Maximizing Linear Functions Subject to Linear Inequalities- The Simplex Method

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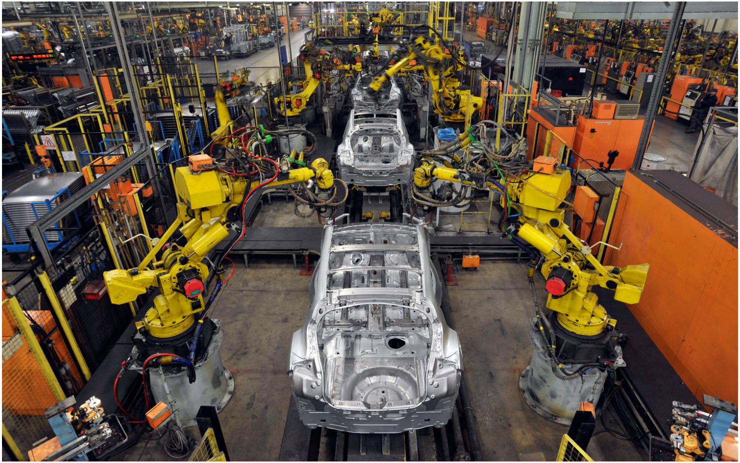
Motivation and Background

- George Dantzig invented the Simplex Method in 1947, and it is still used in many applications of linear programming.
- How do we maximize or minimize given linear restrictions?
- Applications of the simplex method include: economics, industry, and transportation problems.

Applications



Standard form

- For a given linear program, we can convert it to the following form:

$$\max \quad z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n} &= b_1 \\ \ldots \qquad \ldots \qquad \ldots \qquad \ldots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn} &= b_m \end{aligned}$$

for nonnegative x_i $(i = 1, 2 \ldots n)$.

Rewriting LP

We want to maximize

$$\mathbf{c} \cdot \mathbf{x}$$

with

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$$

We can write the LP as

- $$\begin{array}{cc} \left[\begin{array}{ccc} 1 & -\mathbf{c}^T & 0 \\ 0 & \mathbf{A} & \mathbf{b} \end{array}\right] & \begin{array}{l} \text{Row } 0 \\ \text{Row } i \end{array} \end{array}$$
- $i = 1, 2, \ldots m$

Definitions

Feasible Solution

A solution which meets all constraints but may or may not be optimal.

Basic Feasible Solution

A feasible solution with m number of nonzero x_i.

Theorem

If one feasible solution exists, then there exists a basic feasible solution.

Simplex Algorithm

- **Step 1:** If $\mathbf{c}^T \geq 0$ in Row 0, the current basic solution is optimal.

  $\Rightarrow$ Otherwise, pick a variable x_j with a negative coefficient in Row 0.

- **Step 2:** For each Row $i, i \geq 1$, where there is strictly positive "entering variable coefficient", choose the pivot row with

$$\theta = \min \; (b_i / A_{ij}) \quad A_{ij} > 0.$$

Maximization Example

Maximize $z = 4x_1 + 6x_2$, subject to

$$\begin{aligned} -x_1 + x_2 &\leq 11 \\ x_1 + x_2 &\leq 27 \\ 2x_1 + 5x_2 &\leq 90 \end{aligned}$$

with $x_1, x_2 \geq 0$

Maximization Example

Rewriting the LP, we introduce **slack variables** s_1, s_2, and s_3

$$\begin{aligned}
-x_1 + x_2 + s_1 \qquad\qquad &= 11 \\
x_1 + x_2 \qquad + s_2 \qquad &= 27 \\
2x_1 + 5x_2 \qquad\qquad + s_3 &= 90
\end{aligned}$$

with $x_1, x_2, s_1, s_2, s_3 \geq 0$.

Maximization Example

Our tableau becomes

$$\begin{bmatrix} 1 & -4 & -6 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 11 \\ 0 & 1 & 1 & 0 & 1 & 0 & 27 \\ 0 & 2 & 5 & 0 & 0 & 1 & 90 \end{bmatrix}$$

with the basic feasible solution $(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90)$. Then,

$$z = 4(0) + 6(0) + 0(11) + 0(27) + 0(90) = 0$$

Maximization Example

We choose x_2 as "entering variable" and s_1 as "exiting variable" since $\theta = 11$. After pivoting, our tableau becomes

$$\begin{bmatrix} 1 & -10 & 0 & 6 & 0 & 0 & 66 \\ 0 & -1 & 1 & 1 & 0 & 0 & 11 \\ 0 & 2 & 0 & -1 & 1 & 0 & 16 \\ 0 & 7 & 0 & -5 & 0 & 1 & 35 \end{bmatrix}$$

with the basic feasible solution $(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$. Then,

$$z = 4(0) + 6(11) + 0(0) + 0(16) + 0(35) = 66$$

Maximization Example

After carrying out the Simplex Method two more times, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{8}{3} & \frac{2}{3} & 132 \\ 0 & 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & 12 \\ 0 & 0 & 0 & 1 & \frac{7}{3} & -\frac{2}{3} & 14 \\ 0 & 1 & 0 & 0 & \frac{5}{3} & -\frac{1}{3} & 15 \end{bmatrix}$$

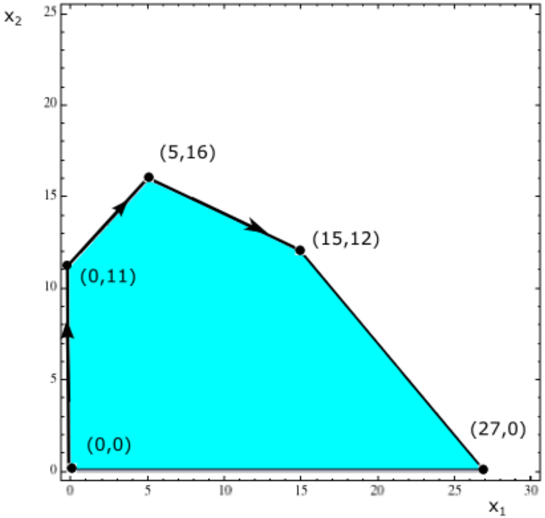
with the basic feasible solution $(x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)$. Then,

$$z = 4(15) + 6(12) + 0(14) + 0(0) + 0(0) = 132$$

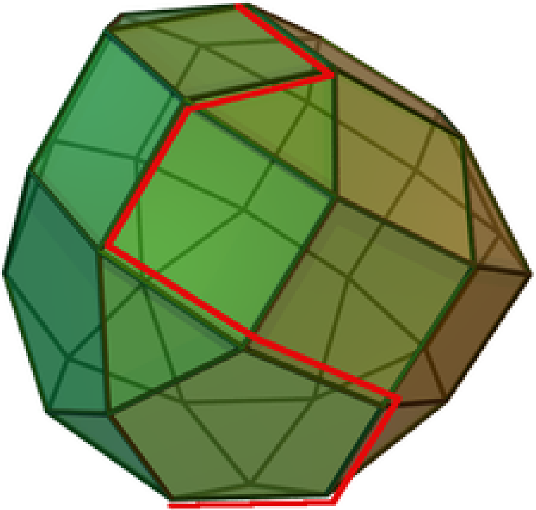
Graphing the Simplex Method

- We can consider the simplex algorithm as moving from one vertex v to another adjacent vertex w of a polytope
- Each step increases the objective function.

Graphical Example: max $\quad z = 4x_1 + 6x_2$



Higher dimensions



Disadvantages of the Method

- The method works well for non-degenerate matrix $\mathbf{A}$.
- A basic feasible solution is needed as a starting point.
- Other methods (i.e. interior point) may be more efficient with large m.

Concluding Remarks

- Introduced the Simplex Method with a specific maximization example.
- Provided intuition for the graphical representation of the method.
- Highlighted a few applications and shortcomings.

References

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