proposed in [29], suppose that

$$z(t) = \left[x_{a_1}(t)\ x_{a_2}(t)\ \cdots\ x_{a_p}(t)\right]^T \in \mathbb{R}^p,$$

where $\mathbf{I} := \{a_1, a_2, \ldots a_p\} \subseteq \{1, 2, \ldots, n\}$ is a set of indexes. Also, let us assume that $\dot{h}_\rho(z(t)),\ \rho \in \mathcal{I}_r$ can be expressed by

$$\dot{h}_\rho(z(t)) = \frac{\partial h_\rho(z(t))}{\partial x(t)}\dot{x}(t) = \sum_{l=1}^{s} v_{\rho l}(x(t))\xi_{\rho l}\dot{x}(t), \quad \rho \in \mathcal{I}_r \tag{16}$$

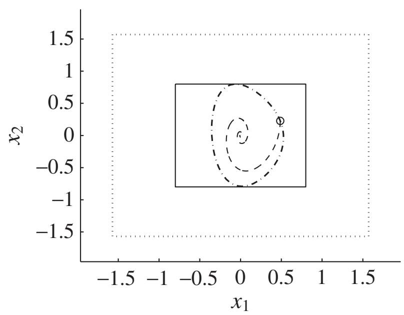

Fig. 4. Example 3. Boundaries of $\mathbf{S}(\delta^*)$ (solid line), $\boldsymbol{\Omega}(P_{z(t)z(t)}, c^*)$ (dashdot line), and $\mathbf{C}_1$ (dotted line), and a converging trajectory (dashed line) initialized at the "○" mark. Theorem 2 with $\lambda = 33$ and $\phi_\rho = 1.8459,\ \rho \in \mathcal{I}_r$ was used.

in the set $\mathbf{H}_2$ of state variables:

$$\mathbf{H}_2 := \left\{ x(t) \in \mathbb{R}^n \left| \frac{\partial h_\rho(z(t))}{\partial x(t)} = \sum_{l=1}^{s} \upsilon_{\rho l}(x(t))\xi_{\rho l},\ \forall \rho \in \mathcal{I}_r \right. \right\},$$

where $\xi_{\rho l}^T \in \mathbb{R}^n$ is a constant vector and $\upsilon_{\rho l}(x(t))$ are nonlinear functions which satisfy the properties $0 \leqslant \upsilon_{\rho l}(x(t)) \leqslant 1$ and $\sum_{l=1}^{s} \upsilon_{\rho l}(x(t)) = 1$. Define a set $\mathbf{C}_2 \subseteq \mathbf{H}_2$ described as $\mathbf{C}_2 := \{x(t) \in \mathbb{R}^n | |x_i(t)| \leqslant \bar{x}_{2,i},\ i \in \mathbf{I}\}$. Then, it is useful to express $\mathbf{C} := \mathbf{C}_1 \cap \mathbf{C}_2$ as $\mathbf{C} = \{x(t) \in \mathbb{R}^n | |x_i(t)| \leqslant \bar{x}_i,\ i \in \mathbf{I}\}$. Based on the assumption and definitions, we derive two sufficient conditions whose solutions can be efficiently obtained by solving EVPs [2]. Moreover, the conditions not only establish whether the unforced system of (1) is asymptotically stable under constraint (3) but allow one to obtain a sharp estimate of $\boldsymbol{\Omega}(c^*)$ as well.

Theorem 3. *If there exist symmetric matrices P_i, M_{ij}, and scalars g_i such that the following EVP has a solution:*

$$\underset{P_i, M_{ij}, g_i}{\text{Minimize}}\ \alpha \text{ subject to}$$

$$\text{LMIs in Theorem 1}, \tag{17}$$

$$\Omega_{ij}^{\rho l} + \Omega_{ji}^{\rho l} \preccurlyeq 0, \quad i \leqslant j, \quad i, j, \rho \in \mathcal{I}_r, \quad l \in \{1, 2, \ldots, s\}, \tag{18}$$

$$\frac{1}{\bar{x}_k^2} e_k e_k^T \prec P_i, \quad i \in \mathcal{I}_r, \quad k \in \mathbf{I}, \tag{19}$$

$$P_i \preccurlyeq \alpha I, \quad i \in \mathcal{I}_r, \tag{20}$$

where

$$\Omega_{ij}^{\rho l} := \begin{bmatrix} -P_i & (*) \\ g_j \xi_{\rho l} A_i & -2g_i + \frac{1}{\phi_\rho^2} \end{bmatrix},$$

then the unforced system of (1) is asymptotically stable. Moreover, $\boldsymbol{\Omega}(P_{z(t)}, 1)$ is an inner estimate of the DA and the boundary of $\boldsymbol{\Omega}(P_{z(t)}, 1)$ is enlarged as close as possible to that of $\boldsymbol{\Omega}(P_{z(t)}, c^*)$, where

$$c^* = \max \left\{ c \in \mathbb{R} \middle| \boldsymbol{\Omega}(P_{z(t)}, c) \subseteq \mathbf{R} \cap \mathbf{C} \right\}.$$

Proof. The proof consists of several parts.

1) **Part 1. Proof of $\boldsymbol{\Omega}(P_{z(t)},\ 1) \subseteq \mathbf{C}$** : If (19) holds, then one has

$$\frac{1}{\bar{x}_k^2} e_k e_k^T \prec P_{z(t)},\ \forall x(t) \in \mathbf{C}, k \in \mathbf{I}.$$

Define

$$\mathbf{Z}_k := \left\{ \zeta \in \mathbf{C} \mid e_k^T \zeta = \bar{x}_k \right\} \cup \left\{ \zeta \in \mathbf{C} \mid e_k^T \zeta = -\bar{x}_k \right\},\ \forall k \in \mathbf{I}.$$

Then, multiplying the above inequality by ζ_k^T on the left and its transpose on the right, where $\zeta_k \in \mathbf{Z}_k$, one gets

$$\zeta_k^T P_{z(t)} \zeta_k > 1,\ \forall x(t) \in \mathbf{C}, \zeta_k \in \mathbf{Z}_k, k \in \mathbf{I}.$$

This implies $\mathbf{Z}_k \subset \boldsymbol{\Omega}(P_{z(t),}^{-1}\ 1)^C \cap \mathbf{C}, \forall k \in \mathbf{I}$, and hence

$$\bigcup_{k \in \mathbf{I}} \mathbf{Z}_k = \partial \mathbf{C} \subset \boldsymbol{\Omega}(P_{z(t)}, 1)^C \cap \mathbf{C},$$

where $\partial \mathbf{C}$ and $\Omega(P_{z(t)}, 1)^C$ denote the boundary of $\mathbf{C}$ and the complement of $\boldsymbol{\Omega}(P_{z(t)}, 1)$, respectively. Therefore, from $\partial \mathbf{C} \subset \boldsymbol{\Omega}(P_{z(t)}, 1)^C \cap \mathbf{C}$, one concludes $\boldsymbol{\Omega}(P_{z(t)}, 1) \subseteq \mathbf{C}$.

2) **Part 2. Proof of $\boldsymbol{\Omega}(P_{z(t)}, 1) \subseteq (\mathbf{R} \cap \mathbf{C})$**: By Lemma 3, satisfying (18) guarantees

$$\sum_{l=1}^{s} \upsilon_{\rho l}(z(t)) \begin{bmatrix} -P_{z(t)} & (*) \\ g_{z(t)} \xi_{\rho l} A_{z(t)} & -2g_{z(t)} + \frac{1}{\phi_\rho^2} \end{bmatrix} \preccurlyeq 0, \quad \forall x(t) \in \mathbf{C}, \quad \rho \in \mathcal{I}_r.$$

Using Lemma 2, we have

$$\begin{aligned} &\frac{1}{\phi_\rho^2} x^T(t) \left(\sum_{l=1}^{s} \upsilon_{\rho l}(z(t)) \xi_{\rho l} A_{z(t)} \right)^T \left(\sum_{l=1}^{s} \upsilon_{\rho l}(z(t)) \xi_{\rho l} A_{z(t)} \right) x(t) - x^T(t) P_{z(t)} x(t) \\ &= \frac{1}{\phi_\rho^2} \left(\sum_{l=1}^{s} \upsilon_{\rho l}(z(t)) \xi_{\rho l} \dot{x}(t) \right)^T \left(\sum_{l=1}^{s} \upsilon_{\rho l}(z(t)) \xi_{\rho l} \dot{x}(t) \right) - x^T(t) P_{z(t)} x(t) = \frac{1}{\phi_\rho^2} \dot{h}_\rho^2(z(t)) - x^T(t) P_{z(t)} x(t) \leqslant 0, \\ &\quad \forall x(t) \in \mathbf{C}, \quad \rho \in \mathcal{I}_r, \end{aligned}$$

which means that

$$(\boldsymbol{\Omega}(P_{z(t)},1)\cap\mathbf{C})\subseteq\left(\left\{x(t)\in\mathbb{R}^n\middle|\frac{1}{\phi_\rho^2}\dot{h}_\rho^2(z(t))\leqslant 1,\rho\in\mathcal{I}_r\right\}\cap\mathbf{C}\right).$$

Since

$$\left\{x(t)\in\mathbb{R}^n\middle|\left(1/\phi_\rho^2\right)\dot{h}_\rho^2(z(t))\leqslant 1,\ \rho\in\mathcal{I}_r\right\}=\left\{x(t)\in\mathbb{R}^n\middle|\left|\dot{h}_\rho(z(t))\right|\leqslant\phi_\rho,\ \rho\in\mathcal{I}_r\right\}=\mathbf{R}$$

and $\boldsymbol{\Omega}(P_{z(t)},1)\subseteq\mathbf{C}$ from Part 1, one can conclude that $\boldsymbol{\Omega}(P_{z(t)},1)\subseteq(\mathbf{R}\cap\mathbf{C})$.

3) **Part 3. Proof of $\boldsymbol{\Omega}(P_{z(t)},1)\subseteq\mathbf{D}$**: If the LMI problem of Theorem 1 holds, then $\boldsymbol{\Omega}(P_{z(t)},c^*)$ is an inner estimate of the DA, where $c^*=\max\{c\in\mathbb{R}|\boldsymbol{\Omega}(P_{z(t)},c)\subseteq\mathbf{R}\cap\mathbf{C}_1\}$. Since $(\mathbf{R}\cap\mathbf{C})\subseteq(\mathbf{R}\cap\mathbf{C}_1)$, one can conclude from Part 2 that $\boldsymbol{\Omega}(P_{z(t)},1)\subseteq\mathbf{D}$.
4) **Part 4. The boundary of $\boldsymbol{\Omega}(P_{z(t)},1)$ is enlarged as close as possible to the boundary of $\boldsymbol{\Omega}(P_{z(t)},c^*)$ with** $c^*=\max\{c\in\mathbb{R}|\boldsymbol{\Omega}(P_{z(t)},c)\subseteq\mathbf{R}\cap\mathbf{C}\}$: If (20) holds, then $x^T(t)P_{z(t)}x(t)\leqslant\alpha x^T(t)x(t),\forall x(t)\in\mathbf{C}$, which means that $\{x(t)\in\mathbb{R}^n|x^T(t)x(t)\leqslant 1/\alpha\}\subseteq\boldsymbol{\Omega}(P_{z(t)},1)$. Hence, minimizing α while imposing constraint $\{x(t)\in\mathbb{R}^n|x^T(t)x(t)\leqslant 1/\alpha\}\subseteq\boldsymbol{\Omega}(P_{z(t)},1)$ makes $\boldsymbol{\Omega}(P_{z(t)},1)$ to be maximized. Moreover, since $\boldsymbol{\Omega}(P_{z(t)},1)\subseteq(\mathbf{R}\cap\mathbf{C})$ from part 2, the boundary of $\boldsymbol{\Omega}(P_{z(t)},1)$ is enlarged as close as possible to the boundary of $\boldsymbol{\Omega}(P_{z(t)},c^*)$ with $c^*=\max\{c\in\mathbb{R}|\boldsymbol{\Omega}(P_{z(t)},c)\subseteq\mathbf{R}\cap\mathbf{C}\}$. This completes the proof.

Theorem 4. *If there exist symmetric matrices P_{ij}, M_{ij}, matrices L_i,R_i, and scalars g_i such that the following EVP has a solution:*

$$\underset{P_{ij},M_{ij},L_i,R_i,g_i}{\text{Minimize}}\ \alpha \text{ subject to}$$

$$\textit{LMIs in Theorem 2},$$

$$\Omega_{ij}^{\rho l}+\Omega_{ji}^{\rho l}\preccurlyeq 0,\quad i\leqslant j,\quad i,j,\rho\in\mathcal{I}_r,\quad l\in\{1,2,\ldots,s\},$$

$$\Psi_{ij}^{k}+\Psi_{ji}^{k}\prec 0,\quad i\leqslant j,\quad i,j\in\mathcal{I}_r,\quad k\in\mathbf{I},$$

$$P_{ij}+P_{ji}\preccurlyeq 2\alpha I,\quad i\leqslant j,\quad i,j\in\mathcal{I}_r,$$

where

$$\Omega_{ij}^{\rho l}:=\begin{bmatrix}-P_{ij} & (*)\\ g_j\xi_{\rho l}A_i & -2g_i+\frac{1}{\phi_\rho^2}\end{bmatrix},\quad \Psi_{ij}^{k}:=\frac{1}{\bar{x}_k^2}e_ke_k^T-P_{ij},$$

then the unforced system of (1) is asymptotically stable. Moreover, $\boldsymbol{\Omega}(P_{z(t)z(t)},1)$ is an inner estimate of the DA and the boundary of $\boldsymbol{\Omega}(P_{z(t)z(t)},1)$ is enlarged as close as possible to that of $\boldsymbol{\Omega}(P_{z(t)z(t)},c^*)$, where

$$c^*=\max\left\{c\in\mathbb{R}\middle|\boldsymbol{\Omega}\left(P_{z(t)z(t)},c\right)\subseteq\mathbf{R}\cap\mathbf{C}\right\}.$$

Proof. The proof is similar to that of Theorem 3, being thus omitted here for brevity. □

Remark 5. Several remarks can be made on the results above.

1) A similar idea was previously introduced in [29], i.e. constraint (3) was converted into LMIs. The major differences between the approaches in [29] and those of this paper is that the conditions of Theorems 3 and 4 are independent of initial states and offer an inner estimate of the DA whereas those of [29] are dependent on the initial states and do not consider whether the initial state identified as stable is inside the boundary of the largest sublevel set $\boldsymbol{\Omega}(\bullet,c^*)$.
2) Although assumption (16) is useful, it may be a strong restriction when dealing with many practical systems.
3) Even though the EVP in Theorems 3 and 4 maximize the size of the level curve $\boldsymbol{\Omega}(\bullet,1)$, it cannot become the largest sublevel set $\boldsymbol{\Omega}(\bullet,c^*)$ in most cases because the conditions are only sufficient for the existence of the Lyapunov functions.

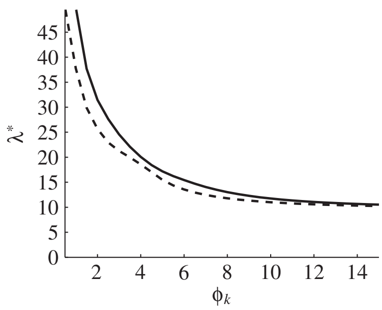

Fig. 5. Example 4. Allowable upper bounds of λ computed by using Theorems 3 (dashed line) and 4 (solid line) for different values of $\phi_1=\phi_2$.

Example 4. Let us consider Example 2 again. From the membership functions, we have

$$\xi_{11} = [0\ 0], \quad \xi_{12} = [0.5\ 0], \quad \xi_{21} = [-0.5\ 0], \quad \xi_{22} = [0\ 0], \tag{21}$$
$$\upsilon_{11}(x(t)) = 1 - \cos x_1(t), \quad \upsilon_{12}(x(t)) = \cos x_1(t), \quad \upsilon_{21}(x(t)) = \cos x_1(t), \quad \upsilon_{22}(x(t)) = 1 - \cos x_1(t), \tag{22}$$
$$\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{C} = \{x(t) \in \mathbb{R}^n \| x_i(t)| \leqslant \pi/2,\ i \in \mathcal{I}_2\}.$$

A detailed description for the derivation process of (21) and (22) can be found in [29], being omitted here for brevity. The maximum values of λ, denoted by λ^*, were computed by using Theorems 3 and 4 for several values of $\phi_1 = \phi_2$, and Fig. 5 shows the results. By comparing Fig. 2(a) with Fig. 5, the results of Theorems 3 and 4 seem to be the same as those of Theorems 1 and 2 with no more apparent conservativeness. Hence, the additional constraints in terms of EVPs involved in Theorems 3 and 4 may not lead to more conservative results. In addition, using Theorem 4 for $\lambda = 30,\ \phi_\rho = 1.4,\ \rho \in \mathcal{I}_r$ and $\lambda = 20,\ \phi_\rho = 3.4,\ \rho \in \mathcal{I}_r$, we can readily obtain the boundary of $\boldsymbol{\Omega}(P_{z(t)z(t)}, 1)$ illustrated in Fig. 6. From the figure, one can conclude that the proposed methods allow us to achieve satisfactorily accurate estimates of $\boldsymbol{\Omega}(P_{z(t)z(t)}, c^*)$ in a computationally efficient fashion. In contrast, approaches in [29] cannot provide a feasible solution for any initial state around the origin, and this demonstrates the less conservativeness of the proposed methods compared with the existing ones.

4. Stabilization

First of all, three sufficient conditions for the solution of stabilization problem are provided. To this end, we apply non-PDC control laws presented in [10] for discrete-time T–S fuzzy systems to the continuous-time ones, and extend them in order to reduce the conservativeness. The following theorem effectively combines Theorem 1 and the non-PDC control law-based stabilization method.

Theorem 5. *Consider assumption (3). If there exist symmetric matrices P_i, M_{ij}, and matrices K_i such that*

$$P_i \succ 0, \quad i \in \mathcal{I}_r, \tag{23}$$

$$\Upsilon_{ij} + \Upsilon_{ji} \prec 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \tag{24}$$

$$\widetilde{\Upsilon}_{ij}^{\rho} + \widetilde{\Upsilon}_{ji}^{\rho} \succ 0, \quad i \leqslant j, \quad i,j,\rho \in \mathcal{I}_r, \tag{25}$$

hold, where $\widetilde{\Upsilon}_{ij}^{\rho} := P_\rho + M_{ij}$ and $\Upsilon_{ij} := A_iP_j + B_iK_j + P_jA_i^T + K_j^TB_i^T + \sum_{\rho=1}^{r}\phi_\rho\widetilde{\Upsilon}_{ij}^{\rho}$, then the closed-loop system (1) with the non-PDC control law [10]

$$u(t) = K_{z(t)}P_{z(t)}^{-1}x(t) \tag{26}$$

is asymptotically stable. Moreover, $\boldsymbol{\Omega}(P_{z(t)}, c^*)$ is an inner estimate of the DA, where

$$c^* = \max\{c \in \mathbb{R} | \boldsymbol{\Omega}(P_{z(t)}, c) \subseteq \mathbf{R} \cap \mathbf{C}_1\}.$$

Proof. Let us consider the FLF candidate $V(x(t)) = x^T(t)P_{z(t)}^{-1}x(t)$ proposed in [10]. From (23), $V(x(t)) > 0$, $\forall x(t) \in \mathbf{C}_1 - \{0_n\}$ is guaranteed. If (24) holds, then by Lemma 3, one gets

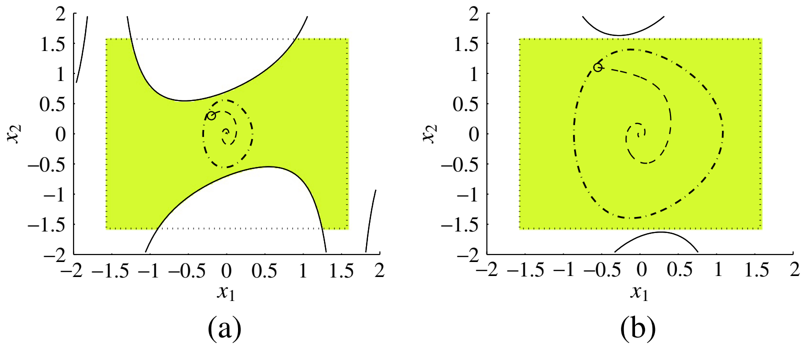

Fig. 6. Example 4. Boundaries of **R** (solid line), $\boldsymbol{\Omega}(P_{z(t)z(t)}, 1)$ (dashdot line), **C** (dotted line), the region of $\mathbf{R} \cap \mathbf{C}$ (colored area), and a converging trajectory (dashed line) initialized at the "∘" mark. (a) The results of Theorem 4 with $\lambda = 30$ and $\phi_\rho = 1.4,\ \rho \in \mathcal{I}_r$. (b) The results of Theorem 4 with $\lambda = 20$ and $\phi_\rho = 3.4,\ \rho \in \mathcal{I}_r$.

$$P_{z(t)}A_{z(t)}^T + K_{z(t)}^T B_{z(t)}^T + A_{z(t)}P_{z(t)} + B_{z(t)}K_{z(t)} + \sum_{\rho=1}^{r} \phi_\rho (P_\rho + M_{z(t)z(t)}) \prec 0, \quad \forall x(t) \in \mathbf{C}_1.$$

Then, by following the same lines as in the proof of Theorem 1, one can prove

$$0 \succ P_{z(t)}A_{z(t)}^T + K_{z(t)}^T B_{z(t)}^T + A_{z(t)}P_{z(t)} + B_{z(t)}K_{z(t)} + \sum_{\rho=1}^{r} \phi_\rho (P_\rho + M_{z(t)z(t)}) \succcurlyeq P_{z(t)}A_{z(t)}^T + K_{z(t)}^T B_{z(t)}^T + A_{z(t)}P_{z(t)} + B_{z(t)}K_{z(t)} - \dot{P}_{z(t)},$$
$$\forall x(t) \in \mathbf{R} \cap \mathbf{C}_1. \tag{27}$$

Using the relation $\dot{P}_{z(t)}^{-1} = -P_{z(t)}^{-1}\dot{P}_{z(t)}P_{z(t)}^{-1}$ and multiplying (27) by $P_{z(t)}^{-1}$ left and right lead

$$\left(A_{z(t)} + B_{z(t)}K_{z(t)}P_{z(t)}^{-1}\right)^T P_{z(t)}^{-1} + P_{z(t)}^{-1}\left(A_{z(t)} + B_{z(t)}K_{z(t)}P_{z(t)}^{-1}\right) + \dot{P}_{z(t)}^{-1} \prec 0, \quad \forall x(t) \in \mathbf{R} \cap \mathbf{C}_1,$$

which implies $\dot{V}(x(t)) < 0$, $\forall x(t) \in (\mathbf{R} \cap \mathbf{C}_1) - \{0_n\}$. In summary, $V : \mathbf{R} \cap \mathbf{C}_1 \to \mathbb{R}$ is a continuously differentiable function, $V(0_n) = 0$, and $V(x(t)) > 0$, $\dot{V}(x(t)) < 0$, $\forall x(t) \in (\mathbf{R} \cap \mathbf{C}_1) - \{0_n\}$. Therefore, the closed-loop system (1) with (26) is asymptotically stable. In addition, from the similar argument as in the proof of Theorem 1, the largest inner estimate of the DA whose shape is fixed by $V(x(t))$ is $\boldsymbol{\Omega}(P_{z(t)}^{-1}, c^*)$, where $c^* = \max\{c \in \mathbb{R} | \ \boldsymbol{\Omega}(P_{z(t)}^{-1}, c) \subseteq \mathbf{R} \cap \mathbf{C}_1\}$. $\square$

By extending $V(x(t)) = x^T(t)P_{z(t)}^{-1}x(t)$, the next theorem generalizes Theorem 5.

Theorem 6. *Consider assumption (3). If there exist symmetric matrices P_{ij}, M_{ij}, matrices K_i, G_i, and L_i such that*

$$P_{ij} + P_{ji} \succ 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \tag{28}$$
$$\Upsilon_{ij} + \Upsilon_{ji} \prec 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \tag{29}$$
$$\widetilde{\Upsilon}_{ij}^{\rho} + \widetilde{\Upsilon}_{ji}^{\rho} \succ 0, \quad i \leqslant j, \quad i,j,\rho \in \mathcal{I}_r \tag{30}$$

hold, where

$$\widetilde{\Upsilon}_{ij}^{\rho} := P_{i\rho} + P_{\rho j} + M_{ij}, \quad \Upsilon_{ij} := \begin{bmatrix} \begin{pmatrix} K_j^T B_i^T + B_i K_j \\ + A_i L_j^T + L_j A_i^T + \sum_{\rho=1}^{r} \phi_\rho \widetilde{\Upsilon}_{ij}^{\rho} \end{pmatrix} & (*) \\ P_{ij} - L_i^T + G_j^T A_i^T & -G_i - G_i^T \end{bmatrix},$$

then the closed-loop system (1) with the non-PDC control law

$$u(t) = K_{z(t)} P_{z(t)z(t)}^{-1} x(t) \tag{31}$$

is asymptotically stable. Moreover, $\boldsymbol{\Omega}(P_{z(t)z(t)}^{-1}, c^*)$ is an inner estimate of the DA, where

$$c^* = \max\left\{c \in \mathbb{R} \middle| \boldsymbol{\Omega}\left(P_{z(t)z(t)}^{-1}, c\right) \subseteq \mathbf{R} \cap \mathbf{C}_1\right\}.$$

Proof. Let us consider the extended FLF candidate $V(x(t)) = x^T(t)P_{z(t)z(t)}^{-1}x(t)$. From (28) and Lemma 3, $V(x(t)) > 0$, $\forall x(t) \in \mathbf{C}_1 - \{0_n\}$ is guaranteed. If (29) holds, then by Lemma 3, we have

$$\begin{bmatrix} \begin{pmatrix} K_{z(t)}^T B_{z(t)}^T + B_{z(t)}K_{z(t)} + A_{z(t)}L_{z(t)}^T + L_{z(t)}A_{z(t)}^T \\ + \sum_{\rho=1}^{r} \phi_\rho \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t))h_j(z(t))(P_{\rho j} + P_{i\rho} + M_{ij}) \end{pmatrix} & (*) \\ P_{z(t)z(t)} - L_{z(t)}^T + G_{z(t)}^T A_{z(t)}^T & -G_{z(t)} - G_{z(t)}^T \end{bmatrix} \prec 0, \quad \forall x(t) \in \mathbf{C}_1.$$

By means of Lemma 1, it follows that

$$P_{z(t)z(t)}A_{z(t)}^T + K_{z(t)}^T B_{z(t)}^T + A_{z(t)}P_{z(t)z(t)} + B_{z(t)}K_{z(t)} + \sum_{\rho=1}^{r} \phi_\rho \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t))h_j(z(t))(P_{\rho j} + P_{i\rho} + M_{ij}) \prec 0, \quad \forall x(t) \in \mathbf{C}_1.$$

Then, by following the same lines as in the proof of Theorem 2, one gets

$$0 \succ P_{z(t)z(t)}A_{z(t)}^T + K_{z(t)}^T B_{z(t)}^T + A_{z(t)}P_{z(t)z(t)} + B_{z(t)}K_{z(t)} + \sum_{\rho=1}^{r} \phi_\rho \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t))h_j(z(t))(P_{\rho j} + P_{i\rho} + M_{ij}) \succcurlyeq P_{z(t)z(t)}A_{z(t)}^T$$
$$+ K_{z(t)}^T B_{z(t)}^T + A_{z(t)}P_{z(t)z(t)} + B_{z(t)}K_{z(t)} - \dot{P}_{z(t)z(t)}, \quad \forall x(t) \in \mathbf{R} \cap \mathbf{C}_1.$$

Using the relation $\dot{P}_{z(t)z(t)}^{-1} = -P_{z(t)z(t)}^{-1}\dot{P}_{z(t)z(t)}P_{z(t)z(t)}^{-1}$ and multiplying left and right by $P_{z(t)z(t)}^{-1}$ yield

$$\left(A_{z(t)} + B_{z(t)}K_{z(t)}P^{-1}_{z(t)z(t)}\right)^T P^{-1}_{z(t)z(t)} + P^{-1}_{z(t)z(t)}\left(A_{z(t)} + B_{z(t)}K_{z(t)}P^{-1}_{z(t)z(t)}\right) + \dot{P}^{-1}_{z(t)z(t)} \prec 0, \quad \forall x(t) \in \mathbf{R} \cap \mathbf{C}_1,$$

which implies $\dot{V}(x(t)) < 0,\ \forall x(t) \in (\mathbf{R} \cap \mathbf{C}_1) - \{0_n\}$. In summary, $V : \mathbf{R} \cap \mathbf{C}_1 \to \mathbb{R}$ is a continuously differentiable function, $V(0_n) = 0$, and $V(x(t)) > 0,\ \dot{V}(x(t)) < 0,\ \forall x(t) \in (\mathbf{R} \cap \mathbf{C}_1) - \{0_n\}$. Therefore, the closed-loop system (1) with (31) is asymptotically stable. In addition, from the similar argument as in the proof of Theorem 2, the largest inner estimate of the DA whose shape is fixed by $V(x(t))$ is $\boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, c^*)$, where $c^* = \max\{c \in \mathbb{R} | \boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, c) \subseteq \mathbf{R} \cap \mathbf{C}_1\}$. □

Remark 6. The condition of Theorem 5 is recovered by setting $P_{ij} = P_i,\ P_{ik} = P_{kj} = P_k,\ L_i^T = P_i$, and $G_i = G_j = \varepsilon I$ with sufficiently small $\varepsilon > 0$ in the condition of Theorem 6.

Finally, the following theorem is an alternative to Theorem 6 for reducing the conservativeness.

Theorem 7. *Let $\mu > 0$ be a given scalar and consider assumption (3). If there exist symmetric matrices P_{ij}, $M^{(11)}_{ij},\ M^{(22)}_{ij}$, matrices $M^{(21)}_{ij} = \left(M^{(12)}_{ij}\right)^T,\ K_i$, and G_i such that*

$$P_{ij} + P_{ji} \succ 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \tag{32}$$

$$\Upsilon_{ij} + \Upsilon_{ji} \prec 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \tag{33}$$

$$\widetilde{\Upsilon}^{\rho}_{ij} + \widetilde{\Upsilon}^{\rho}_{ji} \succ 0, \quad i \leqslant j, \quad i,j,\rho \in \mathcal{I}_r \tag{34}$$

hold, where

$$\widetilde{\Upsilon}^{\rho}_{ij} := \begin{bmatrix} \begin{pmatrix} -G_\rho - G_\rho^T \\ +P_{\rho j} + P_{i\rho} + M^{(11)}_{ij} \end{pmatrix} & (*) \\ -\mu G_\rho + M^{(21)}_{ij} & M^{(22)}_{ij} \end{bmatrix}, \quad \Upsilon_{ij} := \begin{bmatrix} \begin{pmatrix} G_j^T A_i^T + K_j^T B_i^T \\ +A_i G_j + B_i K_j \end{pmatrix} & (*) \\ \mu A_i G_j + \mu B_i K_j + P_{ij} - G_i^T & -\mu G_i - \mu G_i^T \end{bmatrix} + \sum_{\rho=1}^{r} \phi_\rho \widetilde{\Upsilon}^{\rho}_{ij},$$

then the closed-loop system (1) with the non-PDC control law [10]

$$u(t) = K_{z(t)} G^{-1}_{z(t)} x(t) \tag{35}$$

is asymptotically stable. Moreover, $\boldsymbol{\Omega}(G^{-T}_{z(t)} P_{z(t)z(t)} G^{-1}_{z(t)}, c^*)$ is an inner estimate of the DA, where

$$c^* = \max\left\{c \in \mathbb{R} \middle| \boldsymbol{\Omega}\left(G^{-T}_{z(t)} P_{z(t)z(t)} G^{-1}_{z(t)}, c\right) \subseteq \mathbf{R} \cap \mathbf{C}_1\right\}. \tag{36}$$

Proof. First of all, it is necessary to check the existence of $G^{-1}_{z(t)}$. If (33) and (34) hold, then we have $\mu(G_i + G_i^T) \succ \sum_{\rho=1}^{r} \phi_\rho M^{(22)}_{ii} \succ 0,\ i \in \mathcal{I}_r$. Therefore, $G_{z(t)} + G^T_{z(t)} \succ 0,\ \forall x(t) \in \mathbf{C}_1$, which ensures that $G^{-1}_{z(t)}$ exists for all $x(t) \in \mathbf{C}_1$. Let us consider the extended FLF candidate $V(x(t)) = x^T(t) G^{-T}_{z(t)} P_{z(t)z(t)} G^{-1}_{z(t)} x(t)$ which generalizes the FLF proposed in [10]. Since the satisfaction of (32) along with Lemma 3 ensures that $P_{z(t)z(t)} \succ 0$, $\forall x(t) \in \mathbf{C}_1$, one has $G^{-T}_{z(t)} P_{z(t)z(t)} G^{-1}_{z(t)} \succ 0,\ \forall x(t) \in \mathbf{C}_1$, which means that $V(x(t)) > 0$, $\forall x(t) \in \mathbf{C}_1 - \{0_n\}$. If (33) and (34) hold, then by Lemma 3, we have

$$\begin{bmatrix} \begin{pmatrix} G^T_{z(t)} A^T_{z(t)} + K^T_{z(t)} B^T_{z(t)} \\ +A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)} \end{pmatrix} & (*) \\ \mu\left(A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)}\right) + P_{z(t)z(t)} - G^T_{z(t)} & -\mu G_{z(t)} - \mu G^T_{z(t)} \end{bmatrix} + \sum_{\rho=1}^{r} \phi_\rho \widetilde{\Upsilon}^{\rho}_{z(t)z(t)} \prec 0, \quad \forall x(t) \in \mathbf{C}_1 \tag{37}$$

and $\widetilde{\Upsilon}^{\rho}_{z(t)z(t)} \succ 0$, $\forall x(t) \in \mathbf{C}_1$, $\rho \in \mathcal{I}_r$. Thus, from definition (7), one can prove that

$$\sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) \widetilde{\Upsilon}^{\rho}_{z(t)z(t)} \preccurlyeq \sum_{\rho=1}^{r} \phi_\rho \widetilde{\Upsilon}^{\rho}_{z(t)z(t)}, \quad \forall x(t) \in \mathbf{R} \cap \mathbf{C}_1.$$

According to property (4), the following equations hold:

$$\sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) \widetilde{\Upsilon}^{\rho}_{z(t)z(t)} = \begin{bmatrix} -\dot{G}_{z(t)} - \dot{G}^T_{z(t)} + \dot{P}_{z(t)z(t)} & (*) \\ -\mu \dot{G}_{z(t)} & 0 \end{bmatrix} + \sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) \begin{bmatrix} M^{(11)}_{z(t)z(t)} & (*) \\ M^{(21)}_{z(t)z(t)} & M^{(22)}_{z(t)z(t)} \end{bmatrix} = \begin{bmatrix} -\dot{G}_{z(t)} - \dot{G}^T_{z(t)} + \dot{P}_{z(t)z(t)} & (*) \\ -\mu \dot{G}_{z(t)} & 0 \end{bmatrix},$$

$\forall x(t) \in \mathbf{C}_1$.

Thus, it follows from (37) that

$$\begin{aligned}
0 &\succ \begin{bmatrix} G_{z(t)}^T A_{z(t)}^T + K_{z(t)}^T B_{z(t)}^T + A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)} & (*) \\ \mu\left(A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)}\right) + P_{z(t)z(t)} - G_{z(t)}^T & -\mu G_{z(t)} - \mu G_{z(t)}^T \end{bmatrix} \\
&\quad + \sum_{\rho=1}^{r} \phi_\rho \widetilde{\Upsilon}_{z(t)z(t)}^{\rho} \succeq \begin{bmatrix} \begin{pmatrix} \left(A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)} - \dot{G}_{z(t)}\right)^T \\ + \left(A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)} - \dot{G}_{z(t)}\right) + \dot{P}_{z(t)z(t)} \end{pmatrix} & (*) \\ \mu\left(A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)} - \dot{G}_{z(t)}\right) + P_{z(t)z(t)} - G_{z(t)}^T & -\mu G_{z(t)} - \mu G_{z(t)}^T \end{bmatrix} \\
&= \begin{bmatrix} \begin{pmatrix} \left(A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)} - \dot{G}_{z(t)}\right)^T G_{z(t)}^{-T} L_{z(t)}^T \\ + L_{z(t)} G_{z(t)}^{-1} \left(A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)} - \dot{G}_{z(t)}\right) + \dot{P}_{z(t)z(t)} \end{pmatrix} & (*) \\ R_{z(t)}^T G_{z(t)}^{-1} \left(A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)} - \dot{G}_{z(t)}\right) + P_{z(t)z(t)} - L_{z(t)}^T & -R_{z(t)} - R_{z(t)}^T \end{bmatrix}, \quad \forall x(t) \in \mathbf{R} \cap \mathbf{C}_1, \qquad (38)
\end{aligned}$$

where $L_{z(t)}^T = G_{z(t)}^T$ and $R_{z(t)} = \mu G_{z(t)}^T$. Set $\overline{A}_{z(t)z(t)} := G_{z(t)}^{-1}\left(A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)} - \dot{G}_{z(t)}\right)$. Then, multiply (38) by $\begin{bmatrix} I & \overline{A}_{z(t)z(t)}^T \end{bmatrix}$ on the left and by its transpose on the right or simply make use of Lemma 1 to obtain

$$\left(G_{z(t)}^T A_{z(t)}^T + K_{z(t)}^T B_{z(t)}^T - \dot{G}_{z(t)}^T\right) G_{z(t)}^{-T} P_{z(t)z(t)} + P_{z(t)z(t)} G_{z(t)}^{-1} \left(A_{z(t)} G_{z(t)} + B_{z(t)} K_{z(t)} - \dot{G}_{z(t)}\right) + \dot{P}_{z(t)z(t)} \prec 0, \quad \forall x(t) \in \mathbf{R} \cap \mathbf{C}_1. \qquad (39)$$

By using the relation $\dot{G}_{z(t)}^{-1} = -G_{z(t)}^{-1} \dot{G}_{z(t)} G_{z(t)}^{-1}$ and by multiplying (39) by $G_{z(t)}^{-T}$ on the left and by $G_{z(t)}^{-1}$ on the right, it can be shown that

$$\begin{aligned}
&\left(A_{z(t)} + B_{z(t)} K_{z(t)} G_{z(t)}^{-1}\right)^T G_{z(t)}^{-T} P_{z(t)z(t)} G_{z(t)}^{-1} + G_{z(t)}^{-T} P_{z(t)z(t)} G_{z(t)}^{-1} \left(A_{z(t)} + B_{z(t)} K_{z(t)} G_{z(t)}^{-1}\right) + \dot{G}_{z(t)}^{-T} P_{z(t)z(t)} G_{z(t)}^{-1} + G_{z(t)}^{-T} P_{z(t)z(t)} \dot{G}_{z(t)}^{-1} \\
&+ G_{z(t)}^{-T} \dot{P}_{z(t)z(t)} G_{z(t)}^{-1} \prec 0, \quad \forall x(t) \in \mathbf{R} \cap \mathbf{C}_1,
\end{aligned}$$

which implies $\dot{V}(x(t)) < 0,\ \forall x(t) \in (\mathbf{R} \cap \mathbf{C}_1) - \{0_n\}$. In summary, $V : \mathbf{R} \cap \mathbf{C}_1 \to \mathbb{R}$ is a continuously differentiable function, $V(0_n) = 0$, and $V(x(t)) > 0,\ \dot{V}(x(t)) < 0,\ \forall x(t) \in (\mathbf{R} \cap \mathbf{C}_1) - \{0_n\}$. Therefore, the closed-loop system (1) with (35) is asymptotically stable. In addition, similarly to Theorem 1, the largest inner estimate of the DA whose shape is fixed by $V(x(t))$ is $\boldsymbol{\Omega}(G_{z(t)}^{-T} P_{z(t)z(t)} G_{z(t)}^{-1}, c^*)$, where $c^* = \max\{c \in \mathbb{R} | \boldsymbol{\Omega}(G_{z(t)}^{-T} P_{z(t)z(t)} G_{z(t)}^{-1}, c) \subseteq \mathbf{R} \cap \mathbf{C}_1\}$. □

Remark 7. Several remarks can be made on the results above.

1) The condition of Theorem 5 is recovered by setting $\mu = 0,\ G_i = P_i,\ G_j = P_j,\ P_{ij} = P_i,\ M_{ij}^{(11)} = M_{ij},\ M_{ij}^{(21)} = 0$, and $M_{ij}^{(22)} = \varepsilon I$ with sufficiently small $\varepsilon > 0$ in the condition of Theorem 7.
2) Unlike the comparison of Theorems 5 and 7, it is difficult to theoretically compare the degree of conservativeness between Theorems 6 and 7. But later, we will use an example (see Example 5) to show that Theorem 6 can be less conservative than Theorem 7 for certain systems.
3) The condition of Theorem 6 in [18] is recovered by setting $G_i = G_j = G,\ P_{ij} = P_i,\ G_k = 0,\ P_{kj} = P_{ik} = P_k$, $M_{ij}^{(11)} = M,\ M_{ij}^{(21)} = 0$, and $M_{ij}^{(22)} = \varepsilon I$ with sufficiently small $\varepsilon > 0$ in the condition of Theorem 7.
4) A condition to obtain the PDC control law $u(t) = F_{z(t)}x(t)$ can be derived by setting $G_k = 0$ and $G_i = G_j = G$ in (32)–(34). In this case, the fuzzy local feedback gains are $F_i = K_i G^{-1},\ i \in \mathcal{I}_r$. In addition, taking $P_{ij} = P_i,\ P_{kj} = P_{ik} = P_k$, $M_{ij}^{(11)} = M,\ M_{ij}^{(21)} = 0$, and $M_{ij}^{(22)} = \varepsilon I$ with sufficiently small $\varepsilon > 0$, the condition of Theorem 7 reduces to that of Theorem 6 in [18].
5) In Theorem 7, a question that arises naturally is how to choose the parameter μ which can be viewed as a tuning parameter. Up to the authors knowledge, there is no formal procedure to determine the parameter. This may be a limitation of the approach. Typically, it can be determined by a combination of previous expertise and trial and error. In Example 6, we will show how the parameter affects the conservativeness of the condition.

Example 5. Let us consider (1) taken from [18] with

$$A_1 = \begin{bmatrix} 3.6 & -1.6 \\ 6.2 & -4.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -a & -1.6 \\ 6.2 & -4.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.45 \\ -3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -b \\ -3 \end{bmatrix},$$

The stability of the closed-loop system was checked for several values of pairs (a,b), $a \in [0,60]$ and $b \in [0,4]$, where Corollary 3 in [32] (Case 1), Theorem 10 in [4] with $\phi_1 = \phi_2 = 1$ (Case 2), Theorem 5 with $\phi_1 = \phi_2 = 1$ (Case 3), Theorem 6 with $\phi_1 = \phi_2 = 1$ (Case 4), Theorem 7 with $\mu = 0.04$ and $\phi_1 = \phi_2 = 1$ (Case 5), Theorem 6 in [18] with $\mu = 0.04$ and $\phi_1 = \phi_2 = 1$ (Case 6), and Theorem 7 with PDC control law, $\mu = 0.04$, and $\phi_1 = \phi_2 = 1$ (Case 7) were used. Notice that, in this example, the upper bounds for the time-derivatives of the membership functions were chosen arbitrarily and fixed in all cases for the purpose of comparison. Also, Theorem 10 in [4] was implemented with Lemma 3 instead of the relaxation principle presented in [17]. Fig. 7(a) shows the comparisons of the results for Cases 1–5. As can be seen, the methods provided in this paper involve the

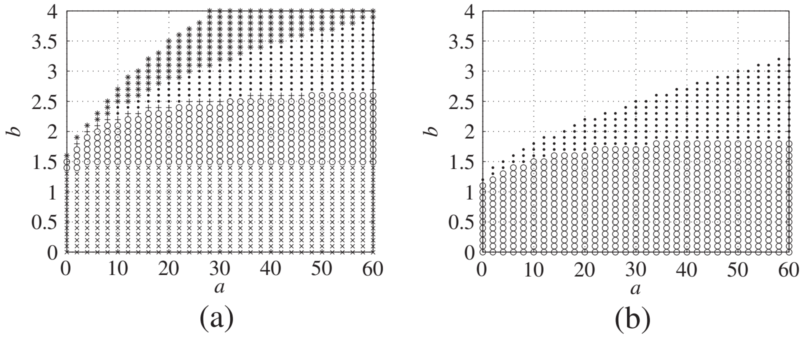

Fig. 7. Example 5. (a) Stabilization region based on Case 1 ($\times$), Case 2 ($\times$ and $\circ$), Case 3 ($\times$, $\circ$, and +), Case 4 ($\times$, $\circ$, +, $\bullet$, and $*$), and Case 5 ($\times$, $\circ$, +, and $\bullet$). (b) Stabilization region based on Case 6 ($\circ$) and Case 7 ($\circ$ and $\bullet$).

previous ones with lager stabilization regions. Also, it is clear that Theorem 6 provides a lager feasible region than Theorems 5 and 7. Finally, Fig. 7(b) gives comparison of results under PDC control law (Cases 6 and 7). This figure shows that the overall area of the domains achieved with Case 7 is always greater than the corresponding one obtained with Case 6.

Example 6. Consider Example 5 again. Here, we aim to show how the parameter μ in Theorem 7 affects the conservativeness. To this end, the maximum values of β, denoted by β^*, were computed by using Theorem 7 for several values of pairs (a,μ), $a \in [0,60]$ and $\mu \in [0,1]$. Fig. 8 shows the obtained upper bounds for different values of α and μ. As it can be seen from this figure, the best results are provided by setting $\mu = 0.1$. Therefore, we can conclude that, for this particular example, it may be more judicious to choose $\mu = 0.1$ to obtain less conservative results.

Example 7. Let us consider (1) with

$$A_1 = \begin{bmatrix} -a & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 10 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ b \end{bmatrix}, \quad h_1(z(t)) = \frac{1+\sin x_1(t)}{2}, \quad h_2(z(t)) = 1 - h_1(z(t)),$$
$$\mathbf{C}_1 = \{x(t) \in \mathbb{R}^n \,|\, |x_i(t)| \leqslant \pi/2, i \in \mathcal{I}_2\},$$

and set $a = -4$ and $b = 1$. For this case, the CQLF-based condition in [31] (the condition of Theorem 1 in [31]) was found infeasible. In contrast, the conditions of Theorems 5–7 with $\mu = 0.04$ and $\phi_\rho = 5, \ \rho \in \mathcal{I}_r$ admitted feasible solutions. This reveals that the proposed methods offer improvements over the CQLF approach. Using Theorem 6 with $\phi_\rho = 5, \ \rho \in \mathcal{I}_r$, the boundaries of the several regions and c^* were computed via a gridding procedure. Fig. 9(a) illustrates the boundaries of $\mathbf{R}$, $\boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, c^*)$, $\mathbf{C}_1$, the region of $\mathbf{R} \cap \mathbf{C}_1$, and a converging trajectory. Also, Fig. 9(b) shows the evolution of the Lyapunov function. It can be observed from these figures that the Lyapunov function always decreases for all $t \geqslant 0$ and the state trajectory of the closed-loop system converges to the equilibrium point.

Once again, as it has been done for the stability problems in the previous section, by employing assumption (16), new sufficient conditions to design non-PDC control laws and estimate of the DA can be derived. In what follows, three sufficient conditions corresponding to Theorems 5–7 are given. The development will be presented in details only for Theorem 8, since the proofs of the other theorems are quite similar to that of Theorem 8.

Theorem 8. *If there exist symmetric matrices P_i, M_{ij}, S_i and matrices K_i such that the following EVP has a solution:*

$$\underset{P_i, M_{ij}, K_i, S_i}{\text{Minimize}} \ \alpha \text{ subject to}$$

$$\textit{LMIs in Theorem } 5, \tag{40}$$

$$\Omega^{\rho l}_{ij} + \Omega^{\rho l}_{ji} \preccurlyeq 0, \quad i \leqslant j, \quad i,j,\rho \in \mathcal{I}_r, \quad l \in \{1,2,\ldots,s\}, \tag{41}$$

$$\begin{bmatrix} -S_i & (*) \\ I & -P_i \end{bmatrix} \preccurlyeq 0, \quad i \in \mathcal{I}_r, \tag{42}$$

$$\begin{bmatrix} -P_i & (*) \\ e_k^T P_i & -\bar{x}_k^2 \end{bmatrix} \prec 0, \quad i \in \mathcal{I}_r, \quad k \in \mathbf{I}, \tag{43}$$

$$S_i \preccurlyeq \alpha I, \quad i \in \mathcal{I}_r \tag{44}$$

where

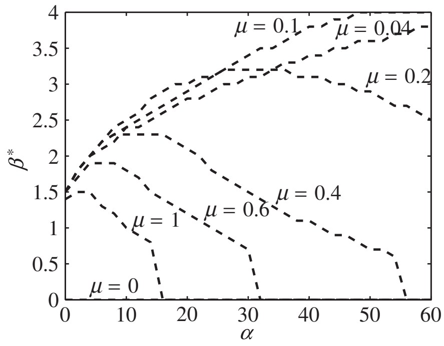

Fig. 8. Example 6. Allowable upper bounds of β computed by using Theorem 7 for different values of α and μ.

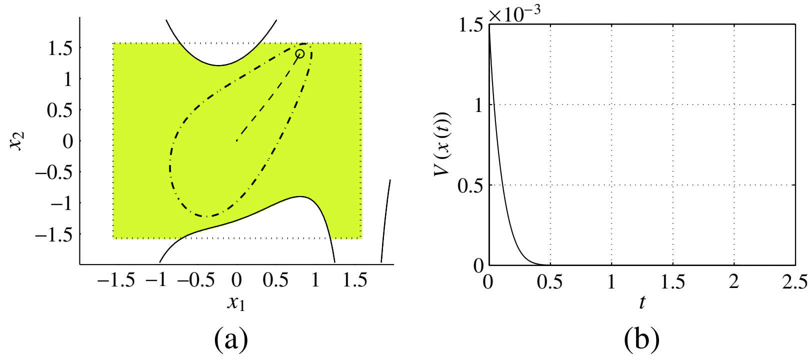

Fig. 9. Example 7. (a) Boundaries of $\mathbf{R}$ (solid line), $\boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, c^*)$ (dashdot line), $\mathbf{C}_1$ (dotted line), the region of $\mathbf{R} \cap \mathbf{C}_1$ (colored area), and a converging trajectory (dashed line) initialized at the "∘" mark, where Theorem 6 with $\phi_\rho = 5,\ \rho \in \mathcal{I}_r$ has been used. (b) The evolution of the Lyapunov function.

$$\Omega_{ij}^{\rho l} := \begin{bmatrix} -P_i & (*) \\ \xi_{\rho l}(A_i P_j + B_i K_j) & -\phi_\rho^2 \end{bmatrix},$$

then the closed-loop system (1) with the non-PDC control law (26) is asymptotically stable. Moreover, $\boldsymbol{\Omega}(P^{-1}_{z(t)}, 1)$ *is an inner estimate of the DA and the boundary of* $\boldsymbol{\Omega}(P^{-1}_{z(t)}, 1)$ *is enlarged as close as possible to that of* $\boldsymbol{\Omega}(P^{-1}_{z(t)}, c^*)$*, where* $c^* = \max\{c \in \mathbb{R} | \boldsymbol{\Omega}(P^{-1}_{z(t)}, c) \subseteq \mathbf{R} \cap \mathbf{C}\}$.

Proof. The proof consists of several parts.

1) **Part 1. Proof of** $\boldsymbol{\Omega}(P^{-1}_{z(t)}, 1) \subseteq \mathbf{C}$: If (43) holds,then one has

$$\begin{bmatrix} -P_{z(t)} & (*) \\ e_k^T P_{z(t)} & -\bar{x}_k^2 \end{bmatrix} \prec 0, \quad \forall x(t) \in \mathbf{C}, k \in \mathbf{I}.$$

Performing a congruence transformation to the above inequality by $\mathrm{diag}(P^{-1}_{z(t),} I)$, it follows that

$$\begin{bmatrix} -P^{-1}_{z(t)} & (*) \\ e_k^T & -\bar{x}_k^2 \end{bmatrix} \prec 0, \ \forall x(t) \in \mathbf{C}, k \in \mathbf{I}. \tag{45}$$

Define

$$\mathbf{Z}_{k+} := \{\zeta \in \mathbf{C} \mid e_k^T \zeta = \bar{x}_k\}, \ \forall k \in \mathbf{I}$$

and

$$\mathbf{Z}_{k-} := \{\zeta \in \mathbf{C} \mid e_k^T \zeta = -\bar{x}_k\}, \ \forall k \in \mathbf{I}.$$

Then, multiplying (45) by $[\zeta_k^T \ 1/\bar{x}_k]$ on the left and its transpose on the right, where $\zeta_k \in \mathbf{Z}_{k+}$, one gets

$$\zeta_k^T P^{-1}_{z(t)} \zeta_k > 1, \ \forall x(t) \in \mathbf{C}, \zeta_k \in \mathbf{Z}_{k+}, k \in \mathbf{I}.$$

In the same vein, for $\zeta_k \in \mathbf{Z}_{k-}$, multiplying (45) by $[\zeta_k^T \ \ -1/\bar{x}_k]$ on the left and its transpose on the right yields

$$\zeta_k^T P_{z(t)}^{-1} \zeta_k > 1, \ \forall x(t) \in \mathbf{C}, \zeta_k \in \mathbf{Z}_{k-}, k \in \mathbf{I}.$$

This implies $\mathbf{Z}_{k+} \cup \mathbf{Z}_{k-} \subset \boldsymbol{\Omega}(P_{z(t)}^{-1}, 1)^C \cap \mathbf{C}, \forall k \in \mathbf{I}$, and hence

$$\bigcup_{k \in \mathbf{I}} (\mathbf{Z}_{k+} \cup \mathbf{Z}_{k-}) = \partial \mathbf{C} \subset \boldsymbol{\Omega}(P_{z(t)}^{-1} \ 1)^C \cap \mathbf{C}$$

where $\partial \mathbf{C}$ and $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1)^C$ denote the boundary of $\mathbf{C}$ and complement of $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1)$, respectively. Therefore, from $\partial \mathbf{C} \subset \boldsymbol{\Omega}(P_{z(t)}^{-1}, 1)^C \cap \mathbf{C}$, one can conclude that $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1) \subseteq \mathbf{C}$.

2) **Part 2. Proof of $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1) \subseteq (\mathbf{R} \cap \mathbf{C})$:** By Lemma 3, satisfying (41) guarantees

$$\begin{bmatrix} -P_{z(t)} & (*) \\ \sum_{l=1}^{s} \upsilon_{\rho l}(z(t)) \xi_{\rho l} \left(A_{z(t)} P_{z(t)} + B_{z(t)} K_{z(t)} \right) & -\phi_\rho^2 \end{bmatrix} \preccurlyeq 0, \quad \forall x(t) \in \mathbf{C}, \quad \rho \in \mathcal{I}_r.$$

After using a congruence transformation with matrix $\mathrm{diag}(P_{z(t)}^{-1}, I)$ and applying Schur complement, one gets

$$\begin{aligned} &\frac{1}{\phi_\rho^2} x^T(t) \left(\sum_{l=1}^{s} \upsilon_{\rho l}(z(t)) \xi_{\rho l} \left(A_{z(t)} + B_{z(t)} K_{z(t)} P_{z(t)}^{-1} \right) \right)^T \left(\sum_{l=1}^{s} \upsilon_{\rho l}(z(t)) \xi_{\rho l} \left(A_{z(t)} + B_{z(t)} K_{z(t)} P_{z(t)}^{-1} \right) \right) x(t) - x^T(t) P_{z(t)}^{-1} x(t) \\ &= \frac{1}{\phi_\rho^2} \left(\sum_{l=1}^{s} \upsilon_{\rho l}(z(t)) \xi_{\rho l} \dot{x}(t) \right)^T \left(\sum_{l=1}^{s} \upsilon_{\rho l}(z(t)) \xi_{\rho l} \dot{x}(t) \right) - x^T(t) P_{z(t)}^{-1} x(t) = \frac{1}{\phi_\rho^2} \dot{h}_\rho^2(z(t)) - x^T(t) P_{z(t)}^{-1} x(t) \leqslant 0, \quad \forall x(t) \in \mathbf{C}, \\ &\rho \in \mathcal{I}_r, \end{aligned}$$

which implies that

$$\left(\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1) \cap \mathbf{C} \right) \subseteq \left(\left\{ x(t) \in \mathbb{R}^n \middle| \frac{1}{\phi_\rho^2} \dot{h}_\rho^2(z(t)) \leqslant 1, \ \rho \in \mathcal{I}_r \right\} \cap \mathbf{C} \right).$$

Since

$$\left\{ x(t) \in \mathbb{R}^n \middle| \left(1/\phi_\rho^2 \right) \dot{h}_\rho^2(z(t)) \leqslant 1, \ \rho \in \mathcal{I}_r \right\} = \left\{ x(t) \in \mathbb{R}^n \middle| \left| \dot{h}_\rho(z(t)) \right| \leqslant \phi_\rho, \ \rho \in \mathcal{I}_r \right\} = \mathbf{R}$$

and $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1) \subseteq \mathbf{C}$ from Part 1, one can conclude that $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1) \subseteq (\mathbf{R} \cap \mathbf{C})$.

3) **Part 3. Proof of $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1) \subseteq \mathbf{D}$:** If the LMI problem of Theorem 5 holds, then $\boldsymbol{\Omega}(P_{z(t)}^{-1}, c^*)$ is an inner estimate of the DA, where $c^* = \max\{c \in \mathbb{R} | \boldsymbol{\Omega}(P_{z(t)}^{-1}, c) \subseteq \mathbf{R} \cap \mathbf{C}_1\}$. Since $(\mathbf{R} \cap \mathbf{C}) \subseteq (\mathbf{R} \cap \mathbf{C}_1)$, one can conclude from Part 2 that $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1) \subseteq \mathbf{D}$.

4) **Part 4. The boundary of $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1)$ is enlarged as close as possible to the boundary of $\boldsymbol{\Omega}(P_{z(t)}^{-1}, c^*)$ with** $c^* = \max\{c \in \mathbb{R} | \boldsymbol{\Omega}(P_{z(t)}^{-1}, c) \subseteq \mathbf{R} \cap \mathbf{C}\}$: If (42) holds,then

$$\begin{bmatrix} -S_{z(t)} & I \\ I & -P_{z(t)} \end{bmatrix} \preccurlyeq 0, \ \forall x(t) \in \mathbf{C}.$$

After applying Schur complement, we find $P_{z(t)}^{-1} \preccurlyeq S_{z(t)}, \ \forall x(t) \in \mathbf{C}$. Also, satisfying (44) ensures $S_{z(t)} \prec \alpha I, \ \forall x(t) \in \mathbf{C}$. Thus, we have $x^T(t) P_{z(t)}^{-1} x(t) < \alpha x^T(t) x(t), \forall x(t) \in \mathbf{C}$, which means that $\{x(t) \in \mathbb{R}^n | x^T(t) x(t) \leqslant 1/\alpha\} \subseteq \boldsymbol{\Omega}(P_{z(t)}^{-1}, 1)$. Hence, minimizing α while imposing constraint $\{x(t) \in \mathbb{R}^n | x^T(t) x(t) \leqslant 1/\alpha\} \subseteq \boldsymbol{\Omega}(P_{z(t)}^{-1}, 1)$ makes $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1)$ to be maximized. Moreover, since $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1) \subseteq (\mathbf{R} \cap \mathbf{C})$ from Part 2, the boundary of $\boldsymbol{\Omega}(P_{z(t)}^{-1}, 1)$ is enlarged as close as possible to the boundary of $\boldsymbol{\Omega}(P_{z(t)}^{-1}, c^*)$ with $c^* = \max\{c \in \mathbb{R} | \boldsymbol{\Omega}(P_{z(t)}^{-1}, c) \subseteq \mathbf{R} \cap \mathbf{C}\}$ This completes the proof.

The next two theorems are the counterparts of Theorems 6 and 7.

Theorem 9. *If there exist symmetric matrices P_{ij}, M_{ij}, S_{ij}, matrices K_i, G_i, L_i, and scalars g_i such that the following EVP has a solution:*

$$\begin{aligned} &\underset{P_{ij}, M_{ij}, S_{ij}, K_i, G_i, L_i, g_i}{\text{Minimize}} \ \alpha \ \text{subject to} \\ &\textit{LMIs in Theorem } 6, \\ &\Omega_{ij}^{k\rho l} + \Omega_{ji}^{k\rho l} \preccurlyeq 0, \quad i \leqslant j, \quad i,j,k,\rho \in \mathcal{I}_r, \quad l \in \{1,2,\ldots,s\} \\ &\widetilde{\Omega}_{ij} + \widetilde{\Omega}_{ji} \preccurlyeq 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \\ &\Psi_{ij}^k + \Psi_{ji}^k \prec 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \quad k \in \mathbf{I}, \\ &S_{ij} + S_{ji} \preccurlyeq 2\alpha I, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \end{aligned}$$

where

$$\Omega_{ij}^{k\rho l} := \begin{bmatrix} -P_{ij} & (*) \\ \xi_{\rho l} (A_i P_{jk} + B_i K_j) & -\phi_\rho^2 \end{bmatrix}, \quad \widetilde{\Omega}_{ij} := \begin{bmatrix} -S_{ij} & (*) \\ I & -P_{ij} \end{bmatrix}, \quad \Psi_{ij}^k := \begin{bmatrix} -P_{ij} & (*) \\ e_k^T P_{ij} & -\bar{x}_k^2 \end{bmatrix},$$

then the closed-loop system (1) with the non-PDC control law (31) is asymptotically stable. Moreover, $\boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, 1)$ *is an inner estimate of the DA and the boundary of* $\boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, 1)$ *is enlarged as close as possible to that of* $\boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, c^*)$*, where* $c^* = \max\left\{c \in \mathbb{R} | \boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, c) \subseteq \mathbf{R} \cap \mathbf{C}\right\}$.

Theorem 10. *Let* $\mu > 0$ *be a given scalar. If there exist symmetric matrices* P_{ij}, $M^{(11)}_{ij}$, $M^{(22)}_{ij}$, S_{ij}*, matrices* $M^{(21)}_{ij} = \left(M^{(12)}_{ij}\right)^T$, K_i*, and* G_i *such that the following EVP has a solution:*

$$\begin{aligned}
&\underset{P_{ij}, M^{(11)}_{ij}, M^{(12)}_{ij}, M^{(21)}_{ij}, M^{(22)}_{ij}, S_{ij}, K_i, G_i}{\text{Minimize}} \quad \alpha \text{ subject to} \\
&\text{LMIs in Theorem 7}, \\
&\Omega^{\rho l}_{ij} + \Omega^{\rho l}_{ji} \preccurlyeq 0, \quad i \leqslant j, \quad i,j,\rho \in \mathcal{I}_r, \quad l \in \{1,2,\ldots,s\}, \\
&\widetilde{\Omega}_{ij} + \widetilde{\Omega}_{ji} \preccurlyeq 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \\
&\Psi^k_{ij} + \Psi^k_{ji} \prec 0, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r, \quad k \in \mathbf{I}, \\
&S_{ij} + S_{ji} \preccurlyeq 2\alpha I, \quad i \leqslant j, \quad i,j \in \mathcal{I}_r,
\end{aligned}$$

where

$$\Omega^{\rho l}_{ij} := \begin{bmatrix} -P_{ij} & (*) \\ \xi_{\rho l}(A_i G_j + B_i K_j) & -\phi^2_\rho \end{bmatrix}, \quad \widetilde{\Omega}_{ij} := \begin{bmatrix} -S_{ij} & (*) \\ I & -G_i - G_i^T + P_{ij} \end{bmatrix}, \quad \Psi^k_{ij} := \begin{bmatrix} -P_{ij} & (*) \\ e_k^T G_i & -\bar{x}_k^2 \end{bmatrix},$$

then the closed-loop system of (1) with the non-PDC control law (35) is asymptotically stable. Moreover, $\boldsymbol{\Omega}(G^{-T}_{z(t)} P_{z(t)z(t)} G^{-1}_{z(t)}, 1)$ *is an inner estimate of the DA and the boundary of* $\boldsymbol{\Omega}(G^{-T}_{z(t)} P_{z(t)z(t)} G^{-1}_{z(t)}, 1)$ *is enlarged as close as possible to that of* $\boldsymbol{\Omega}(G^{-T}_{z(t)} P_{z(t)z(t)} G^{-1}_{z(t)}, c^*)$*, where* $c^* = \max\left\{c \in \mathbb{R} | \boldsymbol{\Omega}(G^{-T}_{z(t)} P_{z(t)z(t)} G^{-1}_{z(t)}, c) \subseteq \mathbf{R} \cap \mathbf{C}\right\}$.

Remark 8. The strengths of the proposed methods are enumerated as:

1) The solutions of the proposed methods offer inner estimates of the DA;
2) In contrast to the conditions in [28], those of Theorems 8–10 do not depend on the initial states;

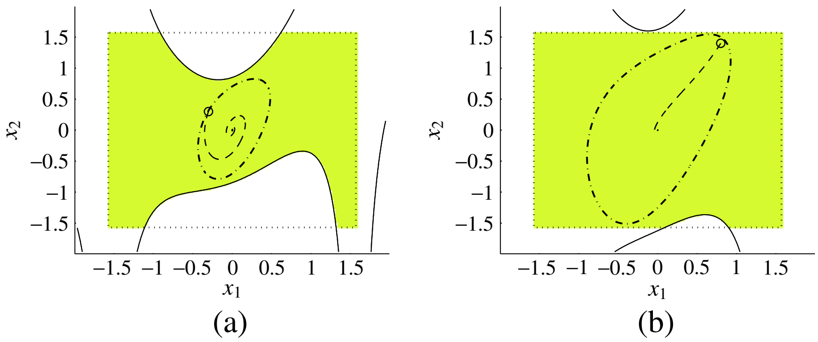

Fig. 10. Example 8. Boundaries of **R** (solid line), $\boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, 1)$ (dashdot line), **C** (dotted line), the region of $\mathbf{R} \cap \mathbf{C}$ (colored area), and a converging trajectory (dashed line) initialized at the "∘" mark. (a) The results of Theorem 9 with $\phi_\rho = 2$, $\rho \in \mathcal{I}_r$. (b) The results of Theorem 9 with $\phi_\rho = 5$, $\rho \in \mathcal{I}_r$.

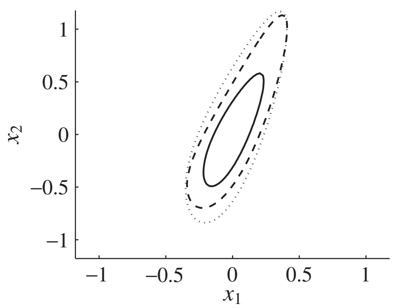

Fig. 11. Example 8. Boundaries of $\boldsymbol{\Omega}(\bullet, 1)$ obtained by using Theorems 8 (solid line), 9 (dotted line), and 10 (dashed line).

3) Although the condition of Theorem 3 in [30] does not require the constraint (3) and does not depend on the initial states, the approach needs to compute the bounds on the time derivatives of the membership functions prior to design a fuzzy controller. It may be impossible to estimate the bounds especially when the derivatives of the premise variables (the state variables) depend on the control input. On the other hand, even though the conditions of Theorems 8–10 require the constraint (3), there is no need to compute the bounds $\phi_\rho,\ \rho \in \mathcal{I}_n$. All that is required to design a fuzzy controller is that one just assumes the bounds $\phi_\rho,\ \rho \in \mathcal{I}_n$ and then obtains solutions to the conditions, from which one can obtain inner estimates of the DA which satisfy the constraint (3) with the bounds $\phi_\rho,\ \rho \in \mathcal{I}_n$ assumed by the designer. Herein, the proposed approaches can be applied even when the derivatives of the premise variables are dependent on the control input. In this sense, the proposed methods may be more beneficial than that of [30].

Example 8. Let us consider the system used in Example 7 with $a = -4$ and $b = 1$. The CQLF-based condition in [31] and initial states dependent condition in [29] for $\phi_\rho = 2,\ \rho \in \mathcal{I}_r$ and $\phi_\rho = 5,\ \rho \in \mathcal{I}_r$ were found infeasible.Moreover, the approach proposed in [30] cannot be applied to this example because the control input affects the time derivatives of the premise variables.On the contrary, the conditions of Theorems 8–10 for $\mu = 0.04,\ \phi_\rho = 2,\ \rho \in \mathcal{I}_r$, and $\phi_\rho = 5,\ \rho \in \mathcal{I}_r$ admitted feasible solutions. Therefore, it can be seen that the proposed methods outperform the existing ones. Fig.10(a) and (b) show the boundaries of $\boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, 1)$ obtained by using Theorem 9 for $\phi_\rho = 2,\ \rho \in \mathcal{I}_r$ and $\phi_\rho = 5, \rho \in \mathcal{I}_r$, respectively. Especially, Fig.10(b) clearly shows that the boundary of $\boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, 1)$ is very close to that of $\boldsymbol{\Omega}(P^{-1}_{z(t)z(t)}, c^*)$. One step further, let us consider the case where $a = -10$ and $b = 1$. For this case, the conditions of Theorems 8–10 for $\phi_\rho = 5,\ \rho \in \mathcal{I}_r$ admit feasible solutions. Fig.11 shows the boundaries of $\Omega(\bullet, 1)$ obtained by using Theorems 8–10, from which it can be observed that, among them, the largest region is provided by Theorem 9. This reveals that the condition of Theorem9 is the least conservative one for this specific case.

5. Conclusion

In this paper, novel stability analysis and controller synthesis stratagies for continuous-time T–S fuzzy systems have been proposed. First, we have derived less conservative LMI conditions by extending the previous ones. To this end, extended FLFs and non-PDC control laws have been employed. Then, the resultant conditions have been extended to those which can be used to estimate the DA. Finally, the effectiveness of the proposed approaches has been demonstrated through some examples.

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