Integration of stiff ordinary differential equations using a forward interpolation scheme

Johannes Brust

UC Merced , California, USA

M 298, Directed group study

Outline

Introduction

What is a stiff ordinary differential equation ?

Numerical method

Questions

Definition

For an ordinary differential equation given in the form :

$$\frac{dy}{dx} = \frac{[y - G(x)]}{a(x,y)} \tag{1}$$

It's discretization is said to be 'stiff' if :

$$\left| \frac{a(x_n, y_n)}{\Delta x} \right| << 1$$

Properties

$$\frac{dy}{dx} = \frac{[y - G(x)]}{a(x, y)}$$

- The slopes of solutions take very large positive and negative values.
- The family of solutions horn out.
- Stiff equations are hard to solve using conventional numerical methods.

Example

A stiff equation with $\Delta x = 1$ and $a(x, y) = \frac{1}{5}$:

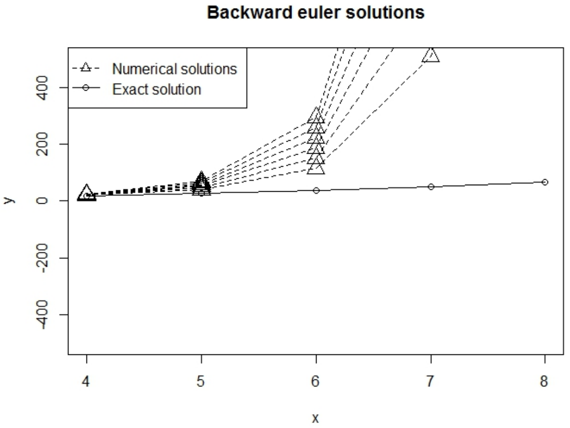
$$\frac{dy}{dx} = 5(y - x^2)$$

It's solution is :

$$Y(x) = 0.08 + 0.4x + x^2$$

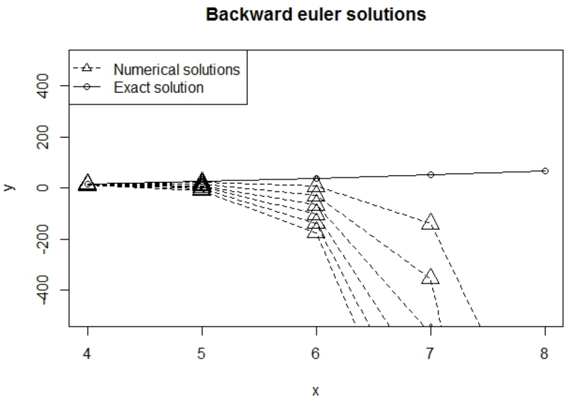
Example, $y_0 = 20, \ldots 25$

$$y_n = y_{n-1} + 5(y_{n-1} - x_{n-1}^2)$$



Example, $y_0 = 13, \dots 18$

$$y_n = y_{n-1} + 5(y_{n-1} - x_{n-1}^2)$$



What is the problem ?

The discrete form of the equation is

$$\Delta y = \frac{y_{n-1} - G(x_{n-1})}{a(x_{n-1}, y_{n-1})} \Delta x$$

- In a stiff equation $\frac{\Delta x}{a(x_{n-1}, y_{n-1})}$ is not a good approximation to $\frac{dx}{a(x,y)}$.
- The forward step does not resolve the equation well.

Occurances

Stiff equations are a particular class of ordinary differential equations and arise naturally in fields with equations that contain different time scales :

- Chemical kinetics.
- Missile guidance.
- Electrical circuit development.

Example, phase-plane

- There seems to be a particular solution $y = Y(x)$ such that the slope has a ' reasonable value '.
- Thus, the goal of the numerical solution becomes to approximate $Y(x)$.

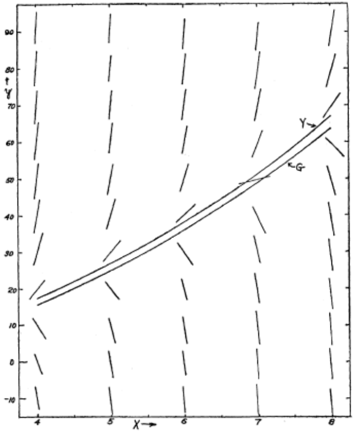


FIGURE 1

Slopes, dy/dx, for a Typical Stiff Equation, $dy/dx = 5(y - x^2)$.

Forward interpolation

- The idea is to approximate y_n from the knowledge of $y_{n-1}, y_{n-2}, \ldots$
- The method searches for the tangent at the next solution point y_n that also goes through y_{n-1} .

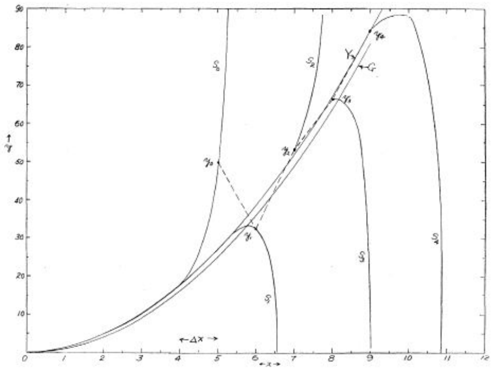


FIGURE 3

Integration of $dy/dx = 5(y - x^2)$ in the positive x direction from $(x = 5, y = 50)$. Here $a/\Delta x = 1/5$, $G(x) = x^2$, $Y(x) = 0.08 + 0.4x + x^2$.

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Numerical method

Applying the linear forward interpolation leads to :

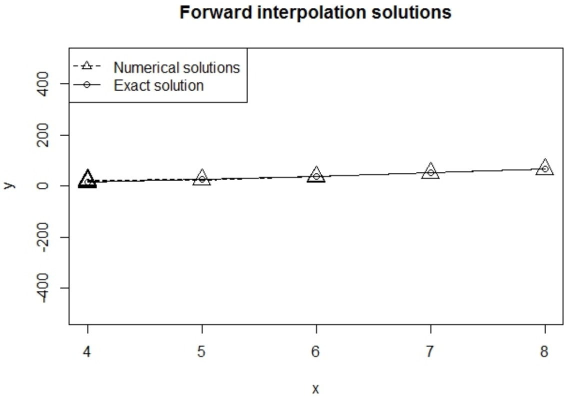
$$\left(\frac{dy}{dx}\right)_n = \frac{y_n - y_{n-1}}{\Delta x} = \frac{y_n - G_n}{a(y_n, x_n)} \qquad (2)$$

This relation can be solved for y_n to give an update formula of the solution.

Example, Numerical method

Applying the linear forward interpolation to the example :

$$\left(\frac{dy}{dx}\right)_n = y_n = \frac{y_{n-1} - 5x_n^2}{1-5} \tag{3}$$



Numerical method, why does it work ?

In order to verify convergence of the numerical solution y_n toward the equilibrium solution Y the asymptotic behavior can be described.
Writing the update formula as :

$$y_n - Y_n = -\left(\underbrace{\frac{\frac{a(x_n)}{\Delta x}}{1 - \frac{a(x_n)}{\Delta x}}}_{=b}\right)[y_{n-1} - Y_{n-1} + \epsilon]$$

By an recursive substitution one obtains the solution :

$$y_n - Y_n = \underbrace{b^n(y_0 - Y_0)}_{=0} - b \underbrace{\sum_{i=0}^{n}(b)^i}_{=\frac{a(x)}{\Delta x} - 1 = \frac{\frac{a(x)}{\Delta x}}{-b}} * \epsilon$$

Thus solving the update formula in the linear case gives an asymptotic formula that is tractable :

$$y_n - Y_n = -b \underbrace{\sum_{i=0}^{n} (b)^i}_{=\frac{\frac{a(x)}{\Delta x}}{-b}} * \epsilon$$

$$y = Y + \frac{a(x)}{\Delta x} * \epsilon$$

- This shows that the convergence of y to Y depends on the behavior of ϵ
- ϵ can be controlled depending on the number of points used in the interpolation.

Improvements

The method can be extended depending on the number of previous values used in the interpolation. The effects of including higher order polynomials in the forward interpolation are twofold :

1. The update formula will change.
2. The error (ϵ) in the asymptotic form takes a different form, obtaining a higher order.

References

Reference : Curtiss, Hirschfelder (1952), "*Integration of Stiff equations*"

Questions ?