Notes on Variational Obstacle Path Planning using the Weighted L_p Norm

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Abstract

In this technical note, an optimal safe path planning for unicycle robot in the planar space is considered using the weighted L_p norm. The nominal trajectory minimizing the cost function composed with curvature, velocity and circular obstacle potential in [1] is used to generate the safe trajectory. In this paper, the geometry of the obstacle is generalized to ellipsoidal and rectangular by using the weighted L_p norm shown in [2]. The nominal trajectory of the robot is then formulated with the boundary value problem (BVP) of twelve dimensional ordinary differential equation. BVP is then solved numerically by using the collocation method instead of the shooting method.

I. Obstacle potential using weighted L_p norm

Suppose that $x \in \mathbb{R}^n$ being positive means element-wise positive, then the weighted L_p norm is defined with the positive vector $\sigma > 0$ for some $\sigma \in \mathbb{R}^n$.

Definition 1 (Weighted L_p norm)**.** Let $\sigma \in \mathbb{R}^n$ be a positive vector, and $0 < p < \infty$, then $||\cdot||_{\sigma,p} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is the σ-weighted L_p norm such that

$$||x||_{(\sigma,p)} := \left(\sum_{i=1}^{n} (|x_i|/\sigma_i)^p \right)^{\frac{1}{p}},$$

The one level set of the weighted L_p approaches to the rectangle with half lengths σ in $\mathbb{R}^2$ as p goes to infinity. In addition, if $p = 2$, then the one level set is equal to ellipse. More detailed boundary design of the obstacle with different p values can be found in [2].

In this paper, rectangular obstacles are considered and approximated by the level set of weighted L_p norm for some p and σ. Remark that the circular or elliptical obstacle can be modeled simply by choosing appropriate p an σ without sacrificing anything.

Suppose that $g_o \in SE(2)$ represents the obstacle frame relative to some world frame, and let $g_o = [x_o, y_o, \theta_o]$. The shape of the robot is defined by the half lengths σ_o and some even p_o. Without loss of generality we define $g_o : SE(2) \to SE(2)$ as a transformation from the obstacle frame to world frame, and g_o^{-1} does the inverse transformation.

In the planar space, the obstacle potential, $V : \mathbb{R}^2 \to \mathbb{R}^2$ is now defined as

$$V(q) := \frac{\tau}{||g_o^{-1}(q)||^{p_o}_{(\sigma_o,p_o)} - 1} \tag{1}$$

for $q \in \mathbb{R}^2$.

The potential for the circular obstacle with radius $r > 0$ in [1] will be a special case when half lengths are chosen by $\sigma = [r, r]$ and $p = 2$.

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II. Optimal path plannign

The above potential is then used to derive the necessary condition for the optimization problem,

$$\min_{g(\cdot)\in SE(2)} \int_0^T ||\ddot{g}||^2 + \sigma||\dot{g}||^2 + V(Pos(g)) \tag{2}$$

with $g(t) = [x(t), y(t), \theta(t)]$ for $t \in [0, T]$ represents the trajectory of the point mass unicycle and $Pos(g) := [x, y]$. The initial and final configuration and the linear and angular velocities of the robot are constraints is given by

$$g(0) = g_i = [x_0, y_0, \theta_0] \tag{3}$$
$$g(T) = g_f = [x_f, y_f, \theta_f] \tag{4}$$
$$\dot{g}(0) = \dot{g}_i = [x_0', y_0', \theta_0'] \tag{5}$$
$$\dot{g}(T) = \dot{g}_f = [x_f', y_f', \theta_f'] \tag{6}$$

In addition, the nonholonomic constraint for the unicycle is given by

$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0. \tag{7}$$

Similar to [1], the nominal trajectory can be achieved by solving the following BVP.

$$\theta^{(4)}(t) = \sigma\theta^{(2)}(t) - 1/J\lambda(t)(\dot{x}(t)\cos(\theta(t)) + \dot{y}(t)\sin(\theta(t))) \tag{8}$$
$$x^{(4)}(t) = \sigma x^{(2)}(t) + \frac{\tau p_o r_x(t)^{p_o-1}}{\sigma_o(1)^{p_o}} \frac{1}{m(||g_o^{-1}(Pos(g))||^{p_o}_{(\sigma_o,p_o)} - 1)^2} + \frac{1}{m}(\dot{\lambda}(t)\sin(\theta(t)) + \lambda(t)\dot{\theta}(t)\cos(\theta(t))) \tag{9}$$
$$y^{(4)}(t) = \sigma y^{(2)}(t) + \frac{\tau p_o r_y(t)^{p_o-1}}{\sigma_o(2)^{p_o}} \frac{1}{m(||g_o^{-1}(Pos(g))||^{p_o}_{(\sigma_o,p_o)} - 1)^2} + \frac{1}{m}(-\dot{\lambda}(t)\cos(\theta(t)) + \lambda(t)\dot{\theta}(t)\sin(\theta(t))) \tag{10}$$

where $[r_x(t), r_y(t)] := g_o^{-1}(Pos(g))$.

III. Solution to BVP

In this section, the above BVP is solved by using the collocation method. The $g(t)$ is approximated by piecewise B-spline polynomials, and the above 12 boundary constraints, 3 dynamics constraints, and 1 nonholonomic constraint is given for the constraints on the coefficients of B-splines. By sampling on the collocation time for each B-splines, the BVP is convereted to the NLP problem with an auxiliary cost function, which we choose as

$$J(X) = \int_0^T (\dot{x}\sin(\theta) - \dot{y}\cos(\theta))^2 \tag{11}$$

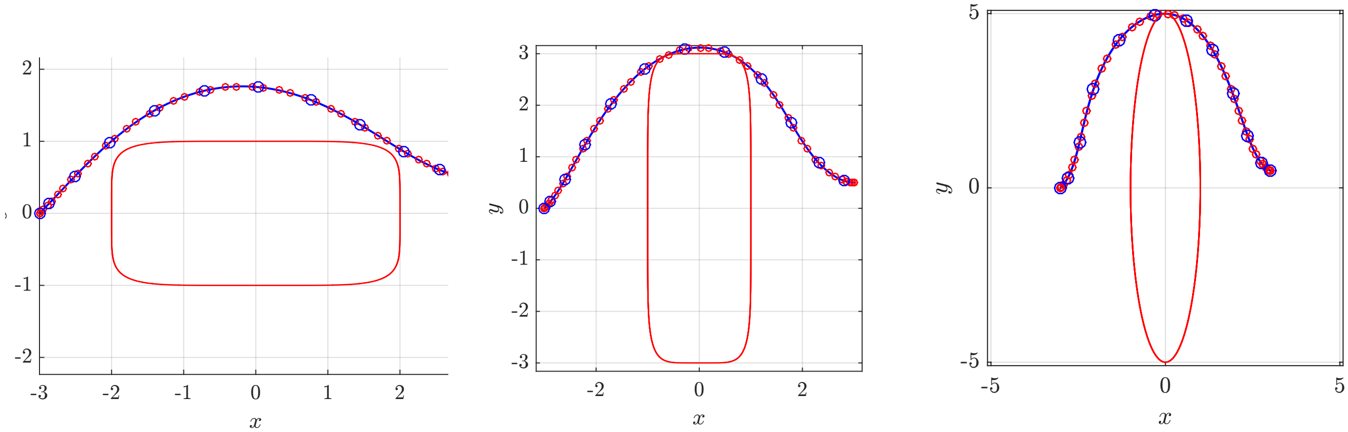
where X represents the free coefficients to optimize.

A. Simulation

The initial and final configuration and velocities are given as

$$g(0) = [-3, 0, \pi/4] \tag{12}$$
$$g(T) = [3, 0.5, 0] \tag{13}$$
$$\dot{g}(0) = [0, 0, 0] \tag{14}$$
$$\dot{g}(T) = [0, 0, 0] \tag{15}$$

The obstacle $g_o = [0, 0, 0]$ where the center is located at $(0, 0)$ and its orientation is aligned to the world frame. In this examples, $T = 1$ is chosen, and 50 points are samples with the Gaussian-Legendre quadrature approximation to each node points and the points in between the node points. 12 piecewise B-splines with order 7 are used to approximated. The error to the all the constraints (boundary values and the dynamics and nonholonomic constraint) are less than 10^{-7}. The computation time are less than 6 seconds.



(a) $\sigma_o = (2, 1)$ and $p = 6$ (b) $\sigma_o = (1, 3)$ and $p = 6$ (c) $\sigma_o = (1, 5)$ and $p = 2$

Fig. 1. Solution to the BVP

IV. Open questions

A. Free final time problem with bounded velocities

Since the above solution to the BVP generates the nominal trajectory for a given $T > 0$. We can denote such solution as $g_T : [0, T] \to SE(2)$. Observe that the solution g_T is continuously differentiable over the compact interval $[0, T]$, and so there exists $B := [B_1, B_2] > 0$ such that $\dot{x}^2(t) + \dot{y}^2(t) \leq B_1$, and $|\dot{\theta}(t)| \leq B_2$ for all $t \in [0, T]$.

Problem 1 (Existence of T over fixed B)**.** Given $B_1 > 0$ and $B_2 > 0$, does there exist $T > 0$ such that the solution g_T of the ODE, (8-10), with boundary value condition (3-6) and nonholonomic constraint (7) satisfies

$$\dot{x}^2(t) + \dot{y}^2(t) \leq B_1 \tag{16}$$
$$|\dot{\theta}(t)| \leq B_2 \tag{17}$$

for all $t \in [0, T]$.

B. Minimum time problem with bounded velocities

Suppose that Problem 1 has a solution for any given $B_1, B_2 > 0$ pair, find the minimum T which achieve the goal.

Problem 2 (Minimum T given the bound for linear and angular speed)**.** Given $B_1 > 0$ and $B_2 > 0$, find the minimum T such that the solution g_T satisfies (16) and (17).

$$\min_T T$$

subject to

$$\begin{aligned} (8-10) &: \text{ dynamic constraints} \\ (3-6) &: \text{ boundary values} \\ (7) &: \text{ nonholonomic constraint} \\ (16-17) &: \text{ bounds for linear and angular velocity} \end{aligned}$$

If the optimal solution exist, then it could be a suboptimal solution to the original minimum time problem but we may have a *smooth* solution as oppose to the bang bang type of solution for velocity control.

C. Kinematic control

What would be the control for the unicycle in this case? Can we derive a state feedback controller?

D. Non point-mass unicycle

How would we incorporate the actual body of the unicycle (asymmetric body- not the circular one). In the original obstacle potential only demonstrates the collision avoidance to the trajectory. However, this trajectory could be a path for a center of mass of actual non point-mass robot. How do we handle the actual geometry of the robot?

References

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[2] N. P. Hyun, P. A. Vela, and E. I. Verriest, "A new framework for optimal path planning of rectangular robots using a weighted l_p norm," *IEEE Robotics and Automation Letters*, vol. 2, no. 3, pp. 1460–1465, July 2017.