Consensus and cooperation in networked multi-agent systems [Olfati, 2007]

George Kontoudis, DevaPrakash Muniraj

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Aerospace and Ocean Engineering Department, Virginia Tech

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Outline

Consensus & Cooperation

This paper provides a framework for analysis of consensus algorithms for multi-agent network systems

- Consensus is defined as reaching an agreement regarding a certain quantity of interest that depends on the state of all agents
- A protocol, also called consensus algorithm, is an interaction rule that specifies the exchange information between an agent and its neighbors on the network
- Networked systems, that are included in agents, are equipped with sensing, computing, and communicating devices

Consensus in Networks

- For a directed graph $G = (V, E)$, with a set of nodes $V = 1, 2, ..., n$ and edges $E \subseteq V \times V$. A simple consensus algorithm of a nth order linear system on a graph is

$$\dot{x}_i = \sum_{j \in N_i} (x_j(t) - x_i(t)) + b_i(t), \quad x_i(0) = z_i \in \mathbb{R}, b_i(t) = 0$$

  with collective dynamics $\dot{x} = -Lx$
- Since all row-sums of the Laplacian are zero, L has always a zero eigenvalue $\lambda_1 = 0$
- The consensus value is the avg of the initial states $\alpha = \frac{1}{n} \sum_i z_i$

The f-Consensus problem & Cooperation

Differences between constrained and unconstrained problems

- In unconstrained problems the state of all agents asymptotically become the same
- In constrained problems (f-consensus problems) the state of all agents asymptotically become $f(z)$

To solve the f-consensus problem we need

- Willingness to participate from all agents
- Cooperation from all agents

Applications (1/2)

Common consensus problems for multi-agent systems

- Synchronization of coupled oscillators which has dynamics

$$\dot{\theta}_i = \kappa \sum_{j \in N_i} sin(\theta_j - \theta_i) + \omega_i,$$

  where ω_i is frequency and θ_i is the phase of the ith oscillator
- Flocking theory of mobile agents with sensing and communication devices, using proximity graphs
- Fast consensus in small-worlds deals with network design problem. The problem is addressed with either design of weights or design of topology.

Applications (2/2)

- Rendezvous in space which reaches a consensus in position by a number of agents
- Distributed sensor fusion in sensor networks to implement or approximate a Kalman-filter, or estimate linear least-squares
- Distributed formation control

Information Consensus in Networked Systems

- Consider the dynamics $\dot{x}_i = u_i$ of a graph $G = (V, E)$, that reaches consensus asymptotically
- The adjacency matrix is $A = [a_{ij}]$, and the set of neighbors $N_i = j \in V : a_{ij} \neq 0$
- A dynamic graph is time-varying $G(t) = (V, E(t))$ with the $A(t)$ and the linear system is a distributed consensus algorithm

$$\dot{x}_i(t) = \sum_{j \in N-i} a_{ij}(x_j(t) - x_i(t))$$

- For undirected graphs $(a_{ij} = a_{ji})$ as $t = \infty$ results the avg of the initial states $\alpha = \frac{1}{n} \sum_i x_i(0)$

Laplacian expression

- A Laplacian representation of the dynamics is given by $\dot{x} = -Lx$, where $L = D - A$
- For undirected graphs the Laplcian satisfies the SoS property $x^{\mathsf{T}} L x = \frac{1}{2} \sum_{(i,j) \in E} a_{ij} (x_j - x_i)^2$
- By setting $\frac{1}{2} x^{\mathsf{T}} L x = \phi(x)$ we get the gradient-descent algorithm $\dot{x} = -\nabla \phi(x)$
- For an undirected graph the algorithm converges asymptotically for all initial values

Algebraic Connectivity & Spectral Properties

- According to Gershgorin theorem, eigenvalues of the Laplacian matrix L are located in a disk centered at $\Delta + 0j$ with radius $\Delta = max_i d_i$
- L is a symmetric graph w/ real eigenvalues for undirected graphs, so the λ can be ordered as

$$0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq 2\Delta$$

- The second smallest eigenvalue λ_2 is called algebraic connectivity of a graph and measures the performance of consensus

Strongly Connected and Balanced Graphs

- Strongly connected graphs have a directed path that connects any two nodes
- For a strongly connected graph $rank(L) = n - 1$ and all non-trivial eigenvalues have positive real parts
- For a strongly connected graph w/ $c \geq 1$ strongly connected components, $rank(L) = n - c$
- Balanced graph is a digraph w/ $\sum_{j \neq i} a_{ij} = \sum_{i \neq j} a_{ji}$, which means that the total weight of edges entering and leaving a node are equal for all nodes
- Another property of balanced digraphs is that $w = \underline{1}$ is a left eigenvector of their Laplacian, $\underline{1}^T L = 0$

Convergence Analysis

- For a strongly connected digraph w/ left eigenvector $\gamma = (\gamma_1, ..., \gamma_n)$ which satisfies $\gamma^\intercal L = 0$ and follows the consensus algorithm

$$\dot{x}_i(t) = \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)), \;\; x(0) = z$$

- A consensus is asymptotically reached for all initial states
- Solves the f-consensus problem with the linear function $f(z) = \frac{\gamma^\intercal z}{\gamma^\intercal \underline{1}}$
- If the digraph is balanced an avg-consensus is asymptotically reached and $\alpha = \frac{\sum_i x_i(0)}{n}$

Discrete-Time Disrtibuted Consensus Algorithm

- An iterative form of the consensus algorithm is

$$x_i(k+1) = x_i(k) + \epsilon \sum_{j \in N_i} a_{ij}(x_j(k) - x_i(k))$$

- Can be also formed as

$$x(k+1) = P(x(k))$$

- P is the Perron matrix of the graph $P = I - \epsilon L$, and $\epsilon > 0$ is the step size

Matrix Theory

Three type of matrices are introduced

1. Irreducible matrix if its associated graph is strongly connected
2. A non-negative matrix is called row (or column) stohastic if all of tis row-sums (or column-sums) are 1
3. An irreducible, stohastic matrix is primitive if it has only one eigenvalue w/ maximum modulus (maximum eigenvalue has a simple root)

Perron Matrix and Step Size Properties

For a digraph G w/ n-nodes and maximum degree $\Delta = max_i(\sum_{j \neq i} a_{ij})$, then the Perron P w/ parameters $\epsilon \in (0, \frac{1}{\Delta})$ satisfies

- P is row stohastic, non-negative matrix w/ trivial eigenvalue 1
- All eigenvalues P are in the unit circle
- If G is a balanced graph then P is a doubly stohastic graph (both row-sums and row-columns are 1)
- If G is strongly connected and the step is $0 < \epsilon < \frac{1}{\Delta}$, then P is a primitive matrix. The condition $\epsilon < \frac{1}{\Delta}$ is necessary, because if the step-size is incorrect then P would no longer be a primitive matrix (multiple eigenvalues of 1)

Convergence in Discrete-Time

Consider a network of agents w/ a strongly connected digraph following the distributed network algorithm

$$x_i(k+1) = x_i(k) + \epsilon \sum_{j \in N_i} a_{ij}(x_j(k) - x_i(k)),$$

where $0 < \epsilon < \frac{1}{\Delta}$

- A consensus is asymptotically reached for all initial states
- The consensus value is $\alpha = \sum_i w_i x_i(0)$, w/ $\sum_i w_i = \underline{1}$
- If the digraph is balanced an avg consensus is asymptotically reached $\alpha = \frac{\sum_i x_i(0)}{n}$

Performance of Consensus Algorithms

Alternative Forms of Consensus Algorithms

Weighted-Average Consensus

Consensus under Communication Time-Delays

Consensus in Dynamic Networks

Cooperation in Networked Control Systems

Collective Dynamics of Multi-Vehicle Formations

Stability of Relative Dynamics of Formations

Consensus in Complex Networks

Multi-vehicle Formation Control

References

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Thank You!