Linear Systems: Matrices, eigen-values, ...

Herramientas de Modelación - Maestría en Gestión y Diseño de Procesos

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Outline

Topic

Example : Bungee-jumping family plan [1]

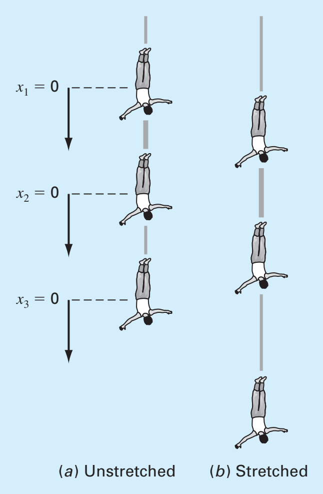

FIGURE 8.1
Three individuals connected by bungee cords.

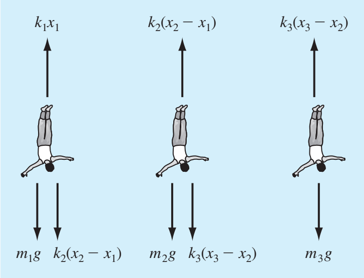

FIGURE 8.2
Free-body diagrams.

Example : System of equations

The second Newton law produces:

$$
\begin{aligned}
m_1 \frac{d^2 x_1}{dt^2} &= m_1 g + k_2(x_2 - x_1) - k_1 x_1 \\
m_2 \frac{d^2 x_2}{dt^2} &= m_2 g + k_3(x_3 - x_2) + k_2(x_1 - x_2) \\
m_3 \frac{d^2 x_3}{dt^2} &= m_3 g + k_3(x_2 - x_3)
\end{aligned}
$$

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m_3\frac{d^2x_3}{dt^2} &= m_3g + k_3(x_2 - x_3)
\end{aligned}$$

And for a system in equilibrium, we get:

$$\begin{aligned}
(k_1 + k_2)x_1 \qquad - k_2x_2 \qquad &= m_1g \\
-k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 &= m_2g \\
-k_3x_2 + k_3x_3 &= m_3g
\end{aligned}$$

A matrix

$$
[A] = \begin{array}{cl}
\begin{array}{ccccc} & & \text{Column 3} & & \\ & & \downarrow & & \end{array} & \\
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix} & \begin{array}{l} \\ \longleftarrow \text{Row 2} \\ \\ \\ \\ \\ \end{array}
\end{array}
$$

Special cases

- A symmetric matrix :

$$[A] = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 7 \\ 2 & 7 & 8 \end{bmatrix}$$

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- Banded :

$$[A] = \begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & a_{32} & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{bmatrix}$$

Matrix definition in python

You can use either numpy/scipy or sympy.

See numpy docs .

```
import numpy as np
a = np.matrix('1 2; 3 4')
print(a)
a = np.matrix([[1, 2], [3, 4]])
print(a)
a = np.array([1, 2, 3,
↪  4]).reshape(2,2)
print(a)
```

```
[[1 2]
 [3 4]]
[[1 2]
 [3 4]]
[[1 2]
```

See sympy docs

```
from sympy.matrices import Matrix,
↪  eye, zeros, ones, diag
M = Matrix([[1,0,4], [0,0,0]])
print(M)
print(eye(4))
print(M[0, 2])
```

```
Matrix([[1, 0, 4], [0, 0, 0]])
Matrix([[1, 0, 0, 0], [0, 1, 0, 0
4
```

```
Matrix([[1, 0, 4], [0, 0, 0]])
Matrix([[1, 0, 0, 0], [0, 1, 0, 0]
4
```

Matrix operations in python

Example in python

```
import numpy as np
a = np.matrix('1 2; 3 4')
b = np.matrix([[5, -1], [-3, 24]])
c = a+b # sum
print(c)
c = a*b # Multiplication
print(c)
c = a/b # multiply by the inverse of b
print(c)
print(c.max())
print(c.min())
d = np.array([[5, -1], [-3, 24]])
print("d:\n",d)
```

Output

```
[[ 6  1]
 [ 0 28]]
[[-1 47]
 [ 3 93]]
[[ 0.2         -2.        ]
 [-1.           0.16666667]]
0.2
-2.0
d:
 [[ 5 -1]
 [-3 24]]
```

You can watch : https://www.youtube.com/watch?v=XkY2DOUCWMU

Topic

Example

Solve the following linear system (See numpy linalg)

$$\begin{bmatrix} 150 & -100 & 0 \\ -100 & 150 & -50 \\ 0 & -50 & 50 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 588.6 \\ 686.7 \\ 784.8 \end{Bmatrix}$$

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Solve the following linear system (See numpy linalg)

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```python
import numpy as np

A = np.array([[150, -100, 0], [-100, 150, -50], [0, -50, 50]])
b = np.array([588.6, 686.7, 784.8])
x = np.linalg.solve(A, b) # magic
print(x)
# confirm
print(A.dot(x) - b)
```

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$$\begin{bmatrix} 150 & -100 & 0 \\ -100 & 150 & -50 \\ 0 & -50 & 50 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 588.6 \\ 686.7 \\ 784.8 \end{Bmatrix}$$

```python
import numpy as np

A = np.array([[150, -100, 0], [-100, 150, -50], [0, -50, 50]])
b = np.array([588.6, 686.7, 784.8])
x = np.linalg.solve(A, b) # magic
print(x)
# confirm
print(A.dot(x) - b)
```

```
[41.202 55.917 71.613]
[1.25055521e-12 6.82121026e-13 2.27373675e-13]
```

Exercise 1

8.3 Write the following set of equations in matrix form:

$$50 = 5x_3 - 6x_2$$

$$2x_2 + 7x_3 + 30 = 0$$

$$x_1 - 7x_3 = 50 - 3x_2 + 5x_1$$

Solve the system.

Exercise 2

Solve the following system

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 10 & -10 & 0 & -15 & -5 \\ 5 & -10 & 0 & -20 & 0 & 0 \end{bmatrix} \begin{Bmatrix} i_{12} \\ i_{52} \\ i_{32} \\ i_{65} \\ i_{54} \\ i_{43} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 200 \end{Bmatrix}$$

Can you measure the time spent?

Exercise 3

Solve this system:

$$\frac{-2.3x_1}{5} + x_2 = 1.1$$
$$-0.5x_1 + x_2 = 1$$

Plot the system of equations and check whether this solution is or not special.

Exercise 4: Simulating temperature

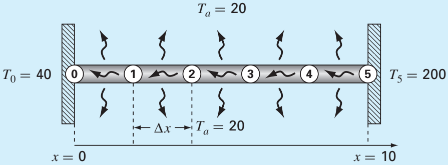


FIGURE 9.7
A noninsulated uniform rod positioned between two walls of constant but different temperature. The finite-difference representation employs four interior nodes.

$$\begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{Bmatrix}$$

Topic

Using `scipy`

See the docs for determinant

Using scipy

See the docs for determinant

```
from scipy import linalg
import numpy as np
A = np.array([[1,2,3], [4,5,6], [7,8,9]])
print(linalg.det(A))
A = np.array([[0,2,3], [4,5,6], [7,8,9]])
print(linalg.det(A))
```

Using scipy

See the docs for determinant

```
from scipy import linalg
import numpy as np
A = np.array([[1,2,3], [4,5,6], [7,8,9]])
print(linalg.det(A))
A = np.array([[0,2,3], [4,5,6], [7,8,9]])
print(linalg.det(A))
```

```
0.0
3.0
```

You can watch : https://www.youtube.com/watch?v=Ip3X9LOh2dk

9.4 Given the system of equations

$$\begin{aligned} 2x_2 + 5x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 1 \\ 3x_1 + x_2 &= 2 \end{aligned}$$

(a) Compute the determinant.

Topic

Using scipy

See inverse with scipy

```
from scipy import linalg
import numpy as np
A = np.array([[1., 2.], [3., 4.]])
B = linalg.inv(A)
print(B)
# verify
print(A.dot(B))
```

```
[[-2.   1. ]
 [ 1.5 -0.5]]
[[1.0000000e+00 0.0000000e+00]
 [8.8817842e-16 1.0000000e+00]]
```

You can watch: https://www.youtube.com/watch?v=uQhTuRlWMxw

Condition number

The number $\kappa = ||A||||A^{-1}||$ is called the condition number of a matrix. Ideally it is 1. If κ is much larger than one, the matrix is ill-conditioned and the solution might have a lot of error.

Compute the condition number of the following matrix:

$$A = \begin{bmatrix} 1.001 & 0.001 \\ 0.000 & 0.999 \end{bmatrix} \tag{1}$$

Topic

LU

You can specifically solve with LU factorization. See docs .

```
from scipy.linalg import lu_factor, lu_solve
import numpy as np
A = np.array([[2, 5, 8, 7], [5, 2, 2, 8], [7, 5, 6, 6], [5, 4, 4, 8]])
b = np.array([1, 1, 1, 1])
lu, piv = lu_factor(A)
x = lu_solve((lu, piv), b)
print(x)
print(A.dot(x) - b)
```

```
[ 0.05154639 -0.08247423  0.08247423  0.09278351]
[0. 0. 0. 0.]
```

Cholesky

Or you can use the Cholesky factorization. See Cholesky docs . The matrix must be positive definite.

```
from scipy.linalg import cho_factor, cho_solve
import numpy as np
A = np.array([[9, 3, 1, 5], [3, 7, 5, 1], [1, 5, 9, 2], [5, 1, 2, 6]])
b = np.array([1, 1, 1, 1])
c, low = cho_factor(A)
x = cho_solve((c, low), b)
print(x)
print(A.dot(x) - b)
```

```
[-0.01749271  0.11953353  0.01166181  0.1574344 ]
[2.22044605e-16 2.22044605e-16 0.00000000e+00 0.00000000e+00]
```

Topic

Definition

The eigen-values and eigen-vectors of a matrix satisfy the equation

$$Ax = \lambda x$$

The eigen-vectors form a basis where the matrix can be diagonalized. In general, computing the eigen vectors and aeigenvalues is hard, and they can also be complex.

You can watch: https://www.youtube.com/watch?v=PFDu9oVAE-g

Implementation in Python

See docs for scipy

```
import numpy as np
from scipy import linalg
A = np.array([[0., -1.], [1., 0.]])
#A = np.array([[1, 0.], [0., 2.]])
#A = np.array([[2, 5, 8, 7], [5, 2, 2, 8], [7, 5, 6, 6], [5, 4, 4, 8]])
#A = np.array([[2, 5, 8, 7], [5, 2, 2, 8], [7, 5, 6, 6], [5, 4, 4, 8]])
sol = linalg.eig(A)
print("Eigen-values: ", sol[0])
print("Eigen-vectors:\n", sol[1])
# verify
print("Verification: ", A.dot(sol[1][:, 0]) - sol[0][0]*sol[1][:, 0])
```

```
Eigen-values:  [0.+1.j 0.-1.j]
Eigen-vectors:
 [[0.70710678+0.j         0.70710678-0.j        ]
 [0.        -0.70710678j 0.        +0.70710678j]]
Verification:  [0.+0.j 0.+0.j]
```

Exercise 1 [2]

[a] **11.** Consider $\boldsymbol{A} = \begin{bmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{bmatrix}$. Find the eigenvalues and accompanying eigenvectors of this matrix, from Gregory and Karney [1969], without using software. *Hint:* The answers can be integers.

Topic

Problem 1

Create a random matrix, with random elements between [-1, 1], and make a histogram for the largest eigenvalue.

Problem 2 [2]

7. (Continuation) A common electrical engineering problem is to calculate currents in an electric circuit. For example, the circuit shown in the figure with R_i (ohms), C_i (microfarads), L (millihenries), and ω (hertz) leads to the system

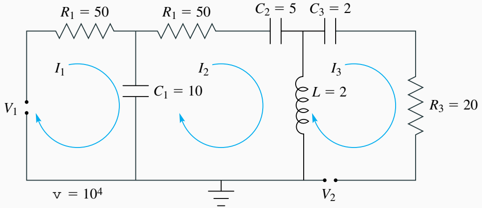
$$\begin{cases} (50-10i)I_1 + (50)I_2 + (50)I_3 = V_1 \\ (10i)I_1 + (10-10i)I_2 + (10-20i)I_3 = 0 \\ -(30i)I_2 + (20-50i)I_3 = -V_2 \end{cases}$$

Select V_1 to be 100 millivolts, and solve two cases:

[a] **a.** The two voltages are in phase; that is, $V_2 = V_1$.

[a] **b.** The second voltage is a quarter of a cycle ahead of the first; that is, $V_2 = iV_1$.

Use the complex arithmetic version of *Naive_Gauss*, and in each case, solve the system for the amplitude (in milliamperes) and the phase (in degrees) for each current I_k. *Hint:* When $I_k = \mathrm{Re}(I_k) + i\,\mathrm{Im}(I_k)$, the amplitude is $|I_k|$, and the phase is $(180°/\pi)\arctan[\mathrm{Im}(I_k)/\mathrm{Re}(I_k)]$. Draw a diagram to show why this is so.



Problem 3 [2]

[a]**4.** The **Hilbert matrix** of order n is defined by $a_{ij} = (i + j - 1)^{-1}$ for $1 \leqq i, j \leqq n$. It is often used for test purposes because of its ill-conditioned nature. Define $b_i = \sum_{j=1}^{n} a_{ij}$. Then the solution of the system of equations $\sum_{j=1}^{n} a_{ij} x_j = b_i$ for $1 \leqq i \leqq n$ is $\boldsymbol{x} = [1, 1, \ldots, 1]^T$. Verify this. Select some values of n in the range $2 \leqq n \leqq 15$, solve the system of equations for $\boldsymbol{x}$ using procedures *Gauss* and *Solve*, and see whether the result is as predicted. Do the case $n = 2$ by hand to see what difficulties occur in the computer.

Thank you

Topic

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