Optimization Case Study

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Outline section 1

Exercises

Fitting problems

ex: a2 (Huber)

Optimal control problems

ex: b3 (Bicycle)

Trajectory generation using invariant representations

ex: c2 (Exact integration), c7 (geometric FS),
c14 (Spline invariants)

Outline section 2

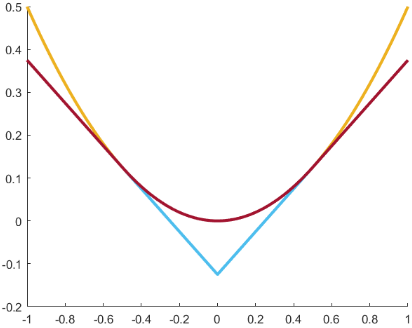
Outline subsection 1

Fitting problems

Exercise a2: Huber norm

Huber norm → combination of L1- and L2-norm

$$h(x) = \begin{cases} \frac{1}{2}x^2 & |x| < \delta. \\ \delta(|x| - \frac{1}{2}\delta) & \text{otherwise.} \end{cases}$$



Fitting problems

Exercise a2: Huber norm

Optimization problem with slack variables:

$$\min \sum s_i$$

$$\text{subject to} - s_i \leq h(e_i) \leq s_i$$

where

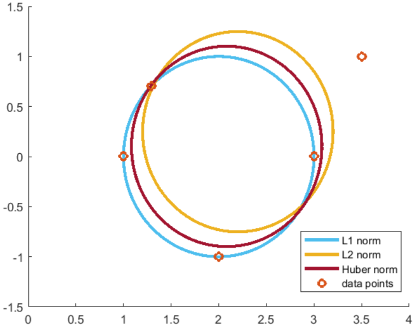
$$e_i = || \begin{bmatrix} x_i - x_c \\ y_i - y_c \end{bmatrix} ||_2 - R$$

Fitting problems

Exercise a2: Huber norm

Depending on 'cut-off' value delta, closer to L1- or L2-norm fit

$$h(x) = (x < \delta) * (\delta(-x - 0.5 * \delta)) +$$
$$(-\delta \leq x) * (x \leq \delta) * (0.5 * x^2) + (x > \delta) * (\delta(x - 0.5 * \delta))$$



Outline subsection 2

1. Exercises
2. Assignments
   - Fitting problems
   - Optimal control problems
   - Trajectory generation using invariant representations

Optimal control problems

Exercise b3: Bicycle

Bicycle model

Ode's
$$\begin{cases} \dot{x} = V * cos(\theta) \\ \dot{y} = V * sin(\theta) \\ \dot{\theta} = \frac{V}{L} * tan(\delta) \\ \dot{V} = a \end{cases}$$
with
$$\begin{cases} L = 1.5m \\ m = 80kg \end{cases}$$

Inputs
$$\begin{cases} \text{acceleration } a \\ \text{steering angle } \delta \end{cases}$$

Optimal control problems

Exercise b3: Bicycle

Optimization problem

$$\text{min } T$$

Subject to

$$\text{Input constraints} \begin{cases} |F| = |m * a| < 100 \ N \\ |\delta| < \pi/4 \end{cases}$$

$$\text{State constraints} \begin{cases} |V| < 5 \ m/s \\ \text{integrated dynamics } (rk4) \end{cases}$$

$$\text{Time constraint} \begin{cases} T > 0 \end{cases}$$

Optimal control problems

Exercise b3: Bicycle

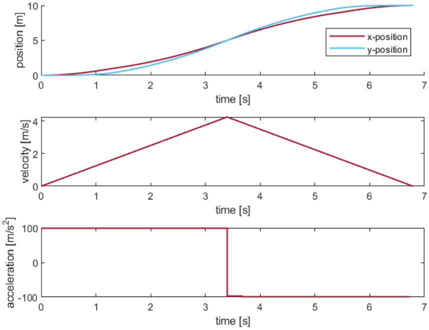


Figure 3: Position, velocity and acceleration.

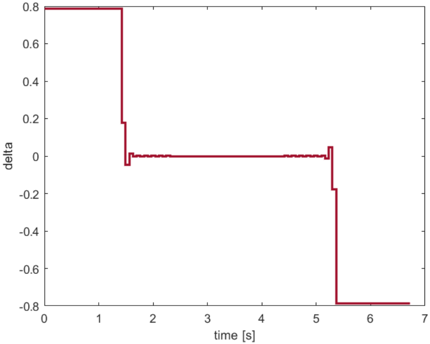


Figure 4: Steering angle.

Optimal control problems

Exercise b3: Bicycle

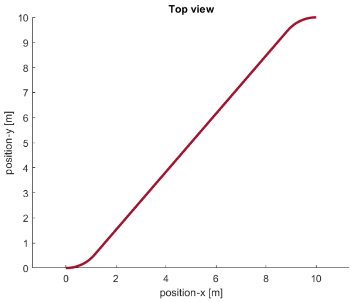


Figure 5: $\theta_{init} = 0^\circ$ & $\theta_{final} = 0^\circ$

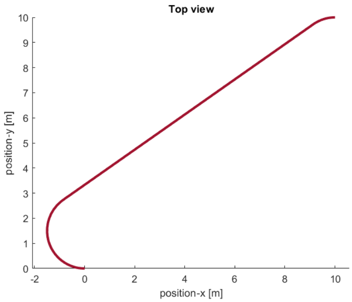


Figure 6: $\theta_{init} = 180^\circ$ & $\theta_{final} = 0^\circ$

Outline subsection 3

Trajectory generation using invariant representations

Exercise c2: Exact integration

Rewrite differential equation representing motion:

$$\dot{p}(t) = R_t(t)\begin{bmatrix} i_1(t) & 0 & 0 \end{bmatrix}^T$$

$$\dot{R}_t(t) = R_t(t) skew(\begin{bmatrix} i_3(t) & i_2(t) & 0 \end{bmatrix}^T)$$

as

$$\dot{X} = HX \tag{1}$$

where $X = \begin{bmatrix} p(t); R_t(t)(:) \end{bmatrix}$

Trajectory generation using invariant representations

Exercise c2: Exact integration

Then

$$H = \begin{bmatrix} 0 & 0 & 0 & \mathbf{i_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{i_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{i_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{i_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{i_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{i_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{i_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{i_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{i_3} \\ 0 & 0 & 0 & \mathbf{i_2} & 0 & 0 & -\mathbf{i_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{i_2} & 0 & 0 & -\mathbf{i_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{i_2} & 0 & 0 & -\mathbf{i_3} & 0 & 0 & 0 \end{bmatrix}$$

and the exact solution to (1) is

$$X(t) = expm(Ht) * X_0$$

Trajectory generation using invariant representations

Exercise c2: Exact integration

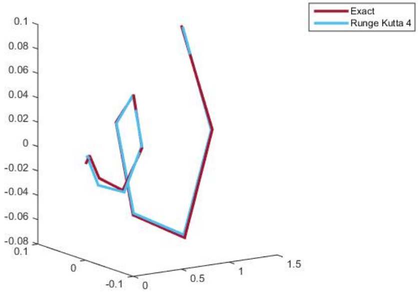


Figure 7: Simulation results of fits: 10 Sample points

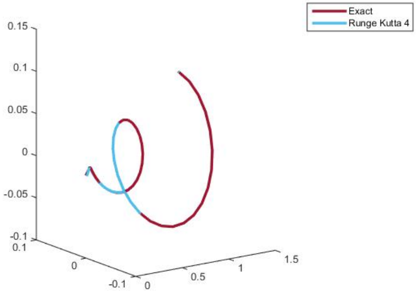


Figure 8: Simulation results of fits: 50 Sample points

Trajectory generation using invariant representations

Exercise c2: Exact integration

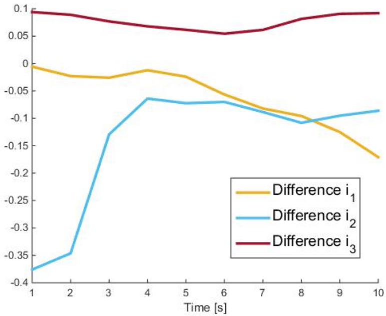


Figure 9: Relative difference Exact - RK4: 10 Sample points

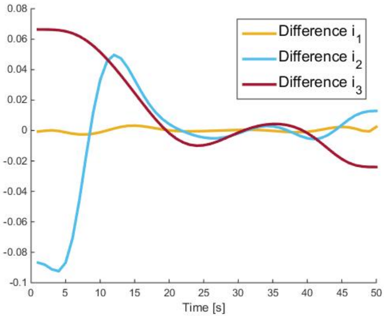


Figure 10: Relative difference Exact - RK4: 50 Sample points

Trajectory generation using invariant representations

Exercise c7: Geometric Frennet-Serret invariants

Revisit the differential equation representing motion, but now as function of the path-length coordinate instead of time:

$$\frac{d}{ds}R(s) = R(s)skew(\begin{bmatrix} i_{3,s}(s) & i_{2,s}(s) & 0 \end{bmatrix}^T)$$

Trajectory generation using invariant representations

Exercise c7: Geometric Frennet-Serret invariants

Now use the chain rule to express the differential equation in the time domain but still with the geometric invariants $i_{2,s}(s)$ and $i_{3,s}(s)$.

$$\begin{aligned}\frac{d}{dt}R(t) &= \left\| \frac{d\mathbf{r}(t)}{dt} \right\| * R(t) * skew(\begin{bmatrix} i_{3,s}(s) & i_{2,s}(s) & 0 \end{bmatrix}^T) \\ &= u * R(t) * skew(\begin{bmatrix} i_{3,s}(s) & i_{2,s}(s) & 0 \end{bmatrix}^T) \\ &= R(t) * skew(\begin{bmatrix} u * i_{3,s}(s) & u * i_{2,s}(s) & 0 \end{bmatrix}^T) \\ &= R(t) * skew(\begin{bmatrix} i_{3,t}(t) & i_{2,t}(t) & 0 \end{bmatrix}^T)\end{aligned}$$

Note that u is equal to $i_{1,t}(t)$.

Trajectory generation using invariant representations

Exercise c7: Geometric Frennet-Serret invariants

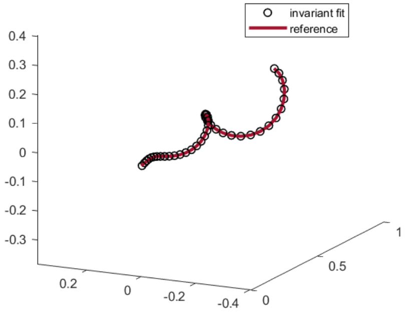


Figure 11: Invariant fit on reference

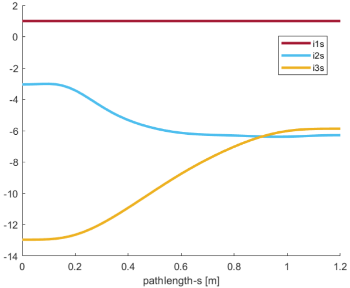


Figure 12: Invariants as a function of s

Trajectory generation using invariant representations

Exercise c7: Geometric Frennet-Serret invariants

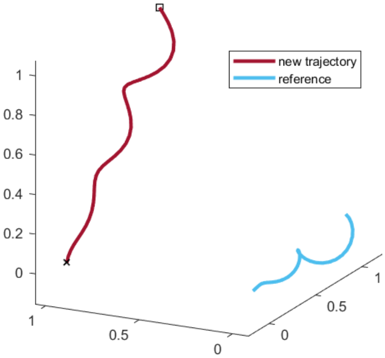


Figure 13: New vs reference trajectory

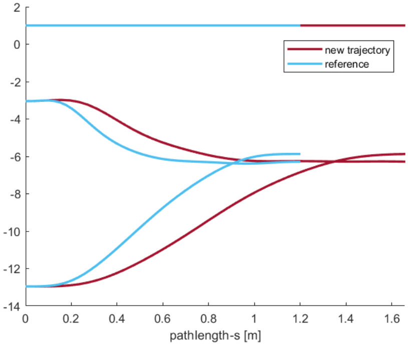


Figure 14: New vs reference trajectory invariants

Trajectory generation using invariant representations

Exercise c14: Spline parametrization of invariants

Instead of describing invariants using variables for each time step $\rightarrow$ describe each invariant as a spline function

Spline parameters

$$\begin{cases} degree = 5 \\ domain = [0, 1] \\ \#knots = 15 \end{cases}$$

Trajectory generation using invariant representations

Exercise c14: Spline parametrization of invariants

Fitting of reference trajectory invariants

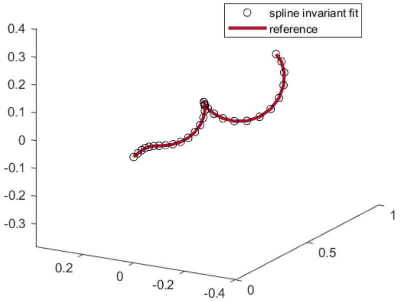


Figure 15: Spline invariant fit

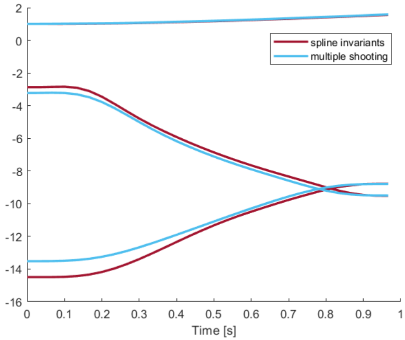


Figure 16: Spline vs multiple shooting invariants

Trajectory generation using invariant representations

Exercise c14: Spline parametrization of invariants

Generate new trajectory with cost function:

$$f(x) = \sum e_i^2$$

where

$$e_i = || \begin{bmatrix} i_{1,new,i} - i_{1,ref,i} \\ i_{2,new,i} - i_{2,ref,i} \\ i_{3,new,i} - i_{2,ref,i} \end{bmatrix} ||_2$$

Trajectory generation using invariant representations

Exercise c14: Spline parametrization of invariants

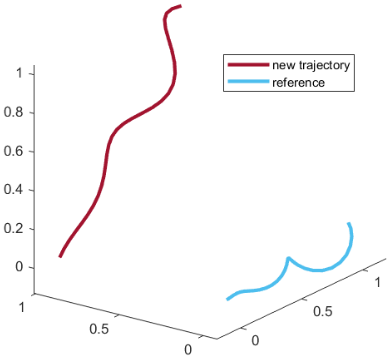


Figure 17: New vs reference trajectory

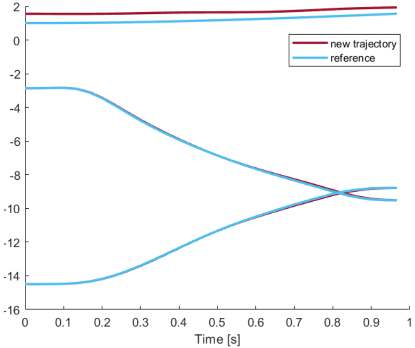


Figure 18: New vs reference trajectory invariants