# Fundamental Concepts in Computational and Applied Mathematics

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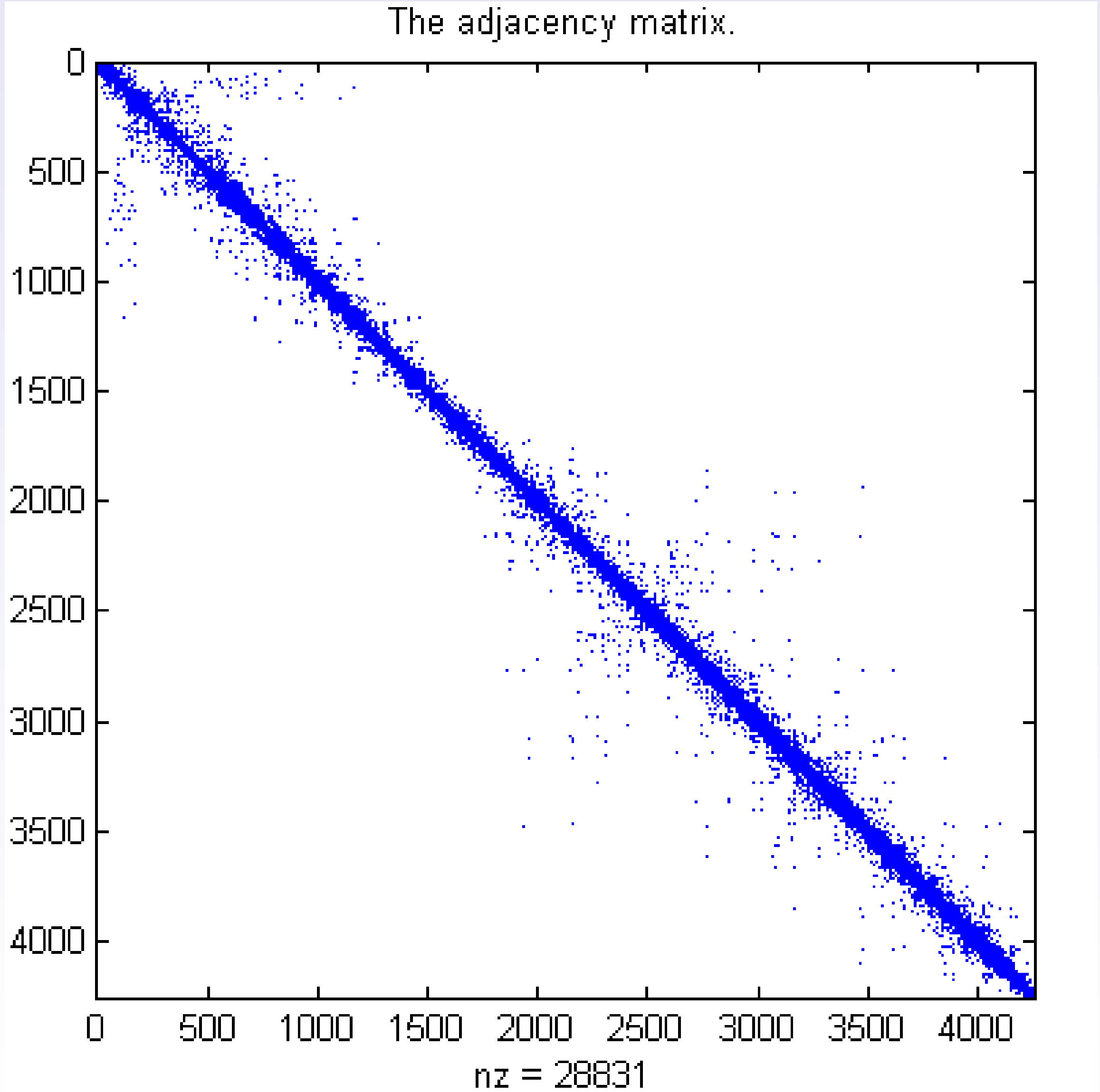
# Sparse Matrix Example

Consider the following matrix:



# Sparse methods

- Sparse matrices arise in many science and engineering applications
- Almost all discretization methods you will encounter will have a special structure
- Sparse methods take advantage of special structure in matrices to improve computational efficiency

# Stationary Iterative Methods

- Based on splitting techniques
- Widely used in engineering, but have been superseded by more modern techniques
- Still useful in certain contexts, e.g. preconditioners

# Basic Method

Consider $Ax = b$

Let $A = L + D + U$ and then "split" the matrix $A$ such that

$$(L+D)x^{k+1} = b - Ux^k, k = 0, \dots$$

or

$$\begin{aligned} Mx^{k+1} &= b + Nx^k, k = 0, \dots && (1) \\ x^{k+1} &= M^{-1}b + M^{-1}Nx^k, && (2) \\ x^{k+1} &= Gx^k + c && (3) \end{aligned}$$

where $G = M^{-1}N, c = M^{-1}b$

# Useful Facts

Fact 1: If we use the SIM in equation (3) $x^k \to \hat{x} \iff \rho(G) < 1$

Fact 2: $e^{k+1} = Ge^k$

Fact 3: Many matrices resulting from problems in S&E satisfy $\rho(G) < 1$

# Finite Precision

- In exact arithmetic the CG method will converge to the solution of $Ax = b$
- Sadly, in finite precision, roundoff error will contaminate the solution
- And even in the exact case, what does this mean for a system with dimension 1M or 1B

# Krylov subspace methods

CG is an example of what we now call a Krylov space method

- Definition: A *Krylov* space is defined by:

$$\mathcal{K}_k(A, b) = span\{b, Ab, A^2b, A^3b, \ldots, A^{k-1}b\}$$

Idea: Choose your iterate, $x^k$, such that it belongs to the Krylov space and it minimizes the distance between it and $\hat{x}$

# CG Convergence Rate

Can show that

$$||x^k - \hat{x}||_A \le 2 \left( \frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^k ||x^0 - \hat{x}||_A \tag{4}$$

Table: Number of iterations to reduce error by $10^{-3}$.

| $\kappa(A)$ | Value | iter |
|---|---|---:|
| 1 | 0.0 | 1 |
| $10^2$ | 0.82 | 15 |
| $10^4$ | 0.98 | 150 |
| $10^8$ | 0.9998 | 15,000 |

# Preconditioning Idea

Discussion

- Why precondition?
- What should the general idea be?
- How would you go about doing that?

# Summary

- Sparse methods can be used to take advantage of special structure
- Iterative methods are the dominant form of solving systems of linear equations in modern day applications
- Most iterative methods must be preconditioned to be effective

# References


- **Methods of Conjugate Gradients for Solving Linear Systems**, Magnus R. Hestenes and Eduard Stiefel, J. Res. of NBS, Vol. 49, No. 6, Dec. 1952.
- **Matrix Computations, 3rd Ed.**, Gene H. Golub and Charles F. Van Loan, Johns Hopkins, 1996.
- **Iterative Methods for Linear and Nonlinear Equations**, C.T. Kelley, SIAM, 1995.