Fundamental Concepts in Computational and Applied Mathematics

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On the Partial Difference Equations of Mathematical Physics [1]

- Discussion of algebraic problems arising in discretization of differential equations
- Behavior of the difference solution as mesh width goes to zero
- BVP and EV problems for elliptic p.d.e.s
- IVP for hyperbolic and parabolic p.d.e.s

[1] R. Courant, K. Friedrichs, H. Lewy, IBM Journal, pp. 215–231, March 1967

Main Results

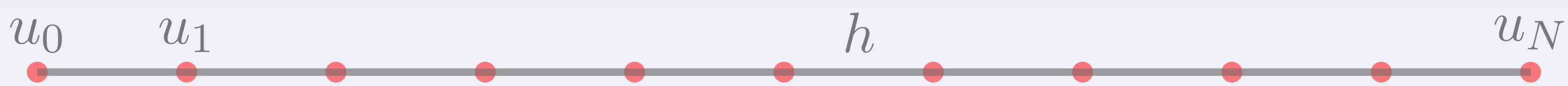
- For elliptic equations
  - difference quotient tends to the corresponding derivative
  - convergence is guaranteed independently of mesh
- For hyperbolic equations
  - convergence is obtained iff certain ratio of mesh width is satisfied
  - something else

Paper Outline

1. Introduction
2. Elliptic equations
   1. Preliminaries
   2. Boundary value and eigenvalue problems
   3. Connections to Random Walk
   4. **Solution of differential equation as a limiting form of solution of the difference equation**
3. Hyperbolic equations
   1. Equation of vibrating string
   2. **Influence of the choice of mesh**
   3. Limiting values for arbitrary rectangular grids
   4. Wave equation in 3 variables

Motivation

- Consider case in 1D on an equally spaced grid $u(x)$
- h is the spatial discretization

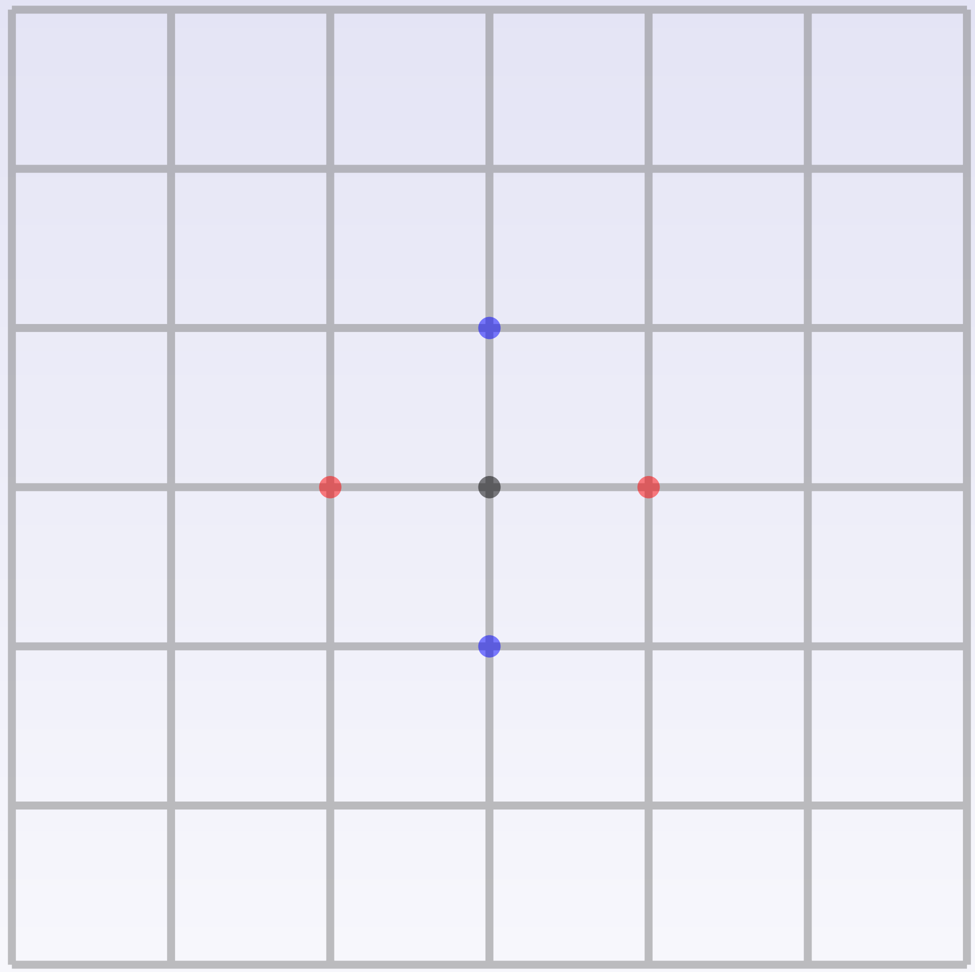


The difference quotient for approximating $\partial u/\partial x$ can be written as:

$$u_x = \frac{u(x_{j+1}) - u(x_{j-1})}{2h}$$

Likewise in 2D

$$\begin{aligned}
u_x &= \frac{u(x+h,y)-u(x,y)}{h}\\
u_y &= \frac{u(x,y+h)-u(x,y)}{h}\\
u_{\bar{x}} &= \frac{u(x,y)-u(x-h,y)}{h}\\
u_{\bar{y}} &= \frac{u(x,y)-u(x,y-h)}{h}
\end{aligned}$$



Corresponding second derivatives

$$u_{xx} = \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}$$

Equations

- Elliptic equations:
$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$$
or
$$\Delta u = u_{xx} + u_{yy}$$
- Hyperbolic equations:
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

both on a simply connected region G with boundary ∂G, and appropriate B.C.'s and I.C.'s

Main Results: Part 1 (Elliptic Case)[2]

Assume

- $f(x, y)$ is a given continuous function,
- $f(x, y)$ has continuous first and second partial derivatives in a region containing G,
- a mesh G_h, with mesh width h,
- $u_h(x, y)$ is the solution of the difference equation $\Delta u = 0$.

Then

- As $h \to 0$, $u_h(x, y)$ converges to $u(x, y)$ satisfying $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on the domain G and equals f on the boundary
- For any interior region within G the difference quotients, u_h tend to the corresponding partial derivatives of $u(x, y)$

[2] Section 4. pp. 221

Main Results: Part 2 (Hyperbolic Case) [3]

Assume

- Hyperbolic equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

- a rectangular mesh, with mesh size h in the time direction, and kh, in the spatial, i.e. x direction
- For $k < 1$, as $h \to 0$, the solution to the difference equation **cannot converge** to the solution of the differential equation (**??**)

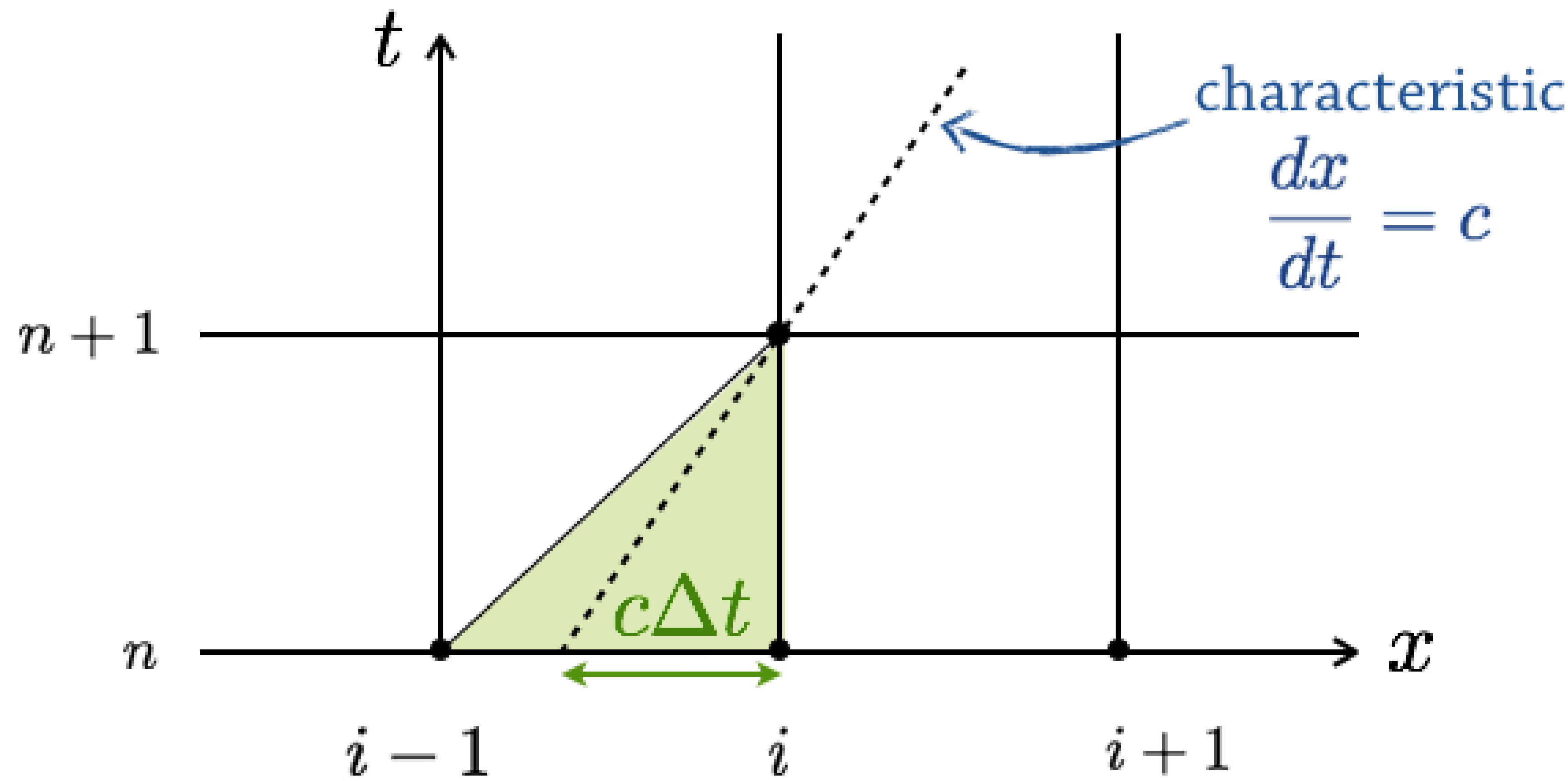
In other words for:

$$k = \frac{\Delta x}{\Delta t} < 1$$

the difference quotient solution will not converge!!!

[3] Section 2, pp. 228

Main Results: CFL Condition Pictorially [4]



Main Results: CFL Condition

For the general wave equation with velocity c:

$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x^2} = 0,$$

today we say that the time step Δt must be chosen so that the CFL condition is met, i.e.:

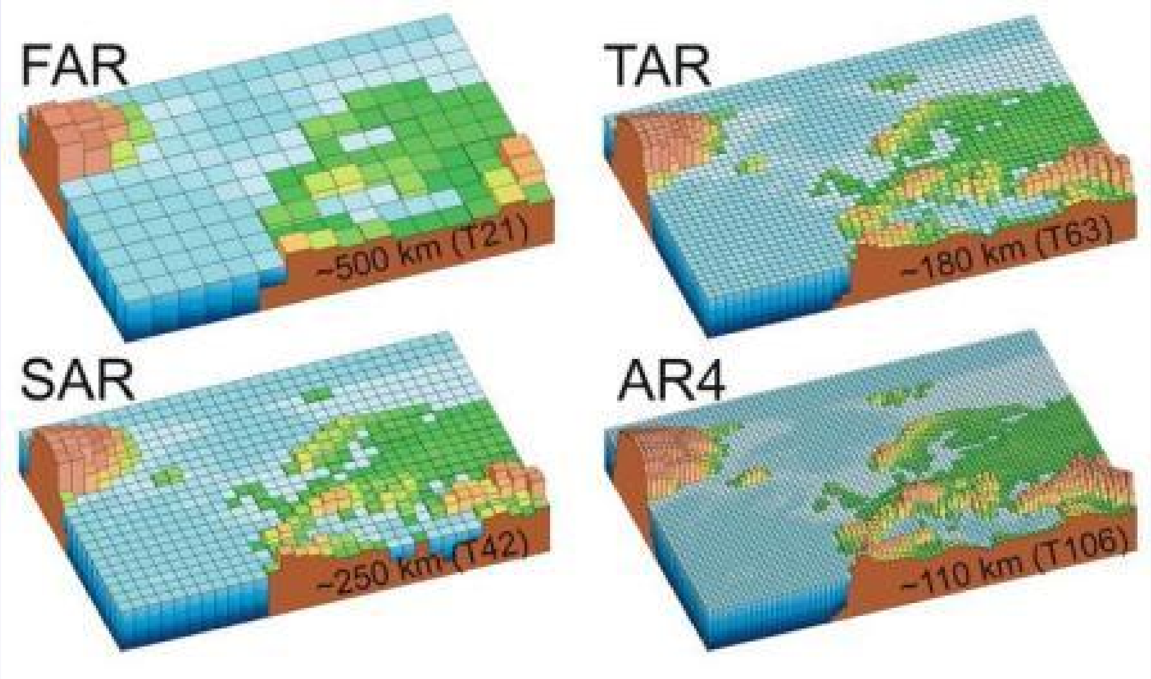
$$\sigma = c\frac{\Delta t}{\Delta x} \leq \sigma_{max}$$

Note:

The value of σ_{max} will vary according to the numerical method used. For an explicit method, σ_{max} is typically 1.

Example: Climate Modeling

- Resolution has increased by factor of almost 5 in the last 20 years
- Assuming an explicit method, what is the time step needed to maintain stability?
- What is the resulting change in computational work?



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