1 RLMPC proof

Main idea: Stack laps together (e.g. laps 1+2, 2+3, ...) and construct stacked iteration costs and Q functions. Then use the initial state (e.g. x_0^2 of laps 2+3) of one stack, which is the "middle state" of the previous stack (of laps 1+2), to prove that the stacked iteration cost of two consecutive stacks is non-decreasing.

Define the single iteration cost as usual:

$$J^j = J^j_{0\to\infty} = \sum_{t=0}^{\infty} h(x_t^j, u_t^j) \tag{1}$$

$$h(x,u) = 0 \;\forall\; x > P \tag{2}$$

Additionally, define cost of two consecutive iterations:

$$J^{jk} = J^j + J^k = \sum_{t=0}^{\infty} h(x_t^j, u_t^j) + \sum_{t=0}^{\infty} h(x_t^k, u_t^k) \tag{3}$$

with $k = j + 1$.
Also define the Q function as usual and additionally the Q function of two consecutive iterations:

$$Q^{jk}(x) = \begin{cases} Q^j(x) + J^k, & \text{if } 0 \le x < P. \\ Q^k(x-P), & \text{if } P \le x < 2P. \end{cases} \tag{4}$$

with P = periodicity and two consecutive iterations j and k. See fig. 1 for the illustration of these stacked functions.
This stacked Q function just adds the Q functions of two consecutive iterations (with $Q = 0$ at $x = 2P$).
Then we can write the optimal LMPC cost for two consecutive laps 2 and 3 as follows (use 3rd plot in figure, iterations 2 and 3):

$$J^{*,23}_{0\to N}(x_0^2) = \min_u \left[\sum_{t=0}^{N-1} h(x,u) + Q^{01}(x_N) \right] \tag{5}$$

$$J^{*,23}_{0\to N}(x_0^2) = \min_u \left[\sum_{t=0}^{N-1} h(x,u) + Q^{0}(x_N) \right] + J^1 \tag{6}$$

with $x_N \in [0, P]$.
We assume that following equation is still valid:

$$J^{*,23}_{0\to N}(x_0^2) \ge J^{23}. \tag{7}$$

We can also express the iteration cost of laps 1 and 2 as

$$J^{12} = J^1 + J^2 \tag{8}$$

$$J^{12} = J^1 + \sum_{t=0}^{N-1} h(x-P,u) + Q^0(x_N - P) \tag{9}$$

with $x_N > P$ (see 2nd plot in figure, iterations 1 and 2). *Note:* This comes from the fact that, in iteration 12, we use $Q^{-1,0}$ which is not illustrated in the

figure.
Comparing eq. 6, 7, and 9 leads us to

$$J^{12} \geq J^{*,23}_{0\to N}(x^2_0) \geq J^{23} \quad \square \tag{10}$$

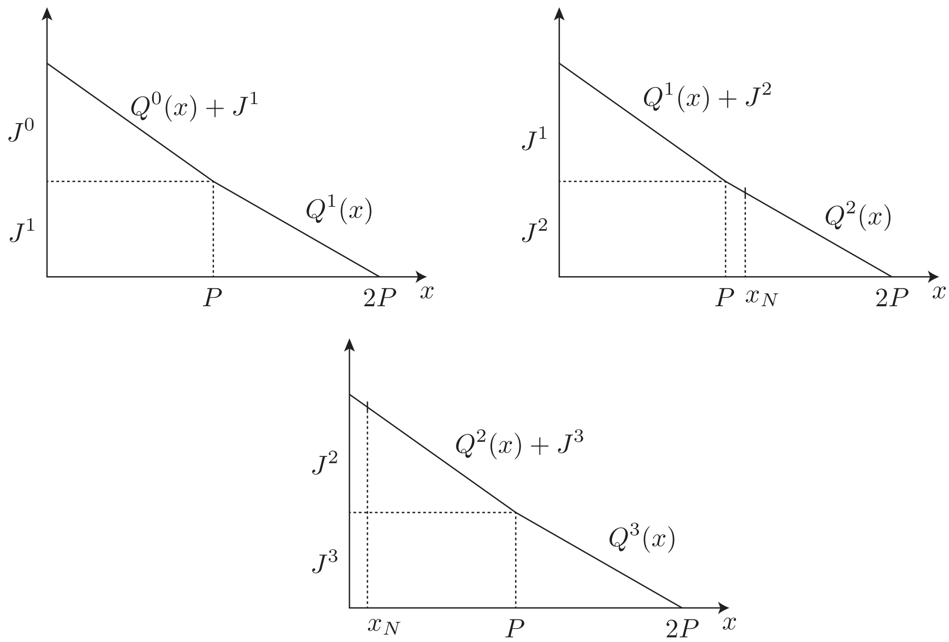

Figure 1: Illustration