Strapdown Inertial Navigation. A different approach to vehicle navigation that doesn't rely on an on-board inertial platform is a *strapdown inertial navigation system*. Interestingly, this approach does not require an analytical model of the vehicle, and that's a good thing because vehicle models are notoriously suspect due to poorly known aerodynamic forces and torques and other model errors. The approach does require, however, an on-board computer.

To begin, consider the translational kinematic vectors of a vehicle mass center coordinatized in an inertial frame.

$$\begin{array}{ll} \text{pos.} & \boldsymbol{p} = x_1\hat{\boldsymbol{n}}_1 + x_2\hat{\boldsymbol{n}}_2 + x_3\hat{\boldsymbol{n}}_3 \\ \text{vel.} & \boldsymbol{v} = \dot{x}_1\hat{\boldsymbol{n}}_1 + \dot{x}_2\hat{\boldsymbol{n}}_2 + \dot{x}_3\hat{\boldsymbol{n}}_3 = v_1\hat{\boldsymbol{n}}_1 + v_2\hat{\boldsymbol{n}}_2 + v_3\hat{\boldsymbol{n}}_3 \\ \text{acc.} & \boldsymbol{a} = \ddot{x}_1\hat{\boldsymbol{n}}_1 + \ddot{x}_2\hat{\boldsymbol{n}}_2 + \ddot{x}_3\hat{\boldsymbol{n}}_3 = \dot{v}_1\hat{\boldsymbol{n}}_1 + \dot{v}_2\hat{\boldsymbol{n}}_2 + \dot{v}_3\hat{\boldsymbol{n}}_3 = a_1\hat{\boldsymbol{n}}_1 + a_2\hat{\boldsymbol{n}}_2 + a_3\hat{\boldsymbol{n}}_3 \end{array}$$

If the vehicle accelerations can be directly measured through on-board accelerometers, then the vehicle velocities and positions at time t_j can be computed through numerical integration. The index k takes on values 1, 2 or 3.

$$v_k(t_j) = \int_{t_{j-1}}^{t_j} a_k(t)\,\mathrm{d}t + v_k(t_{j-1}) \quad ; \quad x_k(t_j) = \int_{t_{j-1}}^{t_j} v_k(t)\,\mathrm{d}t + x_k(t_{j-1}) \qquad (115.1)$$

Importantly, this approach is not concerned with the forces that *cause* the accelerations because the accelerations are simply measured. That is, analytical knowledge of the forces and dynamic model is traded for measurement knowledge of the accelerations.

At this point we encounter our first issue: the on-board accelerometers will provide acceleration measurements in a body fixed reference frame whereas the integrations in eq. (115.1) require acceleration measurements in an inertial reference frame.

$$\begin{array}{ccccc} \text{Needed} & & & & \text{Measured} \\ a_1\hat{\boldsymbol{n}}_1 + a_2\hat{\boldsymbol{n}}_2 + a_3\hat{\boldsymbol{n}}_3 & = & \boldsymbol{a} & = & \tilde{a}_1\hat{\boldsymbol{b}}_1 + \tilde{a}_2\hat{\boldsymbol{b}}_2 + \tilde{a}_3\hat{\boldsymbol{b}}_3 \end{array}$$

We've learned, however, that the vector components in one frame can be related to the vector components in another through the direction cosine matrix (see the page titled *The Direction Cosine Matrix* or the page titled *An Orientation Matrix*). We'll deal with this on the next page.

Rotational Needs. The on-board accelerometers provide acceleration measurements in a body fixed reference frame but what's needed are acceleration measurements in an inertial reference frame. Therefore, the orientation matrix between the two frames must be determined.

$$[\boldsymbol{a}]_b = [C][\boldsymbol{a}]_n \qquad ; \qquad [\boldsymbol{a}]_n = [C]^{\mathsf{T}}[\boldsymbol{a}]_b \tag{116.1}$$

The matrix $[C]$ can be constructed if a set of attitude coordinates can be determined; and a set of attitude coordinates can be determined if the vehicle angular velocity vector can be measured; and the vehicle angular velocity vector can be measured from on-board rate gyroscopes (or something better). We're familiar with this type of computation from rigid body attitude kinematics.

$$[\dot{\theta}_1,\, \dot{\theta}_2,\, \dot{\theta}_3]^{\mathsf{T}} = [A(\theta_1,\, \theta_2,\, \theta_3)]\, [\omega_1,\, \omega_2,\, \omega_3]^{\mathsf{T}} \tag{116.2}$$

Here, the angles $(\theta_1,\, \theta_2,\, \theta_3)$ are the three attitude coordinates of the vehicle whereas $(\omega_1, \omega_2, \omega_3)$ are components of the vehicle angular velocity in the body fixed reference frame. These three equations can be numerically integrated using an on-board computer to produce the time evolution of the attitude coordinates. The integral form of eq. (116.2) can be written in a succinct form by expressing the right side as a function of the instantaneous time rather than the instantaneous attitude and angular velocity.

$$\theta_k(t_j) = \int_{t_{j-1}}^{t_j} f_k(t)\, \mathrm{d}t + \theta_k(t_{j-1}) \quad ; \quad k = 1, 2, 3 \tag{116.3}$$

Knowing the attitude means the orientation matrix can be determined, which in turn means the components of the inertial acceleration in the inertial frame can be computed from the right-most expression in eq. (116.1).

$$[\boldsymbol{a}]_n = [C]^{\mathsf{T}}[\boldsymbol{a}]_b \qquad \rightarrow \qquad [a_1,\, a_2,\, a_3]^{\mathsf{T}} = [C(\theta_1, \theta_2, \theta_3)]^{\mathsf{T}}\, [\tilde{a}_1,\, \tilde{a}_2,\, \tilde{a}_3]^{\mathsf{T}} \tag{116.4}$$

Sensor Placement and Gravity. The accelerometers will naturally measure the inertial acceleration of the mass center if they are placed at the vehicle mass center. If they are placed somewhere else on the vehicle, however, then rotational contributions will need to be subtracted from the measurements. Let's see how.

The inertial position vector $\grave{\boldsymbol{p}}$ of an accelerometer can be given by vector addition, $\grave{\boldsymbol{p}} = \boldsymbol{p} + \boldsymbol{\rho}$. Here, $\boldsymbol{p}$ is the inertial position of the mass center and $\boldsymbol{\rho}$ is the position of the accelerometer relative to the mass center. The vector $\boldsymbol{\rho}$ is known and is constant when viewed in the body fixed reference frame.

The inertial acceleration of the accelerometer location has terms due to rotational motion of the vehicle.

$$\grave{\boldsymbol{a}} = \boldsymbol{a} + \boldsymbol{\alpha} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \tag{117.1}$$

The vector $\grave{\boldsymbol{a}}$ reflects the measurement signal of the on-board accelerometer, but it's $\boldsymbol{a}$, the acceleration of the mass center, that we seek.

$$\boldsymbol{a} = \grave{\boldsymbol{a}} - \boldsymbol{\alpha} \times \boldsymbol{\rho} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \tag{117.2}$$

The terms $\boldsymbol{\alpha} \times \boldsymbol{\rho}$ and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$ are rotational contributions. Keeping in mind that $\boldsymbol{\rho}$ is known, the first of these requires a way to measure the vehicle angular acceleration vector $\boldsymbol{\alpha}$. This can be done by using and properly arranging pairs of displaced accelerometers. The second of these requires the vehicle angular velocity vector $\boldsymbol{\omega}$, which can be measured from on-board rate gyroscopes (or something better).

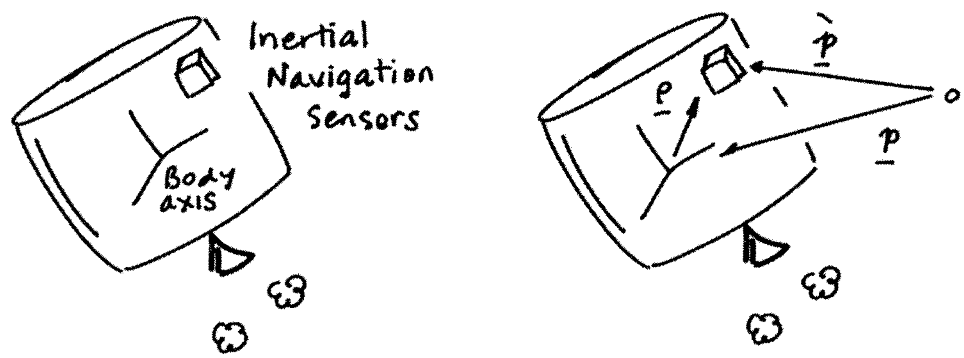
One final issue is that an on-board accelerometer measures the sum of vehicle accelerations and gravity. Thus, the contribution due to gravity should truly be subtracted at some point in the calculation.

$$\boldsymbol{a} = \grave{\boldsymbol{a}} - \boldsymbol{a}_g - \boldsymbol{\alpha} \times \boldsymbol{\rho} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \tag{117.3}$$

It is important to mention that $\boldsymbol{a}_g$ is naturally expressed in the inertial reference frame whereas the other terms on the right side of eq. (117.3) are naturally expressed in the body fixed reference frame.

Diagram of a Displaced Accelerometer. Below is a diagram for an accelerometer that is not placed at the vehicle mass center. The inertial position vector of the accelerometer is $\grave{\boldsymbol{p}}$; the inertial position of the vehicle mass center is $\boldsymbol{p}$; and the position of the accelerometer relative to the mass center is $\boldsymbol{\rho}$.

The puffs of smoke are part of the special effects.



Strapdown Inertial Navigation Steps. A process to perform strapdown inertial navigation is outlined in these four steps.

1. Read the outputs of the on-board accelerometers and rate gyros. This gives the instantaneous translational acceleration at the accelerometer location, $\grave{\boldsymbol{a}}$, and instantaneous angular velocity $\boldsymbol{\omega}$. These measurements will be the body fixed components of these inertial vectors.
2. Subtract the rotational contributions from the accelerometer readings to give the inertial translational acceleration at the vehicle mass center, $\boldsymbol{a} = \grave{\boldsymbol{a}} - \boldsymbol{\alpha} \times \boldsymbol{\rho} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$. This result will have the added affect of gravity.
3. Integrate the rotational and translational kinematic equations together to compute increments to the attitude and inertial velocity and position vector components in the inertial frame. (See the past few pages for clarifications to the variable meanings.) Note that the gravity contribution is subtracted in the velocity calculation, where it is assumed that the gravity magnitude is constant and collinear to the $\hat{\boldsymbol{n}}_3$ direction.

$$\Delta\boldsymbol{\theta}(t_j) \equiv \boldsymbol{\theta}(t_j) - \boldsymbol{\theta}(t_{j-1}) = \int_{t_{j-1}}^{t_j} \boldsymbol{f}(t)\,\mathrm{d}t \tag{119.1}$$

$$[\Delta\boldsymbol{v}(t_j)]_n \equiv [\boldsymbol{v}(t_j) - \boldsymbol{v}(t_{j-1})]_n = \int_{t_{j-1}}^{t_j} [C(t)]^{\mathsf{T}}[\boldsymbol{a}(t)]_b\,\mathrm{d}t - g(t_j - t_{j-1})[0\ 0\ 1]^{\mathsf{T}} \tag{119.2}$$

$$[\Delta\boldsymbol{p}(t_j)]_n \equiv [\boldsymbol{p}(t_j) - \boldsymbol{p}(t_{j-1})]_n = \int_{t_{j-1}}^{t_j} [C(t)]^{\mathsf{T}}[\boldsymbol{v}(t)]_b\,\mathrm{d}t \tag{119.3}$$

4. Add the increments to the current inertial states to compute the updated inertial states.

Coning & Sculling & Scrolling. The previous page listed some integrations to compute increments to the attitude and inertial velocity and position vector components in the inertial frame.

$$\Delta\boldsymbol{\theta}(t_j) = \int_{t_{j-1}}^{t_j} \boldsymbol{f}(t)\,\mathrm{d}t \tag{120.1}$$

$$[\Delta\boldsymbol{v}(t_j)]_n = \int_{t_{j-1}}^{t_j} [C(t)]^\mathsf{T}[\boldsymbol{a}(t)]_b\,\mathrm{d}t - g(t_j - t_{j-1})[0\ 0\ 1]^\mathsf{T} \tag{120.2}$$

$$[\Delta\boldsymbol{p}(t_j)]_n = \int_{t_{j-1}}^{t_j} [C(t)]^\mathsf{T}[\boldsymbol{v}(t)]_b\,\mathrm{d}t \tag{120.3}$$

These increments are subsequently added to the current inertial states to compute the updated inertial states.

Although rigorous and correct, the calculations in eqs. (120.1) through (120.3) are not what are actually done in practice. For efficient numerical integration, some simplifications are made in the integrand of eq. (120.1) that are later partially accounted for. This accounting is called *coning*. Moreover, again in the name of numerical efficiency, the integrations in eqs. (120.2) and (120.3) are done on the body components. Corrections are made before those results are mapped to inertial velocity and position components, and those corrections are called *sculling* and *scrolling*, respectively. Here's a little bit of an explanation of things.

Consider eq. (120.2) without the gravity term (it simply gets in the way of the discussion) and componentized in a body fixed frame.

$$[\Delta\boldsymbol{v}(t_j)]_b = [C] \int_{t_{j-1}}^{t_j} [C(t)]^\mathsf{T}[\boldsymbol{a}(t)]_b\,\mathrm{d}t \tag{120.4}$$

Importantly, this is not the same as integrating the body fixed components of the acceleration.

$$[\Delta\boldsymbol{v}(t_j)]_b \neq \int_{t_{j-1}}^{t_j} [\boldsymbol{a}(t)]_b\,\mathrm{d}t \equiv [\delta\boldsymbol{v}(t_j)]_b \tag{120.5}$$

Clearly, the time-varying attitude in the integrand of eq. (120.4) is the difference.[15] Thus, because calculations in practice involve $[\delta\boldsymbol{v}(t_j)]_b$, a (sculling) correction is needed before this result is mapped to the inertial frame.

[15] $[\Delta\boldsymbol{v}(t_j)]_b = [\delta\boldsymbol{v}(t_j)]_b$ only if the attitude is constant from t_{j-1} to t_j.

It's Called Strapdown Because ... We can see why this kind of device is called a strapdown inertial navigation system.

1. The accelerometers and rate gyroscopes (or something better) are *strapped down* to the vehicle.
2. The accelerometers and rate gyroscopes and on-board computer work together to produce the vehicle *inertial* states (i.e., vehicle *navigation*) at any desired time.
3. The accelerometers and rate gyroscopes and on-board computer constitute the *system*.

A MEMS Gyroscope. A rate gyroscope is a classic way to measure vehicle angular rates, but it's not the only way. Recently, new gyroscope ideas have been developed because of advancements in MEMS (Micro-machined Electro-Mechanical Systems) technology. One idea is the *vibrating structure gyroscope.*[16] A vibrating structure gyroscope measures an angular rate using the basic principles of point mass kinetics. The development is shown here.

A primary component of this device is a micro-machined mass that is connected to an inner frame by a set of springs. The inner frame, in turn, is connected to a fixed housing by a different set of springs that are orthogonal to the first set. A set of dampers are used too. The small mass is driven in a known sinusoidal manner along the direction of the first set of springs, and the inner frame is allowed to move in an orthogonal direction. Both of these motions take place in a plane that can undergo general tumbling motion.

The position, velocity, and acceleration vectors along body fixed axis can be written in terms of Cartesian coordinates and their derivatives and the elements of the angular velocity vector.

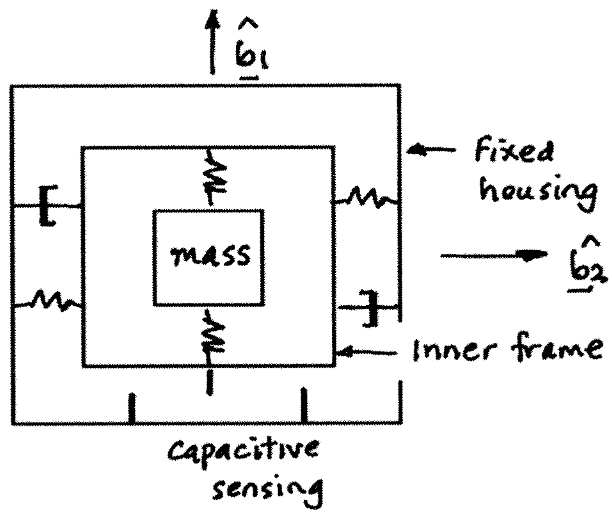
$$\begin{aligned}
\boldsymbol{p} &= x\hat{\boldsymbol{b}}_1 + y\hat{\boldsymbol{b}}_2 \\
\boldsymbol{v} &= (\dot{x} - y\omega_3)\hat{\boldsymbol{b}}_1 + (\dot{y} + x\omega_3)\hat{\boldsymbol{b}}_2 + (y\omega_1 - x\omega_2)\hat{\boldsymbol{b}}_3 \\
\boldsymbol{a} &= (\ddot{x} - y\dot{\omega}_3 - 2\dot{y}\omega_3 + y\omega_1\omega_2 - x\omega_2^2 - x\omega_3^2)\hat{\boldsymbol{b}}_1 \\
&\quad + (\ddot{y} + x\dot{\omega}_3 + 2\dot{x}\omega_3 + x\omega_1\omega_2 - y\omega_3^2 - y\omega_1^2)\hat{\boldsymbol{b}}_2 \\
&\quad + (y\dot{\omega}_1 - x\dot{\omega}_2 + 2\dot{y}\omega_1 - 2\dot{x}\omega_2 + x\omega_1\omega_3 + y\omega_2\omega_3)\hat{\boldsymbol{b}}_3
\end{aligned}$$

Recall, x and its derivatives are prescribed whereas the coordinate y and its time derivatives, along with the angular velocity components ω_k and their time derivatives, are unknown.

It is straightforward to form the equations of motion for the point mass using Newton's second law. Among the three equations, it is the scalar equation of motion along the $\hat{\boldsymbol{b}}_2$ direction that is important to us because capacitive sensors are used to measure the y motion.

[16] http://www.sensorwiki.org/doku.php/sensors/gyroscope

Diagram of a MEMS Gyroscope.



The Output of a Single MEMS Gyroscope. A MEMS gyroscope uses the motion of a small mass to determine angular velocity. The relevant equation is along a single direction in a plane that can undergo general tumbling motion.

$$-c\dot{y} - ky = m(\ddot{y} + x\dot{\omega}_3 + 2\dot{x}\omega_3 + x\omega_1\omega_2 - y\omega_3^2 - y\omega_1^2) \tag{124.1}$$

The plane containing this motion is perpendicular to the $\hat{\boldsymbol{b}}_3$ axis, and it is the angular velocity about this axis, ω_3, that we wish to detect.

Going forward, eq. (124.1) can be rearranged to isolate a linear ordinary differential equation in y on one side and the Coriolis acceleration on the other. The non Coriolis terms are gathered together and labeled r.

$$m\ddot{y} + c\dot{y} + ky = -2m\dot{x}\omega_3 + r \quad \text{where} \quad r \equiv -m(x\dot{\omega}_3 + x\omega_1\omega_2 - y\omega_3^2 - y\omega_1^2)$$

Importantly, each term in r can be artfully minimized through the MEMS design or through device calibration. This leaves a simplified expression.

$$m\ddot{y} + c\dot{y} + ky = -2m\dot{x}\omega_3 \tag{124.2}$$

The small mass is driven in a known sinusoidal manner, for example $x = A\sin\sigma t$. In turn, this excites the equation for y. Differential equations like eq. (124.2) will be extensively studied in the upcoming chapters. We will learn that the steady-state solution is especially important because that is the part that remains when the natural motions have gone away. For now, it is sufficient to know that the steady-state solution can be determined for the case when the drive frequency of the small mass equals the undamped natural frequency in the equation for y, $\sigma = \sqrt{k/m} \equiv \omega_n$.

$$y = -\frac{x\omega_3}{\zeta\omega_n} \qquad \rightarrow \qquad \omega_3 = -\frac{y\zeta\omega_n}{x} \tag{124.3}$$

Here, $\zeta \equiv c/(2\sqrt{km}) < 1$ is the dimensionless damping ratio.

More on MEMS. A single MEMS gyroscope uses the motion of a small mass to excite a small structural system to determine angular velocity.

$$\omega_3 = -\frac{y\zeta\omega_n}{x} \tag{125.1}$$

This angular velocity is perpendicular to the plane containing the x and y motions of the mass and structural system. A set of three MEMS gyroscope devices arranged in orthogonal planes and strapped down to a vehicle can be used to measure the three body fixed components of the angular velocity vector. Thus, these MEMS devices can replace classical rate gyroscopes. Modern inertial navigation systems use this kind of MEMS technology to provide accurate and precise vehicle navigation.[17]

The strapdown inertial navigation system that we've discussed in the past several pages is *un-aided*. That is, there is no mention of an additional, independent sensor to correct for accelerometer and gyroscope sensor errors or numerical integration errors. Unfortunately, it is not currently possible to achieve navigation-grade performance with a MEMS-based un-aided strapdown inertial navigation system. Consequently, it is common to use a global navigation satellite system as an aid. In this case, the signals from a global navigation satellite system are used together with the measurements from a strapdown inertial navigation system in an overall computational framework called a *Kalman Filter*. Kalman Filtering is the stuff of advanced estimation.[18]

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[17] http://www.vectornav.com/. For example, see the VN-100 and VN-200 series.

[18] I am happy to acknowledge my conversations with James Doebbler, Director, Research and Development, VectorNav Technologies. Mr. Doebbler contributed significantly to the strapdown inertial navigation system content and the MEMS gyroscope content.