Chapter 2
First-Order Ordinary Differential Equations

First-order ordinary differential equations have some rather special properties, which result for the most part because they can only contain a limited number of terms. In fact, all linear first-order ordinary differential equations can easily be solved. This is in contrast to higher-order ordinary differential equations that become much more difficult to solve when, for example, they contain variable coefficients. Furthermore, some methods exist to solve a pretty large class of nonlinear first-order ordinary differential equations, which is generally not the case for higher-order nonlinear equations.

This chapter considers methods to solve first-order ordinary differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(x,t). \tag{2.1}$$

If the first-order ordinary differential equation that must be solved is not of the form of Equation (2.1) it must be transformed into that form, and in the process care must be taken that solutions are neither gained nor lost. Theorem D.1 gives conditions for existence and uniqueness of solutions to problems involving equations in the form of Equation (2.1). For the purposes of this chapter, note that as long as $f(x,t)$ is infinitely differentiable in both x and t, then the theorem is satisfied. All the equations considered in this chapter have the properties necessary for existence and uniqueness of solutions.

Even if a reader is familiar with ordinary first-order differential equations, this chapter should not be skipped. Detailed coverage of the method of undetermined coefficients and solutions to constant-coefficient linear equations of any order is in this chapter and serves as a basis for material in subsequent chapters.

2.1 Motivational Examples

The first example of a first-order differential equation comes from heat transfer.

B. Goodwine, *Engineering Differential Equations: Theory and Applications*,
DOI 10.1007/978-1-4419-7919-3_2,

Example 2.1. Consider the problem of determining the temperature of an object placed in an oven (or conversely, a refrigerator). If the inside of the oven is at temperature T_a, and is constant, and the initial temperature of the body is $T(0)$, we want to determine $T(t)$.

Although a complete exposition of heat transfer requires an entire course, a couple of relevant concepts can be introduced here. First, temperature can be considered as a measure of the amount of thermal energy that a body contains. Second, heat transfer, then, is a measure of how much energy is transferred between systems in a given amount of time. Let q denote the rate of heat transfer. The units for q are energy per unit time $\mathrm{J/s}$ or watts W.

Considering an energy balance on the body, we have that the rate of change of the internal energy of the body must be equal to the rate of energy transfer into (or out of) the body from the surrounding air. A basic result from heat transfer is that the heat transfer from a surrounding fluid to a body is given by

$$q(t) = hA\left(T_a - T(t)\right), \tag{2.2}$$

where A is the surface area of the body and h is the *convection heat transfer coefficient* which will have units of $\mathrm{W \cdot K/m^2}$. Equation (2.2) should make perfect sense. The rate at which energy is transferred from the body to the fluid, or vice versa is proportional to the difference in their temperatures and the amount of area over which it may occur.

Inasmuch as temperature is a measure of the amount of thermal energy contained in the body, the rate of change of temperature should be proportional to the rate at which energy is transferred into the body. This is true, and in particular,

$$q(t) = \rho V c_p \frac{\mathrm{d}T}{\mathrm{d}t}(t), \tag{2.3}$$

where ρ is the density of the body, V is the volume, and c_p is the *specific heat* of the material, which has units of $\mathrm{J \cdot K/kg}$.

Conservation of energy requires that the rate of heat transfer into the body must equal the rate of change of its internal energy, thus Equations (2.2) and (2.3) must be equal, so we have

$$hA\left(T_a - T(t)\right) = \rho V c_p \dot{T}(t).$$

If we let $\theta(t) = T(t) - T_a$, then

$$-hA\theta(t) = \rho V c \dot{\theta}(t)$$

or

$$\dot{\theta}(t) + \frac{hA}{\rho V c}\theta(t) = 0. \tag{2.4}$$

Usually, this equation is written in the form

$$\dot{\theta}(t) + \frac{1}{RC}\theta(t) = 0, \tag{2.5}$$

where R is the resistance to convective heat transfer and C is called the lumped thermal capacitance.[1] Equation (2.5) is a linear, first-order, ordinary, constant-coefficient, homogeneous differential equation.

The next section outlines how to solve various forms of first-order equations. As it turns out, there are multiple ways to solve Equation (2.5), and in particular, the two different methods from Section 2.3 may be used to solve this problem.

The next examples come from the field of bioengineering, but first we need to consider some basic reaction rate concepts. The *Michaelis–Menton equation* describes many physiological processes: among other things, biological process catalyzed by enzymes and protein facilitated diffusion of substances into or out of cells. The form of the equation is

$$v_o = \frac{v_{\max}[s]}{k_m + [s]} \tag{2.6}$$

where v_o is the reaction rate or uptake rate, $[s]$ is the concentration of some substrate, and $v_{\max}$ and k_m are constants that depend upon the particular process under consideration. A plot of v_o versus $[s]$ for various values of $v_{\max}$ and k_m is illustrated in Figures 2.1 and 2.2.

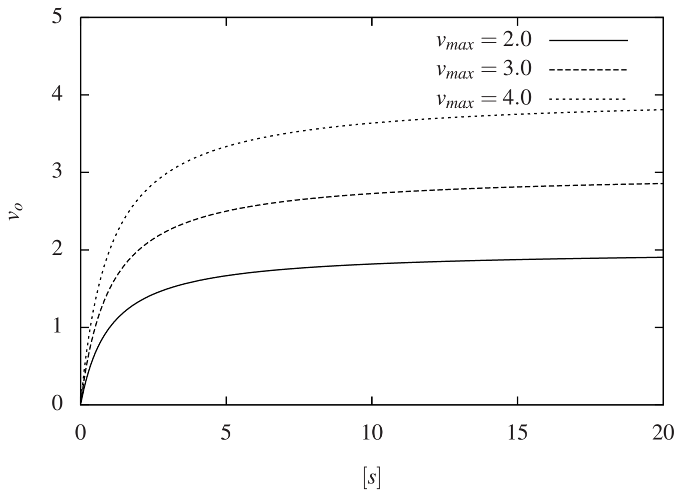

Fig. 2.1 Reaction rate for various $v_{\max}$ and $k_m = 1.0$.

[1] A careful reader, or one with a background in heat transfer, will recognize that fact that when we use $T(t)$ to represent the temperature of the body, it is implicitly assuming that the temperature distribution in the body is uniform. This is intuitively appropriate in some cases, and is rigorously justified by considering the *Boit number*, which is defined as the dimensionless quantity $Bi = hk/L$, where k is the *thermal conductivity* of the body and L is a characteristic length of the body. When $Bi \ll 1$, then the approach taken in this example problem, which is called the *lumped capacitance method*, is a justified approximation. See [25] for a complete exposition.

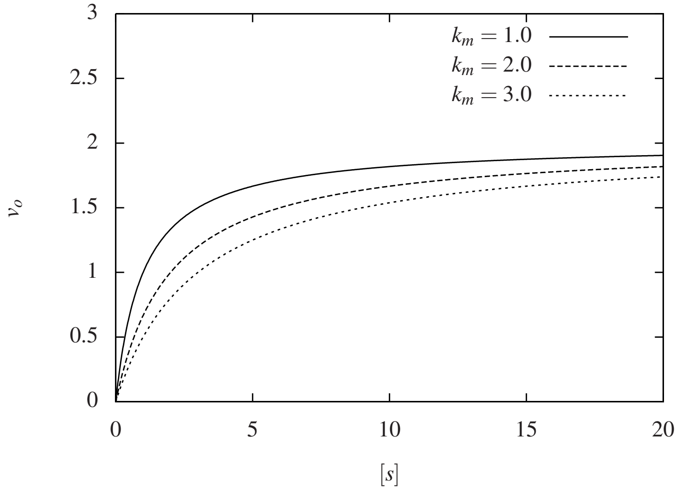

Fig. 2.2 Reaction rate for various k_m and $v_{\max} = 2.0$.

Example 2.2. The rate of uptake of blood plasma glucose into skeletal muscle, the brain, liver, and other organs for oxidation (use for energy) is regulated by hormones such as insulin and facilitated in the different organs by the GLUT family of proteins. Thus if we let g represent the plasma glucose concentration, the change in plasma glucose concentrations due to uptake by, say in, skeletal muscle, is given by

$$\dot{g} = -\frac{v_{\max} g}{k_m + g}$$

or

$$k_m \dot{g} + g\dot{g} + v_{\max} g = 0. \tag{2.7}$$

This is an ordinary, first-order, nonlinear differential equation.

Example 2.3. The rates of metabolism of many drugs are described by Equation (2.6) as well. In some cases, the constant k_m is either very large or very small compared to the blood concentration of the drug so that some simplifications are possible.

For example, alcohol is such that if x represents blood alcohol concentrations, x is always much larger than k_m. In this case the denominator of Equation (2.6) can be approximated by $k_m + x \approx x$, and then the equation describing the blood alcohol concentration as a function of time is

$$\dot{x} = -v_{\max}. \tag{2.8}$$

This is an ordinary, first-order, constant-coefficient, inhomogeneous, linear differential equation.

Example 2.4. For other drugs, cocaine is an illicit example but there are many pharmaceutical examples; metabolism is such that the constant k_m is very large compared to the drug concentration levels. In that case, the denominator of Equation (2.6) can be approximated simply by k_m ($k_m + x \approx k_m$) and the blood drug concentration as a function of time is given by

$$k_m \dot{x} = -v_{\max} x \tag{2.9}$$

which is an ordinary, first-order, constant-coefficient, homogeneous, linear differential equation.

2.2 Homogeneous Constant-Coefficient Linear First-Order Ordinary Differential Equations

Because it is the case that the coefficients of the dependent variable terms in engineering differential equations are often parameters that describe the physical properties of a system, and it is also often the case that such parameters are constant (mass, thermal capacitance, etc.), it is thus often the case that differential equations in engineering have constant coefficients. This section presents a method to solve ordinary, first-order, constant-coefficient, linear differential equations.

The following fact regarding ordinary, constant-coefficient, linear, homogeneous differential equations of any order and worthy of repeated emphasis.

If you remember anything from differential equations, remember the following: ordinary, linear, constant-coefficient, homogeneous differential equations of any order have exponential solutions. To emphasize the fact, let us make it a theorem.

Theorem 2.1. *Ordinary, linear, constant-coefficient, homogeneous differential equations with dependent variable x and independent variable t have solutions of the form $x = c\mathrm{e}^{\lambda t}$ where c and λ are constants.*

Proof. Consider an nth-order, ordinary, linear, constant-coefficient, homogeneous differential equation of the form

$$\alpha_n \frac{\mathrm{d}^n x}{\mathrm{d}t^n} + \alpha_{n-1} \frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}} + \cdots + \alpha_1 \frac{\mathrm{d}x}{\mathrm{d}t} + \alpha_0 x = 0. \tag{2.10}$$

To verify the form of the solution, substitute $x = c\mathrm{e}^{\lambda t}$ into Equation (2.10):

$$\alpha_n \lambda^n c\mathrm{e}^{\lambda t} + \alpha_{n-1} \lambda^{n-1} c\mathrm{e}^{\lambda t} + \cdots + \alpha_1 \lambda c\mathrm{e}^{\lambda t} + \alpha_0 \mathrm{e}^{\lambda t} = 0.$$

Note that $c = 0$ results in $x(t) = 0$, which is a solution to Equation (2.10), but only satisfies the initial value problem where $x(0) = 0$. For the case where $c \neq 0$, because $c\mathrm{e}^{\lambda t}$ is never zero, it is legitimate to divide each side of the equation by it which gives

$$\alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0 = 0 \tag{2.11}$$

which is an nth-order polynomial in λ. Because, by the fundamental theorem of algebra, Equation (2.11) has n solutions, there may be, in fact, up to n different solutions of the form $x = e^{\lambda t}$. □

Remark 2.1. The fact bears repeating: ordinary, linear, constant-coefficient, homogeneous differential equations of any order have exponential solutions.

Armed with this knowledge, we now consider solutions to ordinary, first-order, constant-coefficient, linear differential equations. This provides the general solution to Equation (2.9) (because it is already homogeneous) as well as the homogeneous solution to Equations (2.5) and (2.8). It does nothing for us for Equation (2.7) because it is not linear.

Any ordinary, first-order, homogeneous, linear, constant-coefficient differential equation can be written as

$$\dot{x} + \alpha x = 0.$$

Note that there is no restriction that α may not be zero. Assuming a solution of the form

$$x(t) = c e^{\lambda t}$$

and substituting gives

$$\lambda = -\alpha$$

or

$$x(t) = c e^{-\alpha t}. \tag{2.12}$$

This is the general solution because, by Theorem D.1, the solution is unique for a given initial condition, and the constant c in Equation (2.12) may be used to satisfy any initial condition.

Example 2.5. Returning to Example 2.4, we have a general solution of the form

$$x(t) = c e^{-v_{\max} t / k_m}. \tag{2.13}$$

To determine c, we would have to know the initial blood concentration of the drug. Assuming $x(0) = x_0$, substituting $t = 0$ into Equation (2.13) gives $c = x_0$, so the solution to

$$k_m \dot{x} = -v_{\max} x$$

with $x(0) = x_0$, is

$$x(t) = x_0 e^{-v_{\max} t / k_m}.$$

Remark 2.2. It is worth memorizing that the solution to

$$\dot{x} + \alpha x = 0,$$

where $x(0) = x_0$ is

$$\boxed{x(t) = x_0 e^{-\alpha t}.} \tag{2.14}$$

2.3 Inhomogeneous Constant-Coefficient Linear First-Order Ordinary Differential Equations

Now we consider the same case as in the previous section but where the equation is inhomogeneous. Two solution methods are presented. The first is easier, but only works when the inhomogeneous term is in a particular class of functions, and the second is computationally a bit harder, but will always work. Both approaches require that a homogeneous solution be known, so the first order of business is to determine the homogeneous solution in the form of Equation (2.12) (not in the form of Equation (2.14)) as outlined in the previous section.

2.3.1 Undetermined Coefficients

The idea behind undetermined coefficients is relatively simple, as is illustrated by the following example. The approach has two components. First homogeneous and particular solutions are determined separately and then combined for the solution (this is mathematically justified after the example). Second, a specific form of the particular solution is assumed, which is then substituted into the differential equation which gives rise to equations for some undetermined coefficients in the particular solution.

Example 2.6. Solve

$$\dot{x} + 3x = \sin 2t \tag{2.15}$$

where $x(0) = 1$. This is an ordinary, first-order, linear, constant-coefficient, inhomogeneous differential equation. From Equation (2.12), the homogeneous solution (the solution to $\dot{x} + 3x = 0$) is

$$x_h(t) = c\mathrm{e}^{-3t},$$

where c is an arbitrary real number.

To determine the particular solution, consider the following logic. We seek a function $x(t)$ such that if we take its derivative and add it to three times itself we obtain the function $\sin 2t$. A moment's reflection results in the conclusion that the only sorts of functions that can be combined with their derivative to obtain a sine function are sines and cosines that are a function of the same argument. So, it is logical to assume that the particular solution is of the form

$$x_p(t) = c_1 \cos 2t + c_2 \sin 2t,$$

where c_1 and c_2 are coefficients that are yet to be determined, that is, the undetermined coefficients. The manner in which to compute the undetermined coefficients should be obvious: substitute x_p into the differential equation to see if equations for c_1 and c_2 can be derived. So, because $\dot{x}_p(t) = -2c_1 \sin 2t + 2c_2 \cos 2t$, and substituting gives

$$\begin{aligned}\dot{x}+3x &= (-2c_1\sin 2t+2c_2\cos 2t)+3\,(c_1\cos 2t+c_2\sin 2t)\\ &= (2c_2+3c_1)\cos 2t+(-2c_1+3c_2)\sin 2t\\ &= \sin 2t,\end{aligned}$$

where the last $\sin 2t$ term is the inhomogeneous term from Equation (2.15). The second and third lines of the above equation must be true for all time, therefore

$$\begin{aligned}3c_1+2c_2 &= 0\\ -2c_1+3c_2 &= 1,\end{aligned}$$

which gives $c_1 = -2/13$ and $c_2 = 3/13$, so the particular solution is

$$x_p(t) = -\frac{2}{13}\cos 2t+\frac{3}{13}\sin 2t.$$

The final task is to ensure that the initial condition is satisfied; that is, $x(0) = 1$. Note the following facts.

1. The particular solution satisfies Equation (2.15) but does not satisfy the initial condition.
2. The homogeneous solution does not satisfy the differential equation in Equation (2.15), but does have a coefficient that has not yet been specified which perhaps may be used in some way to satisfy the initial condition.

Now observe that inasmuch as x_h is a homogeneous solution, by definition when it is substituted into Equation (2.15) the result will be zero. So, *because the equation is linear*, it may be added to the particular solution and the sum will still satisfy the differential equation. In particular, using $x = x_h + x_p$ and substituting gives

$$\begin{aligned}\dot{x}+3x &= (\dot{x}_h+\dot{x}_p)+3\,(x_h+x_p)\\ &= (\dot{x}_h+3x_h)+(\dot{x}_p+3x_p)\\ &= 0+(\dot{x}_p+3x_p)\\ &= \sin 2t.\end{aligned}$$

Because $x = x_h + x_p$ satisfies Equation (2.15) and also contains a coefficient that has not yet been specified (the c in x_h), evaluating $x(0)$ and setting it equal to the initial condition gives an equation for c. So,

$$\begin{aligned}x(0) &= x_h(0)+x_p(0)\\ &= c-\frac{2}{13}.\end{aligned}$$

The initial condition was $x(0) = 1$, thus $c = 15/13$ and the solution to the differential equation is

$$x(t) = \frac{15}{13}\mathrm{e}^{-3t}-\frac{2}{13}\cos 2t+\frac{3}{13}\sin 2t.$$

At first glance, the main idea behind the undetermined coefficients approach may seem to be simply educated guesswork. However, the method is actually guaranteed to work if the right conditions are met. Insight into the method is obtained by observing that certain functions have only a finite number of linearly independent derivatives.[2]

Example 2.7. Returning to Example 2.6, we computed that if

$$x(t) = c_1 \cos 2t + c_2 \sin 2t,$$

then

$$\dot{x}(t) + 3x(t) = (3c_1 + 2c_2)\cos 2t + (-2c_1 + 3c_2)\sin 2t.$$

The critical observation is that we started with a function of the form

$$x(t) = c_1 \cos 2t + c_2 \sin 2t,$$

and after substituting it into the differential equation obtained a function of the form

$$x(t) = k_1 \cos 2t + k_2 \sin 2t.$$

Specifically, a linear combination of the function $x(t)$ and its derivative, which results when it is substituted into the differential equation, is exactly the same form as the original function albeit with different coefficients.

As the following theorem shows that if the inhomogeneous term $g(t)$ is such that it only has a finite number of linearly independent derivatives, then, assuming a solution that is a linear combination of $g(t)$ and its derivatives will always lead to a set of equations that will give a solution for the undetermined coefficients. First we need to define what it means for functions to be linearly independent.

Definition 2.1. A set of functions, $\{f_1(t), \ldots, f_n(t)\}$ is *linearly dependent* on an interval $\mathcal{I} = (t_0, t_1)$ if there exists a set of constants, $c_1, \ldots, c_n$ that are not all zero such that

$$c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) = 0, \quad t \in \mathcal{I}. \tag{2.16}$$

If the functions are not linearly dependent, then they are *linearly independent*.

A necessary condition for linear dependence is easy to construct. Differentiating Equation (2.16) $n-1$ times gives the system of algebraic equations

---

[2] By specifying "only a finite number" of linearly independent derivatives, we mean that the largest set of derivatives of the function that is linearly independent does not contain an infinite number of elements.

$$
\begin{aligned}
c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) &= 0 \\
c_1 \frac{\mathrm{d}f_1}{\mathrm{d}t}(t) + c_2 \frac{\mathrm{d}f_2}{\mathrm{d}t}(t) + \cdots + c_n \frac{\mathrm{d}f_n}{\mathrm{d}t}(t) &= 0 \\
c_1 \frac{\mathrm{d}^2 f_1}{\mathrm{d}t^2}(t) + c_2 \frac{\mathrm{d}^2 f_2}{\mathrm{d}t^2}(t) + \cdots + c_n \frac{\mathrm{d}^2 f_n}{\mathrm{d}t^2}(t) &= 0 \\
\vdots &= \vdots \\
c_1 \frac{\mathrm{d}^{n-1} f_1}{\mathrm{d}t^{n-1}}(t) + c_2 \frac{\mathrm{d}^{n-1} f_2}{\mathrm{d}t^{n-1}}(t) + \cdots + c_n \frac{\mathrm{d}^{n-1} f_n}{\mathrm{d}t^{n-1}}(t) &= 0
\end{aligned}
$$

which may be written as

$$
\begin{bmatrix}
f_1(t) & f_2(t) & \cdots & f_n(t) \\
\frac{\mathrm{d}f_1}{\mathrm{d}t}(t) & \frac{\mathrm{d}f_2}{\mathrm{d}t}(t) & \cdots & \frac{\mathrm{d}f_n}{\mathrm{d}t}(t) \\
\frac{\mathrm{d}^2 f_1}{\mathrm{d}t^2}(t) & \frac{\mathrm{d}^2 f_2}{\mathrm{d}t^2}(t) & \cdots & \frac{\mathrm{d}^2 f_n}{\mathrm{d}t^2}(t) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\mathrm{d}^{n-1} f_1}{\mathrm{d}t^{n-1}}(t) & \frac{\mathrm{d}^{n-1} f_2}{\mathrm{d}t^{n-1}}(t) & \cdots & \frac{\mathrm{d}^{n-1} f_n}{\mathrm{d}t^{n-1}}(t)
\end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
$$

From basic linear algebra, in order for this to have a nonzero solution, the determinant of the matrix must be zero. Hence, if the determinant is nonzero, then the set of functions is not linearly dependent; that is, the set of functions is linearly independent. This determinant is called the *Wronskian*.

Definition 2.2. Given n functions, $f_1(t), f_2(t), \ldots, f_n(t)$ define the *Wronskian*, W as the following determinant

$$
W(f_1, f_2, \ldots, f_n)(t) =
\begin{vmatrix}
f_1(t) & f_2(t) & \cdots & f_n(t) \\
\frac{\mathrm{d}f_1}{\mathrm{d}t}(t) & \frac{\mathrm{d}f_2}{\mathrm{d}t}(t) & \cdots & \frac{\mathrm{d}f_n}{\mathrm{d}t}(t) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\mathrm{d}^{n-1} f_1}{\mathrm{d}t^{n-1}}(t) & \frac{\mathrm{d}^{n-1} f_2}{\mathrm{d}t^{n-1}}(t) & \cdots & \frac{\mathrm{d}^{n-1} f_n}{\mathrm{d}t^{n-1}}(t)
\end{vmatrix}.
$$

Be careful about the logic. If the Wronskian is nonzero, then the set of functions is linearly independent. If the Wronskian is zero, it does not necessarily mean that the set of functions is linearly dependent. As is emphasized in the first few examples, the main utility of knowing that a set of functions is linearly independent is that it justifies equating coefficients of the functions on either side of an equality.

Theorem 2.2. *An nth-order, linear, ordinary, constant-coefficient, inhomogeneous differential equation of the form*

$$
\alpha_n \frac{\mathrm{d}^n x}{\mathrm{d}t^n}(t) + \alpha_{n-1} \frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}}(t) + \cdots + \alpha_1 \frac{\mathrm{d}x}{\mathrm{d}t}(t) + \alpha_0 x(t) = g(t), \tag{2.17}
$$

where $g(t)$ has only a finite number of linearly independent derivatives has a particular solution of the form

$$x_p(t) = c_0 g(t) + c_1 \frac{\mathrm{d}g}{\mathrm{d}t}(t) + c_2 \frac{\mathrm{d}^2 g}{\mathrm{d}t^2}(t) + \cdots + c_m \frac{\mathrm{d}^m g}{\mathrm{d}t^m}(t)$$

where m is the maximum number of linearly independent derivatives and there does not exist a combination of coefficients, c_i, where not all the c_i are zero such that $x_p(t)$ is a homogeneous solution to Equation (2.17).

Proof. Consider the vector space

$$V = \left\{ c_0 g(t) + c_1 \frac{\mathrm{d}g}{\mathrm{d}t}(t) + \cdots + c_m \frac{\mathrm{d}^m g}{\mathrm{d}t^m}(t) \middle| c_i \in \mathbb{R}, i \in \{1, \ldots, m\} \right\}.$$

The functions g, $\mathrm{d}g/\mathrm{d}t$, …, $\mathrm{d}^m g/\mathrm{d}t^m$ are the basis elements for V and the operator $\mathrm{d}/\mathrm{d}t$ is a linear operator on V. Consequently

$$\mathrm{D} = \alpha_0 + \alpha_1 \frac{\mathrm{d}}{\mathrm{d}t} + \cdots + \alpha_m \frac{\mathrm{d}^m}{\mathrm{d}t^m}$$

is also a linear operator on V. The null space of D contains only the zero function because by assumption no element of V is a homogeneous solution to Equation (2.17). This implies that the set of functions $\mathrm{D}\,g, \mathrm{D}\,(\mathrm{d}g/\mathrm{d}t), \ldots, \mathrm{D}\,(\mathrm{d}^m g/\mathrm{d}t^m)$ also is a basis for V. Hence,

$$\mathrm{D}\,x_p(t) = c_0\,\mathrm{D}\,g(t) + \cdots + c_m\,\mathrm{D}\,\frac{\mathrm{d}^m g}{\mathrm{d}t^m}(t) = g(t)$$

is satisfied by a unique set of coefficients. □

In general, computing the derivatives of $g(t)$ and at each time checking whether the set is linearly independent is the general approach. However, the number of functions that are commonly encountered in engineering that have this property is somewhat limited and the functions along with what to assume for a particular solution are listed in Table 2.1.

If the inhomogeneous term, $g(t)$ is	Then assume for $x_p(t)$
$\hat{c}\cos\omega t$	$c_1\cos\omega t + c_2\sin\omega t$
$\hat{c}\sin\omega t$	$c_1\cos\omega t + c_2\sin\omega t$
$\hat{c}\mathrm{e}^{\lambda t}$	$c\mathrm{e}^{\lambda t}$
$\hat{c}_n t^n + \cdots + \hat{c}_1 t + \hat{c}_0$	$c_n t^n + \cdots + c_1 t + c_0$
sum of above terms	sum of corresponding terms
product of above terms	product of corresponding terms

Table 2.1 Forms to assume for x_p depending on the inhomogeneous term $g(t)$

Example 2.8. Determine the particular solution to the ordinary, first-order, linear, constant-coefficient, inhomogeneous differential equation

$$3\dot{x}+6x=9\mathrm{e}^{t}.$$

Because $g(t)=\mathrm{e}^t$ does not have any linearly independent derivatives, assume $x_p(t)=c_0\mathrm{e}^t$. Then $\dot{x}_p(t)=c_1\mathrm{e}^t$ and substituting gives

$$3c_1\mathrm{e}^t+6c_1\mathrm{e}^t=9\mathrm{e}^t \quad\Longrightarrow\quad c_1=1.$$

Hence

$$x_p(t)=\mathrm{e}^t.$$

Example 2.9. Determine the particular solution to

$$\dot{x}-x=\cos t+\mathrm{e}^{-t}. \tag{2.18}$$

Computing the first few derivatives of $g(t)$ gives

$$\begin{aligned}
g(t)&=\cos t+\mathrm{e}^{-t}\\
\frac{\mathrm{d}g}{\mathrm{d}t}(t)&=-\sin t-\mathrm{e}^{-t}\\
\frac{\mathrm{d}^2g}{\mathrm{d}t^2}(t)&=-\cos t+\mathrm{e}^{-t}\\
\frac{\mathrm{d}^3g}{\mathrm{d}t^3}(t)&=\sin t-\mathrm{e}^{-t}.
\end{aligned}$$

Either by computing the Wronskian or by simply observing that the third derivative is equal to -1 times the sum of $g(t)$ and its first two derivatives, shows that the particular solution is of the form

$$x_p(t)=c_0\left(\cos t+\mathrm{e}^{-t}\right)+c_1\left(-\sin t-\mathrm{e}^{-t}\right)+c_2\left(-\cos t+\mathrm{e}^{-t}\right)$$

as long as there is no set of constants which make it a homogeneous solution. Because the homogeneous solution is $x_h(t)=\mathrm{e}^t$, no combination of coefficients makes $x_p(t)$ a homogeneous solution.

Substituting x_p into Equation (2.18) gives

$$\begin{aligned}
&\left[c_0\left(-\sin t-\mathrm{e}^{-t}\right)+c_1\left(-\cos t+\mathrm{e}^{-t}\right)+c_2\left(\sin t-\mathrm{e}^{-t}\right)\right]\\
&\qquad-\left[c_0\left(\cos t+\mathrm{e}^{-t}\right)+c_1\left(-\sin t-\mathrm{e}^{-t}\right)+c_2\left(-\cos t+\mathrm{e}^{-t}\right)\right]=\cos t+\mathrm{e}^{-t},
\end{aligned}$$

where the first term in square braces is $\dot{x}_p$ and the second term in square braces is x_p. As pointed out, the third term in the $\dot{x}_p$ term can be expressed as a sum of the other terms, so

$$\begin{aligned}
\cos t+\mathrm{e}^{-t}=&\left[c_0\left(-\sin t-\mathrm{e}^{-t}\right)+c_1\left(-\cos t+\mathrm{e}^{-t}\right)\right.\\
&\left.+c_2\left(-\cos t-\mathrm{e}^{-t}+\sin t+\mathrm{e}^{-t}+\cos t-\mathrm{e}^{-t}\right)\right]\\
&-\left[c_0\left(\cos t+\mathrm{e}^{-t}\right)+c_1\left(-\sin t-\mathrm{e}^{-t}\right)+c_2\left(-\cos t+\mathrm{e}^{-t}\right)\right].
\end{aligned}$$

Equating the coefficients of $g(t)$, $\dot{g}(t)$, and $\ddot{g}(t)$ gives the system of equations

$$\begin{aligned} -c_2 - c_0 &= 1 \\ c_0 - c_2 - c_1 &= 0 \\ c_1 - c_2 - c_2 &= 0, \end{aligned}$$

which gives

$$c_0 = -\frac{3}{4}, \quad c_1 = -\frac{1}{2}, \quad c_2 = -\frac{1}{4}$$

or

$$x_p(t) = -\frac{3}{4}\left(\cos t + \mathrm{e}^{-t}\right) - \frac{1}{2}\left(-\sin t - \mathrm{e}^{-t}\right) - \frac{1}{4}\left(-\cos t + \mathrm{e}^{-t}\right).$$

2.3.2 Complication: When the Assumed Solution Contains a Homogeneous Solution

By the proof of Theorem 2.2, the method is not guaranteed to work if the function $g(t)$ is not in the range of the operator D. This can happen if D is not full rank, which will be the case if there exists a linear combination of $g(t)$ and its derivatives that is a homogeneous solution of Equation (2.17). If there is a homogeneous solution of this form, then the question is whether $g(t)$ is in the range of D, and if it is not, what can be done about it? First, consider an example.

Example 2.10. Use the method of undetermined coefficients to determine the general solution to

$$\dot{x} + 3x = \mathrm{e}^{-3t} + \sin 2t.$$

Referring to Table 2.1, it is logical to assume

$$x_p(t) = c_1\mathrm{e}^{-3t} + c_2 \sin 2t + c_3 \cos 2t.$$

Differentiating and substituting gives

$$\left(-3c_1\mathrm{e}^{-3t} + 2c_2\cos 2t - 2c_3 \sin 2t\right) + 3\left(c_1\mathrm{e}^{-3t} + c_2 \sin 2t + c_3 \cos 2t\right) = \mathrm{e}^{-3t} + \sin 2t.$$

Equating coefficients of $\mathrm{e}^{-3t}, \sin 2t$ and $\cos 2t$, respectively, gives the following set of equations

$$\begin{aligned} -3c_1 + 3c_1 &= 1 \\ -2c_3 + 3c_2 &= 1 \\ 2c_2 + 3c_3 &= 0. \end{aligned}$$

Note that the first equation is $0 = 1$; that is, there does not exist any c_1 that will satisfy the equations, and hence the assumed form for the particular solution is incorrect.

This problem is due to the fact that e^{-3t} is, in addition to being a component of the inhomogeneous term, a homogeneous solution to the differential equation. When it is substituted into the differential equation it must evaluate to zero, by definition.

To determine a method to deal with this case, first consider

$$\dot{x} + \alpha x = e^{-\alpha t}. \tag{2.19}$$

Table 2.1 would indicate to choose $x_p(t) = ce^{-\alpha t}$; however, this is also the homogeneous solution. Using a technique that actually foreshadows the method of variation of parameters presented subsequently, assume a particular solution of the form

$$x_p(t) = \mu(t)e^{-\alpha t},$$

substitute into Equation (2.19) and use the result to (one hopes) determine $\mu(t)$.[3] Differentiating $x_p(t)$ and substituting gives

$$\left(\dot{\mu}(t)e^{-\alpha t} - \alpha\mu(t)e^{-\alpha t}\right) + \alpha\mu(t)e^{-\alpha t} = e^{-\alpha t},$$

which simplifies to

$$\dot{\mu}(t) = 1$$

or

$$\mu(t) = t + c.$$

Hence,

$$x_p(t) = (t + c)e^{-\alpha t}.$$

Note that inasmuch as the term $ce^{-\alpha t}$ is actually a homogeneous solution, it is not necessary to add it to the particular solution at this stage in the process of determining the solution as it will be added to it subsequently anyway. So the simplest form for the particular solution is

$$x_p(t) = te^{-\alpha t}.$$

Hence, when the assumed form of the particular solution is also the homogeneous solution to the differential equation, the approach is to multiply the assumed form by the independent variable.

Example 2.11. Continuing from Example 2.10, instead of assuming

$$x_p(t) = c_1e^{-3t} + c_2\sin 2t + c_3\cos 2t$$

[3] Assuming a solution of the form of an unknown function times a homogeneous solution is somewhat common and should make some sense that it perhaps might work. Given the relationship between a homogeneous solution and the differential equation, it is definitely plausible that it could be used in combination with other functions to generate a different type of solution.

assume

$$x_p(t) = c_1 t\mathrm{e}^{-3t} + c_2 \sin 2t + c_3 \cos 2t.$$

Differentiating and substituting gives

$$\begin{aligned}&\left(c_1\mathrm{e}^{-3t} - 3c_1 t\mathrm{e}^{-3t} + 2c_2\cos 2t - 2c_3 \sin 2t\right)\\ &+3\left(c_1 t\mathrm{e}^{-3t} + c_2 \sin 2t + c_3\cos 2t\right) = \mathrm{e}^{-3t} + \sin 2t.\end{aligned}$$

Collecting terms now gives

$$\begin{aligned} c_1 &= 1\\ -2c_3 + 3c_2 &= 1\\ 2c_2 + 3c_3 &= 0\end{aligned}$$

which has the solution

$$c_1 = 1, \quad c_2 = \frac{3}{13}, \quad c_3 = -\frac{2}{13},$$

and hence

$$x_p(t) = t\mathrm{e}^{-3t} + \frac{3}{13}\sin 2t - \frac{2}{13}\cos 2t,$$

and the general solution is

$$x(t) = x_h(t) + x_p(t) = c\mathrm{e}^{-3t} + t\mathrm{e}^{-3t} + \frac{3}{13}\sin 2t - \frac{2}{13}\cos 2t.$$

The next example illustrates that it may be the case that the homogeneous solution is "hidden" in some linear combination of $g(t)$ and some of its derivatives.

Example 2.12. Consider

$$\dot{x} + x = t\mathrm{e}^{-t}. \tag{2.20}$$

Computing the first two derivatives of $g(t)$ gives

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \mathrm{e}^{-t} - t\mathrm{e}^{-t}, \quad \frac{\mathrm{d}^2 g}{\mathrm{d}t^2} = -2\mathrm{e}^{-t} + t\mathrm{e}^{-t}.$$

By inspection, we may think that the second derivative is possibly a linear combination of the first two. In fact it is because

$$-1g(t) - 2\frac{\mathrm{d}g}{\mathrm{d}t}(t) = -t\mathrm{e}^{-t} - 2\mathrm{e}^{-t} + 2t\mathrm{e}^{-t} = -2\mathrm{e}^{-t} + t\mathrm{e}^{-t}.$$

The fact that the set $\{g, \mathrm{d}g/\mathrm{d}t\}$ is linearly independent is verified by the Wronskian

$$\begin{vmatrix} t\mathrm{e}^{-t} & \mathrm{e}^{-t} - t\mathrm{e}^{-t}\\ \mathrm{e}^{-t} - t\mathrm{e}^{-t} & -2\mathrm{e}^{-t} + t\mathrm{e}^{-t}\end{vmatrix} = -\mathrm{e}^{-2t} \neq 0.$$

Assuming a solution of the form

$$x_p(t) = c_0 t\mathrm{e}^{-t} + c_1\left(\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right)$$

gives

$$\dot{x}_p(t) = c_0\left(\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right) + c_1\left(-2\mathrm{e}^{-t} + t\mathrm{e}^{-t}\right).$$

Substituting into Equation (2.20) gives

$$\left[c_0\left(\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right) + c_1\left(-2\mathrm{e}^{-t} + t\mathrm{e}^{-t}\right)\right] + \left[c_0 t\mathrm{e}^{-t} + c_1\left(\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right)\right] = (c_0 - c_1)\,\mathrm{e}^{-t},$$

and hence there is no combination of c_0 and c_1 that will allow x_p to satisfy the differential equation because we want this to equal $g(t) = t\mathrm{e}^{-t}$. If, instead, the assumed form of the particular solution is multiplied by t,

$$x_p(t) = t\left(c_0 t\mathrm{e}^{-t} + c_1\left(\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right)\right),$$

then

$$\dot{x}_p(t) = \left(c_0 t\mathrm{e}^{-t} + c_1\left(\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right)\right) + t\left(c_0\left(\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right) + c_1\left(-2\mathrm{e}^{-t} + t\mathrm{e}^{-t}\right)\right).$$

Substituting into the differential equation and simplifying gives

$$\begin{aligned}\dot{x}_p + x_p = &\left(c_0 t\mathrm{e}^{-t} + c_1\left(\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right)\right)\\ &+ t\left(c_0\left(\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right) + c_1\left(-2\mathrm{e}^{-t} + t\mathrm{e}^{-t}\right)\right)\\ &+ t\left(c_0 t\mathrm{e}^{-t} + c_1\left(\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right)\right),\end{aligned}$$

which, after a tedious bit of work, simplifies to

$$\dot{x}_p + x_p = 2\,(c_0 - c_1)\,t\mathrm{e}^{-t} + c_1\mathrm{e}^{-t}.$$

Equating coefficients with

$$g(t) = t\mathrm{e}^{-t}$$

gives

$$c_0 = \frac{1}{2}, \quad c_1 = 0.$$

Hence

$$x_p(t) = \frac{1}{2}t^2\mathrm{e}^{-t}.$$

One way to think of multiplying by t if the inhomogeneous term is a homogeneous solution is that because of the way the product rule for differentiation works, it is "plugging the hole" in the assumed form of the particular solution caused by the homogeneous solution being part of it. It may arise that the inhomogeneous term contains some terms that combine to be a homogeneous solution and some terms that do not. In such a case it would be incorrect to multiply the terms that are not part of the homogeneous solution by t. In such a case there are two approaches.

- It will always work to assume

$$x_p(t) = \left(c_1 g(t) + c_2 \frac{\mathrm{d}g}{\mathrm{d}t}(t) + \cdots c_m \frac{\mathrm{d}^m g}{\mathrm{d}t}(t)\right)$$
$$+ t\left(d_1 g(t) + d_2 \frac{\mathrm{d}g}{\mathrm{d}t}(t) + \cdots d_m \frac{\mathrm{d}^m g}{\mathrm{d}t}(t)\right),$$

even if there is not a homogeneous solution in the first term. In such a case, all the d_i coefficients will be zero. If there are some terms that combine to be homogeneous solutions and some that do not, then some of the c_i and some of the d_i coefficients will be not zero.

- Although the above approach is nice in that it will always work, it is more work to compute all the coefficients. A smarter approach is to try to identify which terms are combining to make a homogeneous solution, and multiply only those by t.

In general, other than being more work, it is not wrong to assume more terms in the particular solution. It will just work out that the coefficients must be zero. If, after substituting an assumed form for the particular solution into the differential equation, it is not possible to determine one or more of the coefficients, it is not possible to solve for them, it generally is due to the fact that they are combining as a homogeneous solution. Clearly, it is advisable to always compute the homogeneous solution first, so that if the homogeneous solution appears explicitly in the assumed form for the particular solution they can be multiplied by the independent variable right away.

The following example illustrates both approaches.

Example 2.13. Determine the general solution to

$$\dot{x} + 3x = \mathrm{e}^{-3t} + \sin 2t.$$

According to Table 2.1, we should assume

$$x_p(t) = c_1 \mathrm{e}^{-3t} + c_2 \sin 2t + c_3 \cos 2t.$$

It should be apparent that this will not work because the exponential is also a homogeneous solution. So, one approach would be to assume

$$x_p(t) = \left(c_1 \mathrm{e}^{-3t} + c_2 \sin 2t + c_3 \cos 2t\right) + t\left(d_1 \mathrm{e}^{-3t} + d_2 \sin 2t + d_3 \cos 2t\right).$$

In this case, it will work out that $d_2 = d_3 = 0$ and c_1 will be arbitrary. Because it is a lot of work to deal with six equations and six coefficients, a more insightful assumption for the particular solution would be

$$x_p(t) = c_1 t \mathrm{e}^{-3t} + c_2 \sin 2t + c_3 \cos 2t,$$

where only the problematic term is multiplied by t.

2.3.3 Variation of Parameters

This method will always work for linear first-order ordinary differential equations. As long as one is willing to evaluate the integrals required, it will yield the solution.

The idea behind the variation of parameters method is that if a homogeneous solution for a differential equation is known, denoted by x_h, then assume a solution of the form $x(t) = \mu(t)x_h(t)$. Substituting the assumed form of the solution into the differential equation will yield an equation for μ that, if it can be solved, will give the solution. Unlike the method for undetermined coefficients, this method will work for a variable-coefficient equation as well, but this section limits the coverage to the constant-coefficient case. Also unlike the case for undetermined coefficients, no special form of the inhomogeneous term is necessary.

Consider the ordinary, first-order, linear, constant-coefficient, inhomogeneous differential equation

$$\dot{x} + \alpha x = g(t),$$

where $x(t_0) = x_0$. From before, $x_h(t) = c\mathrm{e}^{-\alpha t}$. Assume $x(t) = c\mu(t)\mathrm{e}^{-\alpha t}$. Substituting into the differential equation gives

$$c\dot{\mu}(t)\mathrm{e}^{-\alpha t} - c\mu(t)\alpha\mathrm{e}^{-\alpha t} + \alpha c\mu(t)\mathrm{e}^{-\alpha t} = c\dot{\mu}(t)\mathrm{e}^{-\alpha t} = g(t).$$

Hence

$$\dot{\mu}(t) = \frac{1}{c}\mathrm{e}^{\alpha t}g(t)$$

which can be directly integrated. So

$$\mu(t) - \mu(t_0) = \int_{t_0}^{t} \frac{1}{c}\mathrm{e}^{\alpha s}g(s)\mathrm{d}s$$

or

$$\mu(t) = \int_{t_0}^{t} \frac{1}{c}\mathrm{e}^{\alpha s}g(s)\mathrm{d}s + \mu(t_0),$$

where $\mu(t_0)$ is arbitrary. So

$$\begin{aligned} x(t) &= \mu(t)c\mathrm{e}^{-\alpha t} \\ &= c\mathrm{e}^{-\alpha t}\int_{t_0}^{t} \frac{1}{c}\mathrm{e}^{\alpha s}g(s)\mathrm{d}s + \mu(t_0)c\mathrm{e}^{-\alpha t} \\ &= \mathrm{e}^{-\alpha t}\int_{t_0}^{t} \mathrm{e}^{\alpha s}g(s)\mathrm{d}s + c_1\mathrm{e}^{-\alpha t}, \end{aligned}$$

where $c_1 = \mu(t_0)c$. Evaluating $x(t_0)$ gives

$$x(t_0) = \mathrm{e}^{-\alpha t}\int_{t_0}^{t_0} \mathrm{e}^{\alpha s}g(s)\mathrm{d}s + c_1\mathrm{e}^{-\alpha t_0} = c_1\mathrm{e}^{-\alpha t_0} = x_0.$$

Thus $c_1 = x_0\mathrm{e}^{\alpha t_0}$ and

$$\boxed{x(t) = \mathrm{e}^{-\alpha t} \int_{t_0}^{t} \mathrm{e}^{\alpha s} g(s)\mathrm{d}s + x_0 \mathrm{e}^{\alpha t_0} \mathrm{e}^{-\alpha t}.} \tag{2.21}$$

Remark 2.3. If the initial condition were not specified and a general solution were desired, the integral in the above method would become an indefinite integral and a constant of integration would be necessary. It is left as an exercise to prove that the general solution to the ordinary, first-order, linear, constant-coefficient, inhomogeneous differential equation

$$\dot{x} + \alpha x = g(t) \tag{2.22}$$

is

$$\boxed{x(t) = \mathrm{e}^{-\alpha t} \int \mathrm{e}^{\alpha t} g(t)\mathrm{d}t + c\mathrm{e}^{-\alpha t}.} \tag{2.23}$$

2.4 Variable-Coefficient Linear First-Order Ordinary Differential Equations: Variation of Parameters

The same procedure as above may be used in the case of ordinary, first-order, linear, variable-coefficient, differential equations (regardless of whether it is homogeneous or inhomogeneous). Consider the initial value problem

$$\dot{x} + h(t)x = g(t) \tag{2.24}$$

$$x(t_0) = x_0. \tag{2.25}$$

The procedure is the same as before: find a homogeneous solution, $x_h(t)$, assume the solution of the form $x(t) = \mu(t)x_h(t)$, substitute to determine an equation for $\mu(t)$, and if possible, solve for $\mu(t)$. The first task is to determine the homogeneous solution, which is not simply $x_h(t) = c\mathrm{e}^{\lambda t}$ in the case of a variable-coefficient equation.

First consider the corresponding homogeneous equation

$$\frac{\mathrm{d}x_h}{\mathrm{d}t}(t) + h(t)x_h(t) = 0.$$

Rearranging gives

$$\frac{1}{x_h(t)} \frac{\mathrm{d}x_h}{\mathrm{d}t}(t) = -h(t).$$

Integrating each side with respect to t gives

$$\int \frac{1}{x_h(t)} \frac{\mathrm{d}x_h}{\mathrm{d}t}(t)\mathrm{d}t = \int \frac{\mathrm{d}}{\mathrm{d}t} \left(\ln\left(x_h(t)\right)\right) \mathrm{d}t = \ln\left(x_h(t)\right) + c = -\int h(t)\mathrm{d}t.$$

Hence

$$x_h(t) = k\mathrm{e}^{-\int h(t)\mathrm{d}t}, \tag{2.26}$$

where $k = -\mathrm{e}^{-c}$.

Remark 2.4. This procedure to find the homogeneous solution is a special case of the method for separable equations considered subsequently in Section 2.5.1.

Now armed with the homogeneous solution, assume a solution of the form

$$x(t) = \mu(t)x_h(t) = \mu(t)k\mathrm{e}^{-\int h(t)\mathrm{d}t}.$$

Substituting gives

$$\begin{aligned}(\dot{\mu}(t)x_h + \mu(t)\dot{x}_h) + h(t)\,(\mu(t)x_h) &= \left(\dot{\mu}(t)k\mathrm{e}^{-\int h(t)\mathrm{d}t} - \mu(t)h(t)k\mathrm{e}^{-\int h(t)\mathrm{d}t}\right)\\ &\quad + h(t)\left(\mu(t)k\mathrm{e}^{-\int h(t)\mathrm{d}t}\right)\\ &= \dot{\mu}(t)k\mathrm{e}^{-\int h(t)\mathrm{d}t}\\ &= g(t).\end{aligned}$$

Hence

$$\dot{\mu}(t) = \frac{1}{k}g(t)\mathrm{e}^{\int h(t)\mathrm{d}t} \quad \Longrightarrow \quad \mu(t) = \int \left(\frac{1}{k}g(t)\mathrm{e}^{\int h(t)\mathrm{d}t}\right)\mathrm{d}t + c.$$

and

$$x(t) = \left(\int \left(\frac{1}{k}g(t)\mathrm{e}^{\int h(t)\mathrm{d}t}\right)\mathrm{d}t + c\right)\left(k\mathrm{e}^{-\int h(t)\mathrm{d}t}\right)$$

or

$$\boxed{x(t) = \left(\int \left(g(t)\mathrm{e}^{\int h(t)\mathrm{d}t}\right)\mathrm{d}t + c\right)\left(\mathrm{e}^{-\int h(t)\mathrm{d}t}\right).} \tag{2.27}$$

Remark 2.5. Even though the integrals in Equation (2.27) are indefinite, when you are evaluating them you should not include the constants of integration because they were included in the derivation of the solution.

Remark 2.6. Occasionally it is convenient to combine arbitrary constants but not change the name of the variable, as was done in Equation (2.27). The constant k was distributed across both terms in the left side of the equation, so the constant term c is now actually ck; however, because both c and k are arbitrary, it is most convenient just to keep the variable name as c.

At this point it is worth observing that Equation (2.27) is the solution to Equation (2.24). The only possible complication is that sometimes the integrals may not have a closed-form solution, or may simply be difficult to evaluate.

Example 2.14. Determine the general solution to

$$\dot{x} + \frac{3}{t}x = \sin t.$$

This equation is of the form of Equation (2.24), so the general solution is given by Equation (2.27) where $h(t) = 3/t$ and $g(t) = \sin t$. Substituting into the solution gives

$$\begin{aligned}
x(t) &= \left(\int \left(g(t)\mathrm{e}^{\int h(t)\mathrm{d}t}\right)\mathrm{d}t + c\right)\left(\mathrm{e}^{-\int h(t)\mathrm{d}t}\right) \\
&= \left(\int \left((\sin t)\,\mathrm{e}^{\int \frac{3}{t}\mathrm{d}t}\right)\mathrm{d}t + c\right)\left(\mathrm{e}^{-\int \frac{3}{t}\mathrm{d}t}\right) \\
&= \left(\int \left((\sin t)\,\mathrm{e}^{3\ln t}\right)\mathrm{d}t + c\right)\left(\mathrm{e}^{-3\ln t}\right) \\
&= \left(\int t^3 \sin t\,\mathrm{d}t + c\right)\frac{1}{t^3} \\
&= \frac{1}{t^3}\left[3\left(t^2-2\right)\sin t - t\left(t^2-6\right)\cos t + c\right].
\end{aligned}$$

For most people, completing the last step without an integral table or computer program would probably be quite difficult.

2.5 Ordinary First-Order Nonlinear Differential Equations

Unfortunately, it is generally the case that nonlinear differential equations are difficult to solve and often do not even have solutions that can be expressed in terms of elementary functions. In the case of first-order equations, however, there is one case in which a solution may be obtained, and that case is the so-called exact equation. Before presenting the theory and method of exact equations, the next section presents a simplified special case of exact equations, namely, separable equations.

2.5.1 Separable Equations

A notationally simplistic, yet nonetheless useful, description of the idea behind separable equations is that if it is possible to put all the terms that are a function of the dependent variable on one side of the equation and all the terms that are a function of the independent variable on the other side of the equation the equation is separable. In such a case, both sides may be directly integrated.

Example 2.15. Find the general solution to

$$(x+1)\left(t^2+5t+3\right) = x\dot{x}.$$

This may be rearranged as

$$t^2+5t+3 = \frac{x}{x+1}\frac{\mathrm{d}x}{\mathrm{d}t}$$

and each side may be integrated with respect to t

$$\int t^2 + 5t + 3\mathrm{d}t = \int \frac{x(t)}{x(t)+1} \frac{\mathrm{d}x}{\mathrm{d}t}(t)\mathrm{d}t.$$

Recall from calculus the substitution rule for integration, namely,

$$\int_{t_0}^{t} f(x(s)) \frac{\mathrm{d}x}{\mathrm{d}s}(s)\mathrm{d}s = \int_{x(t_0)}^{x(t)} f(x)\mathrm{d}x.$$

Using this fact,

$$\int t^2 + 5t + 3\mathrm{d}t = \int \frac{x(t)}{x(t)+1} \frac{\mathrm{d}x(t)}{\mathrm{d}t}\mathrm{d}t = \int \frac{x}{x+1}\mathrm{d}x,$$

so

$$\frac{t^3}{3} + \frac{5t^2}{2} + 3t = x(t) - \ln(x(t)+1) + c.$$

Note that one problem is that the solution $x(t)$ may be, as is the case in this example, only determined as an implicit function of the dependent variable.

The preceding example was rather precise and in practice the approach is a bit more informal. In words, the simplest way to approach the problem is to notationally treat $\dot{x}$ as $\mathrm{d}x/\mathrm{d}t$ and try to manipulate the equation so that all the x terms are on one side of the equation along with the $\mathrm{d}x$ term and all the t terms are on the other side with the $\mathrm{d}t$ term. This casual use of notational convenience works correctly in this case, however, it is important to recognize that what is actually going on is an integration by substitution on the x side of the equation. Another example illustrates this point and completes the treatment of separable equations. It also illustrates the slight variation in the approach when the problem is an initial value problem rather than finding a general solution, the only difference being that data are now available to make the integrals definite integrals.

Example 2.16. Determine the solution to

$$\dot{x} + \sin(t)x = 0,$$

where $x(1) = 2$. Note this can perhaps be more easily solved by directly using Equation (2.27) with $g(t) = 0$; however, just for the fun of it, this example solves it by recognizing it is separable.

A bit of manipulation gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \sin(t)x = 0 \quad \Longleftrightarrow \quad \frac{\mathrm{d}x}{x} = -\sin(t)\mathrm{d}t,$$

so

$$\int_{x(t_0)}^{x(t)} \frac{1}{x}\mathrm{d}x = -\int_{t_0}^{t} \sin(s)\mathrm{d}s$$

or

$$\int_2^x (t)\frac{1}{x}\mathrm{d}x = \int_1^t \sin(t)\mathrm{d}t \quad \Longleftrightarrow \quad \ln x - \ln 2 = \cos t - \cos 1,$$

which gives, upon taking the exponential of each side

$$x(t) = 2\mathrm{e}^{\cos t - \cos 1}.$$

2.5.2 Exact Equations

Although actually using it is another matter, the idea behind exact equations is actually quite simple. Consider a function $\psi(x(t),t)$ (as usual, t is the independent variable and x is the dependent variable) and consider the level sets of ψ; namely, $\psi(x(t),t) = c$. Differentiating ψ constrained to the level set with respect to t gives

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\partial \psi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \psi}{\partial t} = 0.$$

Note that this is of the form

$$f(x,t)\dot{x} + g(x,t) = 0, \tag{2.28}$$

where f and g are functions of both the independent variable t and the dependent variable x. If a differential equation just so happens to be of the form of Equation (2.28) such that there exists a $\psi(x(t),t)$ such that $\partial\psi/\partial x = f(x,t)$ and $\partial\psi/\partial t = g(x,t)$, then solving Equation (2.28) is simply a matter of determining ψ and setting $\psi(x,t) = c$ for the general solution. The correct value of c is determined from the initial condition in the case of the initial value problem.

Because the order of differentiating the partial derivatives does not matter, that is,

$$\frac{\partial^2\psi}{\partial x \partial t} = \frac{\partial^2\psi}{\partial t \partial x}$$

and because

$$\frac{\partial\psi}{\partial x} = f(x,t), \qquad \frac{\partial\psi}{\partial t} = g(x,t)$$

the following are equivalent

$$\frac{\partial^2\psi}{\partial x \partial t} = \frac{\partial^2\psi}{\partial t \partial x} \qquad \Longleftrightarrow \qquad \frac{\partial f}{\partial t} = \frac{\partial g}{\partial x}.$$

In other words, this proves the following theorem.

Theorem 2.3. *For the ordinary, first-order differential equation*

$$f(x,t)\dot{x} + g(x,t) = 0, \tag{2.29}$$

if

$$\frac{\partial f}{\partial t} = \frac{\partial g}{\partial x}$$

then there exists a function $\psi(x(t),t)$ *such that*

$$\frac{\partial\psi}{\partial x} = f(x,t) \quad \textit{and} \quad \frac{\partial\psi}{\partial t} = g(x,t).$$

The general solution to Equation (2.29) *is given* implicitly *by*

$$\psi(x(t),t) = c.$$

So far, so good, but although the theory is nice and tidy, there are still two practical problems. First, the solution is only given implicitly by ψ. Second, we still need to determine a way to find ψ. The first problem is inherent in the method and is unavoidable. The second problem is addressed subsequently. First, we have an example.

Example 2.17. Consider

$$2x\dot{x} = -2t - 1.$$

In this case $f(x,t) = 2x$ and $g(x,t) = 2t + 1$.

$$\frac{\partial f}{\partial t} = 0, \qquad \frac{\partial g}{\partial x} = 0,$$

the equation is exact.[4] Note that

$$\psi(x,t) = x^2 + t^2 + t$$

is such that

$$\dot{\psi} = 0 \quad \Longleftrightarrow \quad 2x\dot{x} + 2t + 1 = 0,$$

so

$$x^2 + t^2 + t = c$$

gives the solution $x(t)$ implicitly.

Determining $\psi(x,t)$ is actually rather straightforward. Inasmuch as

$$\frac{\partial\psi}{\partial x} = f(x,t)$$

then

$$\psi(x,t) = \int f(x,t)\mathrm{d}x + h(t),$$

and

$$g(x,t) = \frac{\partial\psi}{\partial t} = \frac{\partial}{\partial t}\left(\int f(x,t)\mathrm{d}x + h(t)\right) = \frac{\partial}{\partial t}\left(\int f(x,t)\mathrm{d}x\right) + \dot{h}(t).$$

Thus,

[4] Observe that it is also separable, but that is not the fact we exploit to solve it.

$$h(t) = \int \left(g(x,t) - \frac{\partial}{\partial t} \left(\int f(x,t)\mathrm{d}x \right) \right) \mathrm{d}t$$

and the general solution is given by

$$\boxed{\psi(x,t) = \int f(x,t)\mathrm{d}x + \int \left(g(x,t) - \frac{\partial}{\partial t} \left(\int f(x,t)\mathrm{d}x \right) \right) \mathrm{d}t = c.}$$

2.5.3 Integrating Factors

From the preceding section it may seem that whether a first-order differential equation is exact is simply a matter of luck. In fact, it is possible to convert any first-order equation of the form

$$f(x,t)\dot{x} + g(x,t) = 0 \tag{2.30}$$

that has a solution of the form

$$\psi(x,t) = c$$

into one that is exact. Unfortunately, doing this generally involves solving a partial differential equation. To see this, if there exists a solution to Equation (2.30) of the form $\psi(x,t) = c$, then, as before

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = 0 \quad \Longrightarrow \quad \frac{\partial \psi}{\partial x}\dot{x} + \frac{\partial \psi}{\partial t} = 0. \tag{2.31}$$

In Section 2.5.2 we simply equated $\partial\psi/\partial x$ with f and $\partial\psi/\partial t$ with g, and used the fact that mixed partials were equal as the basis for Theorem 2.3. But this is too restrictive because we could multiply Equation (2.30) by some function $\mu(x,t)$, and as long as it is not zero and is defined, then the solution $x(t)$ is the same. To determine such a function, $\mu(x,t)$, solve both Equation (2.30) and the equation on the right in Equation (2.31) for $\dot{x}$, because x is what ultimately interests us. Doing so and equating them gives

$$\frac{g}{f} = \frac{\frac{\partial \psi}{\partial t}}{\frac{\partial \psi}{\partial x}}$$

or

$$\frac{1}{g(x,t)} \frac{\partial \psi}{\partial t}(x,t) = \frac{1}{f(x,t)} \frac{\partial \psi}{\partial x}(x,t).$$

If we set

$$\mu(x,t) = \frac{1}{g(x,t)} \frac{\partial \psi}{\partial t}(x,t) = \frac{1}{f(x,t)} \frac{\partial \psi}{\partial x}(x,t),$$

then

$$\frac{\partial \psi}{\partial x}(x,t) = \mu(x,t)f(x,t), \qquad \frac{\partial \psi}{\partial t}(x,t) = \mu(x,t)g(x,t),$$

which is exactly what is needed for

$$\mu(x,t)f(x,t)\dot{x}(t) + \mu(x,t)g(x,t) = 0 \tag{2.32}$$

to be exact. So, even if Equation (2.30) is not exact, Equation (2.32) will be as long as we can find $\mu(x,t)$.

These equations were based on computations using the solution $\psi(x,t)$, which is not known. To find an equation for $\mu(x,t)$, expand the equation it must satisfy from Theorem 2.3, which is

$$\frac{\partial(\mu f)}{\partial t} - \frac{\partial(\mu g)}{\partial x} = 0,$$

which, expanding using the product rule gives the partial differential equation for μ

$$f(x,t)\frac{\partial \mu}{\partial t}(x,t) + \mu(x,t)\frac{\partial f}{\partial t}(x,t) - g(x,t)\frac{\partial \mu}{\partial x}(x,t) - \mu(x,t)\frac{\partial g}{\partial x}(x,t) = 0. \tag{2.33}$$

This, unfortunately, is not easy to solve, except in certain special cases.

The following example illustrates the fact that an integrating factor works to make an equation that is not exact into one that is exact. It does not show, however, how to find the integrating factor.

Example 2.18. Consider

$$\left(\frac{1+\sin t}{x+1}\right)\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x\cos t}{x+1} = 0. \tag{2.34}$$

This is not exact because

$$\frac{\partial f}{\partial t} = \frac{\cos t}{x+1} \neq \frac{\partial g}{\partial x} = \frac{\cos t}{(x+1)^2}.$$

However, if we multiply Equation (2.34) by $\mu = x+1$, then

$$(1+\sin t)\frac{\mathrm{d}x}{\mathrm{d}t} + x\cos t = 0$$

is exact because

$$\frac{\partial}{\partial t}(1+\sin t) = \cos t$$

and

$$\frac{\partial}{\partial x}(x\cos t) = \cos t.$$

Doing the necessary computations to find the solution gives

$$x\sin t + x = c.$$

Because finding the integrating factor involves solving a partial differential equation, most approaches depend on special cases or iterative guesswork. A good review of the special cases, such as when the integrating factor only depends on x or t but not both, and how to exploit them are given in [44].

2.6 Summary

Ordinary first-order differential equations are solved using the following methods.

- If the equation is linear, constant-coefficient, and homogeneous, then assuming a solution of the form $x(t) = ce^{\lambda t}$ is probably the easiest method.
- If the equation is linear, variable-coefficient, and homogeneous, then using Equation (2.26) is probably the easiest method.
- If the equation is linear, constant-coefficient, and inhomogeneous with an inhomogeneous term of the form given in Table 2.1, then the method of undetermined coefficients outlined in Section 2.3.1 is probably the easiest.
- If the equation is linear, constant-coefficient, and inhomogeneous the method of variation of parameters with a solution given by Equation (2.23) will work. If the inhomogeneous term is not given in Table 2.1 then this is probably the easiest method.
- If the equation is linear, variable-coefficient, and inhomogeneous the method of variation of parameters with a solution given by Equation (2.27) will work.
- If the equation is nonlinear, first check if it is separable; if it is not, then check if it is exact. If it is not exact, attempt to determine an integrating factor.

2.7 Exercises

2.1. Based on Theorem D.1, which of the following differential equations are guaranteed to have solutions that exist and are unique?

1. $\dot{x} = x$ where $x(0) = 0$.
2. $\dot{x}^2 = x$ where $x(0) = 0$.
3. $\dot{x} = \begin{cases} -1, & x \geq 0, \\ 1, & x < 0, \end{cases}$
   where $x(0) = 0$.
4. $\dot{x} = \begin{cases} -1, & t \geq 5, \\ 1, & t < 5, \end{cases}$
   where $x(0) = 0$.
5. $\dot{x} = x^2$ where $x(0) = 0$.
6. $\dot{x} = x^{1/2}$ where $x(0) = 0$.
7. $\dot{x} = -|x|$ where $x(0) = 0$.
8. $\dot{x} = \sqrt{x^2 + 9}$ where $x(0) = 0$.

2.2. Determine the solution to $\dot{x} = \alpha x$ where $x(0) = 1$. On the same graph, sketch the solution for $\alpha = -1$, $\alpha = 0$, and $\alpha = 1$.

2.3. In dead organic matter, the C^{14} isotope decays at a rate proportional to the amount of it that is present. Furthermore, it takes approximately 5600 years for half of the original amount present to decay.

1. If $x(0)$ denotes the amount present when the organism is alive, determine a differential equation that describes the amount of the C^{14} isotope present if $x(t)$ represents the amount present after time t elapses after the organism dies.
2. In contrast to C^{14}, the C^{12} isotope does not decay and the ratio of C^{12} to C^{14} is constant while an organism is alive. Hence, one should be able to compare the ratio of the two isotopes in a dead specimen to that of a live specimen. Determine how many years have elapsed if the ratio of the amount of C^{14} to C^{12} is 30% of the original value.

Do not look up the formula for half-life and exponential decay problems. The point is to derive the equation in order to relate it to the problem, and then to solve it.

2.4. Consider the first-order, linear, variable-coefficient, homogeneous ordinary differential equation

$$\dot{x}+tx=0.$$

Does assuming a solution of the form $x(t)=\mathrm{e}^{\lambda t}$ where λ is a constant work? Why or why not?

2.5. Consider the first-order, nonlinear, ordinary differential equation $\dot{x}+x^2=0$. Does assuming a solution of the form $x(t)=\mathrm{e}^{\lambda t}$ work? Why or why not?

2.6. As part of a fabrication process, you encounter the following scenario. A vat contains 100 liters of water. In error someone pours 100 grams of a chemical into the vat instead of the correct amount, which is 50 grams. To correct this condition, a stopper is removed from the bottom of the vat allowing 1 liter of the mixture to flow out each minute. At the same time, 1 liter of fresh water per minute is pumped into the vat and the mixture is kept uniform by constant stirring.

1. Show that if $x(t)$ represents the number of grams of chemical in the solution at time t, the equation governing x is

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{x}{100},$$

   where $x(0)=100$. How long will it take for the mixture to contain the desired amount of chemical?
2. Determine the equation governing $x(t)$ if the amount of water in the vat is W liters, the rate at which the mixture flows out is F liters/minute (and the same amount of fresh water is added), and the amount of the chemical initially added is C grams.

2.7. Determine the general solution to Equation (2.5).

1. Determine the temperature of a body for which $R=1$ and $C=10$ if it is initially at 100° and is plunged into a medium held at a constant temperature of $T_a=20^\circ$.
2. If two hot objects with equal masses are dropped into the ocean, which will cool faster, the object that has the shape of a sphere or the object that has the shape of a cube? Justify your answer by referring to Equation (2.4).

2.8. The rate by which people are infected by the zombie plague is proportional to the number of people already infected. Let x denote the number of people infected.[5]

1. What is the differential equation describing the number of people infected? Denote the proportionality constant by k. What are the units for k in the differential equation? What is the general solution to this equation? What are the units for k in the solution?
2. If at time $t = 0$ there are 100,000 people infected and at time $t = 1$ (the next day) there are 150,000 people infected, what is the numerical value of k?
3. For the value of k determined in the previous part, if at time $t = 0$, one person is infected, how long will it take for the zombie plague to infect every person on earth?

2.9. Assume that the rate of loss of a volume of a substance, such as dry ice or a moth ball, due to evaporation is proportional to its surface area.

1. If the substance is in the shape of a sphere, determine the differential equation describing the radius of the ball and solve it to find the radius as a function of time.
2. If the substance is in the shape of a cube, determine the differential equation describing the length of an edge of the cube and solve it to find the length of the edge as a function of time.
3. Use the answers from the previous two parts to determine which shape would be better for a given quantity of material if it is desired for it to take as long as possible to evaporate.

2.10. Use undetermined coefficients to determine the general solution to the following first-order ordinary differential equations.

1. $\dot{x} + x = \cos t$.
2. $\dot{x} + x = \cos 2t$.
3. $\dot{x} + x = \cos t + 2\sin t$.
4. $\dot{x} + 5x = \cos t + 2\sin t$.
5. $5\dot{x} + x = \cos t + 2\sin t$.
6. $\dot{x} + 3x = t^2 + 2t + 1$.
7. $\dot{x} + 3x = t^2 + 2t$.
8. $\dot{x} + 3x = 3t^2$.
9. $\dot{x} + 3x = 3t^2 + \cos 2t$.
10. $\dot{x} + 2x = \mathrm{e}^{-3t}$.
11. $\dot{x} + 2x = \mathrm{e}^{-2t}$.
12. $\dot{x} + 2x = 2\mathrm{e}^{-2t}$.
13. $\dot{x} + 2x = 2\mathrm{e}^{-2t} + \cos t$.
14. $\dot{x} + 2x = 2\mathrm{e}^{-2t} + \cos t + t^3$.

2.11. Show that the set of functions $\left\{t^0, t^1, t^2, t^3, t^4, \ldots, t^n\right\}$ is linearly independent.

[5] In this problem, let x be a real number and do not restrict it to be an integer.

2.12. From Example 2.13, substitute all three particular solutions into the differential equation to verify the conclusions from that example.

2.13. Determine the general solution to

$$\dot{x} + x/t = \cos 5t.$$

2.14. Use two different methods to determine the general solution to $\dot{x} + x = \sin 5t$. Also, find the solution if $x(0) = 0$.

2.15. Use two different methods to determine the general solution to

$$\dot{x} + 5x = \mathrm{e}^{-5t}.$$

Also, find the solution if $x(0) = 1$ and plot the solution versus time for a length of time that is appropriate to demonstrate the qualitative nature of the solution.

2.16. Determine the solution to

$$t\dot{x} + 2x = t^2 - t + 1$$

where $x(1) = 1/2$ and $t > 0$.

2.17. Prove that Equation (2.23) is the solution to Equation (2.22).

2.18. Determine the general solution to

$$\dot{x} + t^2 x = 0$$

using two different methods.

2.19. You are in desperate need to determine (as in make up), by hand, 100 different exact first-order differential equations in less than one hour. What would be a good way to do that? Determine 10 different exact first-order ordinary differential equations using your method.

2.20. Determine the general solution to

$$(2x + 1)\dot{x} = 3t^2.$$

If necessary, you may express the solution as an implicit function.

2.21. Use two different methods to determine the general solution to

$$3t^2\dot{x} + 6tx + 5 = 0.$$

2.22. Prove that all separable first-order ordinary differential equations are exact. In other words, show that separable first-order differential equations are a special case of exact first-order ordinary differential equations.

2.23. A special type of nonlinear first-order ordinary differential equation that can be converted into one that is separable is called a *homogeneous equation*.[6] A function $f(x,y)$ is *homogeneous of order n* if it can be written $f(x,y) = x^n g(u)$ where $u = y/x$. A first-order ordinary differential equation is homogeneous if it can be written as

$$P(x,y) + Q(x,y)\frac{\mathrm{d}x}{\mathrm{d}y} = 0,$$

where $P(x,y)$ and $Q(x,y)$ are homogeneous of the same order and hence may be written $P(x,y) = x^n p(u)$ and $Q(x,y) = x^n q(u)$. Because $y = ux$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$$

then the differential equation is of the form

$$x^n p(u) + x^n q(u)\left(x\frac{\mathrm{d}u}{\mathrm{d}x} + u\right) = 0$$

which is separable, can be solved for u as an implicit function of x, and then $u = y/x$ may be substituted to obtain the answer. Alternatively, the substitution $u = x/y$ may be used and the variable x eliminated. Which is better for any particular problem requires some experience or trial and error.

Verify that each of the following equations is homogeneous, and use this fact to solve them. Some of the integrals may be tricky, so resorting to a table or symbolic mathematics computer package may be necessary.

1. Show that $y^3 + 2x^2 y = c$ implicitly defines the general solution $y(x)$ to $2xy + \left(x^2 + y^2\right)\mathrm{d}y/\mathrm{d}x = 0$ by using the substitution $u = x/y$ and eliminating x.
2. Show that $\tan^{-1}(y/x) - 1/2\ln\left(x^2 + y^2\right) = c$ is the general solution to $(x+y) + (y-x)\mathrm{d}y/\mathrm{d}x = 0$ by using the substitution $u = y/x$.

2.24. An interesting class of nonlinear first-order ordinary differential equations arises in so-called *trajectory problems*. Given a family of curves, the problem is to find an orthogonal family of curves that are orthogonal at every point to the family of curves. If $y_f(x)$ is the given family of curves, then the orthogonal family will have a slope of $-1/y_f'(x)$ at any point $\left(x, y_f(x)\right)$. If we denote the orthogonal family by $y_o(x)$, then the orthogonality condition requires

$$\frac{\mathrm{d}y_f}{\mathrm{d}x}(x)\frac{\mathrm{d}y_o}{\mathrm{d}x}(x) = -1.$$

For example, consider the family of curves given by$y_f(x) = cx^5$, which is illustrated in Figure 2.3 for various values of c. The slope is given by $\mathrm{d}y_f/\mathrm{d}x(x) = 5cx^4$. Hence, for a given point (x,y), the value of c is $c = y/x^5$ so the slope at a given point (x,y) is given by $\mathrm{d}y_f/\mathrm{d}x = 5y/x$. So, finally, a curve orthogonal to $y_f(x)$ at the point (x,y) satisfies

[6] This is a homogeneous equation, not to be confused with a homogeneous solution.

$$\frac{\mathrm{d}y_o}{\mathrm{d}x}(x) = -\frac{1}{5}\frac{x}{y}.$$

Determine the general solution to this equation. Plot it for various values of the parameter that appears in the general solution and also plot $y_f(x)$ for various values of c. Are the curves orthogonal at every point? *Hint:* The orthogonal curves should be ellipses.

1. Repeat this problem for $y_f(x) = cx^4$.
2. Repeat this problem for $y_f(x) = cx$.

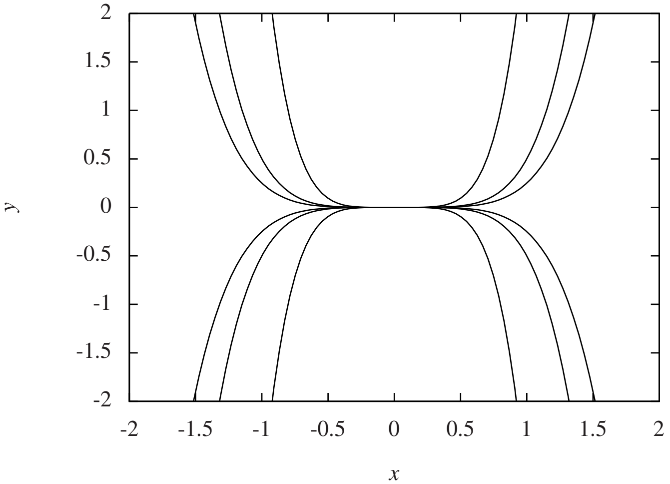

Fig. 2.3 Family of curves for Exercise 2.24.

2.25. For each of the following first-order, ordinary differential equations, indicate

- If the equation is linear
- If the equation is not linear but it is separable
- If the equation is neither linear nor separable but it is exact
- If the equation is neither linear, separable nor exact.

1. $\dot{x} + 5tx = \cosh t$.
2. $\dot{x}\sqrt{x^3}\cos t + 35t^2 = t$.
3. $\dot{x}\left(2t^2x + t\right) + 2tx^2 = -x$.
4. $\dot{x} + x = \cosh t$.
5. $\dot{x}\left(1 - x^2\right) = t$.
6. $\left(2tx^2 + 2x\right) = -\dot{x}\left(2t^2x + 2t\right)$.
7. $\left(2tx^2 + 2x\right) = \dot{x}\left(2t^2x + 2t\right)$.
8. $t^2x^3 + t\left(1 + x^2\right)\dot{x} = 0$.

9. $\dot{x}(\cos t + t) - x\sin t + x = 0.$
10. $\dot{x} + (\cos t)x = 0.$

2.26. Determine the general solution to each of the differential equations in Exercise 2.25 using the first method from the list that is applicable. If it is not linear and not exact, then determine an approximate numerical solution. You may choose your own initial conditions in such a case. Be careful not to pick an initial condition that is a singularity, for example, using $x = 0$ or $t = 0$ if that causes a term in the equation to be undefined.

2.27. Show that

$$\left(2t^2 + 3x\sin^2 t\right)\frac{\mathrm{d}x}{\mathrm{d}t} + 2x\left(t + x\sin t\cos t\right) = 0$$

is not exact, but when multiplied by $\mu(x,t) = x$, it is exact. Find the solution. Leaving the solution in implicit form is fine.

2.28. Show that

$$\left(2x^2t^2 + 3x^3\sin^2 t\right)\frac{\mathrm{d}x}{\mathrm{d}t} + \left(2x^3t + 2x^4\sin t\cos t\right) = 0$$

is not exact, but when multiplied by $\mu(x,t) = 1/x$, it is exact. Find the solution. Leaving the solution in implicit form is fine.

2.29. One special case where it is possible to determine an integrating factor is when it only depends on t. In such a case, Equation (2.33) reduces to

$$f(x,t)\frac{\mathrm{d}\mu}{\mathrm{d}t}(t) + \mu(t)\frac{\partial f}{\partial t}(x,t) - \mu(t)\frac{\partial g}{\partial x}(x,t) = 0$$

which gives

$$\frac{1}{\mu(t)}\frac{\mathrm{d}\mu}{\mathrm{d}t}(t) = \frac{\frac{\partial g}{\partial x}(x,t) - \frac{\partial f}{\partial t}(x,t)}{f(x,t)}. \tag{2.35}$$

1. Show that if we additionally have that the right-hand side of Equation (2.35) is only a function of t, then $\mu(t)$ is given by

$$\mu(t) = \exp\left(\int \frac{\frac{\partial g}{\partial x}(x) - \frac{\partial f}{\partial t}(x)}{f(x)}\mathrm{d}x\right).$$

2. Show that $(\mathrm{e}^t - \sin x) + \cos x(\mathrm{d}x/\mathrm{d}t) = 0$ is not exact, but that the above method to determine an integrating factor applies. Use that to find the solution.

2.30. For each of the first-order differential equations listed in Problem 1.8, determine which, if any, of the following solution methods apply based upon what has been covered in this book so far.

1. Assuming exponential solutions

2. Undetermined coefficients
3. Variation of parameters
4. Using the fact that the equation is separable
5. Using the fact that the equation is exact
6. Determining an approximate numerical solution

It may be the case that no method, one method, or more than one method may apply.

2.31. For relatively high velocities[7] the drag due to the motion of the body through air is proportional to the square of the velocity of the body. Hence, if the direction of positive velocity is down, Newton's law on the body can be represented as

$$m\frac{\mathrm{d}v}{\mathrm{d}t}(t) = mg - kv^2(t). \tag{2.36}$$

1. If a body falls from a sufficiently high altitude, it will reach its *terminal velocity* which is the velocity at which it will stop accelerating. Determine an expression for the terminal velocity, v_{term} from Equation (2.36).
2. Determine the general solution to Equation (2.36).
3. For re-entry of a 5000 kg space vehicle into the atmosphere, it is determined experimentally that the terminal velocity at low altitude is 300 kilometers per hour. If the velocity at time $t = 0$ is 600 kilometers per hour, determine $v(t)$ of the vehicle and plot it versus time. Plot the deceleration (g-force) experienced by the payload versus time.[8]

2.32. You work for the ACME parachute company. A person and parachute weigh 192 lb. Assume that a safe landing velocity is 16 ft/sec and that air resistance is proportional to the square of the velocity, equaling 1/2 lb for each square foot of cross-sectional area of the parachute when it is moving at 20 ft/sec. What must the cross-sectional area of the parachute be in order for the paratrooper to land safely?[9]

[7] More precisely, for certain geometries with Reynolds' numbers between approximately 10^3 and 10^5 [52].

[8] In this problem you are using a constant value for k. For a real vehicle re-entering the atmosphere, due to variation in the density of the atmosphere, k varies significantly.

[9] This problem is adapted from [50].

Springer

http://www.springer.com/978-1-4419-7918-6

Engineering Differential Equations
Theory and Applications
Goodwine, B.
2011, XIX, 745 p., Hardcover
ISBN: 978-1-4419-7918-6