Resilient Asymptotic Consensus in Robust Networks [LeBlanc, 2013]

Jedediah Fry, George Kontoudis

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Aerospace and Ocean Engineering Department, Virginia Tech

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Outline

Introduction

Problem Formulation

Consensus Algorithm

Robust Network Topologies

Resilient Consensus Analysis

- W-MSR always satisfies safety condition for RAC
- $M[t]$, $m[t]$ maximum & minimum values of normal nodes $i \in N$

Lemma: Regardless of network topology, for each normal node $i \in N$ we get

$$x_i[t+1] \in [m(t), M(t)],$$

when using W-MSR algorithm w/ parameter F or f under F-local, F-total or f-fraction (Byzantine or malicious) models.

F-Total Malicious Model

- Characterize networks by necessary & sufficient conditions of W-MSR to succeed

Time-	Threat Models	Scope	Necessary	Sufficient
Invariant	Malicious	F-Total	(F+1,F+1)-robust	(F+1,F+1)-robust
Variant	Malicious	F-Total	-	(2F+1)-robust

F-Local Malicious Model

- Employ F-Local when total number of adversaries is large in large-scale networks

Time-	Threat Models	Scope	Necessary	Sufficient
Invariant	Malicious	F-Local	(F+1)-robust	(2F+1)-robust
Variant	Malicious	F-Local	-	(2F+1)-robust

- For every $F \in \mathbb{Z}_{>0}$ there exists a 2F-robust network that fails to reach consensus w/ W-MSR

f-Fraction Local Malicious Model

- Characterize networks by necessary & sufficient conditions of W-MSR to succeed

Time-	Threat Models	Scope	Necessary	Sufficient
Invariant	Malicious	f-Fraction Local	p'-fraction robust, $2f \leq p \leq 1$	p-fraction robust, $2f \leq p \leq 1$
Variant	Malicious	f-Fraction Local	-	p-fraction robust, $2f < p \leq 1$

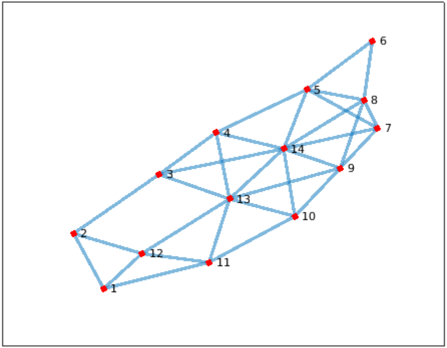
Byzantine Threat Models

Time-	Threat Models	Scope	Necessary	Sufficient
Invariant	Byzantine	F-Total & F-Local	Normal Network (F+1)-robust	Normal Network (F+1)-robust
Invariant	Byzantine	f-Fraction Local	Normal Network f-robust	Normal Network p-robust, $p > f$
Variant	Byzantine	F-Total & F-Local	-	Normal Network (F+1)-robust
Variant	Byzantine	f-Fraction Local	-	Normal Network p-robust, $2f < p \leq 1$

- Normal network D_N, induced by normal nodes

Simulation Network Topology

- For a given (2,2)-robust network we can sustain 1-Total malicious threat model

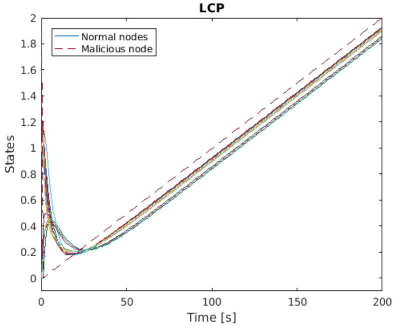


LCP algorithm

The Linear Consensus Protocol (LCP) is given by

$$x_i(t+1) = \sum_{j \in J_i(t)} w_{ij}(t) x_i^j(t)$$

- $\sum_{j=1}^{n} w_{ij}(t) = 1$
- $J_i(t) = V_i(t) \cup \{i\}$
- $x_1(t+1) = w_{1,1}x_1 + w_{1,2}x_2 + w_{1,11}x_{11} + w_{1,12}x_{12}$

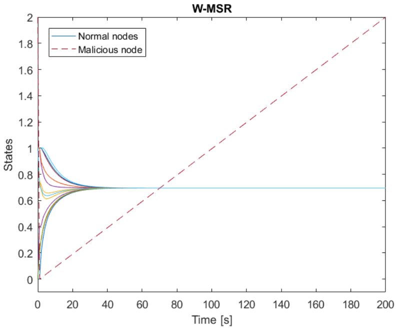


W-MSR algorithm

The W-MSR algorithm is given by

$$x_i(t+1) = \sum_{j \in J_i(t) \setminus R_i(t)} w_{ij}(t) x_i^j(t)$$

- $R_i(t)$ denotes the reduced nodes
- Using F=1 because we have 1-Total model



Node removal W-MSR

The following update law applies w/ W-MSR algorithm

Node	Sort List	Step	Values	Removed
1	{1, 2, 11, 12}	0	{0, 0, 0, 0}	-
1	{1, 2, 11, 12}	1	{0, 0.25, 0.4, 0.2}	{11}
1	{1, 2, 11, 12}	6	{0.4453, 0.4322, 0.4649, 0.4553}	{2, 11}
10	{9, 10, 11, 13, 14}	0	{1, 1, 0, 1, 2}	{11, 14}
10	{9, 10, 11, 13, 14}	1	{1.167, 1, 0.4, 0.87, 0}	{9, 14}
10	{9, 10, 11, 13, 14}	30	{0.7, 0.6921, 0.68, 0.6935, 0.28}	{13, 14}

Construction of Robust Digraphs

Theorem: Let $D = (V, \mathcal{E})$ be an (r, s)-robust digraph, $S \in \mathbb{Z}_{>0}$. Then, $D' = (V \cup \{v_{new}\}, \mathcal{E} \cup \{\varepsilon_{new}\})$ is (r, s)-robust if

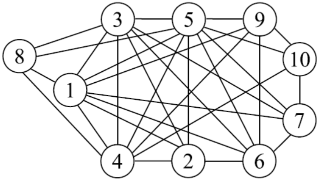
$$d^{in}_{v_{new}} \geq r + s - 1,$$

where v_{new} is the new vertex added to D, and ε_{new} is the directed edge set related to v_{new},

- Start w/ an (r, s)-robust digraph
- Add new nodes w/ incoming edges at least $r + s - 1$
- Arbitrary node selection, scale-free networks

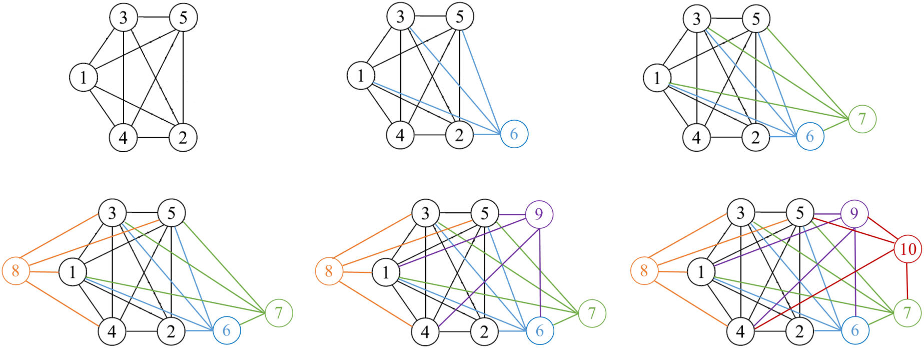
Robust Digraph Example

- Start w/ a K_5 graph (complete-fully connected on 5 nodes)
- K_5 is the only $(3, 2)$-robust 5 node digraph
- Add new nodes w/ incoming edges w/ at least $r + s - 1 = 3 + 2 - 1 = 4$
- End up w/ a $(3, 2)$-robust 10 node digraph, also 4-robust



[LeBlanc, 2013]

Robust Digraph Illustration



Properties of Robust Networks (1/2)

Observation: $(r, 1)$-robustness $\equiv$ r-robustness

Lemma: Every (r, s)-robust digraph $D = (V, \mathcal{E})$ is also (r', s')-robust when $0 \leq r' \leq r$, $1 \leq s' \leq s$.

Lemma: Suppose digraph $D = (V, \mathcal{E})$ is (r, s)-robust, spanning $D' = (V, \mathcal{E}')$, where $\mathcal{E}' = \mathcal{E} \cup \mathcal{E}''$ and $|\mathcal{E}''| \leq 0$. Then D' is (r, s)-robust.

Lemma: No digraph $D = (V, \mathcal{E})$ on n nodes is $([n/2] + 1)$-robust. On the other hand, the complete digraph $K_n = (V, \mathcal{E}_{K_n})$ is the only $([n/2], s)$-robust for $1 \leq s \leq n$ and n odd.

Properties of Robust Networks (2/2)

Lemma: Given an (r, s)-robust digraph $D = (V, \mathcal{E})$ w/ $0 \leq r \leq [n/2]$ the minimum in-degree of D is at least

$$\delta^{in}(D) \geq \begin{cases} r + s - 1 & if \;\; s < r; \\ 2r - 2 & if \;\; s \geq r. \end{cases}$$

Lemma: Given an (r, s)-robust (p-fraction robust) digraph D, let D' be the digraph produced by removing up to k incoming edges from each node in D, where $0 \leq k \leq r$ ($0 \leq q < p \leq 1$). Then D' is $(r - k, s)$-robust ($(p - q)$-fraction robust).

Theorem: Suppose $D = (V, \mathcal{E})$ is an r-robust (or (r, r)-robust) digraph, w/ $0 \leq r \leq [n/2]$ (or $3 \leq r \leq [n/2]$). Then the underlying graph G_D is at least r-connected (or $[3r/2] - 1$-connected).

References

H. LeBlanc, H. Zhang, X. Koutsoukos, S. Sundaram (2013)
Resilient asymptotic consensus in robust networks
IEEE Journal on Selected Areas in Communications, 31(4), 766-781, 2013.

Thank You!