LQR-Trees [Tedrake, 2010]

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Outline

Motivation

- Evaluate regions of stability for nonlinear systems
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Lyapunov Functions

For a given dynamical system

$$\dot{x} = f(x), f(0) = 0$$

a Lyapunov function is $V(x)$, $V \in C$ where

- $V(x) > 0$, positive definite
- $\dot{V}(x) = \frac{dV}{dx}\frac{dx}{dt} < 0$, negative definite

If conditions met in some state space ball B_r, then origin is a.s.

Sums of Squares

We want to check inequalities using sums-of-squares (SoS) method

For a given polynomial $x^4 + 2x^3 + 3x^2 - 2x + 2 \geq 0, \ \ \forall x \in \mathbb{R}$, we can employ SoS

$$x^4 + 2x^3 + 3x^2 - 2x + 2 = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^\intercal \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = X^\intercal A X$$

The eigenvalues of A are $\lambda_1 = 3.88$, $\lambda_2 = 1.65$, $\lambda_1 = 0.47$, so the inequality stands $\forall x \in \mathbb{R}$

Sums of Squares Properties

The general structure of (SoS) for a 4-*th* order polynomial is

$$fx^4 + 2ex^3 + (d+2c)x^2 + 2bx + a = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^{\intercal} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

- Extend to multivariable polynomials
- Check non-negativity by searching positive semidefinite matrix

Feedback Synthesis by SoS Optimization

Given a system $\dot{x} = f(x) + g(x)u$ we want to simultaneously generate

- Feedback control law $u = \pi(x)$
- Lyapunov fcn $V(x)$, s.t. $V(x) > 0$, $\dot{V}(x) = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x}\dot{x}$ be negative definite

BUT, this is a difficult problem as the set of $V(x)$, $\pi(x)$ may not be convex sets

- Rely on LQR synthesis
- Design a series of locally-valid controllers
- Compose these controllers utilizing feedback motion planning

Non-linear System

- For a given non-linear dynamical system

$$\dot{x} = f(x, u), \;\; x \in \mathbb{R}^n, \;\; u \in \mathbb{R}^m$$

- Set goal state x_G, where $f(x_G, u_G) = 0$, and $\bar{x} = x - x_G$, $\bar{u} = u - u_G$
- Linearize around (x_G, u_G), $\bar{x} \approx A\bar{x}(t) + B\bar{u}(t)$
- Define infinite horizon LQR minimum energy cost-to-go fcn

$$J(\bar{x}') = \int_0^\infty [\bar{x}^\mathsf{T}(t) Q \bar{x}(t) + \bar{u}^\mathsf{T}(t) R \bar{u}(t)] dt,$$

$$Q = Q^\mathsf{T} \geq 0, R = R^\mathsf{T} > 0, \bar{x}(0) = \bar{x}'$$

Stabilizing Goal State

- Solution of inf horizon LQR problem results the optimal cost

$$J^*(\bar{x}) = \bar{x}^\intercal S \bar{x}$$

- S is a positive definite matrix, w/ the solution given from ARE

$$0 = Q - SBR^{-1}B^\intercal S + SA + A^\intercal S$$

- The optimal feedback closed loop control policy is given

$$\bar{u}^* = -R^{-1}B^\intercal S\bar{x} = -Kx$$

Time-Invariant LQR Verification

Conclusions

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Summary

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References

Tedrake Russ, Manchester Ian, Tobenkin Mark and Roberts John
LQR-trees: Feedback motion planning via sums-of-squares verification
International Journal of Robotics Researchs, 1038–1052, SAGE Publications, 2010.

Thank You!