Automatica 45 (2009) 1580–1583

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Reducing conservativeness in recent stability conditions of TS fuzzy systems☆

L.A. Mozelli, R.M. Palhares *, F.O. Souza, E.M.A.M. Mendes

Department of Electronics Engineering, Federal University of Minas Gerais, Av. Antônio Carlos 6627, 31270-010, Belo Horizonte, MG, Brazil

ARTICLE INFO

Article history:
Received 26 March 2008
Received in revised form
11 December 2008
Accepted 13 February 2009
Available online 27 March 2009

Keywords:
Stability analysis
Fuzzy systems
Lyapunov function
Nonlinear systems
Linear matrix inequalities (LMIs)

ABSTRACT

In this correspondence a new simple strategy for reducing the conservativeness in stability analysis of continuous-time Takagi–Sugeno fuzzy systems based on fuzzy Lyapunov functions is proposed. This new strategy generalizes previous results.

1. Introduction

Over the last two decades, the combination of LMIs and convex optimization techniques to cope with stability analysis and control design in several areas, including fuzzy control, has been increasingly popular. This allows one to systematically construct functions that guarantee global stability in the sense of Lyapunov. The standard approach for Takagi–Sugeno (TS) fuzzy continuous-time systems (Takagi & Sugeno, 1985) is to use the common quadratic Lyapunov function (CQLF) (Tanaka & Wang, 2001; Teixeira, Assunção & Avellar, 2003). However, due to some conservatism, alternative classes of Lyapunov function candidates have been proposed as, e.g. the piecewise Lyapunov function (PLF) (Cao, Rees, & Feng, 1997; Johansson, Rantzer, & Årzén, 1999) and the fuzzy Lyapunov function (FLF) (Tanaka, Hori, & Wang, 2003). This paper deals with the latter issue and its conservatism in stability analysis via LMIs.

2. Preliminaries

Consider the unforced Takagi–Sugeno fuzzy model

$$\dot{x} = \sum_{i=1}^{r} h_i A_i x, \tag{1}$$

where h_i is the normalized grade of membership of each local system A_i, which satisfies the properties

$$h_i \geq 0, \quad \sum_{i=1}^{r} h_i = 1, \quad \sum_{i=1}^{r} \dot{h}_i = 0. \tag{2}$$

The search for a common P matrix to construct a Lyapunov function $V(x) = x'Px$ can be summed up by the following Lemma (Tanaka & Wang, 2001). Here $\mathcal{R}$ is the set $\{1, 2, \ldots, r\}$.

Lemma 1. *The TS fuzzy system (1) is asymptotically stable if there exists $P = P' \succ 0$ satisfying:*

$$A_i'P + PA_i \prec 0, \quad i \in \mathcal{R}. \tag{3}$$

Most of the subsequent works relying on CQLFs (Tanaka & Wang, 2001; Teixeira et al., 2003) present roughly the same constraints, however they introduce quadratic form relaxations (i.e. right-hand side slack variables) into the LMI formulation. Extra variables may provide advantage to LMI solvers, allowing them to find solutions when the conventional conditions fail due to numeric reasons. As a tradeoff, an increase in computational cost occurs.

Johansson et al. (1999) have demonstrated that a CQLF does not exist for some TS systems. An alternative is to consider the FLF introduced, e.g., by Tanaka et al. (2003):

$$V(x) = x' \sum_{i=1}^{r} h_i P_i x, \tag{4}$$

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Mikael Johansson under the direction of Editor André L. Tits.

* Corresponding author. Tel.: +55 31 3409 3457; fax: +55 31 3409 4850.
E-mail addresses: mozelli@cpdee.ufmg.br (L.A. Mozelli), palhares@cpdee.ufmg.br (R.M. Palhares), oliveira@cpdee.ufmg.br (F.O. Souza), emmendes@cpdee.ufmg.br (E.M.A.M. Mendes).

0005-1098/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2009.02.023

which is a fuzzy blending of quadratic functions parameterized by the same membership functions used in (1).

To consider this type of function it is necessary that $h_i \in \mathcal{C}^1$, which is met for TS fuzzy systems constructed by the sector nonlinearity approach (Tanaka & Wang, 2001), which is a global (semiglobal) method to obtain TS models that exactly represent nonlinear dynamics. Furthermore the time-derivative of (4) contains information on the time-derivative of the membership functions, which may lead to some conservatism in the LMI stability analysis. This particular feature is the focus of the forthcoming sections.

3. Analysis with fuzzy Lyapunov functions

The following Lemma gives sufficient conditions to guarantee the stability of (1) when FLF candidate (4) is considered (Tanaka et al., 2003, Theorem 1).

Lemma 2. *Assume that $|\dot{h}_k| \leq \phi_k, k \in \mathcal{R}$. TS fuzzy system (1) is stable if the following LMIs are satisfied*

$$P_i = P_i' \succ 0, \quad i \in \mathcal{R}, \tag{5}$$

$$P_\phi + \frac{1}{2}\left(A_i'P_j + P_jA_i + A_j'P_i + P_iA_j\right) \prec 0, \quad i \leq j, \tag{6}$$

where $i, j \in \mathcal{R}$, $P_\phi = \sum_{k=1}^{r} \phi_k P_k$ and ϕ_k are scalars.

Remark 3. In order to recast the stability conditions based on (4) into LMIs, it is necessary to consider ϕ_k, $k \in \mathcal{R}$ in advance, scalars denoting upper bounds to the time-derivative of the membership functions.

From the theoretical point of view, it is clear that the fuzzy Lyapunov function is a much richer class of function than the common quadratic one. The use of structural information on the TS system to construct a Lyapunov function is an advantage in comparison to an approach that disregards the time-derivative of the membership functions. One may even expect that conditions based on the CQLF are particular cases of the FLF. In other words, Lemma 2 should always provide better results, or at least the same, than Lemma 1. From the numeric point of view, this is not true in every case as illustrated by the following example.

Example 4. Consider a TS system as in (1) with

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & a \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -4 & -4 \\ \frac{1}{5}(3b-2) & \frac{1}{5}(3a-4) \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -3 & -4 \\ \frac{1}{5}(2b-3) & \frac{1}{5}(2a-6) \end{bmatrix}, \qquad A_4 = \begin{bmatrix} -2 & -4 \\ b & -2 \end{bmatrix},$$

where

$$\alpha_i(x_i) = \begin{cases} (1-\sin(x_i))/2, & \text{for } |x_i| \leq \pi/2, \\ 0, & \text{for } x_i > \pi/2, \\ 1, & \text{for } x_i < -\pi/2, \end{cases}$$

$$\beta_i(x_i) = 1 - \alpha_i(x_i),$$

$$h_1 = \alpha_1(x_1)\alpha_2(x_2), \qquad h_2 = \alpha_1(x_1)\beta_2(x_2),$$

$$h_3 = \beta_1(x_1)\alpha_2(x_2), \qquad h_4 = \beta_1(x_1)\beta_2(x_2).$$

The stability of this system is checked using Lemmas 1 and 2, for several values of pairs (a, b), $a \in [-10, 0]$, $b \in [0, 200]$ and $|\dot{h}_i| < 0.85$. The results are depicted in Fig. 1 and reveal that the FLF promotes a larger stability margin than the CQLF, however, the opposite occurs for specific sets of parameters.

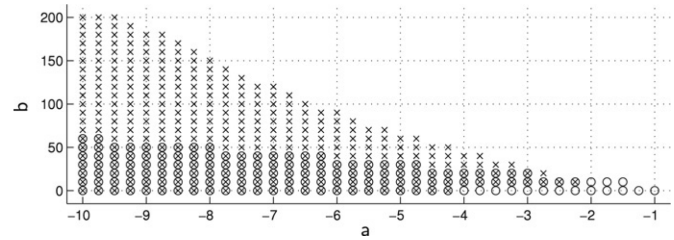

Fig. 1. Stability analysis with Lemmas 1 (∘) and 2 (×).

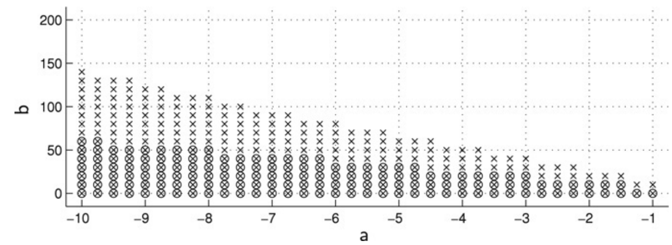

Fig. 2. Stability analysis with Lemmas 1 (∘) and 5 (×).

At first glance, LMIs (5)–(6) are less restrictive than (3), by the presence of more variables, P_i. It seems to be an easier task to guarantee (6), using r variables, than to guarantee the similar condition (3), using a single variable instead. Nevertheless, there is another difference between LMIs (6) and (3). Because FLF candidate (4) generates information about the time-derivative of membership functions into the Lyapunov time-derivative, the term P_ϕ appears into LMIs (6), whereas in (3) there is no such term. Due to (5), P_ϕ always will be positive definite, because it consists of a sum of positive definite matrices P_k multiplied by positive scalars ϕ_k. The matrix sum in (6) must have negative eigenvalues, or a negative sign, without the contribution of P_ϕ. Although the FLF is more general and feasible for TS systems where the CQLF does not exist, the presence of the term P_ϕ may be a drawback from the numerical point of view. Then, P_ϕ may be pointed out as the main source of conservatism when considering FLF analysis. To reinforce this conjecture, consider the following Lemma (Tanaka et al., 2003, Theorem 2), which intends to provide a less conservative version of Lemma 2.

Lemma 5. *Assume that $|\dot{h}_k| \leq \phi_k, k \in \mathcal{R}$. TS fuzzy system (1) is stable if the following LMIs are satisfied*

$$P_i = P_i' \succ 0, \quad i \in \mathcal{R}, \tag{7}$$

$$P_i - P_r \succeq 0, \quad i \in \mathcal{R} - \{r\}, \tag{8}$$

$$\bar{P}_\phi + \frac{1}{2}\left(A_i'P_j + P_jA_i + A_j'P_i + P_iA_j\right) \prec 0, \quad i \leq j, \tag{9}$$

where $i, j \in \mathcal{R}$, $\bar{P}_\phi = \sum_{k=1}^{r-1} \phi_k (P_k - P_r)$ and ϕ_k are scalars.

Repeating Example 4 with Lemmas 1, 2 and 5 results in the stability analysis shown in Figs. 2 and 3. In Fig. 2, note that the result of Lemma 5 involves the result of the common quadratic condition, which was not accomplished by Lemma 2 (recall Fig. 1). In Lemma 5, constraints (8) allow the matrices P_i to be equal, and therefore the term $\bar{P}_\phi$ may vanish and the conditions in Lemma 5 reduce to the conditions in Lemma 1.

However, the stable margins from Lemmas 2 and 5 do not include each other as seen in Fig. 3. The relaxation imposed over P_ϕ allows Lemma 5 to recover the result of the quadratic stability based approaches and thus expanding the stability margin. The price is the contraction of the stability margin for a different set of parameters. Although Lemma 5 partially solves the problem of relaxing Lemma 2 a general condition that involves all the previous conditions is yet to be found. This is exactly what will be presented in the next section.

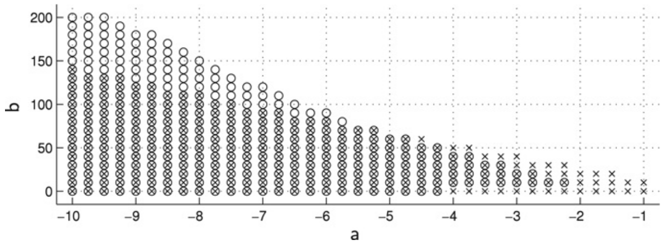

Fig. 3. Stability analysis with Lemmas 2 ($\circ$) and 5 ($\times$).

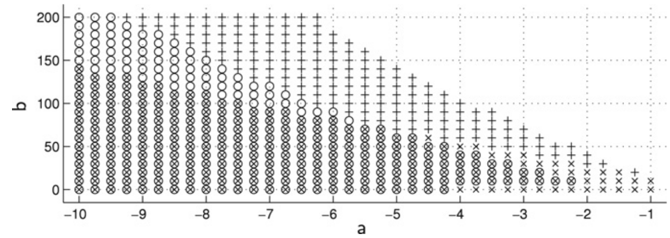

Fig. 4. Stability analysis with Lemmas 2 ($\circ$) and 5 ($\times$) and with Theorem 6 ($\circ$, $\times$, and $+$).

4. Main result

In this section, a less conservative condition is proposed, towards a generalization of the previous results. Since the tests showed that the relaxation over P_ϕ shifts the stability margin, special attention must be paid to this term. This new condition is presented in the sequel.

Theorem 6. *Assume that $|\dot{h}_k| \le \phi_k$, $k \in \mathcal{R}$. TS fuzzy system (1) is stable if the following LMIs are satisfied*

$$P_i = P_i' \succ 0, \quad i \in \mathcal{R}, \tag{10}$$

$$P_i + X \succeq 0, \quad i \in \mathcal{R}, \tag{11}$$

$$\tilde{P}_\phi + \frac{1}{2}\left(A_i'P_j + P_jA_i + A_j'P_i + P_iA_j\right) \prec 0, \quad i \le j, \tag{12}$$

where $i, j \in \mathcal{R}$, $\tilde{P}_\phi = \sum_{k=1}^{r} \phi_k (P_k + X)$, ϕ_k are scalars, and $X = X'$.

Proof. Consider (4) as a Lyapunov function candidate. Then:

$$\dot{V}(x) = \sum_{k=1}^{r} \dot{h}_k x' P_k x + \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \times x'\left(A_i'P_j + P_jA_i + A_j'P_i + P_iA_j\right)x. \tag{13}$$

Based on (2), it follows that $\sum_{k=1}^{r} \dot{h}_k X = \bar{X} = 0$, where X is any symmetric matrix of proper dimension. Adding $\bar{X}$ to (13), assuming that $|\dot{h}_k| \le \phi_k$, $k \in \mathcal{R}$ and that (11) hold it follows that:

$$\begin{aligned}\dot{V}(x) &\le \sum_{k=1}^{r} \phi_k x'(P_k + X)x + \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \times x'\left(A_i'P_j + P_jA_i + A_j'P_i + P_iA_j\right)x \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j x'\left[\tilde{P}_\phi + \frac{1}{2}\left(A_i'P_j + P_jA_i + A_j'P_i + P_iA_j\right)\right]x. \end{aligned}\tag{14}$$

If (12) holds, then $\dot{V}(x) < 0$ and (1) is globally stable. □

Remark 7. It is easy to see that if $X = 0$, then Lemma 2 is recovered, and if $X = -P_r$ so is Lemma 5. Therefore, Lemmas 2 and 5 are particular cases of Theorem 6.

Consider Example 4 again. Fig. 4 shows the comparison between the previous conditions and Theorem 6. Note that the proposed condition involves all previous ones with a larger stability region. With the introduction of a single variable X, a new less conservative condition is obtained. From the point of view of the computational cost, the contribution of this extra variable can be neglected, specially for large number of rules. It is interesting to note how X promotes less conservative conditions. In most cases when the previous stability conditions are prone to fail it is observed that X is negative definite or semidefinite. This situation reinforces the conjecture that P_ϕ is a source of conservativeness.

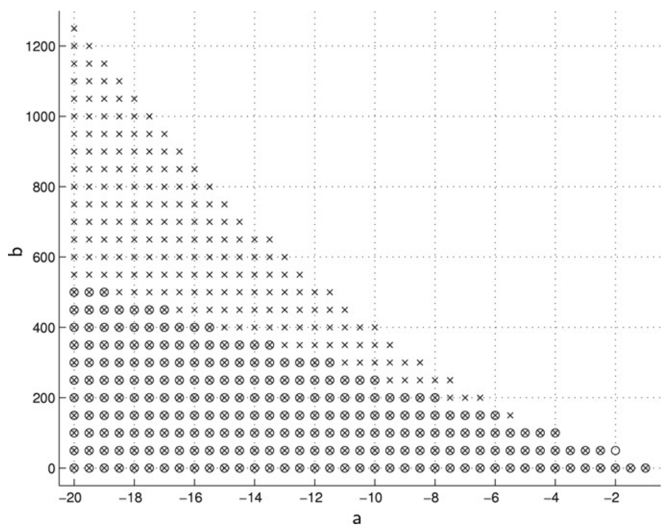

Fig. 5. Stability analysis with Rhee and Won (2006, Theorem 3, $\circ$) and Theorem 6 ($\times$).

5. Further comparisons

In this section two examples are given to illustrate how the use of the information of the upper bound to $\dot{h}_k$ together with the proposed strategy can be less conservative than a recent method in the literature. For instance, Rhee and Won (2006) present a special structure to eliminate the time-derivative of the membership functions which may be less conservative than results presented in Tanaka et al. (2003) are.

Continuing Example 4, Fig. 5 shows a comparison between Theorem 6 and Rhee and Won (2006, Theorem 3) with $\phi_k < 0.85$. It is clear that Theorem 6 provides a much larger feasible region than (Rhee & Won, 2006). The computational implementations have been carried out using SeDuMi (Sturm, 1999) with YALMIP (Löfberg, 2004). For the purpose of comparing computational efforts, consider $a \in [-10, -5]$ and $b \in [0, 20]$ which is feasible for all approaches. Thus the elapsed time normalized by Lemma 1 are: Lemma 1 (1); Lemma 2 (1.33); Lemma 5 (1.50); Theorem 6 (1.52); and Rhee and Won (2006, Theorem 3) (1.54).

As another example, consider a TS system as in (1) with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -(2+\lambda) & -1 \end{bmatrix}.$$

The maximum value of λ for which the stability is guaranteed was checked for several values of ϕ_k. This result is illustrated in Fig. 6 for Theorem 6 (solid), Lemma 1 (dash), and Rhee and Won (2006, Th. 3) (dash-dot). Notice that depending on the upper bound selection, ϕ_k, Theorem 6 provides a larger value of λ.

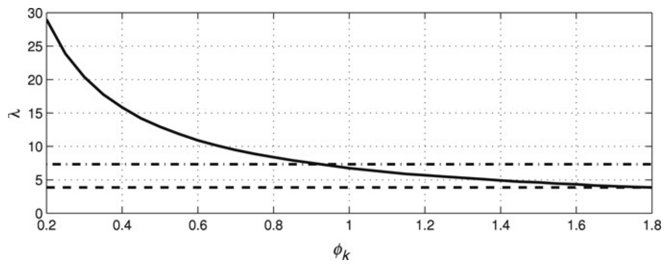

Fig. 6. Stability region for different ϕ_k.

6. Extension to control synthesis

The extension of Theorem 6 to control synthesis is straightforward following the same rationale in Tanaka et al. (2003). However notice that the control synthesis in Tanaka et al. (2003) as well as the one presented in Rhee and Won (2006) fall into bilinear matrix inequalities which may be somewhat harder to solve. In addition to that, the implementation of the stabilization approach in Rhee and Won (2006) is rather cumbersome. The idea, for future work, is to obtain LMI formulations using the congruence transformations as presented in Ebihara and Hagiwara (2004). Notice that Ebihara and Hagiwara (2004, Ineq. 2) is similar to (12).

Acknowledgements

The work has been supported by the Brazilian agencies CNPq and FAPEMIG.

References

Cao, S. G., Rees, N. W., & Feng, G. (1997). Analysis and design for a class of complex control systems part II: Fuzzy controller design. *Automatica*, *33*(6), 1029–1039.

Ebihara, Y., & Hagiwara, T. (2004). New dilated LMI characterizations for continuous-time multiobjective controller synthesis. *Automatica*, *40*(11), 2003–2009.

Johansson, M., Rantzer, A., & Årzén, K.-E. (1999). Piecewise quadratic stability of fuzzy systems. *IEEE Transactions on Fuzzy Systems*, *7*(6), 713–722.

Löfberg, J. (2004). YALMIP: A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD conference* (pp. 284–289).

Rhee, B.-J., & Won, S. (2006). A new fuzzy Lyapunov function approach for a Takagi-Sugeno fuzzy control system design. *Fuzzy Sets and Systems*, *157*(9), 1211–1228.

Sturm, J. F. (1999). Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, *11*(1), 625–653.

Takagi, T., & Sugeno, M. (1985). Fuzzy identification of systems and its applications to modeling and control. *IEEE Transactions on Systems, Man, and Cybernetics*, *15*(1), 116–132.

Tanaka, K., Hori, T., & Wang, H. O. (2003). A multiple Lyapunov function approach to stabilization of fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, *11*(4), 582–589.

Tanaka, K., & Wang, H. O. (2001). *Fuzzy control systems design and analysis: A linear matrix inequality approach*. John Wiley and Sons.

Teixeira, M. C. M., Assunção, E., & Avellar, R. G. (2003). On relaxed LMI-based designs for fuzzy regulators and fuzzy observers. *IEEE Transactions on Fuzzy Systems*, *11*(5), 613–623.