LQR-Trees [Tedrake, 2010]

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Outline

Motivation

Direct Computation of Lyapunov Functions
- Lyapunov Functions
- Sum of Squares Validation
- Complementary - Pontryagin's Principle

Linear Feedback Design and Verification
- Continuous Time-Invariant LQR
- State LQR Verification
- Trajectory Optimization
- Continuous Time-Variant LQR

LQR-Tree Algorithm

Conclusions

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Motivation

- Design robust algorithms for non-linear feedback motion planning
- Non-linear underactuated systems such as robot manipulator or bipedal walking
- Computation of planning regions of attraction (funnels) for non-linear underactuated dynamical systems
- Applicable to real robots

Definition of Lyapunov Functions

For a given dynamical system

$$\dot{x} = f(x), f(0) = 0$$

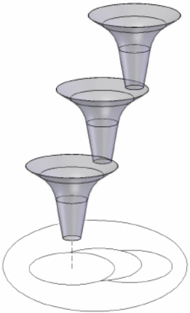
a Lyapunov function is $V(x)$, $V \in C$ where

- $V(x) > 0$, positive definite
- $\dot{V}(x) = \frac{dV}{dx}\frac{dx}{dt} < 0$, negative definite

If conditions met in some state space ball B_r, then origin is a.s.

Sequential Composition of Lyapunov Functions

- Each funnel acts like a valid Lyapunov function
- A.s. of each Lyapunov falls in the region of attraction of the next lower level
- The lowest function stabilizes in the goal point



Sequential composition of funnels [Burridge, 1999]

Sums of Squares

We want to check inequalities and validate Lyapunov functions using sums-of-squares (SoS) method [Parrilo, 2000]

- $x^4 + 2x^3 + 3x^2 - 2x + 2 \geq 0, \;\; \forall x \in \mathbb{R}$, by employing SoS

$$x^4+2x^3+3x^2-2x+2 = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^\intercal \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = X^\intercal A X$$

- Eigenvalues of A are $\lambda_1 = 3.88$, $\lambda_2 = 1.65$, $\lambda_1 = 0.47$, so the inequality stands $\forall x \in \mathbb{R}$

Sums of Squares Properties

General structure of (SoS) for a 4-*th* order polynomial is

$$fx^4 + 2ex^3 + (d + 2c)x^2 + 2bx + a = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

- Extend to multivariable polynomials
- Check non-negativity by searching positive semidefinite matrix

Feedback Synthesis by SoS Optimization

Given a system $\dot{x} = f(x) + g(x)u$ we want to generate

- Feedback control law $u = \pi(x)$
- Lyapunov fcn $V(x)$, s.t. $V(x) > 0$, $\dot{V}(x) = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x}\dot{x} < 0$

BUT, this is a difficult problem as the set of $V(x)$, $\pi(x)$ may not be convex sets

- Rely on LQR synthesis
- Design a series of locally-valid controllers
- Compose these controllers utilizing feedback motion planning

The Minimum Principle

The first order or necessary condition for optimality is called *Maximum (Minimum) Principle*

- Given a function we want to minimize $f(x, y, z)$ on a level surface (constraint) $g(x, y, z)$ we get

$$\nabla f = \lambda \nabla g$$

- To convert a constrained problem to an unconstrained we construct the Hamiltonian function

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^{\mathsf{T}} f(x, u, t)$$

Goal Stabilization

- For a given non-linear dynamical system

$$\dot{x} = f(x, u, t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

- Set goal state x_G, where $f(x_G, u_G) = 0$, and $\bar{x} = x - x_G$, $\bar{u} = u - u_G$
- Linearize around (x_G, u_G), $\bar{x} \approx A\bar{x}(t) + B\bar{u}(t)$
- Infinite horizon LQR minimum energy cost-to-go fcn (performance index)

$$J_\infty = \int_0^\infty [\bar{x}^\intercal(t) Q \bar{x}(t) + \bar{u}^\intercal(t) R \bar{u}(t)] dt,$$

$$Q = Q^\intercal \geq 0, R = R^\intercal > 0$$

Hamiltonian System

Set the Hamiltonian

$$H(x, u, t) = L(x, u, t) + \lambda^\intercal f(x, u, t)$$

- State equation

$$\dot{x} = \frac{\partial H}{\partial \lambda}$$

- Costate equation

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \frac{\partial f^\intercal}{\partial x}\lambda + \frac{\partial L}{\partial x}$$

- Stationarity condition

$$0 = \frac{\partial H}{\partial u} = \frac{\partial f^\intercal}{\partial u}\lambda + \frac{\partial L}{\partial u}$$

Riccati equation and Optimal Control Law

- Infinite horizon LQR problem results the optimal cost

$$J^*(\bar{x}) = \bar{x}^\intercal S \bar{x}$$

- $S > 0$, w/ the solution given from ARE

$$0 = Q - SBR^{-1}B^\intercal S + SA + A^\intercal S$$

- Optimal feedback closed loop control policy

$$\bar{u}^* = -R^{-1}B^\intercal S\bar{x} = -K\bar{x}$$

Goal State Convergence

The domain of attraction of the LQR over some sub-level set

$$B_G(\rho) = \{x | 0 \leq V(x) \leq \rho\}$$

To guarantee a.s. we require $V(x)$ to be a valid Lyapunov function

- $V(x) > 0, \;\; x \in B_G(\rho)$
- $\dot{V}(x) < 0, \;\; x \in B_G(\rho)$

Assign $V(x) = J^*(\bar{x}) = \frac{1}{2}\bar{x}^\intercal S \bar{x}$

- By definition positive definite
- $\dot{V}(x) = \dot{J}(\bar{x}) = \frac{dV}{dx}\frac{dx}{dt} = 2\bar{x}^\intercal S \dot{x} = 2\bar{x}^\intercal S f(x_G + \bar{x}, u_G - K\bar{x})$

Lyapunov Verification Using SoS

We require

$$\dot{J}^*(\bar{x}) < 0, \quad \forall \bar{x} \neq 0 \in B_G(\rho), \quad \dot{J}^*(0) = 0$$

- First, modify the inequality from negative to non-positive

$$\dot{J}^*(\bar{x}) \leq -\epsilon ||\bar{x}||_2^2, \quad \forall \bar{x} \in B_G(\rho), \quad \epsilon \in \mathbb{R}^+$$

- Second, include the constraint with Lagrange multiplier $h(\cdot)$

$$\dot{J}^*(\bar{x}) + h(\bar{x})(\rho - J^*(\bar{x})) \leq -\epsilon ||\bar{x}||_2^2$$

Lagrange Multiplier Searching

- If $f^{(cl)}(x,u) = f(x, u_G - K(x - x_g))$, search for $h(\cdot)$ polynomial with sufficient order for $\dot{J}^*(\bar{x})$, using SoS

$$\begin{aligned} \text{find} \quad & h(\bar{x}) \\ \text{subject to} \quad & \dot{J}^*(\bar{x}) + h(\bar{x})(\rho - J^*(\bar{x})) \leq -\epsilon||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \end{aligned}$$

- If $f^{(cl)}(x) \approx \hat{f}^{(cl)}(\bar{x})$, where $\hat{f}^{(cl)}(\bar{x})$ is the Taylor expansion (algebraic approximation) and $\hat{\dot{J}}(\bar{x}) = 2\bar{x}^{\intercal} S \hat{f}^{(cl)}(\bar{x})$

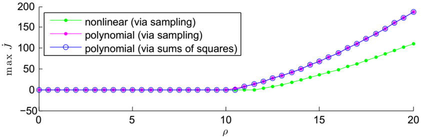
$$\begin{aligned} \text{find} \quad & h(\bar{x}) \\ \text{subject to} \quad & \hat{\dot{J}}(\bar{x}) + h(\bar{x})(\rho - \hat{J}^*(\bar{x})) \leq -\epsilon||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \end{aligned}$$

Optimization for ρ

Set a convex optimization problem for the region of attraction

$$\begin{aligned} \max \quad & \rho \\ \text{subject to} \quad & \hat{J}^*(\bar{x}) + h(\bar{x})(\rho - \hat{J}^*(\bar{x})) \leq -\epsilon||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \\ & \rho > 0 \end{aligned}$$

- At each step the Lagrange multiplier searching is performed
- If the program is feasible ρ increased



Polynomial verification of damped single pendulum [Tedrake, 2010]

Trajectory Optimization

- Trajectory design w/ RRT or other motion planning technique
  1. Don't guarantee stability w/ any initial condition
  2. Need to design a new trajectory every time
  3. Can deal problems w/ up to 5 states
- Increase set of states out of ρ to reach goal
- Stabilize the trajectory using LQR optimal controller
- Initialize out of the domain of attraction and optimize the cost function

$$J = \int_{t_0}^{t_f} [1 + u_0^T R u_0] dt$$

Trajectory Stabilization

- For a given non-linear dynamical system

$$\dot{x} = f(x, u, t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad t \in [t_0, t_f]$$

- A given trajectory $x_0(t)$, $u_0(t)$,
where $\bar{x}(t) = x(t) - x_0(t)$, $\bar{u}(t) = u(t) - u_0(t)$
- Linearize around the trajectory $(x_0(t), u_0(t))$,
$\bar{x} \approx A(t)\bar{x}(t) + B(t)\bar{u}(t)$
- Finite time horizon LQR minimum energy cost-to-go fcn (performance index)

$$J(\bar{x}', t') = \bar{x}^\intercal(t_f)S(t_f)\bar{x}(t_f) + \int_{t'}^{t_f} [\bar{x}^\intercal(t)Q\bar{x}(t) + \bar{u}^\intercal(t)R\bar{u}(t)]dt,$$

$$Q_f = Q_f^\intercal > 0, \quad Q = Q^\intercal \geq 0, \quad R = R^\intercal > 0, \quad \bar{x}(t)' = \bar{x}'$$

Riccati equation and Optimal Control Law

- Infinite horizon LQR problem results the optimal cost

$$J^*(\bar{x}, t) = \bar{x}^\intercal S(t)\bar{x}$$

- $S(t) = S(t)^\intercal > 0$, w/ the solution given from RE

$$\dot{S} = Q - SBR^{-1}B^\intercal S + SA + A^\intercal S$$

- Optimal feedback closed loop control policy

$$\bar{u}^*(t) = -R^{-1}B^\intercal(t)S(t)\bar{x}(t) = -K(t)\bar{x}(t)$$

TV-LQR Verification

- For trajectory stabilization a bounded goal domain is defined (not a.s.)

$$B_f = \{x|0 \leq V(x, t_f) \leq \rho_f\}$$

- Search for time-varying domains

$$B(\rho(\cdot), t) = \{x|0 \leq V(x, t) \leq \rho(t)\}$$

- This sublevel set should guarantee for the closed-loop system

$$x(t) \in B(\rho(\cdot), t) \Rightarrow x(t_f) \in B_G \quad \forall t \in [t_0, t_f]$$

Function of Region of Attraction $\rho(t)$

Time-Invariant case

$$J^*(\bar{x}) \leq \rho, \quad \rho \in \mathbb{R}^+$$

$$\dot{J}(\bar{x}) \leq 0, \quad \dot{J}(0) = 0$$

- $J^*(\bar{x}) = V(x)$: Lyapunov fcn
- ρ: domain of attraction (T-I)

Time-Variant case

$$J^*(\bar{x}, t) \leq \rho(t)$$

$$\dot{J}(\bar{x}, t) \leq \dot{\rho}(t), \quad \dot{J}(x_0, t) = 0$$

- $J^*(\bar{x}, t) = V(x, t)$: Lyapunov fcn
- At every time instant we assign a Lyapunov fcn
- $\rho(t)$: domain of attraction (T-V)
- Conditions assure that $V(x, t)$ decreases faster than $\rho(t)$ along the trajectory

Time-Varying Lyapunov Function

- We assign the positive definite j^* as our Lyapunov fcn

$$V(x,t) = J^*(\bar{x},t) = \bar{x}^\mathsf{T} S(t)\bar{x}$$

- We get the bounded goal domain

$$B_f = \{x|0 \leq \bar{x}^\mathsf{T} S(t)\bar{x} \leq \rho_f\}$$

- The time derivative of the assigned Lyapunov fcn yields

$$\dot{J}^*(\bar{x},t) = \bar{x}^\mathsf{T}\dot{S}(t)\bar{x} + 2\bar{x}^\mathsf{T} S(t) f(\hat{x}_0(t) + \bar{x}, \hat{u}_0(t) - K(t)\bar{x})$$

Selection of $\rho(t)$

- We desire the largest domain of attraction $\rho(t)$
- Initially we approximate $\rho(t)$ w/ a linear polynomial

$$\rho_k(t) = \beta_{1k} t + \beta_{0k}$$

$$\rho(t) = \begin{cases} \rho_k(t), & \forall t \in [t_k, t_{k+1}) \\ \rho_f, & t = t_f \end{cases}$$

- We require for the approximation of domain of attraction

$$\rho_k(t_{k+1}) = \beta_{1k} t_{k+1} + \beta_{0k} \leq \rho(t_{k+1})$$

$$J^*(\bar{x}, t) = \rho_k(t) \Rightarrow \dot{\hat{J}}^*(\bar{x}, t) \leq \dot{\rho}_k(t) = \beta_{1k} \quad \forall t \in [t_k, t_{k+1}),$$

where $\dot{\hat{J}}^*$ is the Taylor expansion of the dynamics

Lagrange Multiplier Searching

Approximately verify the second condition of Lyapunov fcns w/ SoS

$$
\begin{aligned}
\text{find} \quad & h_1(\bar{x}, t), h_2(\bar{x}, t), h_3(\bar{x}, t), \\
\text{subject to} \quad & \dot{\hat{J}}^*(\bar{x}, t) - \dot{\rho}_k(t) + h_1(\bar{x}, t)(\rho_k(t) - J^*(\bar{x}, t)) + \\
& \qquad + h_2(\bar{x}, t)(t - t_k) + h_3(\bar{x}, t)(t_{k+1} - t) \leq 0, \\
& h_2(\bar{x}, t) \geq 0, \\
& h_3(\bar{x}, t) \geq 0
\end{aligned}
$$

- $h_1(\bar{x}, t)$ should be eliminated if the equality constraint holds
- The Lagrange multipliers should be polynomials of sufficient order to counteract $\dot{\hat{J}}^*(\bar{x}, t)$

Optimization for $\rho(t)$

Set a convex optimization problem for the region of attraction

$$\begin{aligned}
\max_{\beta_{\cdot k}} \quad & \rho_k(t_k) = \beta_{1k} t + \beta_{0k}, \qquad k = N-1, \ldots, 1 \\
\text{subject to} \quad & \rho_k(t_{k+1}) \leq \rho(t_{k+1}) \\
& \dot{\hat{J}}^*(\bar{x}, t) - \dot{\rho}_k(t) + h_1(\bar{x}, t)(\rho_k(t) - J^*(\bar{x}, t)) + \\
& \qquad + h_2(\bar{x}, t)(t - t_k) + h_3(\bar{x}, t)(t_{k+1} - t) \leq 0, \\
& h_2(\bar{x}, t) \geq 0, \\
& h_3(\bar{x}, t) \geq 0
\end{aligned}$$

LQR-Tree Algorithm

Conclusions

References

R. Tedrake, I. Manchester, M. Tobenkin, and J. Roberts
LQR-trees: Feedback motion planning via sums-of-squares verification
International Journal of Robotics Researchs, 1038–1052, SAGE Publications, 2010.

R. Burridge, A. Rizzi, and D. Koditschek
Sequential composition of dynamically dexterous robot behaviors
International Journal of Robotics Researchs, 534–555, SAGE Publications, 1999.

P.Parrilo
Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization
Ph.D. Theis, MIT, 2000.

Thank You!