Fundamental Concepts in Computational and Applied Mathematics

Juan Meza
School of Natural Sciences
University of California, Merced

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Introduction

- Elementary discussion of algebraic problems arising in discretization of differential equations
- Behavior of the solution as mesh width tends to zero
- BVP and EV problems for elliptic pdes
- IVP for hyperbolic and parabolic pdes

Main Results

- For elliptic equations
  - difference quotient tends to the corresponding derivative
  - convergence is guaranteed independently of mesh
- For hyperbolic equations
  - convergence is obtained iff certain ratio of mesh width is satisfied
  - something else

Paper Outline

1. Introduction
2. Elliptic equations
3. Hyperbolic equations

Motivation

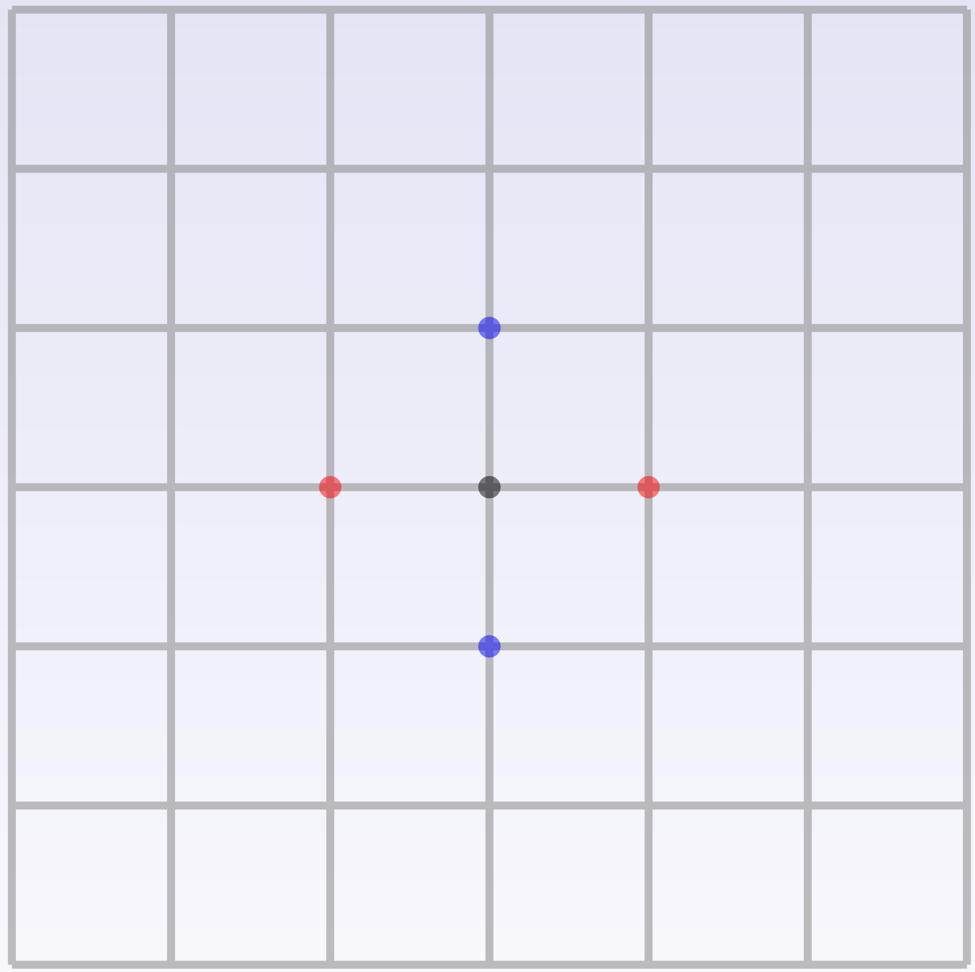
- Consider case in 1D on an equally spaced grid $u(x)$
- h is the spatial discretization

The difference quotient for approximating $\partial u / \partial x$ can be written as:

$$u_x = \frac{u(x_{j+1}) - u(x_{j-1})}{2h}$$

Likewise in 2D

$$
\begin{aligned}
u_x &= \frac{u(x+h,y)-u(x,y)}{h} \\
u_y &= \frac{u(x,y+h)-u(x,y)}{h} \\
u_{\bar{x}} &= \frac{u(x,y)-u(x-h,y)}{h} \\
u_{\bar{y}} &= \frac{u(x,y)-u(x,y-h)}{h}
\end{aligned}
$$



Corresponding second derivatives

$$u_{xx} = \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}$$

Equations

- Elliptic equations:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

  or

$$\Delta u = u_{xx} + u_{yy}$$

- Hyperbolic equations:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

both on a simply connected region G with boundary ∂G, and appropriate B.C.'s and I.C.'s

Main Results: Section 1 (Elliptic Case)

Assume

- $f(x, y)$ is a given continuous function,
- $f(x, y)$ has continuous first and second partial derivatives in a region containing G,
- a mesh G_h, with mesh width h,
- $u_h(x, y)$ is the solution of the difference equation $\Delta u = 0$.

Then

- As $h \to 0$, $u_h(x, y)$ converges to $u(x, y)$ satisfying the pde on the domain G and equals f on the boundary
- For any interior region within G the difference quotients, u_h tend to the corresponding partial derivatives of $u(x, y)$

Main Results: Section 2 (Hyperbolic Case)

Assume

- Hyperbolic equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

- a rectangular mesh, with mesh size h in the time direction, and kh, in the spatial (x) direction

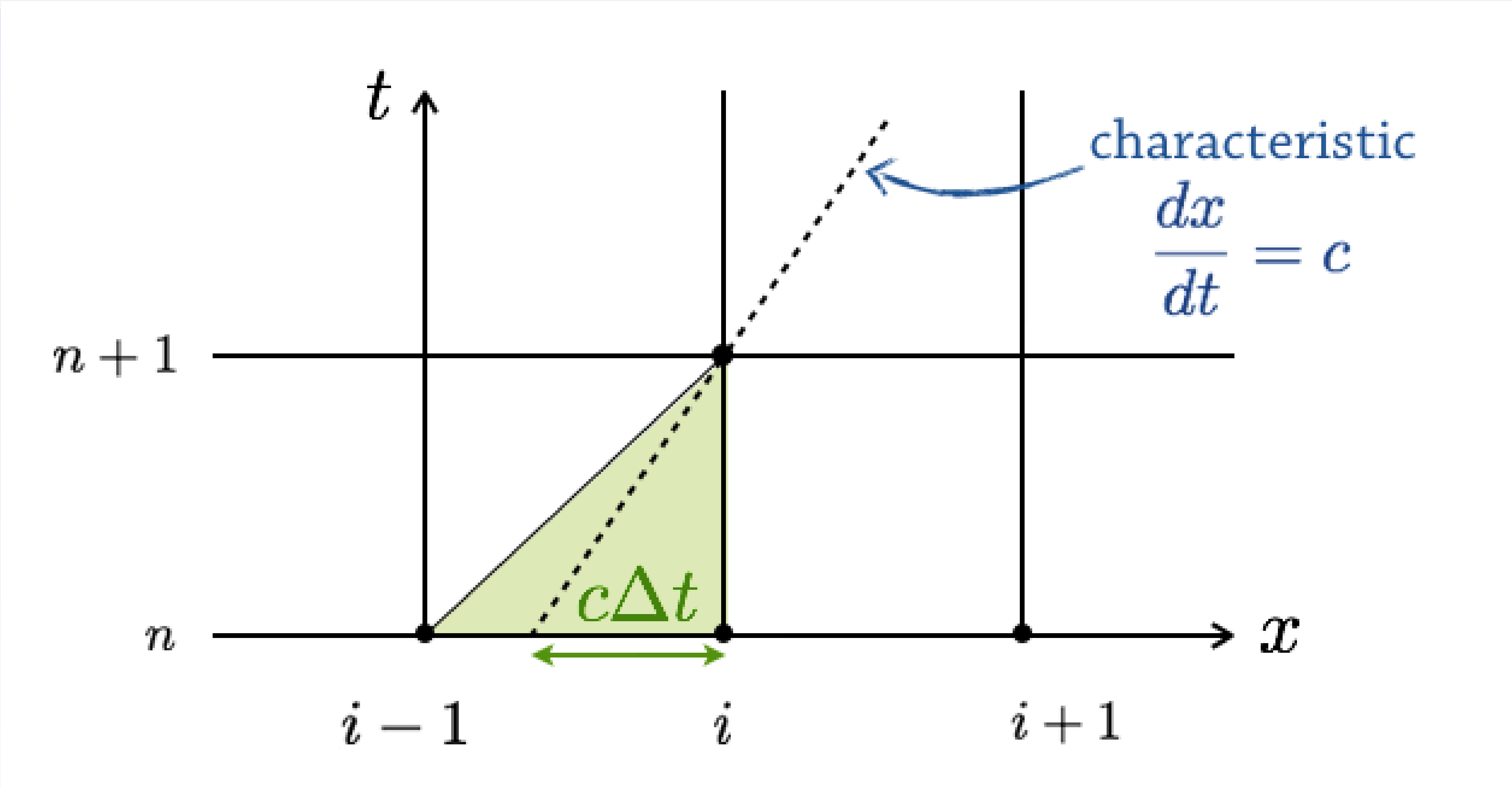
- For $k < 1$, as $h \to 0$, the solution to the difference equation **cannot converge** to the solution of the differential equation (1)

In other words for:

$$k = \frac{\Delta x}{\Delta t} < 1$$

the difference quotient solution will not converge!!!

Main Results: CFL Condition Pictorially [1]



Main Results: CFL Condition

For the general wave equation with velocity c:

$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x^2} = 0,$$

today we say that the time step Δt must be chosen so that the CFL condition is met, i.e.:

$$\sigma = c\frac{\Delta t}{\Delta x} \leq \sigma_{max}$$

Note:

The value of σ_{max} will vary according to the numerical method used. For an explicit method, σ_{max} is typically 1.

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