Improving Lagrange Interpolation

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Motivation and Applications

Population modeling

Motivation and Applications

Spectral collocation for Dirichlet and Neumann condition

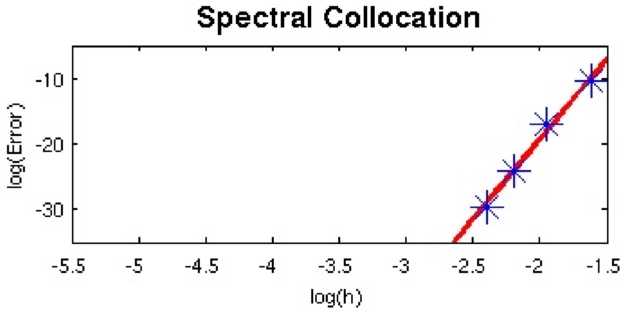

Figure: $u''(x) = e^x$, $u'(1) = -5$ and $u(0) = 3$

Lagrange interpolation

The interpolation polynomial is

$$p(x) = \sum_{j=0}^{n} f_j \ell_j(x), \quad \ell_j(x) = \frac{\prod_{k=0,k\neq j}(x - x_k)}{\prod_{k=0,k\neq j}(x_j - x_k)}.$$

Lagrange interpolation

The Lagrange polynomial

$$\ell_j(x) = \frac{\prod_{k=0,k\neq j}(x - x_k)}{\prod_{k=0,k\neq j}(x_j - x_k)}$$

has the property

$$\ell_j(x_k) = \begin{cases} 1, & j = k \\ 0 & \text{otherwise} \end{cases} \qquad j, k = 0, \ldots, n.$$

Example

Find a polynomial that assumes the values given in the table below.

t (min)	15	40
s (mile)	1/2	3/2

$$p(x) = \frac{1}{2}\underbrace{\left(\frac{x-40}{15-40}\right)}_{\ell_0(x)} + \frac{3}{2}\underbrace{\left(\frac{x-15}{40-15}\right)}_{\ell_1(x)}.$$

Example (continue)

What is $p(30)$?

$$\begin{aligned} p(30) &= \frac{1}{2}\left(\frac{30-40}{15-40}\right) + \frac{3}{2}\left(\frac{x-15}{40-15}\right) \\ &= \frac{11}{10} \quad \text{(mile)} \end{aligned}$$

Example (continue)

Recall

$$p(x) = \frac{1}{2}\underbrace{\left(\frac{x-40}{15-40}\right)}_{\ell_0(x)} + \frac{3}{2}\underbrace{\left(\frac{x-15}{40-15}\right)}_{\ell_1(x)}.$$

$$\ell_0(x_0) = \ell_0(15) = \frac{15-40}{15-40} = 1$$

and

$$\ell_0(x_1) = \ell_0(40) = \frac{40-40}{15-40} = 0$$

Disadvantages of Lagrange's form

1. Each evaluation of $p(x)$ requires $O(n^2)$ additions and multiplications.
2. Adding a new data pair (x_{n+1}, f_{n+1}) requires a new computation from scratch.
3. The computation is numerically unstable.

Runge's phenomenon

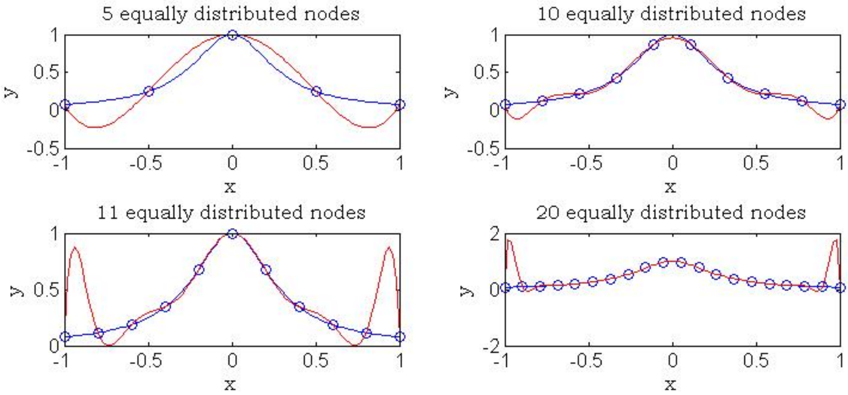

Figure: Runge's phenomenon of the function $f(x) = \frac{1}{1+12x^2}$ on $[-1, 1]$.

Implications of Runge's phenomenon

1. Small changes in the data may cause huge changes in the interpolant.
2. Higher order interpolation is unstable.
3. Uniform convergence is not guaranteed as $n \to \infty$.

Two ways of rewriting the Lagrange representation

1. Modified Lagrange formula (first barycentric interpolation formula)
2. Barycentric formula (second (true) form of the barycentric formula)

Two ways of rewriting the Lagrange representation

$$p(x) = \ell(x) \sum_{j=0}^{n} \frac{w_j}{x - x_j} f_j$$

$$p(x) = \frac{\sum\limits_{j=0}^{n} \frac{w_j}{x-x_j} f_j}{\sum\limits_{j=0}^{n} \frac{w_j}{x-x_j}},$$

where

$$\ell(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$$

and

$$w_j = \frac{1}{\prod_{k \neq j}(x_j - x_k)}, \quad j = 0, \ldots, n.$$

How are the new Lagrange forms differ from the old one?

Methods	Computational Cost for a new node
Modified Lagrange	$O(n)$
Barycentric	$O(n)$

Do better with Chebyshev points

For a well-conditioned polynomial interpolation, we need clustered end-point sets such that asymptotic density is proportional to

$$\frac{1}{\sqrt{1-x^2}}$$

as $n \to \infty$.

Do better with Chebyshev points

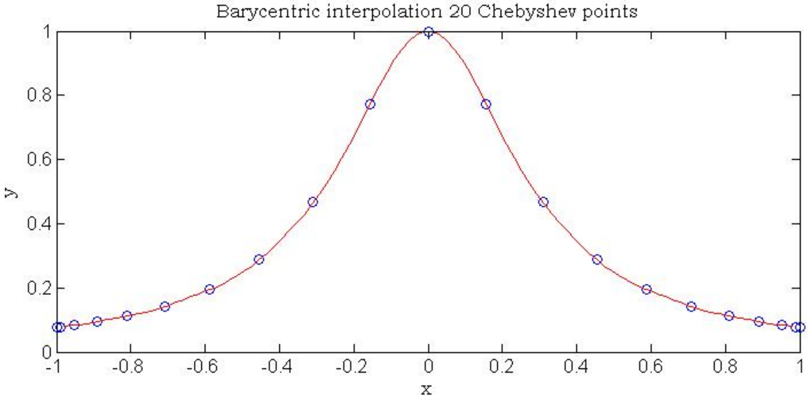

Figure: Barycentric interpolation with 20 Chebyshev points. Notice it is very different from its 20 equidistant 20 interpolants

Convergence Rates

If the function f is analytic in a neighborhood of $[-1, 1]$ with Chebyshev interpolants $\Rightarrow$ fast convergence.

Doubling n gives *quadratic convergence*.

Numerical Stability

Methods	Spacing	Node Arrangement	Stability
Modified Lagrange	equal	increasing	stable
Barycentric	equal	increasing	unstable
Modified Lagrange	equal	decreasing	stable
Barycentric	equal	decreasing	unstable

Numerical Stability

Methods	Spacing	Node Arrangement	Stability
Modified Lagrange	unequal	increasing	stable
Barycentric	unequal	increasing	stable
Modified Lagrange	unequal	decreasing	stable
Barycentric	unequal	decreasing	stable

Numerical Stability with Newton's Method added

Methods	Spacing	Node Arrangement	Stability
Modified Lagrange	equal	increasing	stable
Barycentric	equal	increasing	unstable
Newton	equal	increasing	stable
Modified Lagrange	equal	decreasing	stable
Barycentric	equal	decreasing	unstable
Newton	equal	decreasing	unstable[1]

[1] Unstable around an endpoint

Numerical Stability with Newton's Method added

Methods	Spacing	Node Arrangement	Stability
Modified Lagrange	unequal	increasing	stable
Barycentric	unequal	increasing	stable
Newton	unequal	increasing	unstable[2]
Modified Lagrange	unequal	decreasing	stable
Barycentric	unequal	decreasing	stable
Newton	unequal	decreasing	unstable[3]

[2] Unstable as x approaches one end of the interval

[3] Unstable as x approaches one end of the interval

Comparison between Lagrange, Modified Lagrange, Barycentric, and Newton Interpolation

	Lagrange	Modified Lagrange	Barycentric	Newton
Setup cost	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(n^2)$
Evaluation cost	$O(n^2)$	$O(n)$	$O(n)$	$O(n^2)$

Tie breaker: Number of iterations and accuracy

1. The accuracy of Newton's method suffers when nodes are arranged poorly.
2. But when arranged properly, Barycentric method and Newton's method have about the same accuracy.

Comparison between Lagrange, Barycentric, and Newton Interpolation

	LAGRANGE	BARYCENTRIC	AITKEN/NEVILLE
Setup Costs	$n(n+1)$ subtractions n^2-1 multiplications $n+1$ divisions	$n(n+1)$ subtractions n^2-1 multiplications $n+1$ divisions	$n(n+1)$ subtractions
Evaluation Costs	$n+1$ subtractions n additions $n(n+1)$ multiplications	$n+1$ subtractions $2n$ additions $n+1$ multiplications $n+2$ divisions	$\frac{n+2}{2}(n+1)$ subtractions $n(n+1)$ multiplications $\frac{n}{2}(n+1)$ divisions

References

1. Berrut, J.P. & Trefethen, L.N. (2004) Barycentric Lagrange interpolation. *SIAM Review.*
2. Higham, N.J. (2004) The numerical stability of barycentric Lagrange interpolation. *IMA Journal of Numerical Analysis.*, **24**, 547-556.
3. Winrich, L.B. (1969) Note on a comparison of evaluation schemes for the interpolating polynomial. *Comput. J.* **12**, 154-155.
4. Webb, M., Trefethen, L.N. &Gonnet, P. (2012) Stability of Barycentric interpolation formulas for extrapolation. *SIAM Journal Scientific Computing,* **34**, No. 6, A3009-A3015.