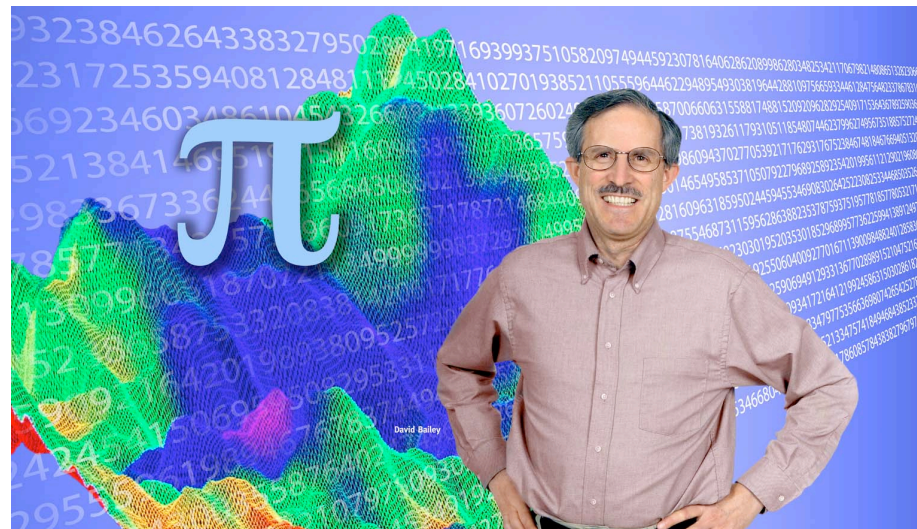


# High-Precision Arithmetic and Mathematical Physics

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# Applications of High-Precision Arithmetic in Modern Scientific Computing

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- ◆ Highly nonlinear computations.
- ◆ Computations involving highly ill-conditioned linear systems.
- ◆ Computations involving data with very large dynamic range.
- ◆ Large computations on highly parallel computer systems.
- ◆ Computations where numerical sensitivity is not currently a major problem, but periodic testing is needed to ensure that results are reliable.
- ◆ Research problems in mathematics and mathematical physics that involve constant recognition and integer relation detection.

Few physicists, chemists or engineers are highly expert in numerical analysis. Thus high-precision arithmetic is often a better remedy for severe numerical round-off error, even if the error could, in principle, be improved with more advanced algorithms or coding techniques.

## Available High-Precision Facilities



Vendor-supported arithmetic:

	Total	Significant	
Type	Bits	Digits	Support
IEEE Double	64	16	In hardware on almost all systems.
IEEE Extended	80	18	In hardware on Intel and AMD systems.
IEEE Quad	128	33	In software from some vendors (50-100X slower than IEEE double).

Non-commercial (free) software:

	Total	Significant	
Type	Bits	Digits	Support
Double-double	128	32	DDFUN90, QD.
Quad-double	256	64	QD.
Arbitrary	Any	Any	ARPREC, MPFUN90, GMP, MPFR.

Commercial software: *Mathematica*, *Maple*.

# LBNL's High-Precision Software

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- ◆ QD: double-double (31 digits) and quad-double (62 digits).
- ◆ ARPREC: arbitrary precision.
- ◆ Low-level routines written in C++.
- ◆ C++ and Fortran-90 translation modules permit use with existing C++ and Fortran-90 programs -- only minor code changes are required.
- ◆ Includes many common functions: sqrt, cos, exp, gamma, etc.
- ◆ PSLQ, root finding, numerical integration.

Available at: **<http://www.experimentalmath.info>**

Authors: Xiaoye Li, Yozo Hida, Brandon Thompson and DHB

# Some Real-World Applications of High-Precision Arithmetic

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- ◆ Supernova simulations (32 or 64 digits).
- ◆ Climate modeling (32 digits).
- ◆ Planetary orbit calculations (32 digits).
- ◆ Coulomb n-body atomic system simulations (32-120 digits).
- ◆ Schrodinger solutions for lithium and helium atoms (32 digits).
- ◆ Electromagnetic scattering theory (32-100 digits).
- ◆ Studies of the fine structure constant of physics (32 digits).
- ◆ Scattering amplitudes of quarks, gluons and bosons (32 digits).
- ◆ Theory of nonlinear oscillators (64 digits).

# Planetary Orbit Calculations



- ◆ A key question of planetary theory is whether the solar system is stable over cosmological time frames (billions of years).
- ◆ Scientists have studied this question by performing very long-term simulations of planetary motions.
- ◆ This problem is well known to exhibit chaos.
- ◆ Simulations typically do well for long periods of time, but then fail at certain key junctures, unless special measures are taken.
- ◆ Researchers have found that double-double or quad-double arithmetic is required to avoid severe numerical inaccuracies, even if other techniques are employed.

“The orbit of any one planet depends on the combined motions of all the planets, not to mention the actions of all these on each other. To consider simultaneously all these causes of motion and to define these motions by exact laws allowing of convenient calculation exceeds, unless I am mistaken, the forces of the entire human intellect.” [Isaac Newton, 1687]

G. Lake, T. Quinn and D. C. Richardson, “From Sir Isaac to the Sloan survey: Calculating the structure and chaos due to gravity in the universe,” *Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, SIAM, Philadelphia, 1997, pg. 1-10.

# High-Precision Arithmetic and Experimental Mathematics



“Experimental” methodology:

- ◆ Compute various mathematical entities (limits, infinite series sums, definite integrals) to high precision.
- ◆ Use algorithms such as PSLQ to recognize these entities in terms of well-known mathematical constants.
- ◆ Use this same process to discover relations between entities.
- ◆ When results are found experimentally, seek to find formal mathematical proofs of the discovered relations.
- ◆ Many results have been found using this methodology, both in pure math and in mathematical physics.

1. J. M. Borwein and DHB, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, A.K. Peters, 2004. Second edition 2008.
2. J. M. Borwein, DHB and R. Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery*, A.K. Peters, 2004.
3. DHB, J. M. Borwein, N. J. Calkin, R. Girgensohn, D. R. Luke, V. Moll, *Experimental Mathematics in Action*, A.K. Peters, 2007.
4. J. M. Borwein and K. Devlin, *The Computer as Crucible: An Introduction to Experimental Mathematics*, A.K. Peters, 2007.

# The PSLQ Integer Relation Algorithm



Let  $(x_n)$  be a given vector of real numbers. An integer relation algorithm finds integers  $(a_n)$  such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

(or within “epsilon” of zero, where  $\text{epsilon} = 10^{-p}$  and  $p$  is the precision).

At the present time the “PSLQ” algorithm of mathematician-sculptor Helaman Ferguson is the most widely used integer relation algorithm. It was named one of ten “algorithms of the century” by *Computing in Science and Engineering*.

1. H. R. P. Ferguson, DHB and S. Arno, “Analysis of PSLQ, an integer relation finding algorithm,” *Mathematics of Computation*, vol. 68, no. 225 (Jan 1999), pg. 351-369.
2. DHB and D. J. Broadhurst, “Parallel integer relation detection: Techniques and applications,” *Mathematics of Computation*, vol. 70, no. 236 (Oct 2000), pg. 1719-1736.

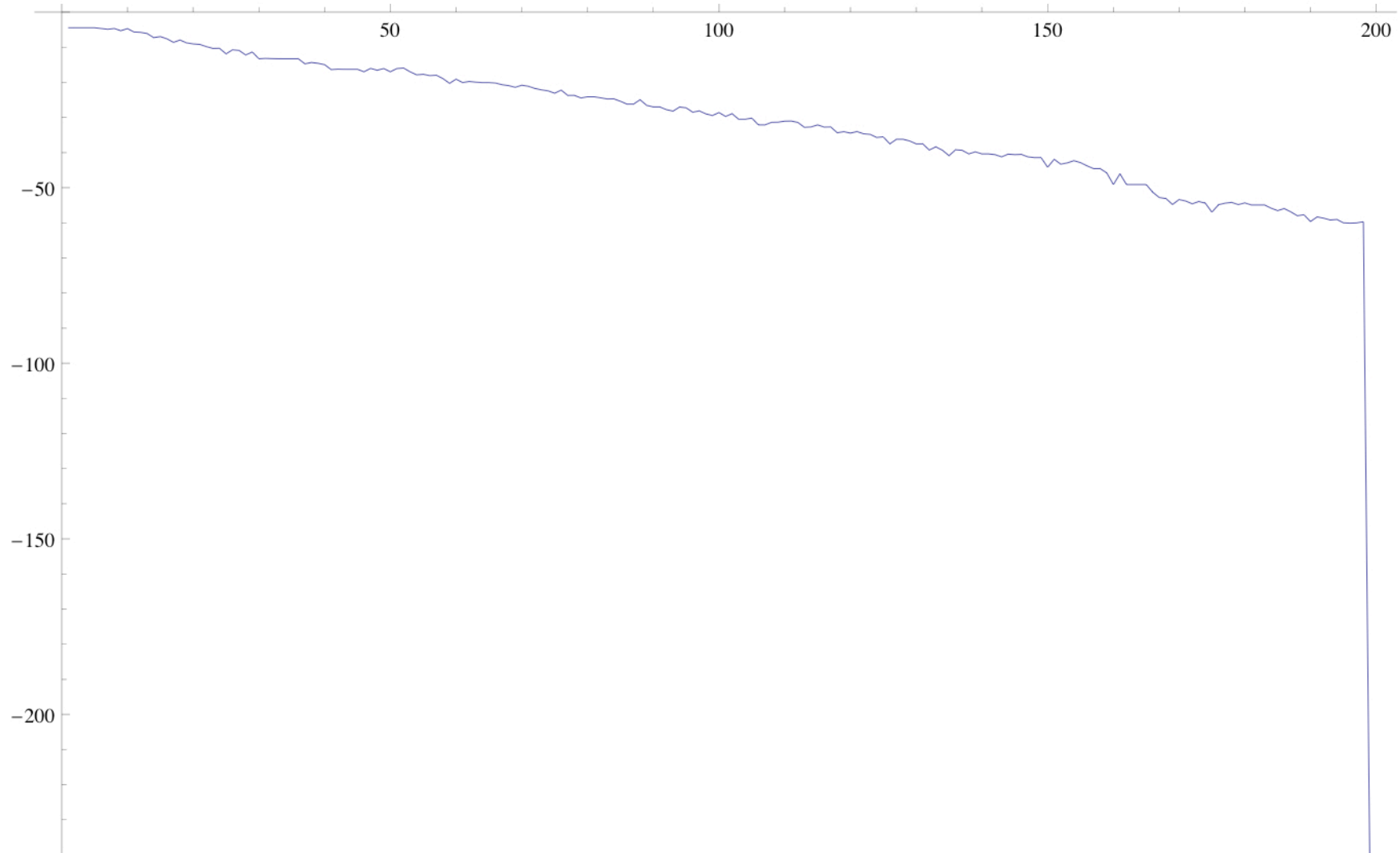


## PSLQ, Continued

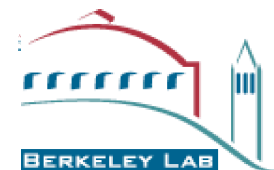


- ◆ PSLQ constructs a sequence of integer-valued matrices  $B_n$  that reduces the vector  $y = x B_n$ , until either the relation is found (as one of the columns of  $B_n$ ), or else precision is exhausted.
- ◆ At the same time, PSLQ generates a steadily growing bound on the size of any possible relation.
- ◆ When a relation is found, the size of smallest entry of the vector  $y$  abruptly drops to roughly “epsilon” (i.e.  $10^{-p}$ , where  $p$  is the number of digits of precision).
- ◆ The size of this drop can be viewed as a “confidence level” that the relation is real and not merely a numerical artifact -- a drop of 20+ orders of magnitude almost always indicates a real relation.
- ◆ PSLQ (or any other integer relation scheme) requires *very high precision arithmetic* (at least  $nd$  digits, where  $d$  is the size in digits of the largest  $a_k$ ), both in the input data and in the operation of the algorithm.

# Decrease of $\log_{10}(\min_k |y_k|)$ as a Function of Iteration Number in a Typical PSLQ Run

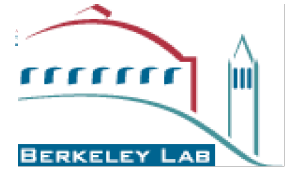


# Methodology for Using PSLQ to Recognize An Unknown Constant $\alpha$



- ◆ Calculate  $\alpha$  to high precision – typically 100 - 1000 digits. This is often the most computationally expensive part of the entire process.
- ◆ Based on experience with similar constants or relations, make a list of possible terms on the right-hand side (RHS) of a linear formula for  $\alpha$ , then calculate each of the  $n$  RHS terms to the same precision as  $\alpha$ .
- ◆ If you suspect  $\alpha$  is algebraic of degree  $n$  (the root of a degree- $n$  polynomial with integer coefficients), compute the vector  $(1, \alpha, \alpha^2, \alpha^3, \dots, \alpha^n)$ .
- ◆ Apply PSLQ to the  $(n+1)$ -long vector, using the same numeric precision as  $\alpha$ , but with a detection threshold a few orders of magnitude larger than “epsilon”– e.g.,  $10^{-480}$  instead of  $10^{-500}$  for 500-digit arithmetic.
- ◆ When PSLQ runs, look for a detection following a drop in the size of the reduced  $y$  vector by at least 20 orders of magnitude, to value near epsilon.
- ◆ If no credible relation is found, try expanding the list of RHS terms.
- ◆ Another possibility is to search for multiplicative relations (i.e., monomial expressions), which can be done by taking logarithms of  $\alpha$  and constants.

# Bifurcation Points in Chaos Theory



Let  $t = B_3$  = the smallest  $r$  such that the “logistic iteration”

$$x_{n+1} = rx_n(1 - x_n)$$

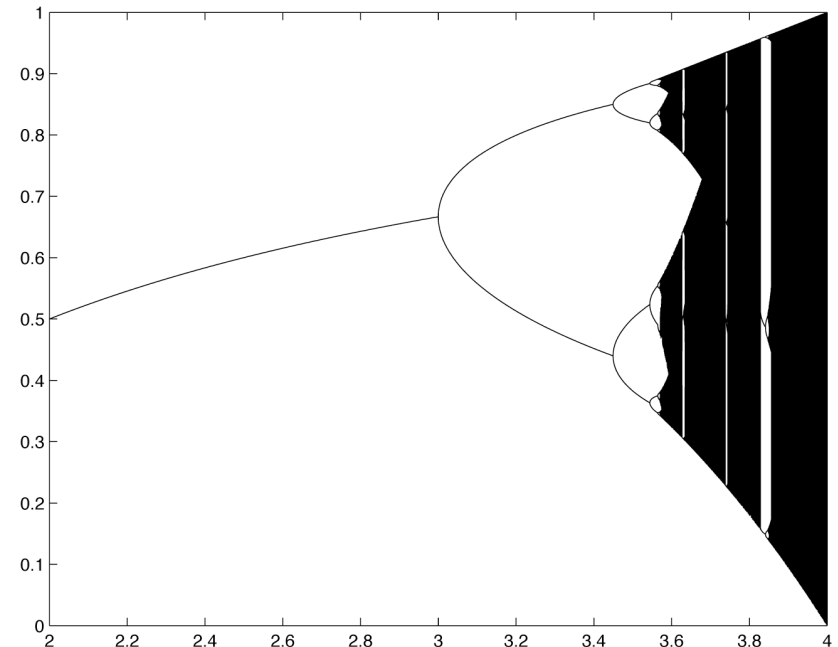
exhibits 8-way periodicity instead of 4-way periodicity.

By means of a sequential approximation scheme, one can obtain the numerical value of  $t$  to any desired precision:

3.544090359551922853615965986604804540583099845444573675457812  
5303058429428588630122562585664248917999626...

Applying PSLQ to  $(1, t, t^2, t^3, \dots, t^{12})$ , we obtained the result that  $t$  is a root of:

$$\begin{aligned} 0 = & 4913 + 2108t^2 - 604t^3 - 977t^4 + 8t^5 + 44t^6 + 392t^7 \\ & - 193t^8 - 40t^9 + 48t^{10} - 12t^{11} + t^{12} \end{aligned}$$



# The BBP Formula for Pi



In 1996, at the suggestion of Peter Borwein, Simon Plouffe used DHB's PSLQ program and arbitrary precision software to discover this new formula for  $\pi$ :

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left( \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

This formula permits one to compute binary (or hexadecimal) digits of  $\pi$  beginning at an arbitrary starting position, using a very simple scheme that can run on any system with standard 64-bit or 128-bit arithmetic.

Recently it was proven that no base- $n$  formulas of this type exist for  $\pi$ , except  $n = 2^m$ .

1. DHB, P. B. Borwein and S. Plouffe, "On the rapid computation of various polylogarithmic constants," *Mathematics of Computation*, vol. 66, no. 218 (Apr 1997), pg. 903-913.
2. J. M. Borwein, W. F. Galway and D. Borwein, "Finding and excluding b-ary Machin-type BBP formulae," *Canadian Journal of Mathematics*, vol. 56 (2004), pg 1339-1342.

# Some Other New BBP-Type Formulas Discovered Using PSLQ



$$\pi^2 = \frac{1}{8} \sum_{k=0}^{\infty} \frac{1}{64^k} \left( \frac{144}{(6k+1)^2} - \frac{216}{(6k+2)^2} - \frac{72}{(6k+3)^2} - \frac{54}{(6k+4)^2} + \frac{9}{(6k+5)^2} \right)$$

$$\pi^2 = \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{729^k} \left( \frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2} \right. \\ \left. - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2} \right)$$

$$\zeta(3) = \frac{1}{1792} \sum_{k=0}^{\infty} \frac{1}{2^{12k}} \left( \frac{6144}{(24k+1)^3} - \frac{43008}{(24k+2)^3} + \frac{24576}{(24k+3)^3} + \frac{30720}{(24k+4)^3} \right. \\ - \frac{1536}{(24k+5)^3} + \frac{3072}{(24k+6)^3} + \frac{768}{(24k+7)^3} - \frac{3072}{(24k+9)^3} - \frac{2688}{(24k+10)^3} \\ - \frac{192}{(24k+11)^3} - \frac{1536}{(24k+12)^3} - \frac{96}{(24k+13)^3} - \frac{672}{(24k+14)^3} - \frac{384}{(24k+15)^3} \\ + \frac{24}{(24k+17)^3} + \frac{48}{(24k+18)^3} - \frac{12}{(24k+19)^3} + \frac{120}{(24k+20)^3} + \frac{48}{(24k+21)^3} \\ \left. - \frac{42}{(24k+22)^3} + \frac{3}{(24k+23)^3} \right)$$

DHB, "A compendium of BBP-type formulas," at <http://crd.lbl.gov/~dhbailey/dhbpapers/bbp-formulas.pdf>.

# History of Numerical Integration (Quadrature)

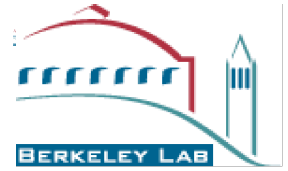
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- ◆ 1670: Newton devises the Newton-Coates integration rule.
- ◆ 1740: Thomas Simpson develops Simpson's rule.
- ◆ 1820: Gauss develops Gaussian quadrature.
- ◆ 1950-1970: Adaptive quadrature, Romberg integration, Clenshaw-Curtis integration, others.
- ◆ 1973: Takashi and Mori develop the tanh-sinh quadrature scheme.
- ◆ 1985-1990: Maple and Mathematica feature built-in numerical quadrature facilities.
- ◆ 2000: Very high-precision quadrature (1000+ digits) methods.

With high-precision numerical values, we can now use PSLQ to obtain analytical evaluations of integrals.

# The Euler-Maclaurin Formula of Numerical Analysis



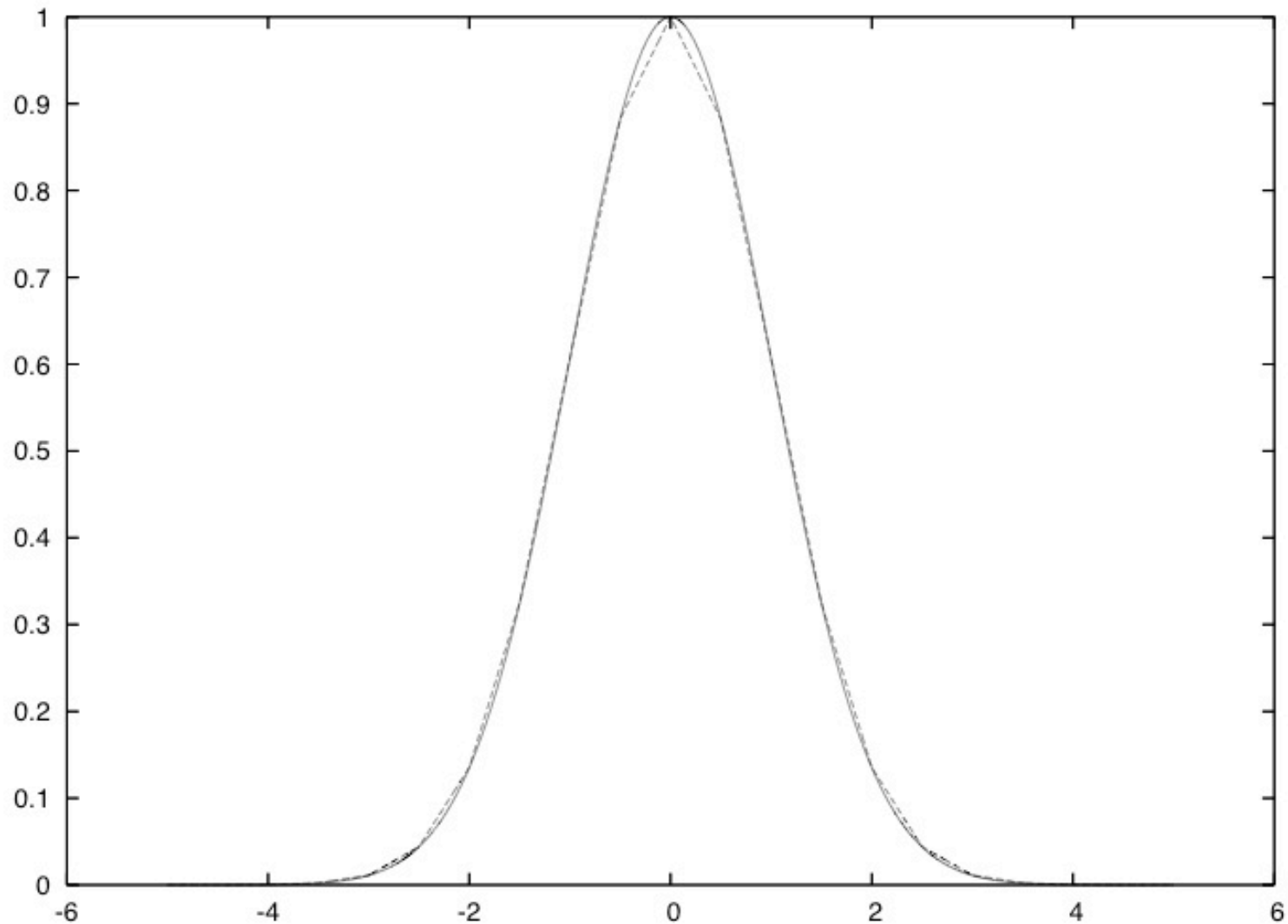
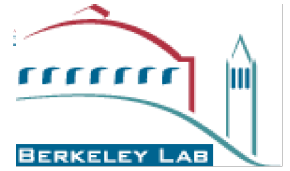
$$\begin{aligned}\int_a^b f(x) \, dx &= h \sum_{j=0}^n f(x_j) - \frac{h}{2}(f(a) + f(b)) \\ &\quad - \sum_{i=1}^m \frac{h^{2i} B_{2i}}{(2i)!} (D^{2i-1} f(b) - D^{2i-1} f(a)) - E(h) \\ |E(h)| &\leq 2(b-a)(h/(2\pi))^{2m+2} \max_{a \leq x \leq b} |D^{2m+2} f(x)|\end{aligned}$$

Here  $h = (b - a)/n$  and  $x_j = a + jh$ ;  $B_{2i}$  are Bernoulli numbers;  $D^m f(x)$  is the  $m$ -th derivative of  $f(x)$ .

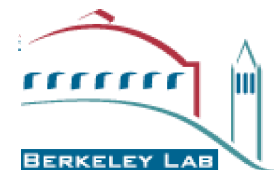
Note when  $f(x)$  and all of its derivatives are zero at the endpoints  $a$  and  $b$  (as in a bell-shaped curve), the error  $E(h)$  of a simple trapezoidal approximation to the integral goes to zero more rapidly than any power of  $h$ .



# Trapezoidal Approximation to a Bell-Shaped Function



# Tanh-Sinh Quadrature



Given  $f(x)$  defined on  $(-1,1)$ , define  $g(t) = \tanh(\pi/2 \sinh t)$ . Then setting  $x = g(t)$  yields

$$\int_{-1}^1 f(x) dx = \int_{-\infty}^{\infty} f(g(t))g'(t) dt \approx h \sum_{j=-N}^N w_j f(x_j),$$

where  $x_j = g(hj)$  and  $w_j = g'(hj)$ . Since  $g'(t)$  goes to zero very rapidly for large  $t$ , the product  $f(g(t))g'(t)$  typically is a nice bell-shaped function for which the Euler-Maclaurin formula implies that the simple summation above is remarkably accurate. Reducing  $h$  by half typically doubles the number of correct digits.

In our experience, we have found that tanh-sinh is the best general-purpose integration scheme for functions with vertical derivatives or singularities at endpoints. It is also best at very high precision ( $> 1000$  digits), because its computation of abscissas and weights is much faster than other schemes.

1. DHB, X.S. Li and K. Jeyabalan, "A comparison of three high-precision quadrature schemes," *Experimental Mathematics*, vol. 14 (2005), no. 3, pg. 317-329.
2. H. Takahasi and M. Mori, "Double exponential formulas for numerical integration," *Publications of RIMS*, Kyoto University, vol. 9 (1974), pg. 721-741.

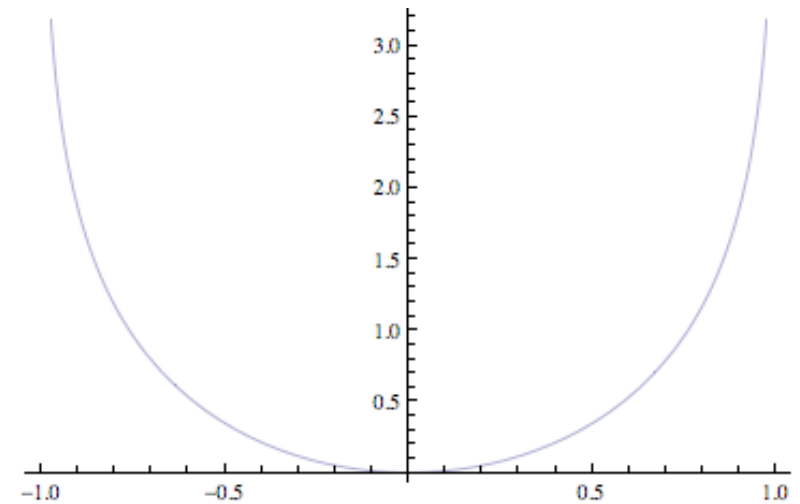
# Original and Transformed Integrand Functions



Original integrand function on  $[-1,1]$ :

$$f(x) = -\log \cos \left( \frac{\pi x}{2} \right)$$

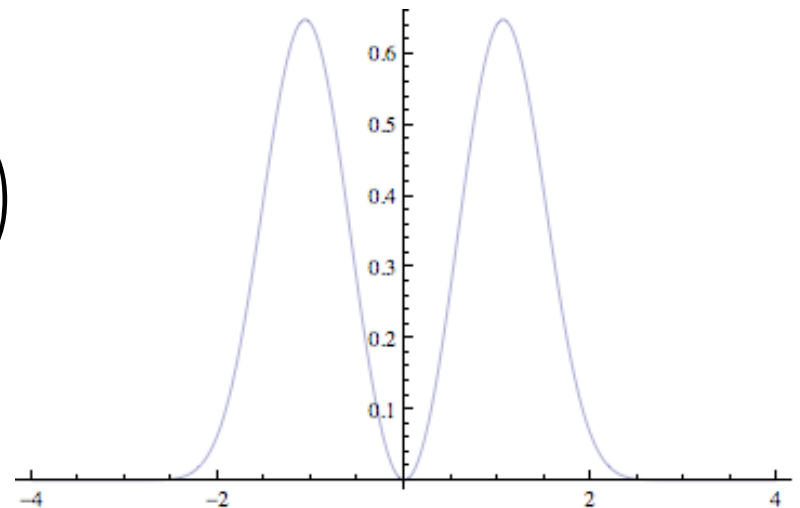
Note the singularities at the endpoints.



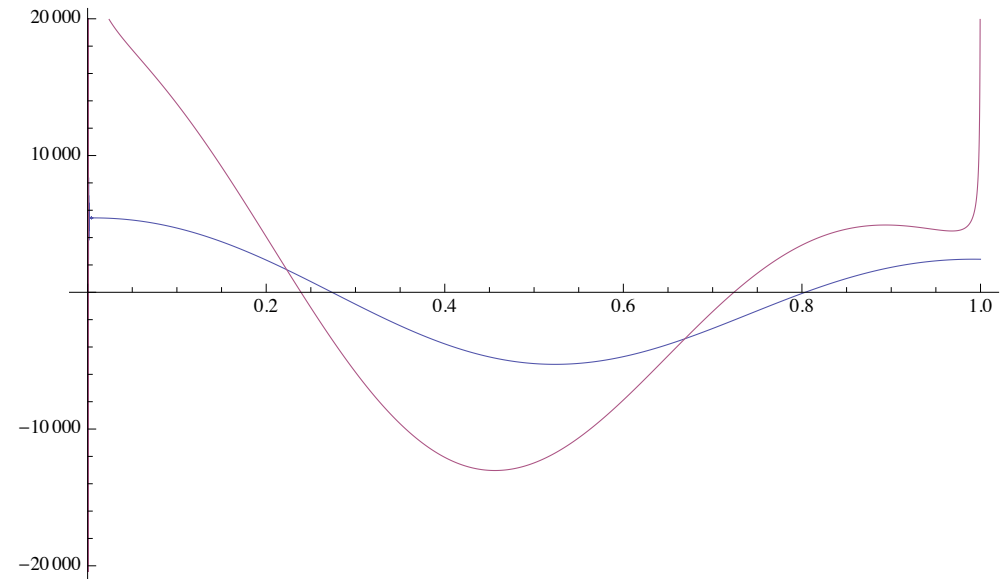
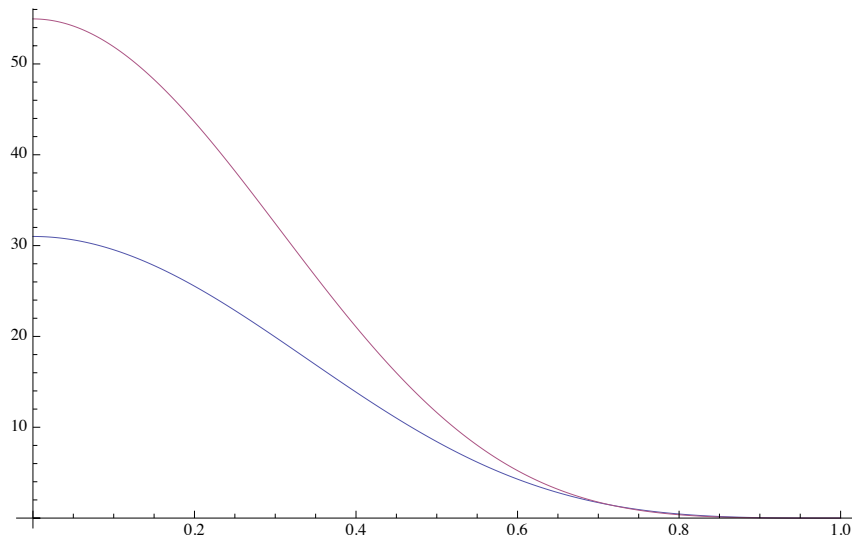
Transformed using  $x = g(t) = \tanh(\sinh t)$ :

$$f(g(t))g'(t) = -\log \cos[\pi/2 \cdot \tanh(\sinh t)] \left( \frac{\cosh(t)}{\cosh(\sinh t)^2} \right)$$

This is now a nice smooth bell-shaped function, so the E-M formula implies that a trapezoidal approximation is very accurate.



# A “Nice” Function That Requires Tanh-Sinh

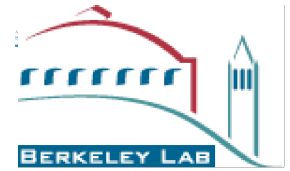


Plots of

$$f(x) = \sin^p(\pi x) \zeta(p, x)$$

and its fourth derivative, for  $p = 3$  (blue) and  $p = 3.5$  (red).

# A Log-Tan Integral Identity

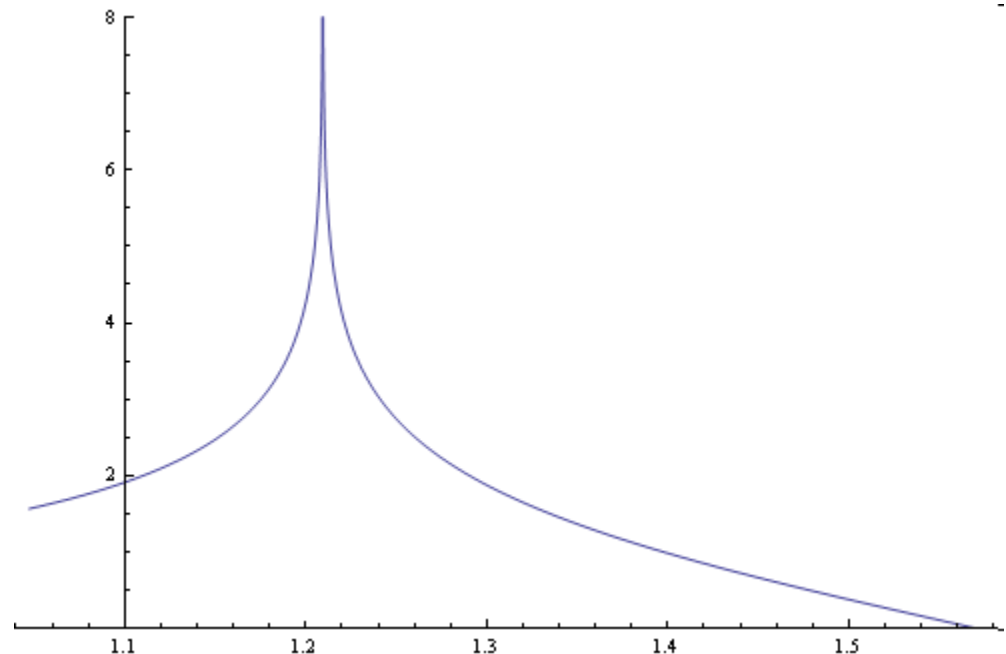


$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt = L_{-7}(2) =$$
$$\sum_{n=0}^{\infty} \left[ \frac{1}{(7n+1)^2} + \frac{1}{(7n+2)^2} - \frac{1}{(7n+3)^2} + \frac{1}{(7n+4)^2} - \frac{1}{(7n+5)^2} - \frac{1}{(7n+6)^2} \right]$$

This identity arises from analysis of volumes of knot complements in hyperbolic space. This is simplest of 998 related identities.

We verified this numerically to 20,000 digits (using highly parallel tanh-sinh quadrature). A proof is now known.

DHB, J. M. Borwein, V. Kapoor and E. Weisstein, "Ten problems in experimental mathematics," *American Mathematical Monthly*, vol. 113, no. 6 (Jun 2006), pg. 481-409 .



# Computing High-Precision Values of Multi-Dimension Integrals



Computing multi-hundred digit numerical values of 2-D, 3-D and higher-dimensional integrals remains a major challenge.

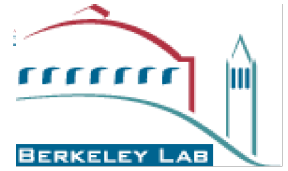
Typical approach:

- ◆ Consider the 2-D or 3-D domain divided into 1-D lines.
- ◆ Use Gaussian quadrature (for regular functions) or tanh-sinh quadrature (if function has vertical derivatives or singularities on boundaries) on each of the 1-D lines.
- ◆ Discontinue evaluation beyond points where it is clear that function-weight products are smaller than the “epsilon” of the precision level (this works better with tanh-sinh).

Even with “smart” evaluation that avoids unnecessary evaluations, the computational cost increases very sharply with dimension:

- ◆ If 1000 evaluation points are required in 1-D for a given precision, then typically 1,000,000 are required in 2-D and 1,000,000,000 in 3-D, etc.

# Ising Integrals



We recently applied our methods to study three classes of integrals that arise in the Ising theory of mathematical physics –  $D_n$  and two others:

$$C_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$D_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j} \left(\frac{u_i - u_j}{u_i + u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$E_n = 2 \int_0^1 \cdots \int_0^1 \left( \prod_{1 \leq j < k \leq n} \frac{u_k - u_j}{u_k + u_j} \right)^2 dt_2 dt_3 \cdots dt_n$$

where in the last line  $u_k = t_1 t_2 \cdots t_k$ .

DHB, J. M. Borwein and R. E. Crandall, "Integrals of the Ising class," *Journal of Physics A: Mathematical and General*, vol. 39 (2006), pg. 12271-12302.

# Computing and Evaluating $C_n$



We observed that the multi-dimensional  $C_n$  integrals can be transformed to 1-D integrals:

$$C_n = \frac{2^n}{n!} \int_0^\infty t K_0^n(t) dt$$

where  $K_0$  is the modified Bessel function. In this form, the  $C_n$  constants appear naturally in quantum field theory (QFT).

We used this formula to compute 1000-digit numerical values of various  $C_n$ , from which the following results and others were found, then proven:

$$C_1 = 2$$

$$C_2 = 1$$

$$C_3 = L_{-3}(2) = \sum_{n \geq 0} \left( \frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right)$$

$$C_4 = \frac{7}{12} \zeta(3)$$



## Limiting Value of $C_n$



The  $C_n$  numerical values appear to approach a limit. For instance,  
 $C_{1024} = 0.63047350337438679612204019271087890435458707871273234 \dots$

What is this limit? We copied the first 50 digits of this numerical value into the online Inverse Symbolic Calculator (ISC):

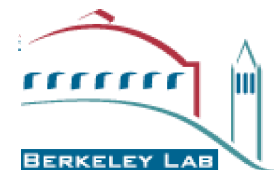
<http://ddrive.cs.dal.ca/~isc>

The result was:

$$\lim_{n \rightarrow \infty} C_n = 2e^{-2\gamma}$$

where gamma denotes Euler's constant. Finding this limit led us to the asymptotic expansion and made it clear that the integral representation of  $C_n$  is fundamental.

## Other Ising Integral Evaluations



$$D_2 = 1/3$$

$$D_3 = 8 + 4\pi^2/3 - 27 L_{-3}(2)$$

$$D_4 = 4\pi^2/9 - 1/6 - 7\zeta(3)/2$$

$$E_2 = 6 - 8 \log 2$$

$$E_3 = 10 - 2\pi^2 - 8 \log 2 + 32 \log^2 2$$

$$E_4 = 22 - 82\zeta(3) - 24 \log 2 + 176 \log^2 2 - 256(\log^3 2)/3 \\ + 16\pi^2 \log 2 - 22\pi^2/3$$

$$E_5 \stackrel{?}{=} 42 - 1984 \operatorname{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) - 1272\zeta(3) \log 2 \\ + 40\pi^2 \log^2 2 - 62\pi^2/3 + 40(\pi^2 \log 2)/3 + 88 \log^4 2 \\ + 464 \log^2 2 - 40 \log 2$$

where  $\operatorname{Li}_n(x)$  is the polylog function.  $D_2$ ,  $D_3$  and  $D_4$  were originally provided to us by mathematical physicist Craig Tracy, who hoped that our tools could help identify  $D_5$ .

# The Ising Integral $E_5$



We were able to reduce  $E_5$ , which is a 5-D integral, to an extremely complicated 3-D integral.

We computed this integral to 250-digit precision, using a highly parallel, high-precision 3-D quadrature program. Then we used a PSLQ program to discover the evaluation given on the previous page.

We also computed  $D_5$  to 500 digits, but were unable to identify it. The digits are available if anyone wishes to further explore this question.

$$E_5 = \int_0^1 \int_0^1 \int_0^1 [2(1-x)^2(1-y)^2(1-xy)^2(1-z)^2(1-yz)^2(1-xyz)^2 \\ (-[4(x+1)(xy+1)\log(2)(y^5z^3x^7 - y^4z^2(4(y+1)z+3)x^6 - y^3z((y^2+1)z^2+4(y+1)z+5)x^5 + y^2(4y(y+1)z^3+3(y^2+1)z^2+4(y+1)z-1)x^4 + y(z(z^2+4z+5)y^2+4(z^2+1)y+5z+4)x^3 + ((-3z^2-4z+1)y^2-4zy+1)x^2 - (y(5z+4)+4)x-1)] / [(x-1)^3(xy-1)^3(xy-1)^3] + [3(y-1)^2y^4(z-1)^2z^2(yz-1)^2x^6 + 2y^3z(3(z-1)^2z^3y^5 + z^2(5z^3+3z^2+3z+5)y^4 + (z-1)^2z(5z^2+16z+5)y^3 + (3z^5+3z^4-22z^3-22z^2+3z+3)y^2 + 3(-2z^4+z^3+2z^2+z-2)y+3z^3+5z^2+5z+3)x^5 + y^2(7(z-1)^2z^4y^6-2z^3(z^3+15z^2+15z+1)y^5+2z^2(-21z^4+6z^3+14z^2+6z-21)y^4-2z(z^5-6z^4-27z^3-27z^2-6z+1)y^3 + (7z^6-30z^5+28z^4+54z^3+28z^2-30z+7)y^2-2(7z^5+15z^4-6z^3-6z^2+15z+7)y+7z^4-2z^3-42z^2-2z+7)x^4-2y(z^3(z^3-9z^2-9z+1)y^6+z^2(7z^4-14z^3-18z^2-14z+7)y^5+z(7z^5+14z^4+3z^3+3z^2+14z+7)y^4+(z^6-14z^5+3z^4+84z^3+3z^2-14z+1)y^3-3(3z^5+6z^4-z^3-z^2+6z+3)y^2-(9z^4+14z^3-14z^2+14z+9)y+z^3+7z^2+7z+1)x^3+(z^2(11z^4+6z^3-66z^2+6z+11)y^6+2z(5z^5+13z^4-2z^3-2z^2+13z+5)y^5+(11z^6+26z^5+44z^4-66z^3+44z^2+26z+11)y^4+(6z^5-4z^4-66z^3-66z^2-4z+6)y^3-2(33z^4+2z^3-22z^2+2z+33)y^2+(6z^3+26z^2+26z+6)y+11z^2+10z+11)x^2-2(z^2(5z^3+3z^2+3z+5)y^5+z(22z^4+5z^3-22z^2+5z+22)y^4+(5z^5+5z^4-26z^3-26z^2+5z+5)y^3+(3z^4-22z^3-26z^2-22z+3)y^2+(3z^3+5z^2+5z+3)y+5z^2+22z+5)x+15z^2+2z+2y(z-1)^2(z+1)+2y^3(z-1)^2z(z+1)+y^4z^2(15z^2+2z+15)+y^2(15z^4-2z^3-90z^2-2z+15)+15] / [(x-1)^2(y-1)^2(xy-1)^2(z-1)^2(yz-1)^2(xy-1)^2] - [4(x+1)(y+1)(yz+1)(-z^2y^4+4z(z+1)y^3+(z^2+1)y^2-4(z+1)y+4x(y^2-1)(y^2z^2-1)+x^2(z^2y^4-4z(z+1)y^3-(z^2+1)y^2+4(z+1)y+1)-1)\log(x+1)] / [(x-1)^3x(y-1)^3(yz-1)^3] - [4(y+1)(xy+1)(z+1)(x^2(z^2-4z-1)y^4+4x(x+1)(z^2-1)y^3-(x^2+1)(z^2-4z-1)y^2-4(x+1)(z^2-1)y+z^2-4z-1)\log(xy+1)] / [x(y-1)^3y(xy-1)^3(z-1)^3] - [4(z+1)(yz+1)(x^3y^5z^7+x^2y^4(4x(y+1)+5)z^6-xy^3((y^2+1)x^2-4(y+1)x-3)z^5-y^2(4y(y+1)x^3+5(y^2+1)x^2+4(y+1)x+1)z^4+y(y^2x^3-4y(y+1)x^2-3(y^2+1)x-4(y+1))z^3+(5x^2y^2+y^2+4x(y+1)y+1)z^2+((3x+4)y+4)z-1)\log(xyz+1)] / [xyz(z-1)^3z(yz-1)^3(xy-1)^3]] / [(x+1)^2(y+1)^2(xy+1)^2(z+1)^2(yz+1)^2(xyz+1)^2] dx dy dz$$

# Recursions in Ising Integrals



Consider the 2-parameter class of Ising integrals (which arises in QFT for odd  $k$ ):

$$C_{n,k} = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^{k+1}} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

After computing 1000-digit numerical values for all  $n$  up to 36 and all  $k$  up to 75 (performed on a highly parallel computer system), we discovered (using PSLQ) linear relations in the rows of this array. For example, when  $n = 3$ :

$$\begin{aligned} 0 &= C_{3,0} - 84C_{3,2} + 216C_{3,4} \\ 0 &= 2C_{3,1} - 69C_{3,3} + 135C_{3,5} \\ 0 &= C_{3,2} - 24C_{3,4} + 40C_{3,6} \\ 0 &= 32C_{3,3} - 630C_{3,5} + 945C_{3,7} \\ 0 &= 125C_{3,4} - 2172C_{3,6} + 3024C_{3,8} \end{aligned}$$

Similar, but more complicated, recursions have been found for all  $n$ .

DHB, D. Borwein, J.M. Borwein and R.E. Crandall, "Hypergeometric forms for Ising-class integrals," *Experimental Mathematics*, vol. 16 (2007), pg. 257-276.

J. M. Borwein and B. Salvy, "A proof of a recursion for Bessel moments," *Experimental Mathematics*, vol. 17 (2008), pg. 223-230.

# Four Hypergeometric Evaluations



$$\begin{aligned}
 c_{3,0} &= \frac{3\Gamma^6(1/3)}{32\pi 2^{2/3}} = \frac{\sqrt{3}\pi^3}{8} {}_3F_2 \left( \begin{matrix} 1/2, 1/2, 1/2 \\ 1, 1 \end{matrix} \middle| \frac{1}{4} \right) \\
 c_{3,2} &= \frac{\sqrt{3}\pi^3}{288} {}_3F_2 \left( \begin{matrix} 1/2, 1/2, 1/2 \\ 2, 2 \end{matrix} \middle| \frac{1}{4} \right) \\
 c_{4,0} &= \frac{\pi^4}{4} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^4}{4^{4n}} = \frac{\pi^4}{4} {}_4F_3 \left( \begin{matrix} 1/2, 1/2, 1/2, 1/2 \\ 1, 1, 1 \end{matrix} \middle| 1 \right) \\
 c_{4,2} &= \frac{\pi^4}{64} \left[ {}_4F_3 \left( \begin{matrix} 1/2, 1/2, 1/2, 1/2 \\ 1, 1, 1 \end{matrix} \middle| 1 \right) \right. \\
 &\quad \left. - {}_3F_3 \left( \begin{matrix} 1/2, 1/2, 1/2, 1/2 \\ 2, 1, 1 \end{matrix} \middle| 1 \right) \right] - \frac{3\pi^2}{16}
 \end{aligned}$$

DHB, J.M. Borwein, D.M. Broadhurst and M.L. Glasser, "Elliptic integral representation of Bessel moments," *Journal of Physics A: Mathematical and Theoretical*, vol. 41 (2008), 5203-5231.

## 2-D Integral in Bessel Moment Study



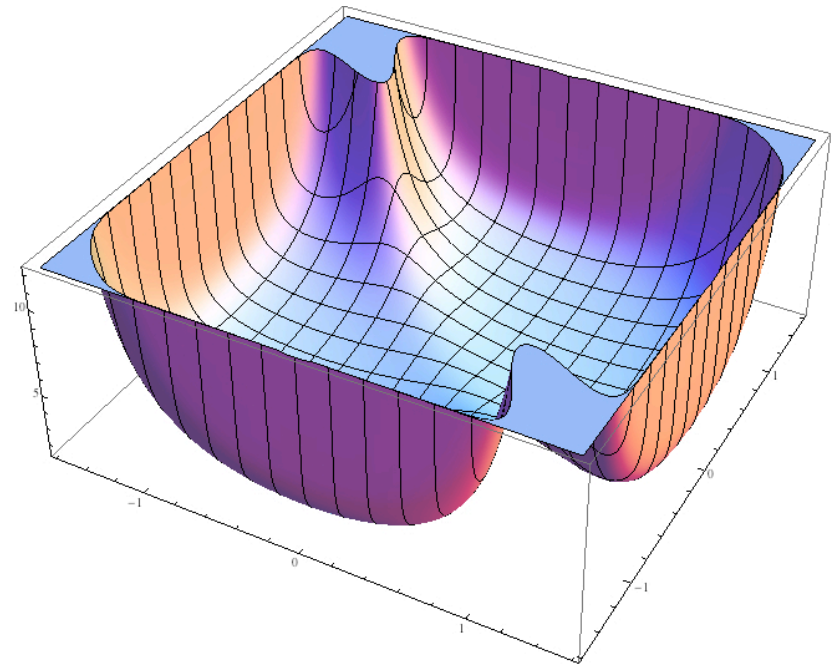
We conjectured (and later proved)

$$c_{5,0} = \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \frac{\mathbf{K}(\sin \theta) \mathbf{K}(\sin \phi)}{\sqrt{\cos^2 \theta \cos^2 \phi + 4 \sin^2(\theta + \phi)}} d\theta d\phi$$

Here  $\mathbf{K}$  denotes the complete elliptic integral of the first kind

Note that the integrand function has singularities on all four sides of the region of integration.

We were able to evaluate this integral to 120-digit accuracy, using 1024 cores of the “Franklin” Cray XT4 system at LBNL.



# Heisenberg Spin Integrals



In another recent application of these methods, we investigated the following “spin integrals,” which arise from studies in mathematical physics:

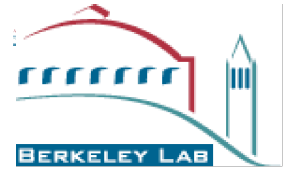
$$P(n) := \frac{\pi^{n(n+1)/2}}{(2\pi i)^n} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} U(x_1 - i/2, x_2 - i/2, \cdots, x_n - i/2) \\ \times T(x_1 - i/2, x_2 - i/2, \cdots, x_n - i/2) dx_1 dx_2 \cdots dx_n$$

where

$$U(x_1 - i/2, x_2 - i/2, \cdots, x_n - i/2) = \frac{\prod_{1 \leq k < j \leq n} \sinh[\pi(x_j - x_k)]}{\prod_{1 \leq j \leq n} i^n \cosh^n(\pi x_j)}$$
$$T(x_1 - i/2, x_2 - i/2, \cdots, x_n - i/2) = \frac{\prod_{1 \leq j \leq n} (x_j - i/2)^{j-1} (x_j + i/2)^{n-j}}{\prod_{1 \leq k < j \leq n} (x_j - x_k - i)}$$

H. E. Boos, V. E. Korepin, Y. Nishiyama and M. Shiroishi, “Quantum correlations and number theory,” *Journal of Physics A: Mathematical and General*, vol. 35 (2002), pg. 4443.

## Evaluations of $P(n)$ Derived Analytically, Confirmed Numerically



$$\begin{aligned}
 P(1) &= \frac{1}{2}, \quad P(2) = \frac{1}{3} - \frac{1}{3} \log 2, \quad P(3) = \frac{1}{4} - \log 2 + \frac{3}{8} \zeta(3) \\
 P(4) &= \frac{1}{5} - 2 \log 2 + \frac{173}{60} \zeta(3) - \frac{11}{6} \zeta(3) \log 2 - \frac{51}{80} \zeta^2(3) - \frac{55}{24} \zeta(5) + \frac{85}{24} \zeta(5) \log 2 \\
 P(5) &= \frac{1}{6} - \frac{10}{3} \log 2 + \frac{281}{24} \zeta(3) - \frac{45}{2} \zeta(3) \log 2 - \frac{489}{16} \zeta^2(3) - \frac{6775}{192} \zeta(5) \\
 &\quad + \frac{1225}{6} \zeta(5) \log 2 - \frac{425}{64} \zeta(3) \zeta(5) - \frac{12125}{256} \zeta^2(5) + \frac{6223}{256} \zeta(7) \\
 &\quad - \frac{11515}{64} \zeta(7) \log 2 + \frac{42777}{512} \zeta(3) \zeta(7)
 \end{aligned}$$

and a much more complicated expression for  $P(6)$ . Run times increase very rapidly with the dimension  $n$ :

$n$	Digits	Processors	Run Time
2	120	1	10 sec.
3	120	8	55 min.
4	60	64	27 min.
5	30	256	39 min.
6	6	256	59 hrs.



# Box Integrals



The following integrals appear in numerous arenas of math and physics:

$$B_n(s) := \int_0^1 \cdots \int_0^1 (r_1^2 + \cdots + r_n^2)^{s/2} dr_1 \cdots dr_n$$

$$\Delta_n(s) := \int_0^1 \cdots \int_0^1 ((r_1 - q_1)^2 + \cdots + (r_n - q_n)^2)^{s/2} dr_1 \cdots dr_n dq_1 \cdots dq_n$$

- $B_n(1)$  is the expected distance of a random point from the origin of  $n$ -cube.
- $\Delta_n(1)$  is the expected distance between two random points in  $n$ -cube.
- $B_n(-n+2)$  is the expected electrostatic potential in an  $n$ -cube whose origin has a unit charge.
- $\Delta_n(-n+2)$  is the expected electrostatic energy between two points in a uniform  $n$ -cube of charged “jellium.”
- Recently integrals of this type have arisen in neuroscience – e.g., the average distance between synapses in a mouse brain.

DHB, J. M. Borwein and R. E. Crandall, “Box integrals,” *Journal of Computational and Applied Mathematics*, vol. 206 (2007), pg. 196-208.

## Recent Result (18 Jan 2009)



$$\begin{aligned}\Delta_3(-1) &= \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{(-1 + e^{-u^2} + \sqrt{\pi} u \operatorname{erf}(u))^3}{u^6} du \\ &= \frac{1}{15} \left( 6 + 6\sqrt{2} - 12\sqrt{3} - 10\pi + 30 \log(1 + \sqrt{2}) + 30 \log(2 + \sqrt{3}) \right)\end{aligned}$$

As in many of the previous results, this was found by first computing the integral to high precision (250 to 1000 digits), conjecturing possible terms on the right-hand side, then applying PSLQ to look for a relation. We now have proven this result.

Dozens of similar results have since been found (see next few viewgraphs), raising hope that all box integrals eventually will be evaluated in closed form.

DHB, J. M. Borwein and R. E. Crandall, "Advances in the theory of box integrals," manuscript, Mar 2009, available at <http://crd.lbl.gov/~dhbailey/dhbpapers/BoxII.pdf>.

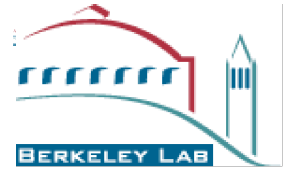
# Recent Evaluations of Box Integrals



$n$	$s$	$B_n(s)$
any 1	even $s \geq 0$ $s \neq -1$	rational, e.g., : $B_2(2) = 2/3$ $\frac{1}{s+1}$
2	-4	$-\frac{1}{4} - \frac{\pi}{8}$
2	-3	$-\sqrt{2}$
2	-1	$2 \log(1 + \sqrt{2})$
2	1	$\frac{1}{3}\sqrt{2} + \frac{1}{3} \log(1 + \sqrt{2})$
2	3	$\frac{7}{5}\sqrt{2} + \frac{3}{20} \log(1 + \sqrt{2})$
2	$s \neq -2$	$\frac{2}{2+s} {}_2F_1\left(\frac{1}{2}, -\frac{s}{2}; \frac{3}{2}; -1\right)$
3	-5	$-\frac{1}{6}\sqrt{3} - \frac{1}{12}\pi$
3	-4	$-\frac{3}{2}\sqrt{2} \arctan \frac{1}{\sqrt{2}}$
3	-2	$-3G + \frac{3}{2}\pi \log(1 + \sqrt{2}) + 3 \operatorname{Ti}_2(3 - 2\sqrt{2})$
3	-1	$-\frac{1}{4}\pi + \frac{3}{2} \log(2 + \sqrt{3})$
3	1	$\frac{1}{4}\sqrt{3} - \frac{1}{24}\pi + \frac{1}{2} \log(2 + \sqrt{3})$
3	3	$\frac{2}{5}\sqrt{3} - \frac{1}{60}\pi - \frac{7}{20} \log(2 + \sqrt{3})$

Here  $F$  is hypergeometric function;  $G$  is Catalan;  $Ti$  is Lewin's inverse-tan function.

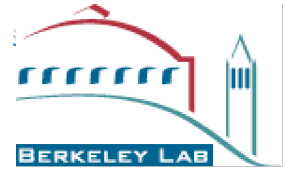
# Recent Evaluations of Box Integrals, Continued



$n$	$s$	$B_n(s)$
4	-5	$-\sqrt{8} \arctan\left(\frac{1}{\sqrt{8}}\right)$
4	-3	$4 G - 12 \operatorname{Ti}_2(3 - 2\sqrt{2})$
4	-2	$\pi \log(2 + \sqrt{3}) - 2 G - \frac{\pi^2}{8}$
4	-1	$2 \log 3 - \frac{2}{3} G + 2 \operatorname{Ti}_2(3 - 2\sqrt{2}) - \sqrt{8} \arctan\left(\frac{1}{\sqrt{8}}\right)$
4	1	$\frac{2}{5} - \frac{G}{10} + \frac{3}{10} \operatorname{Ti}_2(3 - 2\sqrt{2}) + \log 3 - \frac{7\sqrt{2}}{10} \arctan\left(\frac{1}{\sqrt{8}}\right)$
5	-3	$-\frac{110}{9} G - 10 \log(2 - \sqrt{3}) \theta - \frac{1}{8} \pi^2 + 5 \log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{5}{2} \sqrt{3} \arctan\left(\frac{1}{\sqrt{15}}\right)$ $-10 \operatorname{Cl}_2\left(\frac{1}{3} \theta + \frac{1}{3} \pi\right) + 10 \operatorname{Cl}_2\left(\frac{1}{3} \theta - \frac{1}{6} \pi\right)$ $+ \frac{10}{3} \operatorname{Cl}_2\left(\theta + \frac{1}{6} \pi\right) + \frac{20}{3} \operatorname{Cl}_2\left(\theta + \frac{4}{3} \pi\right) - \frac{10}{3} \operatorname{Cl}_2\left(\theta + \frac{5}{3} \pi\right) - \frac{20}{3} \operatorname{Cl}_2\left(\theta + \frac{11}{6} \pi\right)$
5	-2	$\frac{8}{3} B_5(-6) - \frac{1}{3} B_5(-4) + \frac{5}{2} \pi \log(3) + 10 \operatorname{Ti}_2\left(\frac{1}{3}\right) - 10 G$
5	-1	$-\frac{110}{27} G + \frac{10}{3} \log(2 - \sqrt{3}) \theta + \frac{1}{48} \pi^2 + 5 \log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{5}{2} \sqrt{3} \arctan\left(\frac{1}{\sqrt{15}}\right)$ $+ \frac{10}{3} \operatorname{Cl}_2\left(\frac{1}{3} \theta + \frac{1}{3} \pi\right) - \frac{10}{3} \operatorname{Cl}_2\left(\frac{1}{3} \theta - \frac{1}{6} \pi\right)$ $- \frac{10}{9} \operatorname{Cl}_2\left(\theta + \frac{1}{6} \pi\right) + \frac{20}{3} \operatorname{Cl}_2\left(\theta + \frac{4}{3} \pi\right) - \frac{10}{3} \operatorname{Cl}_2\left(\theta + \frac{5}{3} \pi\right) + \frac{20}{9} \operatorname{Cl}_2\left(\theta + \frac{11}{6} \pi\right)$
5	1	$-\frac{77}{81} G + \frac{7}{9} \log(2 - \sqrt{3}) \theta + \frac{1}{360} \pi^2 + \frac{1}{6} \sqrt{5} + \frac{10}{3} \log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{4}{3} \sqrt{3} \arctan\left(\frac{1}{\sqrt{15}}\right) +$ $\frac{7}{9} \operatorname{Cl}_2\left(\frac{1}{3} \theta + \frac{1}{3} \pi\right) - \frac{7}{9} \operatorname{Cl}_2\left(\frac{1}{3} \theta - \frac{1}{6} \pi\right)$ $- \frac{7}{27} \operatorname{Cl}_2\left(\theta + \frac{1}{6} \pi\right) - \frac{14}{27} \operatorname{Cl}_2\left(\theta + \frac{4}{3} \pi\right) + \frac{7}{27} \operatorname{Cl}_2\left(\theta + \frac{5}{3} \pi\right) + \frac{14}{27} \operatorname{Cl}_2\left(\theta + \frac{11}{6} \pi\right)$

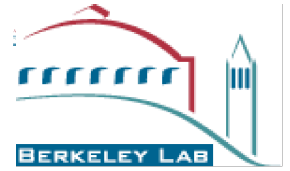
Here  $G$  is Catalan;  $\operatorname{Cl}$  is Clausen function;  $\operatorname{Ti}$  is Lewin function; and  $\theta = \arctan((16-3\sqrt{15})/11)$ .

# Recent Evaluations of Box Integrals, Continued



$n$	$s$	$\Delta_n(s)$
2	-5	$\frac{4}{3} + \frac{8}{9}\sqrt{2}$
2	-1	$\frac{4}{3} - \frac{4}{3}\sqrt{2} + 4\log(1 + \sqrt{2})$
2	1	$\frac{2}{15} + \frac{1}{15}\sqrt{2} + \frac{1}{3}\log(1 + \sqrt{2})$
3	-7	$\frac{4}{5} - \frac{16\sqrt{2}}{15} + \frac{2\sqrt{3}}{5} + \frac{\pi}{15}$
3	-2	$2\pi - 12 G + 12 \operatorname{Ti}_2(3 - 2\sqrt{2}) + 6\pi \log(1 + \sqrt{2}) + 2\log 2 - \frac{5}{2}\log 3 - 8\sqrt{2} \arctan\left(\frac{1}{\sqrt{2}}\right)$
3	-1	$\frac{2}{5} - \frac{2}{3}\pi + \frac{2}{5}\sqrt{2} - \frac{4}{5}\sqrt{3} + 2\log(1 + \sqrt{2}) + 12\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) - 4\log(2 + \sqrt{3})$
3	1	$-\frac{118}{21} - \frac{2}{3}\pi + \frac{34}{21}\sqrt{2} - \frac{4}{7}\sqrt{3} + 2\log(1 + \sqrt{2}) + 8\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$
3	3	$-\frac{1}{105} - \frac{2}{105}\pi + \frac{73}{840}\sqrt{2} + \frac{1}{35}\sqrt{3} + \frac{3}{56}\log(1 + \sqrt{2}) + \frac{13}{35}\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$

# Recent Evaluations of Box Integrals, Continued



$n$	$s$	$\Delta_n(s)$
4	-10	$-\frac{8}{45} \pi \sqrt{3} - \frac{20}{9} \pi \log(2) + \frac{4}{9} \pi^2 + \frac{4}{3} \log(2) + \frac{1}{15} \log(3) + \frac{8}{3} \text{Ti}_2(3 - 2\sqrt{2})$
4	-9	$\frac{16}{5} \pi \sqrt{3} - \frac{32}{3} \pi \log(2) - \frac{2}{3} \pi^2 + \frac{16}{5} \pi + 8\sqrt{2} \arctan(2\sqrt{2}) - 24 \log(2) + \frac{2}{5} \log(3)$ $+ 12 \pi \log(\sqrt{2} - 1) - 64 \text{Ti}_2(3 - 2\sqrt{2}) + \frac{160}{3} G$
4	-3	$-\frac{128}{15} + \frac{1}{63} \pi - 8 \log(1 + \sqrt{2}) - 32 \log(1 + \sqrt{3}) + 16 \log(2) + 20 \log(3)$ $-\frac{8}{5} \sqrt{2} + \frac{32}{5} \sqrt{3} - 32 \sqrt{2} \arctan\left(\frac{1}{\sqrt{8}}\right) - 96 \text{Ti}_2(3 - 2\sqrt{2}) + 32 G$
4	-2	$-\frac{16}{15} \pi \sqrt{3} + \frac{16}{3} \pi \log(1 + \sqrt{3}) - \frac{8}{3} \pi \log(2) + 4 \pi \log(\sqrt{2} + 1) - \frac{2}{3} \pi^2 + \frac{4}{5} \pi$ $+ \frac{8}{5} \sqrt{2} \arctan(2\sqrt{2}) + \frac{2}{5} \log(3) + 8 \text{Ti}_2(3 - 2\sqrt{2}) - \frac{40}{3} G$
4	-1	$\frac{704}{195} - \frac{8}{39} \pi - \frac{100}{13} \log(3) + \frac{120}{13} \log(2) - \frac{8}{65} \sqrt{2} + \frac{128}{65} \sqrt{3}$ $-\frac{140}{13} \log(1 + \sqrt{2}) - \frac{32}{13} \log(1 + \sqrt{3}) + \frac{160}{13} \sqrt{2} \arctan\left(\frac{1}{\sqrt{8}}\right) + \frac{48}{13} G$
4	1	$-\frac{23}{135} - \frac{16}{315} \pi - \frac{52}{105} \log(2) + \frac{197}{420} \log(3) + \frac{73}{630} \sqrt{2} + \frac{8}{105} \sqrt{3}$ $+ \frac{1}{14} \log(1 + \sqrt{2}) + \frac{104}{105} \log(1 + \sqrt{3}) - \frac{68}{105} \sqrt{2} \arctan\left(\frac{1}{\sqrt{8}}\right) - \frac{4}{15} G + \frac{4}{5} \text{Ti}_2(3 - 2\sqrt{2})$
5	-3	$-\frac{12304}{63} - \frac{512}{21} \sqrt{2} + \frac{576}{7} \sqrt{3} + \frac{800}{21} \sqrt{5} - \frac{320}{3} B_2(3) + \frac{448}{3} B_2(5)$ $- 320 B_3(1) + 960 B_3(3) - \frac{1792}{3} B_3(5) - 160 B_4(-1) + \frac{4400}{3} B_4(1) - \frac{20720}{9} B_4(3)$ $+ 896 B_4(5) + 32 B_5(-3) + \frac{800}{3} B_5(-1) - 1488 B_5(1) + \frac{14336}{9} B_5(3) - 448 B_5(5)$
5	-1	$\frac{16388}{189} + \frac{1024}{189} \sqrt{2} - \frac{192}{7} \sqrt{3} - \frac{4000}{189} \sqrt{5} + \frac{64}{3} B_2(5) - \frac{192}{7} B_2(7) + \frac{320}{3} B_3(3)$ $- 256 B_3(5) + \frac{960}{7} B_3(7) + 160 B_4(1) - \frac{6160}{9} B_4(3) + 784 B_4(5) - \frac{1760}{7} B_4(7)$ $+ 32 B_5(-1) - 400 B_5(1) + \frac{8192}{9} B_5(3) - 672 B_5(5) + \frac{1056}{7} B_5(7)$
5	1	$-\frac{1279}{567} G - \frac{4}{189} \pi + \frac{4}{315} \pi^2 - \frac{449}{3465} + \frac{3239}{62370} \sqrt{2} + \frac{568}{3465} \sqrt{3} - \frac{380}{6237} \sqrt{5}$ $+ \frac{295}{252} \log(3) + \frac{1}{54} \log(1 + \sqrt{2}) + \frac{20}{63} \log(2 + \sqrt{3}) + \frac{64}{189} \log\left(\frac{1+\sqrt{5}}{2}\right)$ $-\frac{73}{63} \sqrt{2} \arctan\left(\frac{1}{\sqrt{8}}\right) - \frac{8}{21} \sqrt{3} \arctan\left(\frac{1}{\sqrt{15}}\right) + \frac{104}{63} \log(2 - \sqrt{3}) \theta$ $+ \frac{5}{7} \text{Ti}_2(3 - 2\sqrt{2}) + \frac{104}{63} \text{Cl}_2\left(\frac{1}{3} \theta + \frac{1}{3} \pi\right) - \frac{104}{63} \text{Cl}_2\left(\frac{1}{3} \theta - \frac{1}{6} \pi\right)$ $- \frac{104}{189} \text{Cl}_2\left(\theta + \frac{1}{6} \pi\right) - \frac{208}{189} \text{Cl}_2\left(\theta + \frac{4}{3} \pi\right) + \frac{104}{189} \text{Cl}_2\left(\theta + \frac{5}{3} \pi\right) + \frac{208}{189} \text{Cl}_2\left(\theta + \frac{11}{6} \pi\right)$

## Cautionary Example



These constants agree to 42 decimal digit accuracy, but are NOT equal:

$$\int_0^\infty \cos(2x) \prod_{n=1}^\infty \cos(x/n) \, dx =$$

0.392699081698724154807830422909937860524645434187231595926

$$\frac{\pi}{8} =$$

0.392699081698724154807830422909937860524646174921888227621

Richard Crandall has now shown that this integral is merely the first term of a very rapidly convergent series that converges to  $\pi/8$ :

$$\frac{\pi}{8} = \sum_{m=0}^{\infty} \int_0^{\infty} \cos[2(2m+1)x] \prod_{n=1}^{\infty} \cos(x/n) \, dx$$

1. D. H. Bailey, J. M. Borwein, V. Kapoor and E. Weisstein, “Ten Problems in Experimental Mathematics,” *American Mathematical Monthly*, vol. 113, no. 6 (Jun 2006), pg. 481-409 .

2. R. E. Crandall, “Theory of ROOF Walks, 2007, available at <http://people.reed.edu/~crandall/papers/ROOF.pdf>.

# Summary



- ◆ Numerous state-of-the-art large-scale scientific calculations now require numerical precision beyond conventional 64-bit floating-point arithmetic.
- ◆ The emerging “experimental” methodology in mathematics and mathematical physics often requires hundreds or even thousands of digits of precision.
- ◆ Double-double, quad-double and arbitrary precision software libraries are now widely available (and in most cases are free). High-precision arithmetic is also integrated into *Mathematica* and *Maple*.
- ◆ High-precision evaluation of integrals, followed by constant-recognition techniques, has been a particularly fruitful area of recent research, with many new results in pure math and mathematical physics.
- ◆ There is a critical need to develop faster techniques for high-precision numerical integration in multiple dimensions.