LQR-Trees [Tedrake, 2010]

George Kontoudis, Shriya Shah

ME5984 Motion Planning Analysis
Spring 2017

Mechanical Engineering Department, Virginia Tech

April 15, 2017

Outline

Motivation

- Design robust algorithms for non-linear feedback motion planning
- Non-linear underactuted systems such as robot manipulator or bipedal walking
- Computation of planning regions of attraction (funnels) for non-linear underactuated dynamical systems
- Applicable to real robots

Definition of Lyapunov Functions

For a given dynamical system

$$\dot{x} = f(x), f(0) = 0$$

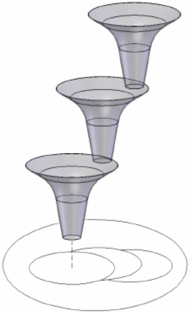
a Lyapunov function is $V(x)$, $V \in C$ where

- $V(x) > 0$, positive definite
- $\dot{V}(x) = \frac{dV}{dx}\frac{dx}{dt} < 0$, negative definite

If conditions met in some state space ball B_r, then origin is a.s.

Sequential Composition of Lyapunov Functions

- Each funnel acts like a valid Lyapunov function
- A.s. of each Lyapunov falls in the region of attraction of the next lower level
- The lowest function stabilizes in the goal point



Sequential composition of funnels [Burridge, 1999]

Sums of Squares

We want to check inequalities and validate Lyapunov functions using sums-of-squares (SoS) method [Parrilo, 2000]

- $x^4 + 2x^3 + 3x^2 - 2x + 2 \geq 0, \;\; \forall x \in \mathbb{R}$, by employing SoS

$$x^4+2x^3+3x^2-2x+2 = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^\intercal \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = X^\intercal A X$$

- Eigenvalues of A are $\lambda_1 = 3.88$, $\lambda_2 = 1.65$, $\lambda_1 = 0.47$, so the inequality stands $\forall x \in \mathbb{R}$

Sums of Squares Properties

General structure of (SoS) for a 4-*th* order polynomial is

$$fx^4 + 2ex^3 + (d + 2c)x^2 + 2bx + a = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^{\intercal} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

- Extend to multivariable polynomials
- Check non-negativity by searching positive semidefinite matrix

Feedback Synthesis by SoS Optimization

Given a system $\dot{x} = f(x) + g(x)u$ we want to generate

- Feedback control law $u = \pi(x)$
- Lyapunov fcn $V(x)$, s.t. $V(x) > 0$, $\dot{V}(x) = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x}\dot{x} < 0$

BUT, this is a difficult problem as the set of $V(x)$, $\pi(x)$ may not be convex sets

- Rely on LQR synthesis
- Design a series of locally-valid controllers
- Compose these controllers utilizing feedback motion planning

The Minimum Principle

The first order or necessary condition for optimality is called *Maximum (Minimum) Principle*

- Given a function we want to minimize $f(x, y, z)$ on a level surface (constraint) $g(x, y, z)$ we get

$$\nabla f = \lambda \nabla g$$

- To convert a constrained problem to an unconstrained we construct the Hamiltonian function

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^{\mathsf{T}} f(x, u, t)$$

Goal Stabilization

- For a given non-linear dynamical system

$$\dot{x} = f(x, u, t), \ \ x \in \mathbb{R}^n, \ \ u \in \mathbb{R}^m$$

- Set goal state x_G, where $f(x_G, u_G) = 0$, and $\bar{x} = x - x_G$, $\bar{u} = u - u_G$
- Linearize around (x_G, u_G), $\bar{x} \approx A\bar{x}(t) + B\bar{u}(t)$
- Infinite horizon LQR minimum energy cost-to-go fcn (performance index)

$$J_\infty = \frac{1}{2}\int_0^\infty [\bar{x}^\intercal(t)Q\bar{x}(t) + \bar{u}^\intercal(t)R\bar{u}(t)]dt,$$

$$Q = Q^\intercal \geq 0, R = R^\intercal > 0$$

Hamiltonian System

Set the Hamiltonian

$$H(x,u,t) = L(x,u,t) + \lambda^{\intercal} f(x,u,t)$$

- State equation

$$\dot{x} = \frac{\partial H}{\partial \lambda}$$

- Costate equation

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \frac{\partial f^{\intercal}}{\partial x}\lambda + \frac{\partial L}{\partial x}$$

- Stationarity condition

$$0 = \frac{\partial H}{\partial u} = \frac{\partial f^{\intercal}}{\partial u}\lambda + \frac{\partial L}{\partial u}$$

Riccati equation and Optimal Control Law

- Infinite horizon LQR problem results the optimal cost

$$J^*(\bar{x}) = \frac{1}{2}\bar{x}^\intercal S\bar{x}$$

- $S > 0$, w/ the solution given from ARE

$$0 = Q - SBR^{-1}B^\intercal S + SA + A^\intercal S$$

- Optimal feedback closed loop control policy

$$\bar{u}^* = -R^{-1}B^\intercal S\bar{x} = -Kx$$

Goal State Convergence

The domain of attraction of the LQR over some sub-level set

$$B_G(\rho) = \{x | 0 \leq V(x) \leq \rho\}$$

To guarantee a.s. we require $V(x)$ to be a valid Lyapunov function

- $V(x) > 0, \;\; x \in B_G(\rho)$
- $\dot{V}(x) < 0, \;\; x \in B_G(\rho)$

Assign $V(x) = J^*(\bar{x}) = \frac{1}{2}\bar{x}^\intercal S \bar{x}$

- By definition positive definite
- $\dot{V}(x) = \dot{J}(\bar{x}) = \frac{dV}{dx}\frac{dx}{dt} = 2\bar{x}^\intercal S \dot{x} = 2\bar{x}^\intercal S f(x_G + \bar{x}, u_G - K\bar{x})$

Lyapunov Verification Using SoS

We require

$$\dot{J}^*(\bar{x}) < 0, \quad \forall \bar{x} \neq 0 \in B_G(\rho), \quad \dot{J}^*(0) = 0$$

- First, modify the inequality from negative to non-positive

$$\dot{J}^*(\bar{x}) \leq -\epsilon ||\bar{x}||_2^2, \quad \forall \bar{x} \in B_G(\rho), \quad \epsilon \in \mathbb{R}^+$$

- Second, include the constraint with Lagrange multiplier $h(\cdot)$

$$\dot{J}^*(\bar{x}) + h(\bar{x})(\rho - J^*(\bar{x})) \leq -\epsilon ||\bar{x}||_2^2$$

Lagrange Mulitplier Searching

- If $f^{(cl)}(x,u) = f(x, u_G - K(x - x_g))$, search for $h(\cdot)$ polynomial with sufficient order for $\dot{J}^*(\bar{x})$, using SoS

$$\begin{aligned} \text{find} \quad & h(\bar{x}) \\ \text{subject to} \quad & \dot{J}^*(\bar{x}) + h(\bar{x})(\rho - J^*(\bar{x})) \leq -\epsilon ||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \end{aligned}$$

- If $f^{(cl)}(x) \approx \hat{f}^{(cl)}(\bar{x})$, where $\hat{f}^{(cl)}(\bar{x})$ is the Taylor expansion (algebraic approximation) and $\dot{\hat{J}}(\bar{x}) = 2\bar{x}^\intercal S \hat{f}^{(cl)}(\bar{x})$

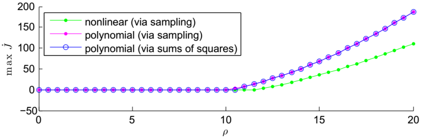
$$\begin{aligned} \text{find} \quad & h(\bar{x}) \\ \text{subject to} \quad & \dot{\hat{J}}(\bar{x}) + h(\bar{x})(\rho - \hat{J}^*(\bar{x})) \leq -\epsilon ||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \end{aligned}$$

Optimization for ρ

Set a convex optimization problem for the region of attraction

$$\begin{aligned} \max \quad & \rho \\ \text{subject to} \quad & \hat{\dot{J}}^*(\bar{x}) + h(\bar{x})(\rho - \hat{J}^*(\bar{x})) \leq -\epsilon ||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \\ & \rho > 0 \end{aligned}$$

- At each step the Lagrange multiplier searching is performed
- If the program is feasible ρ increased



Polynomial verification of damped single pendulum [Tedrake, 2010]

Trajectory Optimization

Time-Invariant LQR Verification

Conclusions

References

R. Tedrake, I. Manchester, M. Tobenkin, and J. Roberts
LQR-trees: Feedback motion planning via sums-of-squares verification
International Journal of Robotics Researchs, 1038–1052, SAGE Publications, 2010.

R. Burridge, A. Rizzi, and D. Koditschek
Sequential composition of dynamically dexterous robot behaviors
International Journal of Robotics Researchs, 534–555, SAGE Publications, 1999.

P.Parrilo
Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization
Ph.D. Theis, MIT, 2000.

Thank You!