1 Definitions

1.1 Problem definition

Solve following problem:

$$J^*_{0\to\infty}(x_S) = \min_{\mathbf{u}} \sum_{k=0}^{\infty} h(x_k, u_k) \tag{1}$$

$$\text{s.t. } x_{k+1} = f(x_k, u_k) \tag{2}$$

$$x_0 = x_S \tag{3}$$

$$x_k \in \mathcal{X}, u_k \in \mathcal{U}, \forall k \geq 0 \tag{4}$$

with

$$h(x_F, 0) = 0 \tag{5}$$

$$h(x_t^j, u_t^j) > 0 \ \forall \ x_t^j \in \mathbb{R}^n \setminus \{x_F\}, u_t^j \in \mathbb{R}^m \setminus \{0\} \tag{6}$$

and x_F a feasible equilibrium:

$$f(x_F, 0) = 0.$$

1.2 General definitions

State evolution: $x_{t+1} = f(x_t, u_t)$
Cost function: $h(x_k^j, u_k^j)$ with k = timestep, j = iteration
Iteration cost: $J^j_{0\to\infty}(x_0^j) = \sum_{k=0}^{\infty} h(x_k^j, u_k^j)$ with x and u are *realized states*.
Q-function: $Q^j(x) = \min J^j_{t\to\infty}(x)$ if $x \in \mathcal{SS}^j$
Optimal cost:

$$J^{*,j}_{0\to N}(x_t^j) = \min_{u} \left[\sum_{k=0}^{N-1} h(x_k, u_k) + Q^{j-1}(x_N) \right]$$

Sampled safe set:

$$\mathcal{SS}^j = \left\{ \bigcup_{i\in M^j} \bigcup_{t=0}^{\infty} x_t^i \right\}$$

and

$$M^j = \left\{ k \in [0, j] : \lim_{t\to\infty} x_t^k = x_F \right\}$$

By definition, there exists a control sequence for every point in the sampled safe set that drives it to the terminal point x_F.

2 Closed lap LMPC

Idea: $x_F^j = x_0^{j+1}$

2.1 Feasibility - proof by induction

Let at time t of the j-th iteration the optimal and feasible solution be:

$$\mathbf{x}_t^* = [x_t^*, x_{t+1}^*, \dots, x_{t+N}^*] \tag{7}$$
$$\mathbf{u}_t^* = [u_t^*, u_{t+1}^*, \dots, u_{t+N}^*] \tag{8}$$

It follows that the optimal state and input are

$$x_t^* = x_t^j \tag{9}$$
$$u_t^* = u_t^j \tag{10}$$

Simply: Whenever the LMPC-prediction is feasible, the receding horizon control is feasible as well (independent of open or closed lap).

2.2 Asymptotic stability

Idea: Show that optimal cost $J_{0\to N}^*$ is a Lyapunov function (decreasing along the closed loop trajectory) for the equilibrium point.

3 Closed loop LMPC

3.1 Definition of periodicity

Introduce periodicity property for the system

$$\dot{x} = f(x, u) \tag{11}$$

so that

$$f(x + P, u) = f(x, u) \tag{12}$$

From eq. 12 one can derive the property for the discrete case:

$$x_{k+1} = x_k + T \cdot f(x_k, u_k) \tag{13}$$
$$x_{k+1} = g(x_k, u_k) \tag{14}$$

so that

$$g(x_k + P, u_k) = x_k + P + T \cdot f(x_k + P, u_k) \tag{15}$$
$$g(x_k + P, u_k) = x_k + P + T \cdot f(x_k, u_k) \tag{16}$$
$$g(x_k + P, u_k) = g(x_k, u_k) + P \tag{17}$$

with P being the period. This also implies that

$$x_0^j = f(x_{end}^{j-1}, u_{end}) - P \tag{18}$$

Also for the system

$$\dot{x} = f(x, u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \tag{19}$$

following equation holds:

$$f(x + P, u) = f(x, u),\ P \in \mathbb{R}^n \tag{20}$$

3.2 Shifted safe set

The shifted safe set collects all states of the current iteration and shifts them by the period.

$$\mathcal{SS}_c^j = \mathcal{X} \cap \left\{ \cup_t \left\{ x_t^j + P \right\} \right\} \tag{21}$$

with $\mathcal{X}$ as the invariant set for which $h(x, u) = 0 \forall x \in \mathcal{X}$. Create extended safe set:

$$\mathcal{SS}^j = \left\{ \bigcup_{i \in M^j} \bigcup_{t=0}^{\infty} \left(x_t^i \cup \left(x_t^i + P \right) \right) \right\}$$