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Local stability analysis of continuous-time Takagi–Sugeno fuzzy systems: A fuzzy Lyapunov function approach

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ABSTRACT

This paper proposes a strategy to estimate invariant subsets of the domain of attraction (DA) for asymptotically stable zero equilibrium points of continuous-time Takagi–Sugeno (T–S) fuzzy systems. Specifically, by using Lyapunov's stability theory and the linear matrix inequality (LMI) technique, sufficient conditions for proving the local stability are provided in terms of single-parameter minimization problems subject to LMI constraints or eigenvalue problems, which are solvable via convex optimizations. The fuzzy Lyapunov functions (FLFs), expressed by the so-called multi-dimensional fuzzy summations, are employed to characterize invariant subsets of the DA as sublevel sets of the FLFs. To compute a larger inner estimate of the DA, an iterative LMI algorithm is also developed. Finally, illustrative examples show the efficacy of the approach.

1. Introduction

This paper focuses on the Takagi–Sugeno (T–S) fuzzy system

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) A_i x(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $z(t) \in \mathbb{R}^p$ is the vector containing premise variables in the fuzzy inference rule, $A_i \in \mathbb{R}^{n\times n}$ is a constant matrix, $i \in \mathcal{I}_r := \{1, 2, \ldots, r\}$ is the rule number, and $h_i\,(z(t))$ is the normalized membership function (NMF) for each rule, which is assumed to be continuously differentiable and satisfies the properties $0 \leqslant h_i(z(t)) \leqslant 1$ and $\sum_{i=1}^{r} h_i(z(t)) = 1$. Stability analysis and control design of (1) have received increasing interest for decades, since the fuzzy systems provide useful frameworks for stability analysis of nonlinear systems in that by means of the Lyapunov theory, several analysis and control problems can be cast as convex linear matrix inequality (LMI) optimization procedures. Approaches based on Lyapunov's stability theory are one of the most popular ways to deal with the problem. Among them, the simplest one is the common quadratic Lyapunov function approach [11,18,23,34–36,42,52,58,62,65]. However, the common quadratic Lyapunov approach may be undesirable in many situations because it entails a considerable conservatism, which mainly stems from the fact that a common Lyapunov matrix should be found for all subsystems of (1).

Substantial efforts have been made to overcome the conservatism of the quadratic stability approach, and recent development can be roughly partitioned into three categories. First, much research attention has been focused on the use of

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information on NMFs' shape to obtain less conservative stability and stabilization conditions. For instance, bounds on the products of the NMFs were used in [40,46] considered bounds on the multiplication of NMFs for the partitioned operating region of NMFs; and [25] employed simplicial partitions of the standard simplex of the NMFs to form a finer mesh. More recently, a membership-function-dependent stability analysis approach was developed in [28], where the membership functions were characterized by some sample points, and by checking the system stability at the sample points, the overall system stability was guaranteed. The second direction is to construct more general classes of Lyapunov functions; for instance, piecewise Lyapunov functions which are piecewise quadratic with respect to some partitions of the state space [1,7,12,21,53], fuzzy Lyapunov functions (FLFs) linearly dependent on the NMFs for the continuous-time systems [38,39,56,57] and discrete-time systems [9,14,17,24,29–31], a class of Lyapunov functions using line-integral [44], and polynomial Lyapunov functions whose dependence on the state variables is expressed as polynomial forms [4,26,27,47,59]. Finally, convergent LMI relaxation techniques for proving positivity of the double or the so-called multi-dimensional (q-dimensional) fuzzy summations were developed in [45] based on Pólya's theorem. Until now, the approach has been applied to construct the multi-dimensional FLFs which are homogeneous polynomial in the NMFs [10,19,29,60,64]. In the most approaches mentioned above, conditions to search for control laws and Lyapunov functions were stated in terms of linear matrix inequalities (LMIs) which can be solved via LMI solvers [13,37,48]. In addition to the LMI framework, recently, the sum-of-squares (SOS) techniques [43] have become important in dealing with stability of polynomial T–S fuzzy systems [59] along with polynomial FLFs.

In this paper, we direct our attention to the FLF approach, which has been broadly applied for various purposes. For example, the fuzzy Lyapunov–Krasovskii function was used in [49,50] to design the fuzzy filter and the dynamic output feedback controller, respectively, for discrete-time T–S fuzzy systems with time-varying delays, where appropriate Lyapunov function was selected using the input–output technique borrowed from the robust control theory. In [51], the filter design method was also applied to a class of sensor networks with time-varying delays and multiple probabilistic packet losses. Although the FLF scheme has proven to be successful in dealing with the discrete-time systems, its use results in difficulties in stability analysis and control design of the "continuous-time" systems. Specifically, what makes the problem challenging from the practical standpoint is that the time-derivative of NMFs $\dot{h}_\rho(z(t)), \forall \rho \in \{1,2,\ldots,r\} =: \mathcal{I}_r$ arising from the Lyapunov inequality contains additional nonlinear terms, which do not have polytopic bounds as the T–S fuzzy systems. To circumvent this obstacle, approaches in [38,39,56,57] assumed that the time-derivative of NMFs is bounded as follows:

$$|\dot{h}_\rho(z(t))| \leqslant \phi_\rho, \quad \rho \in \mathcal{I}_r, \tag{2}$$

where $\phi_\rho > 0, \rho \in \mathcal{I}_r$ are *a priori* given real numbers, while [44] proposes an alternative LMI approach based on a new type of FLFs using line integral to eliminate the terms involving the time-derivative of the NMFs.

On the other hand, extensive research has been undertaken on the local stability problem, which aims at proving the existence of a locally stable region called the domain of attraction (DA) [22], the set of initial conditions for which the state asymptotically converges to the origin, and the problem of estimating the DA. Significant contributions have been made, for example, in the case of polynomial nonlinear systems [6,20,54,55,61] and non-polynomial nonlinear systems [8]. As for the continuous-time T–S fuzzy systems, whenever employing assumption (2), one should consider the fact that the region satisfying (2) is not the whole state space but a narrow limit around the origin, because the time-derivative of the NMFs is, in general, nonlinear in the state variables. Hence, the local stability concept should be taken into consideration when using FLFs. In the last few years, many researchers have investigated the problems of assessing the local stability and estimating the DA for continuous-time T–S fuzzy systems; see, e.g., [2–4,15,16,32,33,41,63]. Specifically, an LMI condition was developed for the first time in the pioneering work [15] by employing the FLFs and considering more structured bounds on the time-derivative of the NMFs. Afterwards, it was generalized in [2] by applying the convergent LMI relaxation technique developed in [45] and employing the multi-dimensional FLFs. Further work was done in [16,41] to deal with local stabilization problems. There, absolute values of partial derivatives of the NMFs with respect to the state variables were bounded by some constants to cast the local stabilization problem into LMIs. More recently in [63], an LMI-based method was developed to compute invariant sets of the state for continuous-time T–S fuzzy systems affected by magnitude bounded or energy bounded disturbances. Besides these, local stability of continuous-time polynomial T–S fuzzy systems was studied in [4] within the framework of the SOS programming. In our prior work [32], based on the FLFs and under assumption (2), sufficient conditions for the local stability and local stabilization were formulated as single-parameter minimization problems subject to LMIs. There, in order to cast assumption (2) into LMI constraints, motivated by [56], we assumed that the gradient of the NMFs varies within a polytopic domain. In [33], the Jacobian of the vector-valued nonlinear function including the NMFs with respect to the state was cast as a convex sum of constant matrices to derive an LMI condition.

Despite the above mentioned progress, there still exists room for further investigation in that most of the existing local stability and stabilization methods using assumption (2) have remained open parameter-tuning problems for the bound assumption (2); one has to determine adequate values of parameters $\phi_\rho > 0, \rho \in \mathcal{I}_r$ so that less conservative estimates of the DA can be computed. In general, the task might not be a trivial one, and therefore, we may be interested in developing alternative methods that do not require assumption (2). This motivates the research of this work.

This paper presents new sufficient conditions to assess the local stability and to estimate invariant subsets of the DA for continuous-time T–S fuzzy systems (1). The way to reach the conditions is different from the existing methods in some aspects. Specifically, the differences and contributions of this study compared with existing work can be summarized as follows: First, we handle the time-derivative of the NMFs in a different way. As in [33], we employ a polytopic-type bound on the Jacobian of the vector $h(z(t)) := [h_1(z(t)) \quad h_2(z(t)) \quad \cdots \quad h_r(z(t))]^T \in \mathbb{R}^r$ containing the NMFs, which is different from the bounding techniques used in [32,41,56] in the sense that existing researches used bounds on each NMF. Then, by means of the Kronecker's product, the Lyapunov inequality, which is a polynomial in $x(t)$, $\dot{x}(t)$, and $\dot{h}(z(t))$, is represented by a quadratic form which depends quadratically on the augmented vector containing the state and the time-derivative of the NMFs. Based on this representation, the local stability condition is derived in terms of single-parameter minimization problems subject to bilinear matrix inequalities (BMIs). With some conservatism, the problem can be reduced to solving single-parameter minimization problems subject to LMI constraints, which can be efficiently solved via a sequence of LMI optimizations such as a bisection algorithm or a line search process, or solving eigenvalue problems (EVPs), which are convex optimizations [5] and hence tractable with the help of LMI solvers [13,37,48]. The most distinguished feature of the proposed approach is that we employ an iterative LMI algorithm, which enables us to obtain reasonably sharper estimates of the DA; as iteration step proceeds, larger and larger DA estimates can be obtained. The main idea of the algorithm stems from the *V-s* iteration algorithm presented in [6,20,54], which can be viewed as a practical way for better estimation of the DA. Second, unlike the most existing methods, the developed approach does not use assumption (2) and hence, does not need the tuning problem for parameters $\phi_\rho > 0, \rho \in \mathcal{I}_r$ in (2). In this regard, the proposed approach can be viewed as an option to the previous local stability methods. Third, we take benefit of the relaxation technique introduced in [45] and rely on the use of the extended FLFs expressed as the multi-dimensional fuzzy summations as in [2,10,29] so as to obtain less conservative results. Finally, examples are given to demonstrate the merit of the proposed strategy.

2. Preliminaries

2.1. Notation and Definitions

The adopted notation is as follows: $\mathbb{N}$ and $\mathbb{R}$: sets of natural and real numbers, respectively; 0_n and $0_{n\times m}$: origin of $\mathbb{R}^n$ and of $\mathbb{R}^{n\times m}$, respectively; A^T: transpose of matrix A; $A \succ 0$, $A \prec 0$, $A \succeq 0$, and $A \preceq 0$: symmetric positive definite, negative definite, positive semi-definite, and negative semi-definite matrix A, respectively; $A \otimes B$: Kronecker's product of matrices A and B; I_n: $n \times n$ identity matrix; $*$ inside a matrix: transpose of its symmetric term; He$\{A\}$: short hand notion for $A^T + A$; co$\{\cdot\}$: convex hull [5]; $\emptyset$: empty set; ∂S: boundary of set S; $\mathcal{I}_r := \{1, 2, \ldots, r\}$ for any $r \in \mathbb{N}$; h_i and x: abbreviations for NMF $h_i(z(t))$ and state $x(t)$, respectively; $h := [h_1 \quad h_2 \quad \cdots \quad h_r]^T \in \mathbb{R}^r$; e_i: $n \times 1$ unit vector with a *1* in the *i*th component and *0*'s elsewhere. Moreover, in order to handle the *q*-dimensional fuzzy summations of matrices $\Upsilon_{(i_1,i_2,\ldots,i_q)}, (i_1, i_2, \ldots, i_q) \in \mathcal{I}_r^q$:

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_q=1}^{r} h_{i_1}h_{i_2}\cdots h_{i_q}\Upsilon_{(i_1,i_2,\ldots,i_q)},$$

the following notation, from [45], will be used. $\mathbf{i}_q := (i_1, i_2, \ldots, i_q)$; $\mathbf{j}_q := (j_1, j_2, \ldots, j_q)$; $\mathbb{I}_q := \{1, 2, \ldots, r\}^q = \mathcal{I}_r^q$; $\mathbb{I}_q^+ := \{\mathbf{i} \in \mathbb{I}_q : i_1 \leqslant i_2 \leqslant \cdots \leqslant i_q\}$; $h_{\mathbf{i}_q} := h_{i_1}h_{i_2}\cdots h_{i_q}$; $\Upsilon_{z^q} := \sum_{\mathbf{i}_q \in \mathbb{I}_q} h_{\mathbf{i}_q}\Upsilon_{\mathbf{i}_q}$; $\mathcal{P}(\mathbf{i}_q)$: set of permutations of a given multi-index $\mathbf{i}_q \in \mathbb{I}_q$. The reader is referred to [45] for a more detailed description and a comprehensive treatment of the multi-dimensional fuzzy summations and the multi-index notation. Finally, for $\mathbf{i}_{q-1} \in \mathbb{I}_{q-1}$, expression $\mathbf{i}_{q-1}(k,l) \in \mathbb{I}_q$, $(k,l) \in \mathcal{I}_q \times \mathcal{I}_r$ is defined as the *q*-dimensional index resulting from inserting *l* before *k*-th index or after $(k-1)$th index of $\mathbf{i}_{q-1}$. For instance, for $r = 4$ and $\mathbf{i}_3 = (1,2,3)$, $\mathbf{i}_3(1,4) = (4,1,2,3)$, $\mathbf{i}_3(2,4) = (1,4,2,3)$, $\mathbf{i}_3(3,4) = (1,2,4,3)$, and $\mathbf{i}_3(4,4) = (1,2,3,4)$.

2.2. T–S fuzzy system and problem formulation

Let us consider the continuous-time nonlinear system $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function such that $f(0_n) = 0_n$, i.e., the origin is an equilibrium point of the system. A wide range of nonlinear systems in the form of $\dot{x} = f(x)$ can be represented by T–S fuzzy system (1) in a compact set $\mathcal{L}$ including the origin, where vector $h := [h_1 \quad h_2 \quad \cdots \quad h_r]^T \in \mathbb{R}^r$ lies in the unit simplex

$$\Lambda := \left\{ \delta \in \mathbb{R}^r : \sum_{i=1}^{r}\delta_i = 1, \quad 0 \leqslant \delta_i \leqslant 1, \quad i \in \mathcal{I}_r \right\}$$

for all $x \in \mathcal{L}$. In this paper, we assume that

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_p \end{bmatrix} = \begin{bmatrix} x_{a_1} \\ x_{a_2} \\ \vdots \\ x_{a_p} \end{bmatrix} = \Pi x \in \mathbb{R}^p, \quad \Pi := \begin{bmatrix} e_{a_1}^T \\ e_{a_2}^T \\ \vdots \\ e_{a_p}^T \end{bmatrix} \in \mathbb{R}^{p\times n},$$

where $\{a_1, a_2, \ldots, a_p\} \subseteq \{1,2,\ldots,n\}$ is a set of indexes, and $e_{a_k} \in \mathbb{R}^n$ is $n \times 1$ unit vector with a *1* in the a_kth component and 0's elsewhere. In addition, region $\mathcal{L}$ satisfying

$$\mathcal{L} \subseteq \left\{ \xi \in \mathbb{R}^n : f(\xi) = \sum_{i=1}^{r} h_i(\omega) A_i \xi, \quad \omega = \Pi\xi, \quad h(\omega) \in \Lambda \right\}$$

is described as $\mathcal{L} := \{\xi \in \mathbb{R}^n : \xi_{a_k} \in [-x_{a_k,\max}, x_{a_k,\max}], k \in \mathcal{I}_p\}$, where $x_{a_k,\max} > 0$ are *a priori* given real numbers.

Remark 1. In this paper, our concern is restricted to the case that premise variables $z_1, z_2, \ldots, z_p$ are the state variables. The proposed approach can be directly extended to the case that the premise variables are linear combinations of the states. Otherwise, the proposed method is invalid.

We assume that the Jacobian matrix of $h := [h_1 \quad h_2 \quad \cdots \quad h_r]^T \in \mathbb{R}^r$:

$$J_h := \begin{bmatrix} \partial h_1 / \partial x \\ \vdots \\ \partial h_r / \partial x \end{bmatrix} \in \mathbb{R}^{r \times n}, \quad \forall x \in \mathcal{L}$$

varies within the convex domain

$$J_h \in \mathrm{co}\{J_1, J_2, \ldots, J_w\}, \quad \forall x \in \mathcal{L}, \tag{3}$$

where $J_\rho \in \mathbb{R}^{r \times n}$ and co$\{\cdot\}$ is the convex hull [5]. Let $V(x) := x^T P_{z^q} x, P_{z^q} \succ 0, \forall x \in \mathcal{L}$ and denote $\Omega(\gamma) := \{\xi \in \mathcal{L} : V(\xi) \leqslant \gamma\}$. The problems addressed in this paper are as follows:

1. Establish if the zero equilibrium point of (1) is locally asymptotically stable.
2. For a given real number $\gamma > 0$, search for a Lyapunov function $V(x)$ for the zero equilibrium point such that $\Omega(\gamma)$ is an invariant subset of the DA [22].
3. Enlarge $\Omega(\gamma)$.

Remark 2. In order to ensure $V(x) = \gamma$, $\forall x \in \partial\Omega(\gamma)$ and use $\Omega(\gamma)$ as an invariant subset of the DA, one needs to impose the constraint $\gamma < V(x), \forall x \in \partial\mathcal{L}$.

Finally, to prove the positivity of the q-dimensional fuzzy summations, the following lemma will be used throughout the paper.

Lemma 1 [45]. *Given symmetric matrices* $\Upsilon_{\mathbf{j}_q}, \mathbf{j}_q \in \mathbb{I}_q$, $\Upsilon_{z^q} \prec 0$ *holds for all* $x \in \mathcal{L}$ *if LMI* $\sum_{\mathbf{j}_q \in \mathcal{P}(\mathbf{i}_q)} \Upsilon_{\mathbf{j}_q} \prec 0$ *is fulfilled for all* $\mathbf{i}_q \in \mathbb{I}_q^+$.

3. Main result

In this section, the main analysis problem is solved. It consists of the estimation of an invariant subset of the DA for system (1) as introduced in the previous section. The next theorem gives a solution to the problem in terms of single-parameter minimization problem subject to BMIs.

Theorem 1. *Let* $q \in \mathbb{N}$ *be given, and let* $\gamma > 0$ *be a given real number. If there exist symmetric matrices* $P_{\mathbf{j}_q} \in \mathbb{R}^{n \times n}, S \in \mathbb{R}^{n \times n}$, *matrices* $Y_{\mathbf{j}_{q-1}} \in \mathbb{R}^{(2n+n^2+rn) \times (np+n+rn)}$, *and a real number* $\beta > 0$ *such that the following optimization is satisfied:*

$$\min_{P_{\mathbf{j}_q}, S, Y_{\mathbf{j}_{q-1}}, \beta} \beta \text{ subject to}$$

$$\sum_{\mathbf{j}_q \in \mathcal{P}(\mathbf{i}_q)} \Upsilon_{\mathbf{j}_q}^{(1)}(\theta, J) \prec 0, \quad \forall (\mathbf{i}_q, \theta, J) \in \mathbb{I}_q^+ \times \mathcal{W} \times \{J_1, J_2, \ldots, J_w\}, \tag{4}$$

$$\sum_{\mathbf{j}_q \in \mathcal{P}(\mathbf{i}_q)} \Upsilon_{(\mathbf{j}_q, k)}^{(2)} \prec 0, \quad \forall (\mathbf{i}_q, k) \in \mathbb{I}_q^+ \times \mathcal{I}_p, \tag{5}$$

$$S \succeq 0, \tag{6}$$

$$\sum_{\mathbf{j}_q \in \mathcal{P}(\mathbf{i}_q)} \Upsilon_{\mathbf{j}_q}^{(3)} \preceq 0, \quad \forall \mathbf{i}_q \in \mathbb{I}_q^+ \tag{7}$$

where $\mathcal{W} := \{-x_{a_1,\max}, x_{a_1,\max}\} \times \{-x_{a_2,\max}, x_{a_2,\max}\} \times \cdots \times \{-x_{a_p,\max}, x_{a_p,\max}\}$,

$$\Upsilon_{\mathbf{j}_q}^{(1)}(\theta, J) := \begin{bmatrix} 0_{n \times n} & * & * & * \\ P_{\mathbf{j}_q} & \gamma S & * & * \\ 0_{n^2 \times n} & 0_{n^2 \times n} & -S \otimes P_{\mathbf{j}_q} & * \\ 0.5\Phi_{\mathbf{j}_{q-1}} & 0_{rn \times n} & 0_{rn \times n^2} & 0_{rn \times rn} \end{bmatrix} + \mathrm{He}\{Y_{\mathbf{j}_{q-1}} C_{j_q}(\theta, J)\} \in \mathbb{R}^{(2n+n^2+rn) \times (2n+n^2+rn)}$$

$$C_{j_q}(\theta,J) := \begin{bmatrix} A_{j_q} & -I_n & 0_{n\times n^2} & 0_{n\times rn} \\ 0_{n\times n} & -\theta_{a_1} I_n & I_n\otimes e_{a_1}^T & 0_{n\times rn} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{n\times n} & -\theta_{a_{p-1}} I_n & I_n\otimes e_{a_{p-1}}^T & 0_{n\times rn} \\ 0_{n\times n} & -\theta_{a_p} I_n & I_n\otimes e_{a_p}^T & 0_{n\times rn} \\ 0_{rn\times n} & 0_{rn\times n} & J\otimes I_n & -I_{rn} \end{bmatrix} \in \mathbb{R}^{(np+n+rn)\times(2n+n^2+rn)}$$

$$\Phi_{\mathbf{j}_{q-1}} := \left[\sum_{k=1}^{q} P_{\mathbf{j}_{q-1}(k,1)} \quad \sum_{k=1}^{q} P_{\mathbf{j}_{q-1}(k,2)} \quad \cdots \quad \sum_{k=1}^{q} P_{\mathbf{j}_{q-1}(k,r)} \right]^T \in \mathbb{R}^{rn\times n}$$

$$\theta := (\theta_{a_1},\theta_{a_2},\ldots,\theta_{a_p}) \in \mathbb{R}^p$$
$$\Upsilon^{(2)}_{(\mathbf{j}_q,k)} := \gamma x_{a_k,\max}^{-2} e_{a_k} e_{a_k}^T - P_{\mathbf{j}_q} \in \mathbb{R}^{n\times n}, \quad \Upsilon^{(3)}_{\mathbf{j}_q} := -\gamma\beta I_n + P_{\mathbf{j}_q} \in \mathbb{R}^{n\times n}$$

then system (1) is locally asymptotically stable. Moreover, an invariant subset of the DA for system (1) is given by $\Omega(\gamma)$.

Proof. The proof consists of several parts.

Part 1. Proof of $V(x) > \gamma, \forall x \in \partial\mathcal{L}$: If (5) holds, then using Lemma 1, one has

$$\gamma x_{a_k,\max}^{-2}\left(e_{a_k} e_{a_k}^T\right) - P_{z^q} \prec 0, \quad \forall (x,k) \in \mathcal{L}\times\mathcal{I}_p. \tag{8}$$

Multiplying (8) by x^T on the left and x on the right, one gets

$$\gamma x_{a_k,\max}^{-2} x_{a_k}^2 < V(x), \quad \forall (x,k) \in \mathcal{L}\setminus\{0_n\}\times\mathcal{I}_p \tag{9}$$

which implies that $\gamma < V(x), \forall x \in \partial\mathcal{L}$, where $\partial\mathcal{L}$ is the boundary of $\mathcal{L}$.

Part 2. Proof that $\Omega(\gamma)$ is an invariant subset of the DA: Inequality (5) or (8) implies $P_{z^q} \succ 0, \forall x \in \mathcal{L}$. Thus, $V(x) > 0, \forall x \in \mathcal{L}\setminus\{0_n\}$ holds. By Lemma 1, satisfying (4) yields

$$\Upsilon^{(1)}_{z^q}(\theta,J) \prec 0, \quad \forall (x,\theta,J) \in \mathcal{L}\times\mathcal{W}\times\{J_1,J_2,\ldots,J_w\}.$$

Since J and θ appear linearly in $\Upsilon^{(1)}_{z^q}(\theta,J)$, using relation (3), we have that the above inequality ensures

$$\Upsilon^{(1)}_{z^q}(z,J_h) =: \Gamma(z) \prec 0, \quad \forall x \in \mathcal{L},$$

where $z := (x_{a_1},x_{a_2},\ldots,x_{a_p})$. The last inequality can be rewritten as

$$\Gamma(z) = \begin{bmatrix} 0_{n\times n} & * & * & * \\ P_{z^q} & \gamma S & * & * \\ 0_{n^2\times n} & 0_{n^2\times n} & -S\otimes P_{z^q} & * \\ 0.5\Phi_{z^{q-1}} & 0_{rn\times n} & 0_{rn\times n^2} & 0_{rn\times rn} \end{bmatrix} + \mathrm{He}\left\{ Y_{z^{q-1}} \begin{bmatrix} A_z & -I_n & 0_{n\times n^2} & 0_{n\times rn} \\ 0_{n\times n} & -z_{a_1} I_n & I_n\otimes e_{a_1}^T & 0_{n\times rn} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{n\times n} & -z_{a_{p-1}} I_n & I_n\otimes e_{a_{p-1}}^T & 0_{n\times rn} \\ 0_{n\times n} & -z_{a_p} I_n & I_n\otimes e_{a_p}^T & 0_{n\times rn} \\ 0_{rn\times n} & 0_{rn\times n} & J_h\otimes I_n & -I_{rn} \end{bmatrix} \right\} \prec 0, \quad \forall x \in \mathcal{L}.$$

Next, define

$$\zeta(x) := \begin{bmatrix} x \\ A_z x \\ A_z x\otimes x \\ J_h A_z x\otimes x \end{bmatrix} \in \mathbb{R}^{2n+n^2+rn}. \tag{10}$$

Then, using relations $\dot{x} = A_z x, \forall x \in \mathcal{L}$ and $\dot{h} = J_h\dot{x} = J_h A_z x, \forall x \in \mathcal{L}$, one concludes that

$$\mathrm{He}\left\{ Y_{z^{q-1}} \begin{bmatrix} A_z & -I_n & 0_{n\times n^2} & 0_{n\times rn} \\ 0_{n\times n} & -z_{a_1} I_n & I_n\otimes e_{a_1}^T & 0_{n\times rn} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{n\times n} & -z_{a_{p-1}} I_n & I_n\otimes e_{a_{p-1}}^T & 0_{n\times rn} \\ 0_{n\times n} & -z_{a_p} I_n & I_n\otimes e_{a_p}^T & 0_{n\times rn} \\ 0_{rn\times n} & 0_{rn\times n} & J_h\otimes I_n & -I_{rn} \end{bmatrix} \right\} \zeta(x) = 0_{2n+n^2+rn}, \quad \forall x \in \mathcal{L}$$

holds. Thus, by direct calculation, it follows that

$$\begin{aligned}
\zeta(x)^T \Gamma(z)\zeta(x) = &\ \zeta(x)^T \begin{bmatrix} 0_{n\times n} & * & * & * \\ P_{z^q} & \gamma S & * & * \\ 0_{n^2\times n} & 0_{n^2\times n} & -S\otimes P_{z^q} & * \\ 0.5\Phi_{z^{q-1}} & 0_{rn\times n} & 0_{rn\times n^2} & 0_{rn\times rn} \end{bmatrix}\zeta(x) \\
= &\ \gamma \dot{x}^T S\dot{x} - (\dot{x}^T S\dot{x})x^T P_{z^q}x + x^T(P_{z^q}A_z + A_z^T P_{z^q})x \\
&+ \sum_{\mathbf{j}_{q-1}\in\mathbb{I}_{q-1}} h_{\mathbf{j}_{q-1}} \left[\sum_{k=1}^{q} x^T P_{\mathbf{j}_{q-1}(k,1)}x \ \cdots \ \sum_{k=1}^{q} x^T P_{\mathbf{j}_{q-1}(k,r)}x \right] J_h A_z x \\
= &\ \left(x^T A_z^T S A_z x\right)(\gamma - V(x)) + x^T\left(P_{z^q}A_z + A_z^T P_{z^q}\right)x \\
&+ \sum_{\mathbf{j}_{q-1}\in\mathbb{I}_{q-1}} h_{\mathbf{j}_{q-1}} \left(\sum_{k=1}^{q} \dot{h}_1 x^T P_{\mathbf{j}_{q-1}(k,1)}x + \cdots + \sum_{k=1}^{q} \dot{h}_r x^T P_{\mathbf{j}_{q-1}(k,r)}x \right) \\
= &\ \left(x^T A_z^T S A_z x\right)(\gamma - V(x)) + \dot{V}(x) \\
&< 0, \quad \forall x\in\mathcal{L}\setminus\{0_n\} \\
&\iff \dot{V}(x) < \left(x^T A_z^T S A_z x\right)(V(x)-\gamma), \quad \forall x\in\mathcal{L}\setminus\{0_n\} \\
\Rightarrow &\ \dot{V}(x) < 0, \quad \forall x\in\Omega(\gamma)\setminus\{0_n\} \\
\Rightarrow &\ \Omega(\gamma)\setminus\{0_n\} \subseteq \left\{\xi\in\mathcal{L} : (\partial V(\xi)/\partial\xi)\sum_{i=1}^{r} h_i(\omega)A_i\xi < 0, \quad \omega = \Pi\xi\right\}
\end{aligned} \tag{11}$$

By the Lyapunov argument, (1) is locally asymptotically stable and $\Omega(\gamma)$ is an invariant subset of the DA [22].

Part 3. Proof of enlargement of $\Omega(\gamma)$: LMIs in (7) with Lemma 1 yields

$$V(x) \leqslant \gamma\beta x^T x, \quad \forall x\in\mathcal{L} \iff \gamma^{-1}V(x) - 1 \leqslant \beta x^T x - 1, \quad \forall x\in\mathcal{L} \iff V(x) - \gamma \leqslant \gamma\beta(x^T x - 1/\beta), \quad \forall x\in\mathcal{L} \tag{12}$$

which implies $\{\xi\in\mathcal{L} : \xi^T\xi \leqslant 1/\beta\} \subseteq \Omega(\gamma)$. Therefore, minimizing β while imposing constraint $\{\xi\in\mathcal{L} : \xi^T\xi \leqslant 1/\beta\} \subseteq \Omega(\gamma)$ makes $\Omega(\gamma)$ to be enlarged. This completes the proof. □

Remark 3. As one can see from the proof of Theorem 1, a distinguished feature of the proposed LMI condition is that while in existing approaches, the time-derivative of the FLF $\dot{V}(x)$ is treated as a function quadratic with respect to x, we consider it a function quadratic with respect to the augmented vector

$$\zeta(x) := \begin{bmatrix} x \\ A_z x \\ A_z x\otimes x \\ J_h A_z x\otimes x \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \dot{x}\otimes x \\ \dot{h}\otimes x \end{bmatrix}$$

including the time-derivative of the NMFs. For this reason, this approach allows us not to assume bounds (2) on the time-derivative of the NMFs, and hence, does not need to tune parameters $\phi_\rho > 0, \rho\in\mathcal{I}_r$ in (2). In the sequel, we will show through an example that the proposed condition combined with an iteration algorithm outperforms the previous approaches in terms of the volume of the DA estimation.

Remark 4. Let us define

$$\mathbf{1}_r := \left[\underbrace{1 \ \cdots \ 1}_{r-\text{times}}\right] \in \mathbb{R}^{1\times r}.$$

For any matrices $M_{\mathbf{j}_q} \in \mathbb{R}^{(2n+n^2+rn)\times n}, \mathbf{j}_q \in \mathbb{I}_q$, $\Upsilon^{(1)}_{\mathbf{j}_q}(\theta, J)$ in Theorem 1 can be replaced by

$$\Upsilon^{(1)}_{\mathbf{j}_q}(\theta, J) + \mathrm{He}\left\{M_{\mathbf{j}_q}\left[0_{n\times(2n+n^2)} \quad \mathbf{1}_r\otimes I_n\right]\right\}$$

without introducing any conservatism. This is because, using relation $\dot{h}_1 + \dot{h}_2 + \cdots + \dot{h}_r = \mathbf{1}_r\dot{h} = 0$, it can be easily seen that

$$\mathrm{He}\left\{M_{z^q}\left[0_{n\times(2n+n^2)} \quad \mathbf{1}_r\otimes I_n\right]\right\}\zeta(x) = 0_{2n+n^2+rn}, \quad \forall x\in\mathcal{L}$$

holds, where vector $\zeta(x)$ is defined in (10). From our own experiment, the additional slack variables can lead to less conservative results when the size of LMI problem is relatively small, otherwise, they can result in more frequent failure in achieving a feasible solution due to higher computational burden.

Remark 5. It is worth mentioning that $\dot{V}(x)$ in the proof of Theorem 1 is a third-degree polynomial in x, and hence, the condition of Theorem 1 can be viewed as proving the negative-definiteness of a third-degree polynomial in x in hypercube $\mathcal{L}$. This concept may remind the reader of what the SOS technique performs. In fact, the proposed approach can be interpreted from the viewpoint of the SOS approach [43]. Here, we briefly discuss the connection of the proposed approach with those in [4], where local stability analysis of polynomial T–S fuzzy systems was studied using the SOS framework. For simplicity, let us consider the case where $p = 1$, $r = 2$, $q = 1$, and $z_1 = x_1$. Then, (11) with $S = 0_{n\times n}$ can be rewritten as

$$\dot{V}(x) = x^T\left(P_zA_z + A_z^TP_z\right)x + \left[x^TP_1x \quad x^TP_2x\right](J_hA_zx).$$

Using relation $h_2 = 1 - h_1, \forall x \in \mathcal{L}$, one has

$$\begin{aligned}\dot{V}(x) &= x^T\left(P_zA_z + A_z^TP_z\right)x + \left[x^TP_1x \quad x^TP_2x\right]\begin{bmatrix}\partial h_1/\partial x_1 & 0\\ \partial h_2/\partial x_1 & 0\end{bmatrix}A_zx\\ &= x^T\left(P_zA_z + A_z^TP_z\right)x + \left[x^TP_1x \quad x^TP_2x\right]\begin{bmatrix}\partial h_1/\partial x_1 & 0\\ -\partial h_1/\partial x_1 & 0\end{bmatrix}A_zx\\ &= x^T\left(P_zA_z + A_z^TP_z\right)x + (x^TP_1x - x^TP_2x)([\,\partial h_1/\partial x_1 \quad 0\,]A_zx)\\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_jx^T(P_iA_j + A_j^TP_i)x + \left([\,\partial h_1/\partial x_1 \quad 0\,]A_jx\right)\left(x^TP_1x - x^TP_2x\right), \quad \forall x \in \mathcal{L},\end{aligned}$$

which can be viewed as a special case of Eq. (10) in [4]. In this respect, the use of convex combination representation (3) is philosophically similar to approaches in [4].

The optimization problem in Theorem 1 is bilinear due to the Kronecker's product term $S \otimes P_{\mathbf{j}_q}$ in $\Upsilon^{(1)}_{\mathbf{j}_q}(\theta, J)$. However, if S is fixed to be I_n, the problem becomes a single-parameter minimization problem subject to LMI constraints, which can be efficiently solved via a sequence of LMI optimization problems such as a bisection algorithm or a line search process, or solving EVPs, which are convex optimizations [5], and hence, can be directly treated with the help of LMI solvers [13,37,48]. The special case is stated as a corollary.

Corollary 1. *Let $q \in \mathbb{N}$ be given. If there exist symmetric matrices $P_{\mathbf{j}_q} \in \mathbb{R}^{n\times n}$, matrices $Y_{\mathbf{j}_{q-1}} \in \mathbb{R}^{(2n+n^2+rn)\times(np+n+rn)}$, and a real number $\beta > 0$ such that the optimization*

$$\min_{P_{\mathbf{j}_q}, Y_{\mathbf{j}_{q-1}}, \beta} \beta \text{ subject to} (4),\ (5) \text{ and } (7)$$

with $S = I_n$ and $\gamma = 1$ is satisfied, then system (1) is locally asymptotically stable. Moreover, an invariant subset of the DA for system (1) is given by $\Omega(1)$.

Proof. The proof straightforwardly follows from Theorem 1. □

Remark 6. The numerical complexity of optimization problems based on LMIs can be estimated from the total number N_D of scalar decision variables and the total row size N_L of the LMIs [5,13]. For Corollary 1 with fixed β, $N_D = (n^2 + n)r^q/2 + (2n + n^2 + rn)(np + n + rn)r^{q-1}$ and $N_L = ((2n + n^2 + rn)w2^p + np + n)(r + q - 1)!/(q!\,(r - 1)!)$.

For comparison sake, the quadratic stability condition is given in the following lemma.

Lemma 2 [58]. *If there exists a symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that $P \succ 0$ and $A_i^TP + PA_i \prec 0$ hold for all $i \in \mathcal{I}_r$, then system (1) is locally asymptotically stable.*

Then, it can be established that the condition of Corollary 1 includes the quadratic condition as a special case when $J_h, \forall x \in \mathcal{L}$ can span all the elements of co$\{J_1, J_2, \ldots, J_w\}$.

Theorem 2. *Suppose that $J_h, \forall x \in \mathcal{L}$ can span all the elements of co$\{J_1, J_2, \ldots, J_w\}$. If the LMI problem of* Lemma 2 *is fulfilled, then the optimization of Corollary 1 corresponding to any $q \in \mathbb{N}$ are also satisfied.*

Proof. If $P \succ 0$ and $A_i^TP + PA_i \prec 0$ hold for all $i \in \mathcal{I}_r$, then it is easy to see by the Schur complement that

$$\Xi_{(j,1)} := \begin{bmatrix}\alpha PA_j + \alpha A_j^TP & A_j^T\\ A_j & -I_n\end{bmatrix} \prec 0, \quad j \in \mathcal{I}_r, \tag{13}$$

$$\Xi_2(J) := \begin{bmatrix}-I_n \otimes \alpha P & J^T \otimes I_n\\ J \otimes I_n & -2I_{rn}\end{bmatrix} \prec 0, \quad J \in \{J_1, J_2, \ldots, J_w\} \tag{14}$$

hold for a sufficiently large real number $\alpha > 0$.

On the other hand, since $h_1 + h_2 + \cdots + h_r = \mathbf{1}_r h = 1$, where

$$\mathbf{1}_r := \left[\underbrace{1 \quad \cdots \quad 1}_{r-\text{times}}\right] \in \mathbb{R}^{1\times r},$$

we have that $\mathbf{1}_r J_h = 0_{1\times n}, \forall x \in \mathcal{L}$, implying

$$\mathbf{1}_r J_h \otimes P = (\mathbf{1}_r \otimes P)(J_h \otimes I_n) = 0_{n\times n^2}, \quad \forall x \in \mathcal{L}.$$

Since it is assumed that $J_h, \forall x \in \mathcal{L}$ spans all the elements of co$\{J_1, J_2, \ldots, J_w\}$, it follows that

$$(\mathbf{1}_r \otimes P)(J \otimes I_n) = 0_{n\times n^2}, \quad \forall J \in \{J_1, J_2, \ldots, J_w\}. \tag{15}$$

By direct calculation and from (13)–(15), it can be seen that

$$\mathrm{diag}(\Xi_{(j,1)}, \Xi_2(J)) = \begin{bmatrix} 0_{n\times n} & * & * & * \\ \alpha P & I_n & * & * \\ 0_{n^2\times n} & 0_{n^2\times n} & -I_n \otimes \alpha P & * \\ 0.5(\mathbf{1}_r^T \otimes \alpha P) & 0_{rn\times n} & 0_{rn\times n^2} & 0_{rn\times rn} \end{bmatrix} + \mathrm{He}\{YC_j(\theta, J)\}$$

$$\prec 0, \quad (j, \theta, J) \in \mathcal{I}_r \times \mathcal{W} \times \{J_1, J_2, \ldots, J_w\} \tag{16}$$

hold with

$$Y = \begin{bmatrix} \alpha P & 0_{n\times n} & \cdots & 0_{n\times n} & 0.5(\mathbf{1}_r \otimes \alpha P) \\ I_n & 0_{n\times n} & \cdots & 0_{n\times n} & 0_{n\times nr} \\ 0_{n^2\times n} & 0_{n^2\times n} & \cdots & 0_{n^2\times n} & 0_{n^2\times nr} \\ 0_{rn\times n} & 0_{rn\times n} & \cdots & 0_{rn\times n} & I_{rn} \end{bmatrix} \in \mathbb{R}^{(2n+n^2+rn)\times(np+n+rn)}.$$

Therefore, LMIs in (16) ensure that (4) with $P_{\mathbf{j}_q} = \alpha P, \Phi_{\mathbf{j}_{q-1}} = \mathbf{1}_r^T \otimes \alpha P, Y_{\mathbf{j}_{q-1}} = Y$ holds for any $q \in \mathbb{N}$. Finally, there exist sufficiently large real numbers $\alpha > 0$ and $\beta > 0$ such that $x_{a_k,\max}^{-2} e_{a_k} e_{a_k}^T \prec \alpha P, \forall k \in \mathcal{I}_p$ and $-\beta I_n + P \preceq 0$ hold, which guarantee that (5) and (7) with $P_{\mathbf{j}_q} = \alpha P$ are fulfilled for any $q \in \mathbb{N}$. This completes the proof. □

In order to further enlarge the estimated DA, the *V-s* iteration algorithm presented in [6,20,54] can be applied.
***V–s* iteration algorithm**

Step 1 (Initialization). Given maximum number of iterations $N_{iter} \in \mathbb{N}$, set $k = 0$, $S = I_n$, $\gamma = 1$, and solve for $P_{\mathbf{j}_q}$

$$\min_{P_{\mathbf{j}_q}, Y_{\mathbf{j}_{q-1}}, \beta} \beta \text{ subject to}(4),\ (5) \text{ and } (7)$$

Step 2 (γ Step). Given $P_{\mathbf{j}_q}$ from the previous step, set k to $k + 1$ and solve for S, γ

$$\max_{S, Y_{\mathbf{j}_{q-1}}, \gamma} \gamma \text{ subject to } (4)\text{—}(6)$$

A line search or a bisection process over γ is needed here due to γS term in (4).
Step 3 (β Step). Given $P_{\mathbf{j}_q}, \gamma$ from the previous step, solve for β

$$\min_{\beta} \beta \text{ subject to } (7)$$

Step 4 (V Step). Given S, β, γ from the previous step, solve for $P_{\mathbf{j}_q}$ satisfying LMIs (4), (5) and (7).
Step 5 (Normalization). Set $P_{\mathbf{j}_q}$ to $\gamma^{-1} P_{\mathbf{j}_q}$. If $k = N_{iter}$, then return $P_{\mathbf{j}_q}$. Otherwise, go to Step 2.

4. Examples

All numerical experiments in the sequel were treated with the help of MATLAB 2008a running on a Windows 7 PC with Intel Core i7-3770 3.4 GHz CPU, 24 GB RAM. The LMI problems were solved with SeDuMi [48] and Yalmip [37].

Example 1. Let us consider system (1) with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -(2+\lambda) & -1 \end{bmatrix}, \tag{17}$$

$$z_1 = x_1, \quad h_1 = \frac{1 + \sin x_1}{2}, \quad h_2 = 1 - h_1, \quad \mathcal{L} = \{\xi \in \mathbb{R}^n : \xi_1 \in [-\pi/2, \pi/2]\}$$

taken from [38]. The Jacobian matrix of h is calculated as follows:

$$J_h = \frac{1}{2}\cos x_1 \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \in \mathrm{co}\{J_1, J_2\}, \quad \forall x \in \mathcal{L},$$

where

$$J_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 0 \end{bmatrix}.$$

We aim to evaluate the conservativeness and numerical complexity of several approaches. To this end, the maximum values of λ, denoted by λ^*, such that asymptotic stability of the system is guaranteed for all $\lambda \in [0, \lambda^*]$ were computed using Theorem 1 (T1) in [58], Theorem 3 (T3) in [44], Theorem 6 (T6) in [38], Theorem 4 (T4) in [39], Theorem 4 (T4) in [32], Theorem 2 (T2) in [2], and Corollary 1 (C1). Remark that assumption (2) with $\phi = \phi_1 = \phi_2$ was used for T6 in [38], and T4 in [32], while $|(\partial h_1/\partial x_1)x_1| \leqslant \phi$ and $|(\partial h_1/\partial x_1)x_2| \leqslant \phi$ were utilized for T2 in [2]. Then, stability bound λ^* were computed for different ϕ, and the results are depicted in Fig. 1. From the figure, it can be seen that Corollary 1 provides the least conservative result except for the case when the bounds on the time-derivative of the NMFs or on the partial derivative of the NMFs with respect to the state variables are small. When the bounds are small, T6 in [38], T4 in [32], and T2 in [2] provide stability results that are superior to those given by our Corollary 1. However, this might not explicitly imply that the previous approaches surpass the proposed one, since when bounds on the time-derivative or partial derivative of the NMFs are small, the DAs estimated by T6 in [38], T4 in [32], and T2 in [2] would be confined within a small region around the origin. For a more fair comparison, it is important to compare them in terms of the volume of the DA estimation. The experiment will be conducted in the next example.

In addition, Table 1 lists the numerical complexity of several approaches in terms of N_D, the total number of scalar decision variables, N_L, the total row size of associated LMI, and the average computational time (in seconds) spent by each test to provide a feasible solution with $\lambda = 1$. Here, $\phi = 0.3$ was used for T6 in [38], T4 in [32], and T2 in [2], and the average computational time for each test was obtained by taking the average of ten measures. From the table, one concludes that Corollary 1 requires the highest computational burden among previous approaches. Thus, one may say that the advantage of Corollary 1 comes at the price of a higher computational effort.

Example 2. Let us consider system (1) with

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix},$$

$$z_1 = x_1, \quad h_1 = (1 + \sin x_1)/2, \quad h_2 = 1 - h_1,$$

borrowed from [57] and consider $\mathcal{L} = \{\xi \in \mathbb{R}^n : \xi_1 \in [-x_{1,\max} \quad x_{1,\max}]\}$, where real number $x_{1,\max} > 0$ can be chosen by the system analyst. Matrices J_1 and J_2 are calculated to be

$$J_1 = \frac{1}{2}\max_{x\in\mathcal{L}}(\cos x_1)\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad J_2 = \frac{1}{2}\min_{x\in\mathcal{L}}(\cos x_1)\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

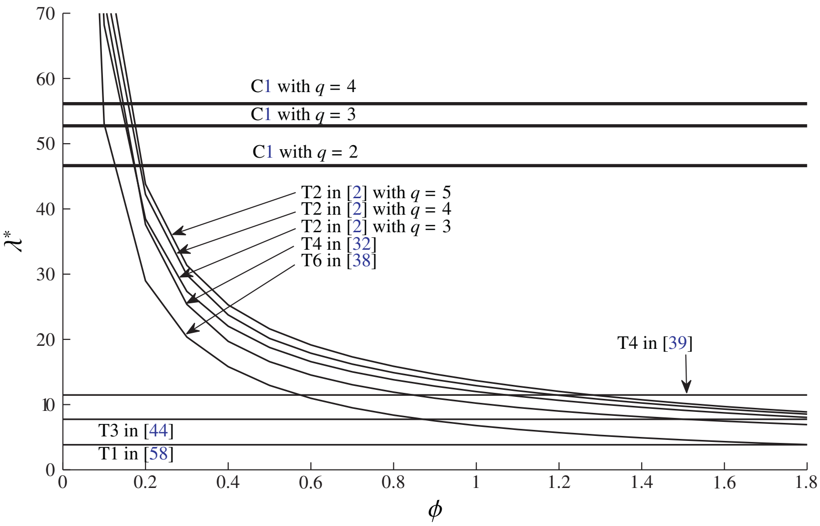

Fig. 1. Example 1. Stability bounds for different ϕ.

Table 1
Example 1. Numerical complexity (N_D total number of scalar decision variables; N_L total row size of corresponding LMI; time in seconds).

Method	N_D	N_L	Time (s)
T1 in [58]	3	6	0.0290
T3 in [44]	7	12	0.0110
T6 in [38]	9	14	0.0110
T4 in [39]	12	12	0.0150
T4 in [32]	42	60	0.0500
T2 in [2] with $q = 3$	120	112	0.0220
T2 in [2] with $q = 4$	240	224	0.0450
T2 in [2] with $q = 5$	480	448	0.1110
Corollary 1 with $q = 2$	204	156	0.1300
Corollary 1 with $q = 3$	408	208	0.3240
Corollary 1 with $q = 4$	816	260	0.5580

For the system, Fig. 2 shows trajectories (dashed lines) and the level set $\Omega(1)$ estimated by using the proposed V-s iteration algorithm with $q = 4$, $x_{1,\max} = 8$, and 20 iterations (blue solid line). The green solid line is the level set computed by T2 in [2] with $q = 5$ and $\phi = 2.23$. In addition, Fig. 3 shows the monotonic decrease of β as the iterations increase. The results suggest that the DA estimated by the proposed approach includes that in [2]. On the other hand, the maximum numerical complexity for each step of the V–s iteration is $N_L = 260$ and $N_D = 816$, while for T2 in [2], $N_L = 448$ and $N_D = 480$. The total computational time of the 20 iterations was nearly 18 min. Thus, one concludes that the proposed method is computationally more demanding than that in [2].

Example 3. Let us consider the chaotic Lorenz system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma x_2 \\ \rho x_1 - x_2 - x_1 x_3 \\ -\beta x_3 + x_1 x_2 \end{bmatrix},$$

which has only one equilibrium point at the origin when $\rho < 1$. Following the same lines as in [34], it can be represented by T–S fuzzy system (1) with

$$A_1 = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & x_{1,\max} \\ 0 & -x_{1,\max} & -\beta \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & -x_{1,\max} \\ 0 & x_{1,\max} & -\beta \end{bmatrix},$$

$$h_1 = \frac{-x_1 + x_{1,\max}}{2x_{1,\max}}, \quad h_2 = \frac{x_1 + x_{1,\max}}{2x_{1,\max}}, \quad z_1 = x_1, \quad \mathcal{L} = \{\xi \in \mathbb{R}^n : \xi_1 \in [-x_{1,\max}, x_{1,\max}]\}.$$

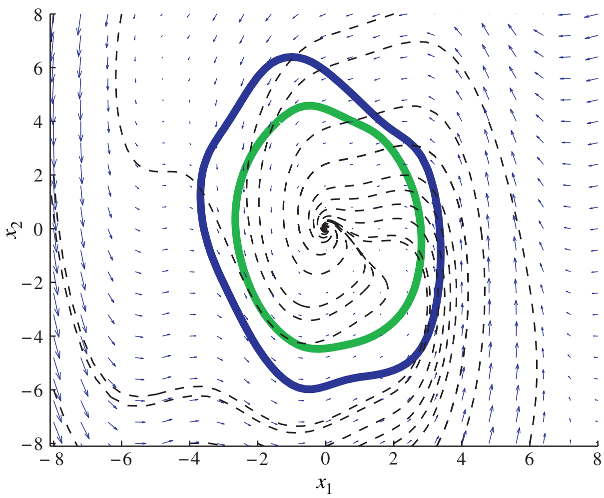

Fig. 2. Example 2. Trajectories (dashed lines) and level set $\Omega(1)$ estimated by using the proposed V–s iteration algorithm (blue solid line) ($q = 4$, $x_{1,\max} = 8$, and $N_{iter} = 20$) and the level set obtained by using Theorem 2 in [2] (green solid line) ($q = 5$). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

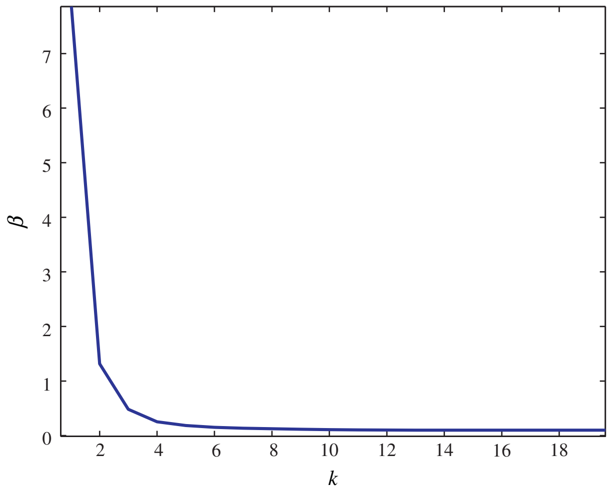

Fig. 3. Example 2. β Versus iteration.

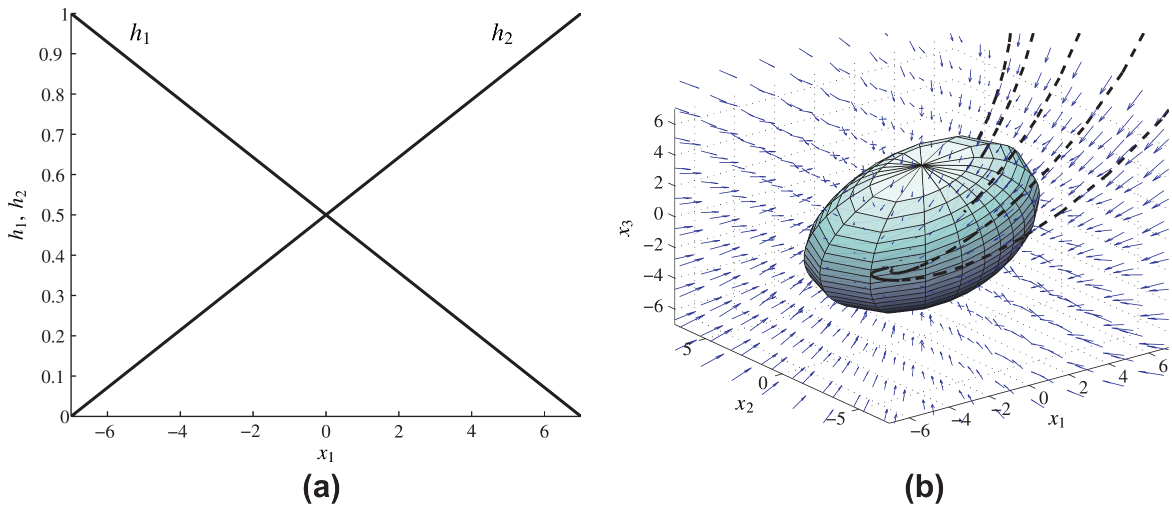

Fig. 4. Example 3. (a) The NMFs for the T–S fuzzy model of the Lorenz system. (b) The inner estimate of the DA computed by the proposed iteration algorithm with $x_{1,\max} = 7$, $q = 3$, $N_{iter} = 20$ and the converging trajectories (dashed lines).

Fig. 4(a) shows the NMFs, which are the widely used triangular type functions. In this case, h and J_h are calculated to be

$$h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{2x_{1,\max}} \begin{bmatrix} -x_1 + x_{1,\max} \\ x_1 - x_{1,\max} \end{bmatrix}, \quad J_h = \frac{1}{2x_{1,\max}} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and we can set

$$J_h \in \mathrm{co}\{J_1\} = \{J_1\}, \quad J_1 = \frac{1}{2x_{1,\max}} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

For the simulation, we set $(\delta, \rho, \beta) = (20.1, 0.2, 23.6)$ for which the origin of the chaotic Lorenz system is the stable equilibrium point. Applying the proposed iteration algorithm with $x_{1,\max} = 7$, $q = 3$, $N_{iter} = 20$, we can obtain an inner estimate of the DA, which is depicted in Fig. 4(b).

5. Conclusion

In this paper, we have presented a strategy to estimate invariant subsets of the DA for continuous-time T–S fuzzy systems. A solution to the problem has been obtained in terms of single-parameter minimization problems subject to BMI or LMI constraints. In order to obtain larger estimation of the DA, an effective V–s iteration algorithm has been introduced. Illustrative examples have demonstrated the validity of the proposed scheme, and showed that the proposed approach can yield better results than existing ones in the literature. Note that the LMI-based methods obtained in this paper are limited to solving the local stability analysis problem, and their application to control synthesis is limited since the extension leads to BMI conditions. In addition, the computational complexity of the proposed optimizations can be an issue for large-scale problems; for instance, large n, r, and q. Generally, a trade-off can be found between computational complexity of the proposed method and its quality in terms of less conservatism. For this reason, extending the proposed analysis scheme to the controller synthesis problem and developing computationally more efficient algorithms are important problems, which are left as future research topics.

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