

LQR-Trees [[Tedrake, 2010](#)]

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Outline

Motivation

Direct Computation of Lyapunov Functions

- Lyapunov Functions

- Sum of Squares Validation

- Complementary - Pontryagin's Principle

Linear Feedback Design and Verification

- Continuous Time-Invariant LQR

- State LQR Verification

- Trajectory Optimization

- Continuous Time-Variant LQR

LQR-Tree Algorithm

- LQR-Tree Algorithm Characteristics

- LQR-Tree Implementation

Conclusions

References

Motivation

- ▶ Design robust algorithms for non-linear feedback motion planning
- ▶ Non-linear underactuated systems such as robot manipulator or bipedal walking
- ▶ Computation of planning regions of attraction (funnels) for non-linear underactuated dynamical systems
- ▶ Applicable to real robots

Definition of Lyapunov Functions

For a given dynamical system

$$\dot{x} = f(x), f(0) = 0$$

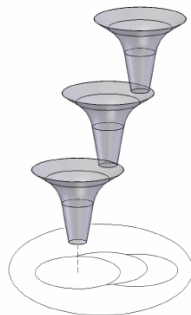
a Lyapunov function is $V(x)$, $V \in C$ where

- ▶ $V(x) > 0$, positive definite
- ▶ $\dot{V}(x) = \frac{dV}{dx} \frac{dx}{dt} < 0$, negative definite

If conditions met in some state space ball B_r , then origin is a.s.

Sequential Composition of Lyapunov Functions

- ▶ Each funnel acts like a valid Lyapunov function
- ▶ A.s. of each Lyapunov falls in the region of attraction of the next lower level
- ▶ The lowest function stabilizes in the goal point



Sequential composition of funnels
[Burridge, 1999]

Sums of Squares

We want to check inequalities and validate Lyapunov functions using sums-of-squares (SoS) method [[Parrilo, 2000](#)]

- $x^4 + 2x^3 + 3x^2 - 2x + 2 \geq 0, \forall x \in \mathbb{R}$, by employing SoS

$$x^4 + 2x^3 + 3x^2 - 2x + 2 = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = X^T A X$$

- Eigenvalues of A are $\lambda_1 = 3.88, \lambda_2 = 1.65, \lambda_3 = 0.47$, so the inequality stands $\forall x \in \mathbb{R}$

Sums of Squares Properties

General structure of (SoS) for a 4-*th* order polynomial is

$$fx^4 + 2ex^3 + (d + 2c)x^2 + 2bx + a = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

- ▶ Extend to multivariable polynomials
- ▶ Check non-negativity by searching positive semidefinite matrix

Feedback Synthesis by SoS Optimization

Given a system $\dot{x} = f(x) + g(x)u$ we want to generate

- ▶ Feedback control law $u = \pi(x)$
- ▶ Lyapunov fcn $V(x)$, s.t. $V(x) > 0$, $\dot{V}(x) = \frac{\partial V}{\partial x} \frac{\partial x}{\partial t} = \frac{\partial V}{\partial x} \dot{x} < 0$

BUT, this is a difficult problem as the set of $V(x)$, $\pi(x)$ may not be convex sets

- ▶ Rely on LQR synthesis
- ▶ Design a series of locally-valid controllers
- ▶ Compose these controllers utilizing feedback motion planning

The Minimum Principle

The first order or necessary condition for optimality is called *Maximum (Minimum) Principle*

- ▶ Given a function we want to minimize $f(x, y, z)$ on a level surface (constraint) $g(x, y, z)$ we get

$$\nabla f = \lambda \nabla g$$

- ▶ To convert a constrained problem to an unconstrained we construct the Hamiltonian function

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^\top f(x, u, t)$$

Goal Stabilization

- For a given non-linear dynamical system

$$\dot{x} = f(x, u, t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

- Set goal state x_G , where $f(x_G, u_G) = 0$, and $\bar{x} = x - x_G$,
 $\bar{u} = u - u_G$
- Linearize around (x_G, u_G) , $\bar{x} \approx A\bar{x}(t) + B\bar{u}(t)$
- Infinite horizon LQR minimum energy cost-to-go fcn
(performance index)

$$J_\infty = \int_0^\infty [\bar{x}^\top(t)Q\bar{x}(t) + \bar{u}^\top(t)R\bar{u}(t)]dt,$$

$$Q = Q^\top \geq 0, R = R^\top > 0$$

Hamiltonian System

Set the Hamiltonian

$$H(x, u, t) = L(x, u, t) + \lambda^\top f(x, u, t)$$

- State equation

$$\dot{x} = \frac{\partial H}{\partial \lambda}$$

- Costate equation

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \frac{\partial f^\top}{\partial x} \lambda + \frac{\partial L}{\partial x}$$

- Stationarity condition

$$0 = \frac{\partial H}{\partial u} = \frac{\partial f^\top}{\partial u} \lambda + \frac{\partial L}{\partial u}$$

Riccati equation and Optimal Control Law

- ▶ Infinite horizon LQR problem results the optimal cost

$$J^*(\bar{x}) = \bar{x}^T S \bar{x}$$

- ▶ $S > 0$, w/ the solution given from ARE

$$0 = Q - SBR^{-1}B^T S + SA + A^T S$$

- ▶ Optimal feedback closed loop control policy

$$\bar{u}^* = -R^{-1}B^T S \bar{x} = -K \bar{x}$$

Goal State Convergence

The domain of attraction of the LQR over some sub-level set

$$B_G(\rho) = \{x | 0 \leq V(x) \leq \rho\}$$

To guarantee a.s. we require $V(x)$ to be a valid Lyapunov function

- ▶ $V(x) > 0, \quad x \in B_G(\rho)$
- ▶ $\dot{V}(x) < 0, \quad x \in B_G(\rho)$

Assign $V(x) = J^*(\bar{x}) = \frac{1}{2}\bar{x}^T S \bar{x}$

- ▶ By definition positive definite
- ▶ $\dot{V}(x) = \dot{J}(\bar{x}) = \frac{dV}{dx} \frac{dx}{dt} = 2\bar{x}^T S \dot{x} = 2\bar{x}^T S f(x_G + \bar{x}, u_G - K\bar{x})$

Lyapunov Verification Using SoS

We require

$$j^*(\bar{x}) < 0, \quad \forall \bar{x} \neq 0 \in B_G(\rho), \quad j^*(0) = 0$$

- First, modify the inequality from negative to non-positive

$$j^*(\bar{x}) \leq -\epsilon \|\bar{x}\|_2^2, \quad \forall \bar{x} \in B_G(\rho), \quad \epsilon \in \mathbb{R}^+$$

- Second, include the constraint with Lagrange multiplier $h(\cdot)$

$$j^*(\bar{x}) + h(\bar{x})(\rho - J^*(\bar{x})) \leq -\epsilon \|\bar{x}\|_2^2$$

Lagrange Multiplier Searching

- If $f^{(cl)}(x, u) = f(x, u_G - K(x - x_g))$, search for $h(\cdot)$ polynomial with sufficient order for $J^*(\bar{x})$, using SoS

$$\begin{aligned} & \text{find} && h(\bar{x}) \\ & \text{subject to} && \hat{J}^*(\bar{x}) + h(\bar{x})(\rho - J^*(\bar{x})) \leq -\epsilon \|\bar{x}\|_2^2 \\ & && h(\bar{x}) \geq 0 \end{aligned}$$

- If $f^{(cl)}(x) \approx \hat{f}^{(cl)}(\bar{x})$, where $\hat{f}^{(cl)}(\bar{x})$ is the Taylor expansion (algebraic approximation) and $\hat{J}(\bar{x}) = 2\bar{x}^\top S \hat{f}^{(cl)}(\bar{x})$

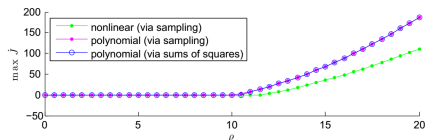
$$\begin{aligned} & \text{find} && h(\bar{x}) \\ & \text{subject to} && \hat{J}(\bar{x}) + h(\bar{x})(\rho - \hat{J}^*(\bar{x})) \leq -\epsilon \|\bar{x}\|_2^2 \\ & && h(\bar{x}) \geq 0 \end{aligned}$$

Optimization for ρ

Set a convex optimization problem for the region of attraction

$$\begin{aligned} & \max \quad \rho \\ & \text{subject to} \quad \hat{J}^*(\bar{x}) + h(\bar{x})(\rho - \hat{J}^*(\bar{x})) \leq -\epsilon \|\bar{x}\|_2^2 \\ & \quad \quad \quad h(\bar{x}) \geq 0 \\ & \quad \quad \quad \rho > 0 \end{aligned}$$

- At each step the Lagrange multiplier searching is performed
- If the program is feasible ρ increased



Polynomial verification of damped single pendulum [Tedrake, 2010]

Trajectory Optimization

- ▶ Trajectory design w/ RRT or other motion planning technique
 1. Don't guarantee stability w/ any initial condition
 2. Need to design a new trajectory every time
 3. Can deal problems w/ up to 5 states
- ▶ Increase set of states out of ρ to reach goal
- ▶ Stabilize the trajectory using LQR optimal controller
- ▶ Initialize out of the domain of attraction and optimize the cost function

$$J = \int_{t_0}^{t_f} [1 + u_0^T R u_0] dt$$

Discarded Sampling Points

- ▶ Trajectory optimization algorithms are local
- ▶ Employed to test validity of sampling points
- ▶ If trajectory optimization fails the point is discarded
- ▶ Algorithms have specific time restriction to improve faster sampling
- ▶ Final tree constraint specifies whether the computed trajectory will connect w/ the tree

$$\min_{x' \in \mathcal{T}_{k-1}} \|x_0(t_f) - x'\|_2 = 0,$$

\mathcal{T}_k : Set of all points in the tree after k-iterations

Trajectory Stabilization

- For a given non-linear dynamical system

$$\dot{x} = f(x, u, t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad t \in [t_0, t_f]$$

- A nominal trajectory $x_0(t)$, $u_0(t)$ yields $\bar{x}(t) = x(t) - x_0(t)$,
 $\bar{u}(t) = u(t) - u_0(t)$
- Linearize around the trajectory $(x_0(t), u_0(t))$,
 $\bar{x} \approx A(t)\bar{x}(t) + B(t)\bar{u}(t)$
- Finite time horizon LQR minimum energy cost-to-go fcn
(performance index)

$$J(\bar{x}', t') = \bar{x}^\top(t_f)S(t_f)\bar{x}(t_f) + \int_{t'}^{t_f} [\bar{x}^\top(t)Q\bar{x}(t) + \bar{u}^\top(t)R\bar{u}(t)]dt,$$

$$Q_f = Q_f^\top > 0, \quad Q = Q^\top \geq 0, \quad R = R^\top > 0, \quad \bar{x}(t)' = \bar{x}'$$

Riccati equation and Optimal Control Law

- ▶ Infinite horizon LQR problem results the optimal cost

$$J^*(\bar{x}, t) = \bar{x}^\top S(t) \bar{x}$$

- ▶ $S(t) = S(t)^\top > 0$, w/ the solution given from RE

$$\dot{S} = Q - SBR^{-1}B^\top S + SA + A^\top S$$

- ▶ Optimal feedback closed loop control policy

$$\bar{u}^*(t) = -R^{-1}B^\top(t)S(t)\bar{x}(t) = -K(t)\bar{x}(t)$$

TV-LQR Verification

- ▶ For trajectory stabilization a bounded goal domain is defined (not a.s.)

$$B_f = \{x | 0 \leq V(x, t_f) \leq \rho_f\}$$

- ▶ Search for time-varying domains

$$B(\rho(\cdot), t) = \{x | 0 \leq V(x, t) \leq \rho(t)\}$$

- ▶ This sublevel set should guarantee for the closed-loop system

$$x(t) \in B(\rho(\cdot), t) \Rightarrow x(t_f) \in B_G \quad \forall t \in [t_0, t_f]$$

Function of Region of Attraction $\rho(t)$

Time-Invariant case

$$J^*(\bar{x}) \leq \rho, \quad \rho \in \mathbb{R}^+$$

$$\dot{J}(\bar{x}) \leq 0, \quad \dot{J}(0) = 0$$

- ▶ $J^*(\bar{x}) = V(x)$: Lyapunov fcn
- ▶ ρ : domain of attraction (T-I)

Time-Variant case

$$J^*(\bar{x}, t) \leq \rho(t)$$

$$\dot{J}(\bar{x}, t) \leq \dot{\rho}(t), \quad \dot{J}(x_0, t) = 0$$

- ▶ $J^*(\bar{x}, t) = V(x, t)$: Lyapunov fcn
- ▶ At every time instant we assign a Lyapunov fcn
- ▶ $\rho(t)$: domain of attraction (T-V)
- ▶ Conditions assure that $V(x, t)$ decreases faster than $\rho(t)$ along the trajectory

Time-Varying Lyapunov Function

- ▶ We assign the positive definite j^* as our Lyapunov fcn

$$V(x, t) = J^*(\bar{x}, t) = \bar{x}^T S(t) \bar{x}$$

- ▶ We get the bounded goal domain

$$B_f = \{x | 0 \leq \bar{x}^T S(t) \bar{x} \leq \rho_f\}$$

- ▶ The time derivative of the assigned Lyapunov fcn yields

$$\dot{J}^*(\bar{x}, t) = \bar{x}^T \dot{S}(t) \bar{x} + 2\bar{x}^T S(t) f(\hat{x}_0(t) + \bar{x}, \hat{u}_0(t) - K(t)\bar{x})$$

Selection of $\rho(t)$

- ▶ We desire the largest domain of attraction $\rho(t)$
- ▶ Initially we approximate $\rho(t)$ w/ a linear polynomial

$$\rho_k(t) = \beta_{1k}t + \beta_{0k}$$

$$\rho(t) = \begin{cases} \rho_k(t), & \forall t \in [t_k, t_{k+1}) \\ \rho_f, & t = t_f \end{cases}$$

- ▶ We require for the approximation of domain of attraction

$$\rho_k(t_{k+1}) = \beta_{1k}t_{k+1} + \beta_{0k} \leq \rho(t_{k+1})$$

$$J^*(\bar{x}, t) = \rho_k(t) \Rightarrow \dot{J}^*(\bar{x}, t) \leq \dot{\rho}_k(t) = \beta_{1k} \quad \forall t \in [t_k, t_{k+1}),$$

where \dot{J}^* is the Taylor expansion of the dynamics

Lagrange Multiplier Searching

Approximately verify the second condition of Lyapunov fcns w/ SoS

$$\begin{aligned} &\text{find} \quad h_1(\bar{x}, t), h_2(\bar{x}, t), h_3(\bar{x}, t), \\ &\text{subject to} \quad \dot{J}^*(\bar{x}, t) - \dot{\rho}_k(t) + h_1(\bar{x}, t)(\rho_k(t) - J^*(\bar{x}, t)) + \\ &\quad \quad \quad + h_2(\bar{x}, t)(t - t_k) + h_3(\bar{x}, t)(t_{k+1} - t) \leq 0, \\ &\quad \quad \quad h_2(\bar{x}, t) \geq 0, \\ &\quad \quad \quad h_3(\bar{x}, t) \geq 0 \end{aligned}$$

- ▶ $h_1(\bar{x}, t)$ should be eliminated if the equality constraint holds
- ▶ The Lagrange multipliers should be polynomials of sufficient order to counteract $\dot{J}^*(\bar{x}, t)$

Optimization for $\rho(t)$

Set a convex optimization problem for the region of attraction

$$\begin{aligned}
 & \max_{\beta \cdot k} \quad \rho_k(t_k) = \beta_{1k}t + \beta_{0k}, \quad k = N - 1, \dots, 1 \\
 & \text{subject to} \quad \rho_k(t_{k+1}) \leq \rho(t_{k+1}) \\
 & \quad \quad \quad \dot{J}^*(\bar{x}, t) - \dot{\rho}_k(t) + h_1(\bar{x}, t)(\rho_k(t) - J^*(\bar{x}, t)) + \\
 & \quad \quad \quad + h_2(\bar{x}, t)(t - t_k) + h_3(\bar{x}, t)(t_{k+1} - t) \leq 0, \\
 & \quad \quad \quad h_2(\bar{x}, t) \geq 0, \\
 & \quad \quad \quad h_3(\bar{x}, t) \geq 0
 \end{aligned}$$

LQR-Tree Algorithm Characteristics

- ▶ Randomized backward growing tree technique in the same fashion as RRT
- ▶ Large web of stabilized controllers for the motion planning, given various initial conditions
- ▶ Information for the funnels allows for immediately discard sample points which are inside the verified region

Objective: *Find the set of stabilizable controllers for all possible initial conditions from which the specific final goal state can be reached.*

LQR-Tree Algorithm

Goal: Stabilize (x_G, u_G)

1. Stabilize x_G, u_G locally w/ TI-LQR
+ Compute region of attraction ρ_c
2. Pick a random point x_s
If already in the funnel discard
3. Perform trajectory optimization w/ constraints $x(0) = x_s$,
 $\text{dist}(x(t_f), (\text{tree})) = 0$
+ Compute feedback cost-to-go TV-LQR
+ Compute largest funnel to land in the goal region
4. Repeat

LQR-Tree Pseudo-code

Algorithm 1 LQR-Tree (f, x_G, u_G, Q, R) [[Tedrake, 2010](#)]

```
1:  $[A, B] \Leftarrow$  linearization of  $f(x, u)$  around  $x_G, u_G$ 
2:  $[K, S] \Leftarrow$  TI-LQR( $A, B, Q, R$ )
3:  $\rho_c \Leftarrow$  domain of attraction computation
4:  $T.\text{init}\{x_G, u_G, S, K, \rho_c, \text{NULL}\}$ 
5: for  $k = 1$  to  $K$  do
6:    $x_{\text{rand}} \Leftarrow$  random sample
7:   if  $x_{\text{rand}} \in C_k$  then
8:     continue
9:   end if
10:   $[t, x_0(t), u_0(t)]$  from trajectory optimization
11:  if  $x_0(t_f) \notin \mathcal{T}_k$  then
12:    continue
13:  end if
14:   $[K(t), S(t)] \Leftarrow$  TV-LQR( $A(t), B(t), Q, R$ )
15:   $\rho_c(t) \Leftarrow$  polynomial domain of attraction computation
16:   $i \Leftarrow$  pointer to branch in tree
17:   $T.\text{add-branch}(x_0(t), u_0(t), S(t), K(t), \rho_c, i)$ 
18: end for
```

Overview

- ▶ Lyapunov functions and SOS
- ▶ LQR, its design and verification
- ▶ Overview of Motion planning
- ▶ RRT algorithm
- ▶ Trajectory Optimization and Time varying LQR
- ▶ LQR trees algorithm
- ▶ LQR trees in action (Simulation)

Conclusions

- ▶ Computation of Lyapunov fcns w/ SoS let us build feedback motion planning algorithms
- ▶ Deal with non-linear systems w/ up to 20 states
- ▶ Offline computation of the problem
- ▶ Need to know the dynamics of the system

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Thank You!