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Optimal observer trajectories for bearings-only tracking by minimizing the trace of the Cramér-Rao lower bound

32nd CDC

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Abstract - This paper compares two performance indices for computing optimal observer paths for the bearings-only source localization and tracking problem, for constant velocity sources. Previous work on this problem is based on maximizing the determinant of the Fisher information matrix (FIM) of the estimation problem. This paper considers minimizing the trace of a weighted sum of the Cramér-Rao lower bound (CRLB) of current or future source position errors, and source velocity errors. Quasi-Newton optimization is used to compare optimal observer paths, given three distinct goals: minimizing current position error, velocity error, and future position error. Significant differences in optimal paths are observed, and the CRLB trace is found to yield smaller estimation ambiguity.

1. Introduction

The problem of locating and tracking of a constant velocity non-maneuvering source using only bearing measurements is known to be dependent on the observer's trajectory. In a typical estimation problem, a freely moving observer records bearing measurements to the source through a passive sensor and estimates the source's unknown dynamics. Although a number of authors have studied the trajectories that make this problem observable [1-3], the problem of finding the optimal observer path has received little attention. Typically it is suggested that the observer executes a extreme change in direction for the bearings-only tracking algorithms to perform well, but more needs to be said on how to use observer trajectories to improve estimation algorithm performance.

This paper addresses the question of how to properly define optimality in the bearings-only tracking problem. The optimality criterion used here is based on minimizing the trace of Cramér-Rao lower bound (CRLB). The CRLB as defined in [4] is a lower bound on the error covariance of the estimation problem, and gives a criterion of optimality that is independent of the estimation algorithm. The CRLB trace approach makes intuitive sense since the trace of the covariance matrix is the measure for defining optimality in Kalman filters [5], and the theoretical covariance matrix for two well known Cartesian estimation algorithms, the Stensfield and the Maximum Likelihood methods [6], are also the same as the CRLB.

Previous work on this problem is based on maximizing the determinant of the Fisher information matrix (FIM) of the estimation problem [7], the inverse of the CRLB. That approach minimizes the area of a confidence ellipsoid around the estimates of the initial position and initial velocity. Using area as a performance measure may favor solutions with highly eccentric confidence ellipsoids. This behavior may be troublesome in source localization or tracking, since the largest ambiguity in the confidence ellipsoid often corresponds to the unknown range variable, while the smaller axis often corresponds to the known bearing measurements. The FIM approach also lacks generality, since it is not possible to design the criterion to minimize different goals such weighted source velocity, position, or predicted position error.

The performance criterion in this paper is a weighted sum of the trace of the source position and velocity CRLB. The position error can correspond to the current time or future time, to provide a prediction error. The trace of the covariance matrix is the sum of its eigenvalues [8], and each eigenvalue corresponds to the square of one axis of the confidence ellipsoid. Therefore, the trace is the sum of the squares of each axis. This performance measure penalizes solutions with a large axis, and, as a result, the optimal paths produced by the CRLB trace approach have smaller ambiguity than those determined using the FIM approach.

The two optimality criteria are compared using the same numerical method as in [7], quasi-Newton optimization. The observer considered has constant instantaneous speed and infinite maneuverability. Optimal trajectories are studied as a function of source direction and observer speed.

The paper is organized as follows. The following section contains a problem statement and definition of the optimality criteria. The third section discusses the method used to calculate the optimal paths. The fourth section presents the optimal paths and gives discussion. The final section presents the conclusions.

2. Problem Statement

The source localization and tracking problem involves estimating the source initial position and velocity through bearing angle measurements. This system is studied extensively in [7], and can be described by the following closed form solution:

$$\mathbf{x}(t) = \Phi(t,0)\mathbf{x}(0) + \mathbf{u}(t)$$
$$\hat{\beta}(t) = \beta(t) + \beta_e(t). \qquad (1)$$

where

$$\mathbf{x} = \begin{bmatrix} r_x & r_y & v_x & v_y \end{bmatrix}^T = \text{observer position and velocity relative to source}$$

$$\Phi(t,0) = \begin{bmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{state transiton matrix}$$

$$\mathbf{u}(t) = \begin{bmatrix} \int_0^t (t-\tau) a_{ox}(\tau) d\tau \\ \int_0^t (t-\tau) a_{oy}(\tau) d\tau \\ \int_0^t a_{ox}(\tau) d\tau \\ \int_0^t a_{oy}(\tau) d\tau \end{bmatrix} = \text{components from observer acceleration}$$

$$\begin{bmatrix} a_{ox}(t) & a_{oy}(t) \end{bmatrix}^T = \text{observer's acceleration}$$

$$\hat{\beta}(t) = \text{bearing measurement}$$

$$\beta(t) = \tan^{-1}\left(\frac{r_y}{r_x}\right) = \text{true bearing}$$

$$\beta_e(t) = \text{zero mean normal bearing error}$$

$$\sigma^2 = \text{variance of bearing error, } \beta_e(t)$$

The CRLB is defined by

$$\mathbf{P} = E[(\mathbf{x} - \hat{\mathbf{x}})^2] \geq \mathbf{J}^{-1} \tag{2}$$

where $\mathbf{P}$ is the error covariance of $\mathbf{x}$, $\hat{\mathbf{x}}$ is the estimate of $\mathbf{x}$, and $\mathbf{J}$ is the FIM defined by

$$\mathbf{J} = -E\left[\frac{\partial^2 \ln p_{\hat{\beta}|\mathbf{x}}(\beta|\mathbf{x})}{\partial \mathbf{x}^2}\right] \tag{3}$$

where

$$p_{\hat{\beta}|\mathbf{x}}(\hat{\beta}|\mathbf{x}) = \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{1}{2\sigma^2}\int_0^T (\hat{\beta} - \beta(\mathbf{x}))^2 d\tau} \tag{4}$$

The FIM for estimating the initial state vector, $\mathbf{x}(0)$, is

$$\mathbf{J} = \frac{1}{\sigma^2}\int_0^T \Phi^T(\tau,0) \frac{\partial\beta}{\partial\mathbf{x}(0)}^T \frac{\partial\beta}{\partial\mathbf{x}(0)} \Phi(\tau,0) d\tau$$

$$= \frac{1}{\sigma^2}\begin{bmatrix} \int_0^T \frac{\sin^2\beta}{r^2} d\tau & -\frac{1}{2}\int_0^T \frac{\sin 2\beta}{r^2} d\tau & \int_0^T \tau \frac{\sin^2\beta}{r^2} d\tau & -\frac{1}{2}\int_0^T \tau \frac{\sin 2\beta}{r^2} d\tau \\ -\frac{1}{2}\int_0^T \frac{\sin 2\beta}{r^2} d\tau & \int_0^T \frac{\cos^2\beta}{r^2} d\tau & -\frac{1}{2}\int_0^T \tau \frac{\sin 2\beta}{r^2} d\tau & \int_0^T \tau \frac{\cos^2\beta}{r^2} d\tau \\ \int_0^T \tau \frac{\sin^2\beta}{r^2} d\tau & -\frac{1}{2}\int_0^T \tau \frac{\sin 2\beta}{r^2} d\tau & \int_0^T \tau^2 \frac{\sin^2\beta}{r^2} d\tau & -\frac{1}{2}\int_0^T \tau^2 \frac{\sin 2\beta}{r^2} d\tau \\ -\frac{1}{2}\int_0^T \tau \frac{\sin 2\beta}{r^2} d\tau & \int_0^T \tau \frac{\cos^2\beta}{r^2} d\tau & -\frac{1}{2}\int_0^T \tau^2 \frac{\sin 2\beta}{r^2} d\tau & \int_0^T \tau^2 \frac{\cos^2\beta}{r^2} d\tau \end{bmatrix} \tag{5}$$

where $r^2 = r_x^2 + r_y^2$.

In the above FIM , both the variables b and r depend on the observer's path. The approach of [7] is to find the observer trajectory that maximizes the determinant of the FIM as shown in the following equation

$$J_{FIM} = |\mathbf{J}|. \tag{6}$$

This criterion of optimality is equivalent to minimizing the volume of the confidence hyper-ellipsoid around the initial state estimate $\mathbf{x}(0)$.

The approach presented in this paper minimizes a weighted sum of the trace of the lower bound of the covariance of the sources position and velocity at a desired time. That time could be the current time or a future time. Let the CRLB of the estimated initial position and velocity be partitioned as follows

$$CRLB = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} = \mathbf{J}^{-1} \tag{7}$$

where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are 2 by 2 matrices. From this it is possible to calculate the CRLB of the source position at a particular time, t_*, and source velocity from the equations

$$\begin{aligned} CRLB_r(t_*) &= \mathbf{A} + t_*(\mathbf{B} + \mathbf{B}^T) + t_*^2\mathbf{C} \\ CRLB_v &= \mathbf{C} \end{aligned} \tag{8}$$

The performance index presented in this paper minimizes the trace of the weighted sum of the position and velocity lower bounds as follows:

$$J_{trace} = tr\left(\alpha_1 CRLB_r(t_*) + \alpha_2 CRLB_v\right) \tag{9}$$

This performance index lets the designer decide whether position or velocity estimates are more important by choosing the weights.

3. Numerical Optimization

The procedure used to find the optimal path for all the performance indices is the same as that used in [7]. A constant velocity source is given a specified direction and speed, and the observer is given a constant speed but infinite maneuverability. To simplify the problem it is assumed that there are only a finite number of course changes over a given time span. The performance indices are a function of this finite set of course changes as in the equation,

$$J(\theta) = J(\beta(\theta), r(\theta)) \tag{10}$$

where θ is a vector that represents the course changes and each element corresponds to a different time.

A quasi-Newton non-linear optimization algorithm is used to find the optimal course vector θ, given the source's initial range, speed and direction and the observer's constant speed. The algorithm used is an unconstrained minimization subroutine from the IMSL [9] package. It should be noted that the optimization computation was difficult for all performance criteria, since there were often more than one local minimum or maximum. The optimal paths are the solutions that consistently give the best performance.

4. Results

Optimum paths are calculated for four different source directions with respect to the x-axis: 45°, 90°, 135° and 180°. Each test spans twenty time units, and the source has the constant velocity of one distance unit per time unit.

For each source direction, four different optimum paths are calculated at the end of the 20 time unit observation period:

1. Minimum current source position error,
2. Minimum predicted source position error 20 time units into the future,
3. Minimum source velocity error,
4. Minimum confidence ellipsoid volume (i.e., maximum FIM determinant).

The observer's path is optimized over twenty course changes over the time span occurring at equal intervals. Each observer is given enough speed to cover 64.71% is of the distance necessary to reach the source. This percentage is chosen to allow a direct comparison with the results in [7].

The paths calculated for the four sources are shown in Figures 1–4. The curves show different optimal paths depending on the performance criterion. Tables 1–4 shows the axes of the confidence ellipsoids of the sources velocity error, current position error and predicted position error. It is assumed that the bearing error's standard deviation is 1°.

For the most part, the optimal paths for the FIM approach correspond well with [7], however, it appears that certain solutions in [7] are only locally optimal. In particular, it is noted that the computation of optimal trajectories can be quite a tricky proposition under all performance criteria studied.

In comparing confidence ellipsoids in Tables 1–4, for every source direction, the CRLB trace approach gives paths yielding minimum ambiguity (i.e. smallest major axis) for the desired performance index, but performance with respect to other errors is sacrificed . For example, the path that minimizes velocity error had larger error in the current and predicted position estimate. The FIM approach compromises between these three goals. Also note that the FIM approach tends to give more eccentric ellipsoids as expected. This behavior is most pronounced when the source is moving away from the observer (direction = 180°). For example, in estimating current position, the CRLB trace method gives a confidence ellipse with major/minor axis ratio 0.9762/0.2219 = 4.32, while the FIM method gives major/minor axis ratio 1.2978/0.1568 = 8.27.

5. Conclusions

This paper discusses two performance criteria for finding the optimal observer paths for bearings-only source localization and tracking. A performance criterion based on minimizing the trace a weighted sum of source velocity and position CRLB is compared to a performance index that maximizes the determinant of the FIM. The motivation for using the CRLB trace approach is to find source estimates with smaller ambiguity and to find the optimal paths for the different goals of estimating source velocity, current position and future position. Optimal observer paths are calculated using a quasi-Newton algorithm. The results suggest significantly different observer paths for the three estimation goals, and the FIM approach suggests a path that is a compromise between these goals. As expected the FIM method does produce more eccentric confidence ellipsoids.

It should be noted that work in this area so far gives only the theoretical optimum observer paths, under the assumption of perfect source information. Study of robustness in the face of source position and velocity uncertainty, and on-line correction of this uncertainty is an important area of future study.

6. References

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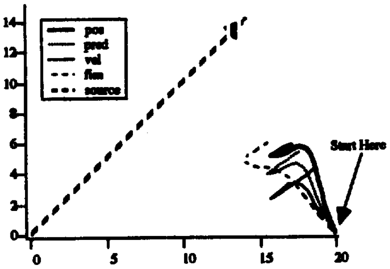

Figure 1. Optimal paths with source direction at 45°

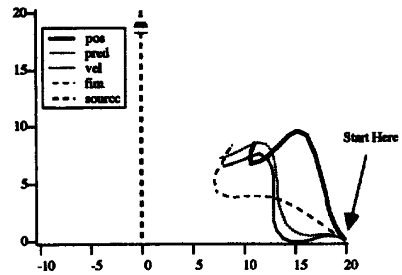

Figure 2. Optimal paths with source direction at 90°

	current position estimate error ellipsoid *major*	*minor*	future position estimate error ellipsoid *major*	*minor*	constant velocity estimate error ellipsoid *major*	*minor*	volume of ellipsoid
trajectory minimizing current position error	**0.4797**	0.1268	1.8077	0.5992	0.0830	0.0181	0.000022
trajectory minimizing future position error	0.5303	0.1244	**1.3689**	0.7042	0.0573	0.0220	0.000019
trajectory minimizing velocity error	0.7326	0.1309	1.5296	0.7562	**0.0519**	0.0275	0.000027
trajectory maximizing determinant of FIM	0.5915	0.1042	1.7434	0.5379	0.0705	0.0178	**0.000014**

Table 1. Axes of confidence ellipsoids with source direction at 45°

	current position estimate error ellipsoid *major*	*minor*	future position estimate error ellipsoid *major*	*minor*	constant velocity estimate error ellipsoid *major*	*minor*	volume of ellipsoid
trajectory minimizing current position error	**0.6081**	0.2564	2.8691	0.9256	0.1408	0.0251	0.000171
trajectory minimizing future position error	0.7543	0.1503	**1.1928**	0.9942	0.0522	0.0280	0.000071
trajectory minimizing velocity error	0.8972	0.1504	1.5214	0.7912	**0.0406**	0.0340	0.000071
trajectory maximizing determinant of FIM	0.9308	0.1659	1.7740	0.9275	0.0490	0.0391	**0.000060**

Table 2. Axes of confidence ellipsoids with source direction at 90°

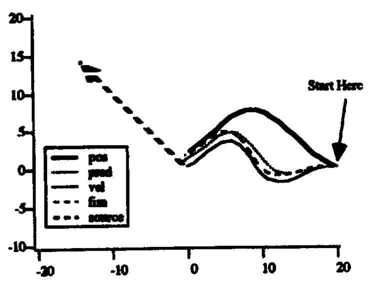

Figure 3. Optimal paths with source direction at 135°

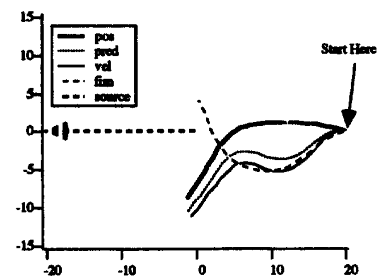

Figure 4. Optimal paths with source direction at 180°

	current position estimate error ellipsoid *major*	*minor*	future position estimate error ellipsoid *major*	*minor*	constant velocity estimate error ellipsoid *major*	*minor*	volume of ellipsoid
trajectory minimizing current position error	**0.6276**	0.2478	3.8338	0.7791	0.1802	0.0226	0.000298
trajectory minimizing future position error	0.9632	0.1794	**1.7463**	0.9824	0.0701	0.0296	0.000174
trajectory minimizing velocity error	1.1697	0.1783	2.0448	0.8646	**0.0589**	0.0337	0.000174
trajectory maximizing determinant of FIM	1.0691	0.1692	1.9585	0.8096	0.0635	0.0287	**0.000167**

Table 3. Axis of confidence ellipsoids with source direction at 135°

	current position estimate error ellipsoid *major*	*minor*	future position estimate error ellipsoid *major*	*minor*	constant velocity estimate error ellipsoid *major*	*minor*	volume of ellipsoid
trajectory minimizing current position error	**0.9762**	0.2219	4.4667	0.8129	0.1996	0.0242	0.000527
trajectory minimizing future position error	1.5552	0.2085	**2.7996**	1.1580	0.1118	0.0347	0.000538
trajectory minimizing velocity error	1.9642	0.2058	3.2891	1.0463	**0.0933**	0.0410	0.000555
trajectory maximizing determinant of FIM	1.2978	0.1568	4.1720	0.4121	0.1495	0.0141	**0.000299**

Table 4. Axis of confidence ellipsoids with source direction at 180°