Airline Scheduling and Routing in a Hub-and-Spoke System

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This paper studies the competitive choice of flight schedules and route prices by airlines operating in a hub-and-spoke system. Airlines choose flight schedules and route prices to maximize profit, considering competitors' decisions. This research makes three contributions. First, an expression is derived calculating demand for each route as a function of the service quality and prices of all routes. Second, a mathematical programming heuristic is developed to find the schedule and prices that maximize an airline's profit against fixed schedules and prices for other airlines. Third, the heuristic is used to study competition by allowing each airline to optimize its schedule and prices against the others' choices and by searching for an equilibrium. The performance of the algorithm is evaluated against alternate heuristics and a five-city sample problem is presented. Finally, two competitive examples are presented and analyzed.

INTRODUCTION

In deregulated markets, passenger airlines have the freedom to choose flight schedules and prices for routes generated by flight schedules. These decisions are very important for an airline that wishes to maximize its profit. The flight schedule fixes a large fraction of an airline's cost, and also the routes the airline can offer. The schedule determines the quality of the airline's service in terms of departure times and transit durations of routes. Quality of service and prices affect the airline's ability to attract travelers.

This paper studies the competitive choice of flight schedules and route prices by an airline in a hub-and-spoke system. The problem is relevant because many airlines in the United States and Europe provide a substantial fraction of their service using hub-and-spoke systems. Clearly, restricting analysis to hub-and-spoke systems is not always appropriate. Many airlines use a combination of hub-and-spokes and direct routings, and JENG[11] has shown that a mixed system is generally more efficient. However, restriction to a hub-and-spoke system simplifies the analysis, because finding the set of routes generated by a flight schedule is much easier than with other routing patterns. For this reason, the paper restricts itself to hub-and-spoke systems.

This research makes three contributions. First, an expression calculating demand for each route as a function of the service quality and price of all routes is derived. Second, a mathematical programming heuristic is developed to find the flight schedule and route prices that maximize an airline's profit against fixed schedules and prices for other airlines. This heuristic allows an airline to optimize its schedule and prices against competing transportation providers that use any type of schedule, hub-and-spokes-based or not. Third, the heuristic is used to study competition in a hub-and-spoke system by allowing each airline to optimize its schedule and prices against the others' choices and by searching for an equilibrium in schedules and prices.

Demand for routings is derived by aggregating individual travelers' preferences. Following recent models of consumer choice for transit services by MCFADDEN[13, 14] and BEN AKIVA and LERMAN,[2] we assume a random utility model such that each individual traveler's demand is given by a logit function. A traveler's demand for each route is a function of the total cost of using that route. A customer's cost has three components: 1) the cost of departing at a time that differs from the customer's most preferred departure time, or, alternately, the cost of arriving at a time that differs from the most preferred arrival time, 2) the cost associated with

the route duration and 3) the actual fare. The total demand for each route is the sum of all travelers' demands for the route.

The profit maximizing heuristic algorithm finds the best flights, routes and prices in a three level hierarchical process. Throughout the paper we use the terms "best" and "optimal" even though our algorithms are heuristic. The third (lowest) level finds optimal route prices that maximize profits. An important assumption made is that airplanes have a finite capacity in seats, which is assumed the same for all airplanes. Route prices are set to maximize the airline's profit subject to this capacity constraint. The second level generates the set of routes considering the flights chosen by the airline. These are the possible routes that can be constructed using the flights. The first (highest) level searches over sets of flights to find the profit maximizing set.

Then, competition between airlines is analyzed by solving for Nash equilibrium flight schedules and route prices for each carrier. At the Nash equilibrium, each airline's flight schedule and route prices maximize its profit against the choices of competitors. For several examples, the set of equilibrium decisions is found.

To simplify the analysis, three assumptions are made. First, there is only one class of customers. Second, planes are assumed to be the same size. Third, no traffic originates or is destined for the hub city. These assumptions are not realistic but are made to reduce the complexity of the problem. It is often assumed that there are at least two different groups of passengers with distinct travel-related costs: business and non-business passengers. If two classes of passengers exist, the algorithm would need to calculate two different prices for each route. This would double the number of price variables and would increase computational time, but not change the model formulation. The presence of two classes of passengers would change the optimal solution, but not the solution methodology. If planes have different sizes, flight scheduling entails choice of flight times and choice of plane types. Although we do not allow mixed fleets, the discussion in Section 4 describes a heuristic that incorporates the fleet mix issue but does not increase problem complexity. The assumption that no passengers begin or end trips at the hub simplified the problem formulation and presentation of the results. Inclusion of these trips would not add to the complexity of the solution procedures described.

The contribution of this research is the integration of competitive scheduling, pricing and consumer choice into one model. Although there have been many papers on airline scheduling, none to our knowledge has studied the interaction of competition, scheduling and pricing issues. The main tool we use is the heuristic algorithm that calculates optimal schedules and prices. We demonstrate that competition can be analyzed for simple examples using the heuristic. An important limitation is that competition is studied only through examples. Clearly, one cannot obtain generalizable insights from the two examples studied. However, by systematically studying more examples, one may find insights about how airlines compete.

This section concludes with a review of the literature. Section 1 describes a model including firm cost, consumer choice and our optimization procedures. Section 2 presents several test problems including performance evaluation against an alternate heuristic. Section 3 models competitive scheduling and pricing in hub-and-spoke systems and presents examples. Section 4 presents extensions, a summary and suggestions for further work.

Literature Review

Relevant literatures include airline scheduling and routing, and transportation economics.

Airline Scheduling and Routing. An excellent recent survey is ETSCHMAIER and MATHAISEL.[8] Most of the literature addresses the problem of minimizing the airline's cost subject to fixed transportation demand. In particular, the effect of airfares on demand and airline profit is not considered.

Work which studied profit maximization with respect to flight schedule choice includes CHAN,[5] which presented an iterative procedure to design a route network. However, prices were fixed and consumer choice was ignored. SOUMIS, FERLAND and ROUSSEAU[18-20] modeled an airline's assignment of passengers to available flights and the selection of profit optimizing flight schedules. There was no customer choice, as the airline assigned passengers to routes, and prices were fixed. The objective was to minimize revenue losses due to overbooked flights plus weighted customer losses subject to the constraint that all passenger demand was satisfied. Excess demand for a flight resulted in lost revenue, not reassignment of passengers to other flights. The assignment of customers to undesirable routes caused customer losses. Flight schedules were optimized by adding or dropping flights if there was an improvement in the objective function. The procedure was partially heuristic. For example, when flights were dropped, not all demands for all origin-destination pairs were computed, just those

for pairs with substantial traffic. Likewise, when adding flights, the algorithm did not recalculate traffic for all origin-destination pairs, just that on pairs that the new routings were most likely to affect.

Transportation Economics. Transportation engineers use techniques of disaggregate choice to model consumer choice. This work assumes a particular form of a random utility model for consumers. Examples are the logit, probit and dogit functions which allow the estimation of consumers' utility functions for different transportation alternatives. See McFadden[13, 14] and Ben-Akiva and Lerman.[2] By observing characteristics of individuals and individuals' choices of transit and properly aggregating individuals into populations with similar characteristics, estimates of total demand for different services are made.

To use this approach, factors which explain customer choice must be specified. WINSTON[21] presents a survey of demand elasticity for different travel modes as a function of service time and transportation price. MORRISON and WINSTON[15] present a detailed econometric analysis of 2325 business trips between 360 city pairs, incorporating modal choice, transportation cost, transit time, and time between departures.

PANZAR[16] and DORFMAN[7] presented theoretical microeconomic models of competition between airlines for service between two cities. These papers ignored the effect of network structure on an airline's choices, assumed a function describing schedule delay, and used very simple demand functions that do not reflect empirically observed patterns of choice.

1. THE MODEL

IN THIS paper the following terms are used. A *flight* is a single plane trip from a spoke city to the hub or from the hub to a spoke city, denoted by the departure and arrival times for this one trip. A *route* is a pair of flights from an origin (spoke) city to a destination (spoke) city. A route is denoted by the departure time of the initial flight when it leaves the origin city, and the arrival time of the second flight at the destination city.

To study competition, a heuristic algorithm must be derived. The objective of the algorithm is to find an airline's profit maximizing flight schedule, routes and prices in a hub-and-spoke system with a single hub. The airline competes against other airlines with fixed routes and prices. It is assumed that the airline can set any flight schedule, but the schedule must repeat daily. The airline must also assign planes to flights so that all scheduled flights are covered by a plane every day. To do this the plane schedule must repeat over some period, but this period may not be one day. Also, the airline can set any non-negative prices on the routes generated by the flights.

To simplify, the following assumptions are made about the airline's cost structure:

A1. Each airplane operated has a fixed cost of F dollars per day.
A2. The planes have a variable cost of V dollars per flying hour.
A3. The variable cost per passenger is zero.
A4. Planes are of the same size and have a finite capacity of M seats.

Assumptions A1, A2 and A3 are realistic. It is difficult for airlines to acquire a plane for less than one day's use. Flying airplanes incurs substantial costs, including fuel, maintenance, and crew staffing. The variable cost per passenger is low if the airline has available seats. Assumption A4 is not realistic, but equality of plane sizes is an assumption made to ease the burden of computation. Optimization that included fleet mix could be handled, although it would require some change in the heuristic and would increase the computation effort.

The algorithm finds the best flights, routes and prices in a three level hierarchical process. The third level, which is the lowest, finds optimal route prices that satisfy capacity constraints and consumer choice behavior. It also determines the airline's total revenue for the routes. The second level generates a set of feasible routes from the set of flights chosen by the airline. These are the possible routes that can be constructed using the flights. The first level, which is the highest, searches over sets of flights to find the profit maximizing set. Flight schedules must have a period of one day and must be covered by a plane schedule having some period.

The next subsections discuss the three levels in detail, beginning with the lowest level.

1.1. Level Three: Profit Maximization with Respect to Prices for Fixed Routes

The lowest level maximizes the airline's profit by adjusting prices. Profit is maximized with respect to prices, while considering consumer choice of routes and airplane capacity constraints. To find profit maximizing prices, demand for routes as a function of route prices must be determined. The

next two subsections discuss consumer choice of routes and the price optimization problem.

1.2. Consumer Choice

Consider n cities $c_1, \ldots, c_n$ having demand for air transport between them. Let the $n \times n - 1$ origin-destination pairs of cities be denoted by OD. For each pair $(c_i, c_j) \in$ OD, consider routes between the cities as provided by a hub-and-spoke airline whose hub location is at city c_o. The air carrier is not the only air carrier providing transport between the cities; the other airlines have fixed schedules and prices for routes between the cities. The chosen set of routes of the airline from c_i to c_j is denoted $R(c_i, c_j)$, while the competitor's routes are denoted $RC(c_i, c_j)$.

Passengers have preferences for routes. Assume that preferences are a function of a most desired travel time, the duration for a route and the price of that route. For the rth route, we will let t_r be the takeoff time of the first flight on the route, we will let l_r be the duration of the route and we will let p_r be the price of the route.

For any c_i, c_j, and $t \in [0, 24)$, let $\beta(c_i, c_j, t)$ denote the density of customers whose most preferred departure time for the trip from c_i to c_j is at time t. It is assumed that this function is piecewise constant on $[0, 24)$. That is, there are $K = K(c_i, c_j)$ constant density intervals over $[0, 24)$: $[0, t_1)$, $[t_1, t_2), \ldots, [t_{K-1}, 24)$. Let $\beta_k(c_i, c_j)$ be the density of the kth interval, $k = 0, \ldots, K - 1$ for travel between c_i, c_j. An example of a density function β is given in Figure 1.

Assume for each origin-destination pair that customers' travel preferences and densities are identical for each day. If this assumption is violated, one can redefine the period of analysis so that densities are identical for each period.

Assume that a customer's utility function is lin-ear and separable in various attributes of routes. Assume money is the numeraire. If a customer whose ideal time for travel is t instead leaves at time t', the passenger's utility declines by $w|t - t'|$ dollars, for some $w > 0$. The term $|t - t'|$ is interpreted to be the difference in hours on a 24-hour clock between t and t'. For example, the difference between 21 and 1 is 4. Customers also suffer a reduction in utility as a function of duration of the routing, in the amount of v dollars per hour for the duration of the trip. Finally, passengers paying p dollars for a route have their utility reduced by p dollars.

Although we have specified that customers have preferences for departure times for flights, the model is easily recast to consider customers with preferences for arrival times. Time t' is then interpreted as the route's arrival time and β as the density function of ideal arrival times. Business travelers are probably most concerned about the arrival time at the destination. Returning business travelers are probably most concerned about departure time to home. Therefore, consumers preferences might be more accurately stipulated in terms of the arrival time on the outbound leg, and the departure time of the return leg. The model can capture these preferences by specifying the share of each market's traffic that corresponds to outbound traffic, and aggregating outbound and returning passengers' demand. Also, customers preferences for departure (or arrival) time need not be specified by a symmetric function such as $|t - t'|$. An asymmetric time metric where equally early and late departures have different utilities could be easily adopted.

Assume for origin-destination pair (c_i, c_j), demand for route r by passengers having ideal departure time t, is given by a logit function. Therefore, if t is in the kth interval of β and (c_i, c_j) is
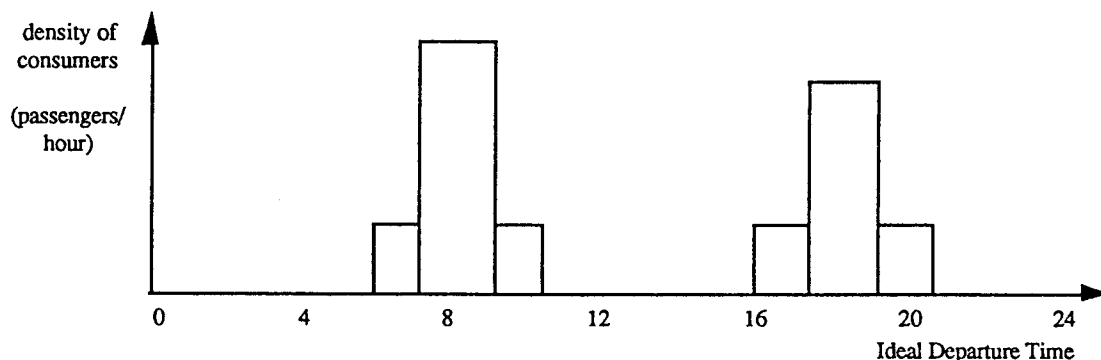

Fig. 1. Example of consumer density.

suppressed for clarity, demand for route r for customers with ideal departure time t is

$$d_r(t) = \beta_k \frac{e^{-\alpha(w|t-t_r|+vl_r+p_r)}}{\sum_{r'=1}^{|R|+|RC|} e^{-\alpha(w|t-t_{r'}|+vl_{r'}+p_{r'})}}.$$

Total demand for route r is $d_r = \int_0^{24} d_r(t)\,dt$. A closed form expression for d_r exists.

To find the expression we need to consider the departure time from city c_i to city c_j for all routes which are the routes in the union of sets: $R(c_i, c_j)$ and $RC(c_i, c_j)$. Suppose that the ordered departure times for these routes are $t_1, \ldots, t_{|R|+|RC|}$ and routes are labeled $r_1, \ldots, r_{|R|+|RC|}$. Some of the departure times may be identical. For each route, r, consider the disutility experienced by the customers with ideal departure time t: $vl_r + p_r + w|t - t_r|$. Figure 2 plots customer disutility for all customer ideal times for route r when $t_r > 12$. For this example, there are three intervals of the 24-hour clock for which the customer disutility has the same slope within the interval: the intervals $[0, a_1]$, $[a_1, t_r]$, and $[t_r, 24]$. If $t_r = 12$, there would be only two intervals. Further, within each interval $w|t - t_r|$ can be written as $w_o(t - t_r) + wu_o$, with $w_o = +w$ or $-w$, and $u_o = 0$ or 24. w_o and u_o can be set using Table I for $t, t_r \in [0, 24]$.

For each route, the 24-hour clock is partitioned into the two or three intervals just as was done for t_r. Considering the intersection of the partitions that all $R + RC$ routes generate, a partition with at most $2(|R| + |RC|) + 1$ intervals results. Within each of these intervals the slope of customer disutility does not change for any route. If the 24-hour clock is further partitioned so that within each interval the density of customer ideal times is con-

TABLE I
$t, t_r \in [0, 24]$

	$|t - t_r| \leqslant 12$	$|t - t_r| > 12$
$t_r \leqslant 12$	$w_o = w\ \mathrm{sign}(t - t_r)$ $u_o = 0$	$w_o = -w$ $u_o = 24$
$t_r > 12$	$w_o = w\ \mathrm{sign}(t - t_r)$ $u_o = 0$	$w_o = +w$ $u_o = 24$

stant, a partition with at most $2(|R| + |RC|) + K + 1$ intervals results. Let the actual number of intervals in this partition be H, and let the partition be given by: $[a_1, a_2), [a_2, a_3), \ldots, [a_H, a_{H+1})$, with $a_1 = 0$, $a_{H+1} = 24$.

Then, for the kth interval, write $w|t - t_r| = w_{rk}(t - t_r) + wu_{rk}$, with $w_{rk} = w_o$ and $u_{rk} = u_o$, properly set by the algorithm. Then

$$d_r = \sum_{k=1}^{H} \beta_k$$

$$\times \int_{a_k}^{a_{k+1}} \frac{e^{-\alpha[w_{rk}(t-t_r)+wu_{rk}+vl_r+p_r]}}{\sum_{r'=1}^{|R|+|RC|} e^{-\alpha[w_{r'k}(t-t_r)+wu_{r'k}+vl_{r'}+p_{r'}]}}\,dt$$

If we define $C_{rk} = e^{-\alpha[-w_{rk}t_r+wu_{rk}+vl_r+p_r]}$, the integral can be rewritten as,

$$d_r = \sum_{k=1}^{H} \beta_k \int_{a_k}^{a_{k+1}} \frac{C_{rk}\,e^{-(\alpha w_{rk}t)}}{\sum_{r'=1}^{|R|+|RC|} C_{r'k}\,e^{-(\alpha w_{r'k}t)}}\,dt.$$

Consider the sets: $P_k = \{r' \in R \cup RC | w_{r'k} = w\}$, and $N_k = \{r' \in R \cup RC | w_{r'k} = -w\}$ and define $CP_k = \sum_{j \in P_k} C_{jk}$ and $CN_k = \sum_{j \in N_k} C_{jk}$. We now evaluate the integrals above by substituting CP_k and CN_k into the denominator. An integral for $r \in P_k$ is $\int_{a_{k+1}}^{a_k} C_{rk} e^{-\alpha wt}/(e^{-\alpha wt}CP_k + e^{\alpha wt}CN_k)\,dt = \int_{a_{k+1}}^{a_k} C_{rk} e^{-2\alpha wt}/(e^{-2\alpha wt}CP_k + CN_k)\,dt$. Make the

disutility

$w_o = +w$

$w_o = -w$

$vl_r + p_r$

0   a_1 t_r 24
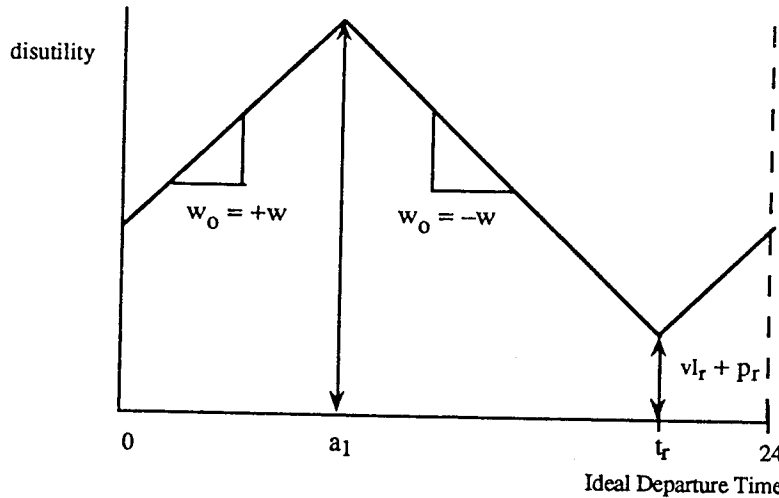
Ideal Departure Time

Fig. 2. Example of consumer disutility curve for departure time t_r when $t_r > 12$.

substitutions $x = e^{-2\alpha wt}$ and $dx = -2\alpha wx\,dt$. We obtain $(-1/2\alpha w)\int_{a_{k+1}}^{a_k} C_{rk}/(xCP_k + CN_k)\,dx = (-C_{rk}/(2\alpha wCP_k))\log(xCP_k + CN_k)|_{e^{-2\alpha wa_k}}^{e^{-2\alpha wa_{k+1}}} = (-C_{rk}/(2\alpha wCP_k))\log(e^{-2\alpha wt}CP_k + CN_{nk})|_{a_k}^{a_{k+1}}$. Summing over all k we obtain

$$d_r = \sum_{k=1}^{H} \frac{-\beta_k}{2\alpha w} \frac{C_{rk}}{CP_k} \log(e^{-2\alpha wt}CP_k + CN_k)|_{a_k}^{a_{k+1}}$$

if $r \in P_k$. (1.2-1a)

An analogous calculation gives

$$d_r = \sum_{k=1}^{H} \frac{\beta_k}{2\alpha w} \frac{C_{rk}}{CN_k} \log(CP_k + e^{2\alpha wt}CN_k)|_{a_k}^{a_{k+1}}$$

if $r \in N_k$. (1.2-1b)

The demand for the rth route is a weighted logit function, where the logit function considers only routes in P_k if r belongs to P_k (or routes in N_k if r belongs to N_k). The "weight" is a function of consumer density and the characteristics of all routes.

1.3. Profit Optimization Subject to Capacity Constraints and Consumer Choice

Equation 1.2-1 states how consumers choose between routes given prices for routes. Using this relationship, it is possible to maximize firm profit with respect to route prices.

Suppose that the airline has N_f flights, $FL = \{f_1, \ldots, f_{N_f}\}$, and there are N_r routes generated by these flights. Let A be the "route-flight" matrix whose r-fth element, a_{rf}, is 1, if route r uses flight f and is 0, otherwise. Let t_f be the total flying time of flight f. Suppose that the schedule uses N_P planes. Order the set of all routes, and let p be the vector of prices for all routes. Let the price of route r be p_r.

The airline's problem of maximizing profit with respect to prices p is:

$$\text{Maximize}_{p} \sum_{(c_i, c_j) \in OD} \sum_{r \in R(c_i, c_j)} p_r d_r$$
$$- V \sum_{\{f \in FL | \exists r \text{ s.t. } a_{rf} = 1\}} t_f - FN_P$$

subject to

(i) d_r is given by (1.2-1)
(ii) $A'd \leqslant Me$, where $e = (1, \ldots, 1) \in R^{N_r}$. (1.3-1)

Recall that M is the capacity of the plane. The firm maximizes profit subject to the logit demand relationship and the capacity constraint for each flight.

The non-linear programming problem of (1.3-1) is equivalent to the problem of maximizing the air-

line's revenue subject to the capacity constraint because the flight schedule and the number of planes are fixed. Recent results by CAPLIN and NALEBUFF[4] and DIERKER[6] show that the objective function is quasi-concave if the airline has a single route for each origin-destination pair, and the density function β falls into a restrictive class, which admits the constant density function. Unless routes share no flights, the constraints (ii) are not quasi-concave. When there is more than one route per origin-destination pair, the objective is not globally quasi-concave. Therefore, optimization methods that rely upon differentiation can at best guarantee a local maximum.

Dualizing constraints (1.3-1) give the Lagrangian for the revenue maximization problem:

$$L(p, \lambda) = \sum_{(c_i, c_j) \in OD} \sum_{r \in R(c_i, c_j)} p_r d_r(p)$$
$$- \sum_{f \in FL} \lambda f$$
$$\times \left[\sum_{\substack{(c_i, c_j) \in OD \\ \text{such that } a_{rf} = 1}} \sum_{r \in R(c_i, c_j)} d_r - M \right].$$

(1.3-2)

The Kuhn-Tucker conditions for this problem are: $\Delta_p L(p, \lambda) = d(p) + p'D_p d(p) - \lambda'A'D_p d(p) = 0$, $\lambda'(A'd - Me) = 0$, $\lambda \geqslant 0$, $p \geqslant 0$, where $\lambda = (\lambda_1, \ldots, \lambda_{N_f})$, and $D_p d(p)$ is the Jacobian of $d(p)$. It is useful to note that $L(p, \lambda)$ can be written as $L(p, \lambda) = \sum_{(c_i, c_j) \in OD} L_{(c_i, c_j)}(p(c_i, c_j), \lambda)$, and $L_{(c_i, c_j)}(p, \lambda)$ only depends on the prices, $p(c_i, c_j)$, for routes for the origin-destination pairs (c_i, c_j).

1.4. Subgradient Optimization

Optimal prices, p^*, satisfying the Kuhn-Tucker conditions must obey

$$L(p^*, \lambda^*)$$
$$= \text{Minimize}_{\lambda \geqslant 0}$$
$$\times \left\{ \sum_{(c_i, c_j) \in OD} \text{Maximize}_{p(c_i, c_j) \geqslant 0} \left\{ L_{(c_i, c_j)}(p(c_i, c_j), \lambda) \right\} \right\}.$$

(1.4-1)

For a fixed set of flights and schedules, a subgradient optimization procedure finds the optimal prices for routes. Subgradient optimization solves (1.4-1) as a two stage iterative process. The first stage assumes a fixed λ and solves the inner maximization by maximizing $n \times (n - 1)$ subproblems. Then,

in the second stage, λ is readjusted to minimize the objective value with the new value of p. The first stage is then repeated. The process terminates when, for fixed λ, the first stage problem is solved, and λ cannot be changed to improve the solution to the second stage.

1.5. Level Two: Constructing Routes from Flights

This subsection describes the procedure that constructs the routes from flight schedules. For each origin-destination pair, the flights determine the routes available for passengers. An important parameter that defines possible routes is the "maximum delay at the hub." The maximum delay at the hub is the maximum time that a passenger traveling from the origin is permitted to wait at the hub for its flight to the destination. For each flight from the origin to the hub, a route exists if there is a flight from the hub to the destination no later than the maximum delay at the hub after the arrival of the flight from the origin. This limits the set of routes considered in the solution.

Given a set of flights, the set of routes for all origin-destination cities is constructed. If the set of flights has a periodic schedule, then the set of routes also has a periodic schedule.

1.6. Level One: Heuristic Procedures for Choosing the Optimal Set of Flights

A heuristic algorithm to find profit maximizing flight times is described next. The heuristic procedure finds the best set of flights that can be implemented by a periodic schedule for the aircraft.

To reduce complexity, the set of possible departure times is discretized, by dividing the 24-hour clock into equal size periods. The end of the last period is understood to correspond to midnight, which is also the beginning of the first period.

The problem of constructing flight schedules that can be implemented by a periodic assignment of aircraft can be thought of as a two-stage process. The first step is that of picking a set of flights, the second step is the assignment of flights to planes so that all flights are "covered" by a plane and the planes have a periodic schedule.

The latter problem can be solved as a minimum cost circulation problem for an appropriately defined network. To do this, create a graph with one node for each city (including the hub) at each time period. Place directed arcs in the graph corresponding to all possible flights. Set a lower bound of 1 unit of flow for the flights which are in the schedule. Impose the following costs on the network: an arc's cost is the variable cost of the flight scheduled on it, and for each arc leaving a node at the first period add the fixed cost of a plane. A minimum cost assignment of the flights to planes with a periodic schedule is the solution of a minimum cost circulation for this network. The plane schedule generated by the covering is periodic, but the period is not necessarily one day. The flight schedule generated by the covering has a one day period.

Although this subproblem can be solved reasonably fast by network techniques, we used a different approach that solved the flight scheduling and the plane scheduling problems concurrently. Our approach is to make the plane assignment problem easier by choosing flight schedules with easily computed periodic plane schedules. Initially, every possible flight between the hub and a spoke city, and between a spoke city and the hub, is assumed to exist. In this case it is easy to compute the number of planes required to cover the flights. For each city we dedicate planes to go to and from that city. If it takes a total of k periods to fly to the city and back, then k planes are required to cover all flights to and from that city for one day. The total number of planes is computed by repeating this calculation at each city and adding the number of planes.

In graph theory terms we start with an Eulerian graph (in-degree = out-degree at each node). Recall that a graph is Eulerian if there exists an Eulerian cycle in the graph, namely a cycle that hits every edge exactly once. Pick an Eulerian cycle and assign a plane to each node at the first period that is crossed by the cycle. Each plane's flight schedule is generated by following a path on the cycle from that node to itself one cycle later. This assignment of planes produces a periodic schedule, whose period is that of the Eulerian cycle.

The procedure then considers three different modifications to the flight schedule that keep the Eulerian property. We use flight removal procedures to eliminate an in-out pair of flights (two consecutive flights, one going from the city into the hub and the other going immediately out to the city again), or an out-in pair (two flights going from the hub to the city and back to the hub again). For an "in-out pair" removal, the pair of flights is removed, and a "flight" from the city to itself is added; this "flight" corresponds to an idle plane at the city. In the second case, out-in pair removal, the pair is removed and a plane is left idle at the hub. Finally a tour elimination procedure eliminates a path of flights whose period is one day. Removing the path removes one plane from the schedule. These three modifications maintain the Eulerian property;

therefore planes can be assigned to the flights with a periodic schedule.

In essence, we are solving the flight scheduling and plane scheduling problems concurrently. Since our flight removal procedure implicitly recognizes the cost of deadheading a plane without passengers, we simply avoid allowing such a schedule from being considered.

Modifications are made to the flight schedule based on the relative profitability of the schedule with the modification. The profitability of a flight schedule is computed as follows. Given a flight schedule, a set of routes between each pair of spoke cities is found using the Level Two procedure. Then, optimal prices are calculated using the Level Three Procedure.

Our procedure is a "greedy close" algorithm. Initially, every possible flight between the hub and a spoke city exists. Optimal prices for this schedule are calculated. Flights are eliminated in a three stage process, with each stage removing flights using one of the flight modifications.

In the first stage, all pairs of flights that involve travel from the hub to a spoke city and back to the hub are considered candidates for elimination. Each candidate pair of flights of this type is evaluated as follows: Given the initial optimal solution (p^*, λ^*) to $L(p, \lambda)$, the change in firm profit after eliminating a candidate pair of flights is estimated. This is done as follows. The Level Two procedure finds the set of routes that exist with the candidate pair eliminated. Then, $d(p)$ is redefined, dropping routes which no longer exist. Eliminate components of the optimal solution, (p^*, λ^*), that do not correspond to existing routes (for p^*) or for existing flights (for λ^*). Denote the reduced vector by $(\hat{p}^*, \hat{\lambda}^*)$. Let $\hat{L}(\hat{p}^*, \hat{\lambda}^*)$ be (1.3-2) evaluated at $(\hat{p}^*, \hat{\lambda}^*)$ for the redefined d. If f and f' are the eliminated flights, compute $\Delta_{ff'} = \hat{L}(\hat{p}^*, \hat{\lambda}^*) + V(t_f + t_{f'}) - L(p^*, \lambda^*)$. $\Delta_{ff'}$ is an estimate of the change in firm profit when the flights are eliminated. Note that $V(t_f + t_{f'})$ is the decrease in cost due to fewer flying hours. If the optimal prices and Lagrange multipliers do not change with elimination of the flights, the estimate is exact. This approximation is used for two reasons. First, it is anticipated that the solution to (1.3-1) does not change much when there are many flights. Second, the time required to recompute the Level Three procedure is substantial.

Next, select the candidate pair that maximizes $\Delta_{ff'}$. Optimize prices for the set of routes with this pair eliminated. If profits increase, search for the next candidate pair of flights from the hub to a spoke city and back to the hub, as described above, at the new prices and Lagrange multipliers. If not, replace the last pair of flights into the solution and continue to the second stage.

The second stage involves eliminating pairs of flights from a spoke city to the hub and returning to the same spoke city at the next available flight. The set of all candidate flight pairs is generated. The effect of eliminating each possible pair of candidate flights is estimated using the same procedure as in the last stage. The best candidate is eliminated, and optimal prices and Lagrange multipliers are computed. If profit increases, the procedure is repeated. If profit declines, the last pair is put back into the solution and the algorithm proceeds to the next stage.

At the completion of the first two stages, if flights have been eliminated, un-utilized airplanes are to be found at spoke cities and/or the hub. It may be profitable to remove some of these planes from the solution by removing "tours" of flights that begin at some city at the beginning of the first period and are at the same city at the end of the last period.

The third stage removes tours. The change in profit associated with eliminating a complete tour of flights is estimated. For each flight in the tour the profit associated with eliminating it is estimated. The Level Two procedure finds the set of routes that exist with the flight eliminated. Then, $d(p)$ is redefined for those routes. Eliminate components of the optimal solution, (p^*, λ^*), that do not correspond to existing routes (for p^*) or for existing flights (for λ^*). Denote the reduced vector by $(\hat{p}^*, \hat{\lambda}^*)$. Let $\hat{L}(\hat{p}^*, \hat{\lambda}^*)$ be (1.3-2) evaluated at $(\hat{p}^*, \hat{\lambda}^*)$ for the redefined d. Finally, compute $\Delta_f = \hat{L}(\hat{p}^*, \hat{\lambda}^*) + V(t_f) - L(p^*, \lambda^*)$. The value Δ_f is an estimate of the change in firm profit when the flight is eliminated.

This value, Δ_f, is used as a measure of the "length" (profit improvement) of each flight. For each city, the Dykstra shortest path algorithm (BAASE[11]) is used to find the longest tour from the city back to the same city at the end of the last period. (Note that since the arcs in the graph always represent a movement forward in time, the graph is acyclic and so use of Dykstra's algorithm is justified.) The tour with the longest "length" is the candidate tour for elimination. The "length" of the longest tour found, plus F, is an estimate of the increase in total profit after removing all flights in the tour and eliminating a plane from the solution.

Level Three procedures are used to find optimal prices with the tour eliminated. If firm profit increases *or* if the schedule profitability prior to the tour elimination was negative, the tour is removed

from the schedule and the tour elimination step is repeated. If firm profit does not increase, *or* profit decreases and the initial schedule profitability was non-negative, the third stage is terminated. Level One procedures are now complete.

2. EVALUATION OF THE HEURISTIC, SAMPLE PROBLEMS AND PERFORMANCE

SEVERAL TEST problems were run using the heuristic techniques described in Section 1. Evaluation of the performance of the heuristic was deemed important in determining its value as a tool for setting schedules and prices.

A natural way to evaluate the heuristic algorithm is to compare it either to optimal solutions or to upper bounds. The difficulty of solving the price optimization made it impossible to consider enumeration of schedules to compute optimal solutions. A small problem with, say, 3 cities and 12 periods, would have at most $12(3)(2)$ flights. Subsets of these flights generate $2^{72} - 1$ different schedules. Thus there are $2^{72} - 1$ possible price optimizations. The time required for price optimization varied with problem parameters, particularly α. The time required could be as short as a few seconds for small values of α (0.0034), but several hours for large values of α (0.034) on a Mac II running Think Pascal version 2. Furthermore, the complexity of the problem makes it difficult to find a non-trivial upper bound. This eliminates both the use of branch and bound and simple comparison of heuristic values and upper bounds. Thus it is necessary to consider alternatives to evaluating the heuristic. We seek to gain some knowledge of the distribution of the objective value of feasible solutions by generating random solutions. We then compare this to the value obtained from the heuristic algorithm.

The following procedure generates k random solutions. Suppose that the maximum number of planes required for the problem is P. The procedure generates approximately k/P random solutions, each with p planes, for each $p = 1, \ldots, P$. To generate a random solution which requires p planes, it routes one plane from city to hub to city for p days. The procedure starts the plane at the hub and randomly picks a city that has not been visited at that time of day, *or* randomly picks the hub. It sends the plane to that city or the hub, then a random procedure either sends the plane back to the hub or it remains idle at the city, and so on. On the pth day it constructs the schedule so that the plane returns to the hub at the end of the day. In this way we generate a tour that one plane can traverse in p days, at the end returning to its starting point. To implement this flight schedule, all that is necessary is to assign a plane to each city the tour crosses at the first period of each of the p days. Optimal prices value can be computed for this flight schedule.

Note that the randomization was over the combinatorial portion of the problem, namely the choice of flights that would exist in the solution, and not over prices. Our method of generating flights was not entirely random, since the method guaranteed the number of planes in the solution and that flights were scheduled from the hub to cities not yet served at that time of day. Thus it is expected that these solutions would be somewhat sensible. We certainly eliminated numerous possible solutions, i.e., multiple flights scheduled from the hub to a spoke city at the same time. Thus we were not comparing completely random solutions to our heuristic.

Also, test problems were chosen for which the random heuristic was expected to perform well. Problems with only a few cities were chosen. The fewer the cities, the less coordination of flights at the hub matters. Problems with symmetrical consumer demands between cities were chosen to raise the probability that a favorable flight schedule would be found. Parameter choice also favorably affected performance of random solutions. Parameters used to run the problem were from Morrison and Winston[15] (Table 5, p. 231) which suggests α should be chosen equal to 0.0034. In running the following examples it was discovered that at this α demand was relatively insensitive to the departure time and duration of the route. Therefore, inconvenient departure times and longer durations affected an airline's demand and optimal revenues less than if α were higher. Consequently, flight schedules that did not coordinate routes did not seriously affect demand, route prices or airline profit. Use of a low α increased the relative profitability of random schedules compared with the heuristic solution.

2.1. Performance of Heuristic versus Random Solutions

Two series of test problems were run to evaluate performance. The first series involved test problems with constant and identical consumer demand density for all origins and destinations for the entire day. The second series involved test problems with constant and identical demand density for all origins and destinations for the first half of a day. During the second half of the day consumer demand density was zero for all origin-destination

pairs. It was anticipated that the heuristic would perform better against random solutions in the second series.

Assumptions about model parameters are: alpha = 0.0034, w = \$23, v = \$14, M = 100 and the number of periods was 12. Estimates of α and v (cost per hour for trip duration) were derived from Morrison and Winston's[15] estimates of these parameters for airline passengers on business trips; w was their estimate of business customers' cost of the time between departures.

Test problems were run with 12 periods, with departures on even hours. Tests were run for 2-, 3-, 4- and 5-city problems. Assumptions for all test problems for all origin-destinations were: competitors had a route every two hours, the price for a route was 100 and the duration of the route was 2 hours. Our flights were one hour long; therefore the minimum duration for routes was 3 hours. Competitors were assumed to use direct flights with shorter route durations. Assumptions for the first series of test problems were: F = \$10,000 per plane, and V = \$10,000 per hour for airline's costs and consumer demand density was $170/(n - 1)$ passengers per hour throughout the day, where n is the number of spoke cities. Note that the number of origin-destination pairs is $n(n - 1)$ and the number of possible flights is $12(2)n$ for each problem. For the second series, parameters were F = 0 and V = \$17,000 and consumer demand density was $280/(n - 1)$ passengers per hour from 00:00 to 12:00, and zero thereafter. For each problem 80 random solutions were generated.

Table II presents the performance of the heuristic and of the random solutions for the two series of tests. For each series, the tables give the objective value of the heuristic's solution, and summary data for the random solutions (mean, standard deviation, maximum and minimum values of solutions) and the ratio of difference of the heuristic value and the minimum for random solutions to the dif-

ference of the maximum and minimum of random solutions. If the maximum was the true optimal profit and the minimum was the worst profit attainable, then this last statistic would represent how well the heuristic did. The minimum and maximum values represent only approximations of the extreme values of the distribution. For both sets, the heuristic performed much better than random solutions on average. The heuristic performed essentially as well as the best random solution found for the first series, finding essentially identical solutions in 3 cases and a slightly worse one in one case. For the second series, as anticipated, the heuristic performed much better than the random heuristic, performing much better in three out of four cases and slightly better in the remaining case.

The small problems analyzed were time consuming. For the symmetric problems the heuristic took 3.0, 1.8, 0.69 and 0.20 hours for the 5-city through 2-city problems, respectively. The corresponding times for generating random solutions were 26, 9.9, 5.2 and 4.6 hours. For the non-symmetric problems the heuristic took 7.6, 3.7, 1.4 and 0.34 hours for the 5-city through 2-city problems respectively. The corresponding times for random solutions were 20, 9.2, 6.5 and 1.9 hours. All times reported are for a Mac II running Think Pascal version 2.

2.2. A Sample Problem

The following larger, sample problem was run for illustrative purposes. It is a 5-city problem, with spoke and hub cities as shown in Figure 3. The duration of trips between spoke and hub cities are marked on each arc. We solved a 12-period problem, with takeoffs permitted on even hours. For this example, the number of possible flights was 120 and the number of origin-destination pairs was 20.

In the performance evaluation tests, we observed that α = 0.0034 resulted in optimal prices that are much above competitors' prices. With that value of

TABLE II

Results for Two Series of Test Problems

| Series	No. of Cities (excluding the Hub)	Heuristic		Random Solution Results 80 Random Solution per Problem			
		$\dfrac{h - min}{max - min}$	Heuristic (\$)	Mean (\$)	Std. Dev. (\$)	Maximum (\$)	Minimum (\$)
1	2	0.9809	20,110	−50,007	60,815	23,616	−159,538
	3	0.9999	30,524	−101,444	64,700	30,536	−185,841
	4	0.9996	45,427	−124,644	79,576	45,523	−213,228
	5	0.9999	54,764	−160,204	95,542	54,782	−301,515
2	2	1.0839	13,723	−91,977	54,557	−3,281	−205,936
	3	1.3518	20,352	−157,143	68,432	−54,736	−268,174
	4	1.0072	30,965	−100,062	65,261	29,417	−186,676
	5	1.3387	38,081	−264,445	117,398	−85,871	−451,815
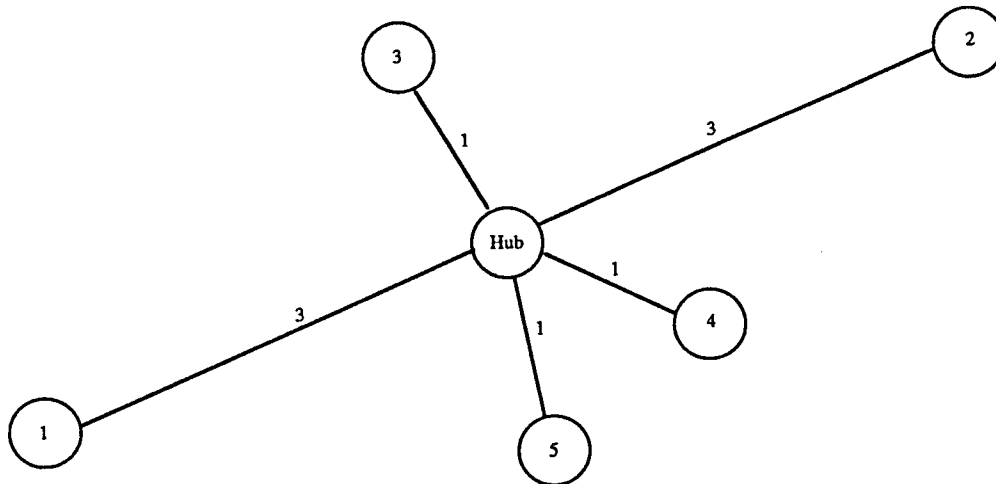
Fig. 3. Hub-and-spoke configuration for sample problem.

α, consumer demand for routes was relatively insensitive to prices, so that one firm could price much above the competitors and still serve many customers. To sharpen the effect of price competition, this sample problem was run with $\alpha = 0.034$. The other parameters were: $w = \$23$, $v = \$14$, $M = 300$, and the number of periods was 12. The airline's costs were assumed to be $F = \$10,000$ per plane and $V = \$17,000$ per hour. Table III reports competitors' routes, prices and route duration along with the airline's route durations. Finally, consumer demand density for all origin-destinations are reported. Consumer demand between cities 1 and 2 had "peaks" in the morning and late afternoon. Competitors chose to schedule flights at peak times. For these longest routings, flight duration for the airline was one hour longer than the competitors, which are assumed to offer direct flights.

Consumer demand between city 1 and cities 3, 4 and 5 was constant throughout the day. Competi-

tors offered two morning and two evening routes. Consumer demand between city 2 and cities 3, 4 and 5 was constant throughout the day with the same demand density as routes between city 1 and cities 3, 4 and 5. Competitors offered three morning and three evening routes.

Consumer demand from cities 3 and 5 to city 4 was a constant from midnight to noon and zero at other times of the day. Competitors scheduled routes every even hour. On the other hand, consumer demand from city 4 to cities 3 and 5 was zero from midnight to noon and a positive constant thereafter. Competitors scheduled routes every even hour.

Consumer demand between cities 3 and 5 was constant throughout the day, with competitors scheduling routes each even hour.

Portions of the solution to the problem are summarized in Figure 4, which reports several flight schedules, and Figure 5, which reports several

TABLE III

Routes, Prices and Durations for Sample Problem

Origin-Destination Pairs	Timings of Competitors' Routes	Competitors' Routes ($)	Duration of Competitors' Routes (hours)	Duration of Airline's Routes (hours)	Consumer Density per Hour
(1, 2), (2, 1)	6:00, 8:00, 10:00, 16:00, 18:00, 20:00	600	6	7	30 from 6:00 to 10:00, 16:00 to 20:00
(1, 3), (3, 1), (1, 4), (4, 1), (1, 5), (5, 1)	6:00, 10:00, 16:00, 20:00	400	5	5	30
(2, 3), (3,2), (2, 4), (4, 2), (2, 5), (5, 2)	6:00, 8:00, 10:00, 16:00, 18:00, 20:00	400	5	5	30
(3, 5), (5, 3)	Every 2 hours	200	3	3	10
(3, 4), (5, 4)	Every 2 hours	200	3	3	60 from 0:00 to 12:00
(4, 3), (4, 5)	Every 2 hours	200	3	3	60 from 12:00 to 24:00

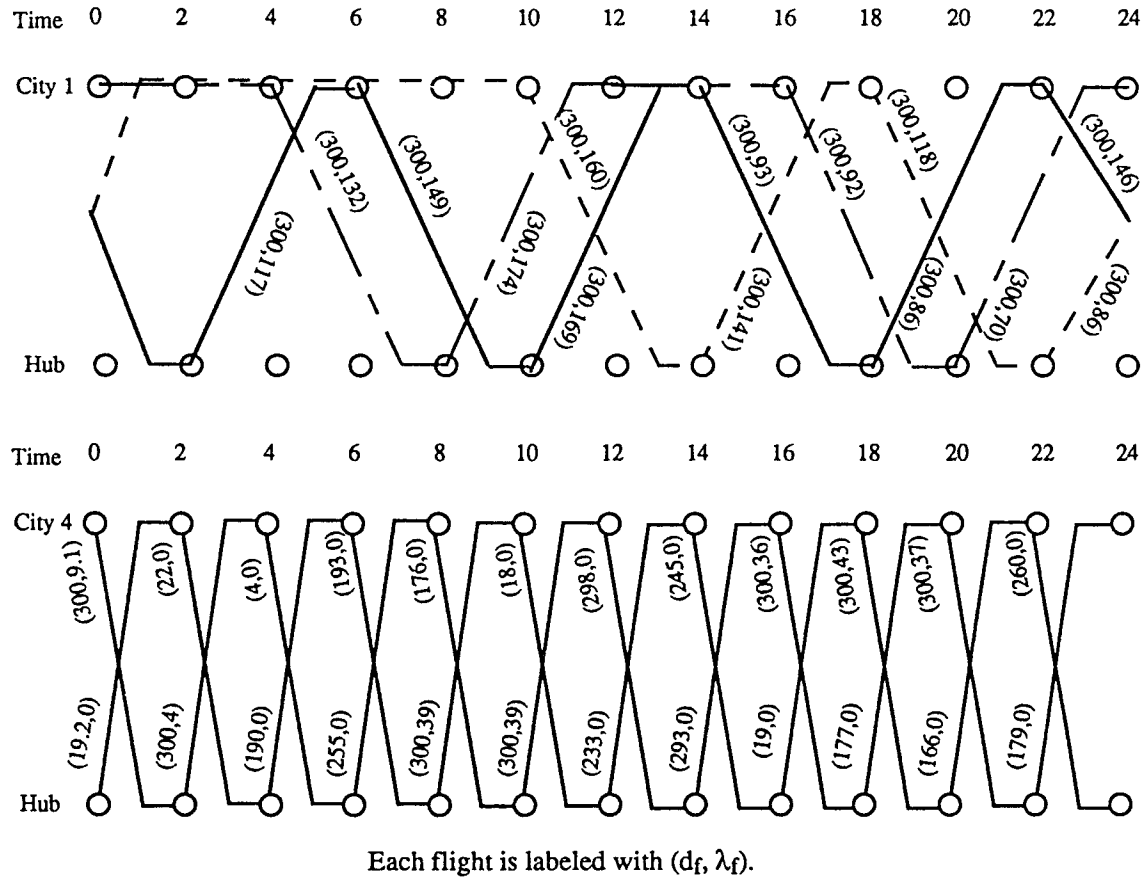Each flight is labeled with (d_f, λ_f).

Fig. 4. Flight schedule from/to cities 1 and 4 for sample problem.

routes' schedules. Figure 4 indicates the volume of passengers on each flight and the shadow price of transit on each capacitated flight. Figure 5 indicates the price, volume and "average schedule delay" for passengers for each route. The average schedule delay is defined as the average time difference between the passenger's ideal departure time and the actual departure time. It is calculated as: $w_r = (\int_0^{24} |t - t_r| d_r(t)\, dt)/(\int_0^{24} d_r(t)\, dt)$. The number of planes in the solution was 14, and their utilization was 46%.

Figure 4 exhibits some representative flight schedules. These include flights to and from cities 1 and 4. Flights to/from city 1 are covered by the 3-plane schedule indicated. Flights to/from city 4 are covered by 2 planes shuttling back and forth between city 4 and the hub.

The solutions for origin-destinations (1, 2) and (2, 1) were very similar, the top panel of Figure 5 displays only (1, 2). Routes tend to occur at peak hours. The airline priced $44 to $141 less than its competitors, capturing 91% of all demand. At 14:00, the airline priced lowest, at $459, to attract customers from the peak periods. The average schedule delay for passengers ranged from 1.1 hours

during peak periods to 4.3 hours during the slow midday period.

Routes found by the heuristic solution for origin-destinations (1, 3), (1, 4), (1, 5), (3, 1), (4, 1) and (5, 1) are similar; Figure 5 shows only (1, 3) in the middle panel. Similarly, solutions for (2, 3), (2, 4), (2, 5), (3, 2), (4, 2) and (5, 2) are similar and only (2, 3) is shown, also in the middle panel. For all of these origin-destinations, 7 routes were scheduled. For both sets, the airline priced routes $12 to $65 less than competitors'. Faced with a higher service level by competitors, the airline priced 3% less on average in the latter set. It is interesting to note that the airline's profit was $18,100 (8.6%) higher when faced by competition scheduling 2 (33%) fewer flights. Said another way, the airline's profit increased by 8.6% when its share of routes increased 20%. Customer's average schedule delay was low and ranged from 1.3 to 1.8 hours.

Routes for origin-destinations (3, 4), (4, 5) were similar to each other, and routes for (4, 3) and (5, 4) were similar to each other. However, in the former set, the heaviest demand occurred before noon, and in the latter set, the heaviest demand occurred after noon. Prices for routes were from $150-$160

(compared to $200 for competitors), except at times when the flights to/from city 4 were capacitated. When this occurred, route prices rose to $172–$176. Routes for (3,4) and (4,5) are shown. Customer's average schedule delay was about 1 hour on routes departing in the peak times, and were as high as 7 hours for routes departing in off-peak times. However, almost no passengers traveled in off-peak hours.

Routes for origin-destinations (3,5) and (5,3) used uncapacitated flights. The routes for both are similar, and only (5,3) is shown in the bottom panel of Figure 5. Prices and demand were the same for all routes. Customer's schedule delay was slightly more than one hour for each route.

The interrelationship of demand in different markets drives the structure of the flight schedule. In the Sample Problem, demand peaks resulted in
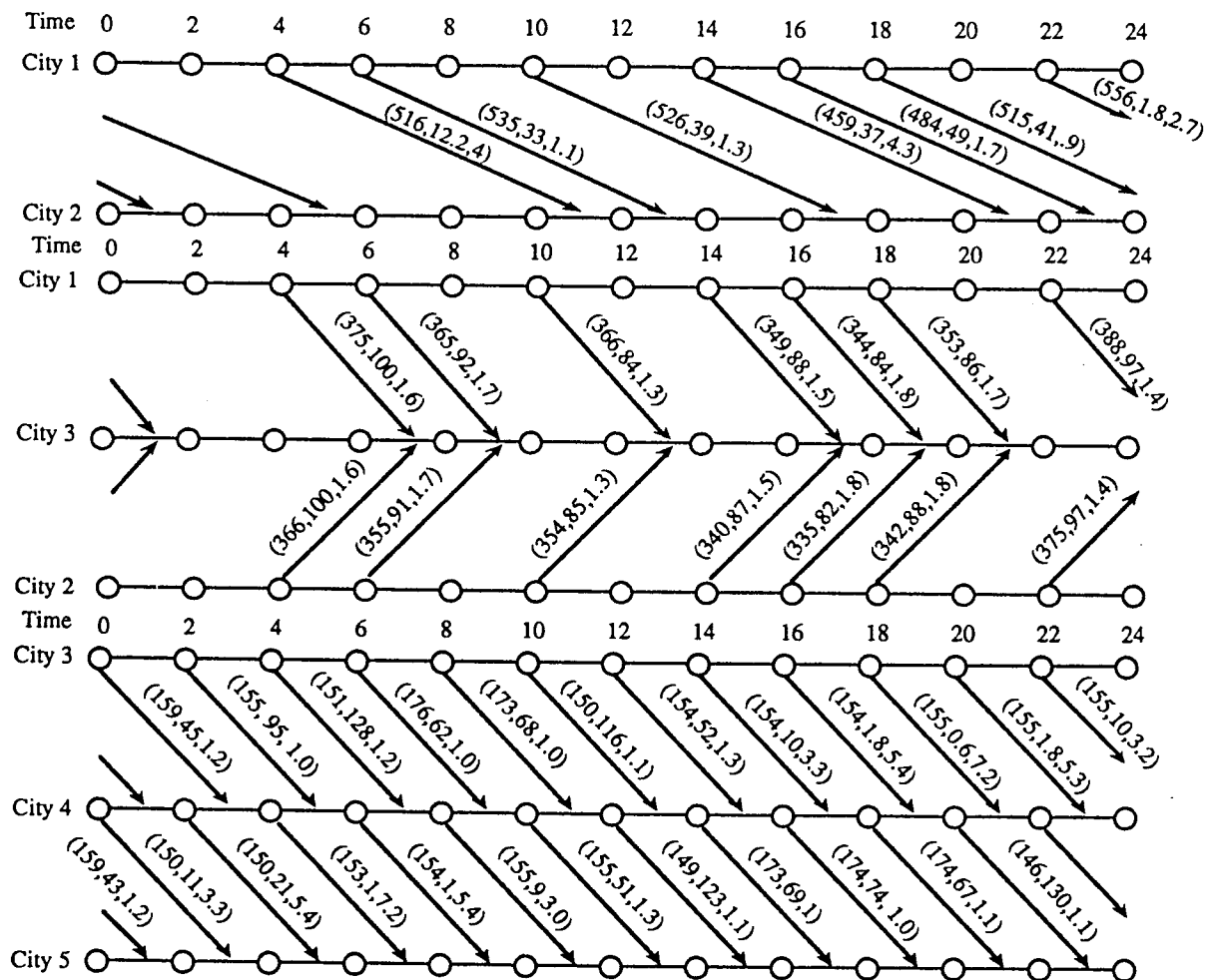
Each route is labeled with (p_r, d_r, w_r).
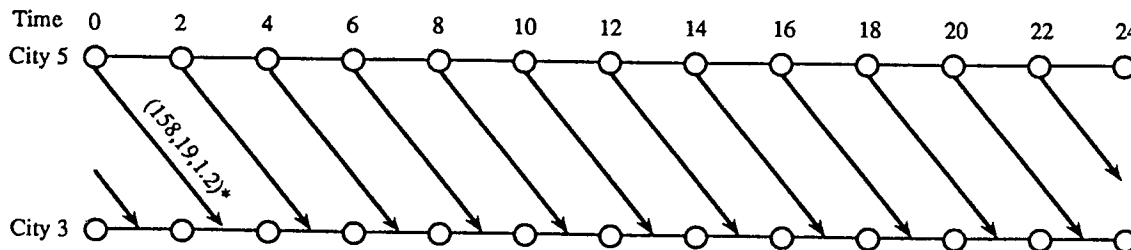
*One route is labeled with (p_r, d_r, w_r); all routes were identical.

Fig. 5. Routes between indicated cities for sample problem.

route bunching for cities $(1,2)$ and $(2,1)$, causing flights between the hub and these cities to be primarily in the morning and afternoon. These flights caused peaked routings to/from other cities to cities 1 and 2 as well. For example, routings between $(1,3)$ were peaked even though demands between these cities were not.

Computation time was long, about 22 hours. Again the bottleneck was price computation, particularly the first price optimization which required two and one half hours. It was found that all price optimization steps were significantly more time consuming than the random problems solved in the last section because price sensitivity (α) was higher for this problem.

3. COMPETITIVE ANALYSIS USING THE HEURISTIC

COMPETITION IN both schedules and prices between airlines that share a common hub can be explored using the heuristic. We model competition using game theory ideas, and find actual solutions using the heuristic by having airlines optimize schedules and prices against each other's.

The following assumptions are made to model competition:

A5. The set of airlines that compete for service is known. The airlines consider serving an identical set of spoke cities using the same hub. Each airline chooses its flight schedules and route prices.

A6. Customer demand is given by the expression (1.2-1).

A7. Each airline is free to alter its prices and schedule at any time. The cost of changing either is zero.

Assumption A5 states that there is a known set of competitors. Although the airlines consider service to the same spoke cities, an airline may choose to serve some cities, or no city. Assumption A6 states that the airlines are differentiated only by route prices, durations and schedules. That is, the airlines are not differentiated by other attributes, for example, advertising or schedule reliability. Assumption A7 assumes that airlines have the ability to begin or end service to a city at zero cost. Airlines may also add or remove aircraft from the schedule at no cost in excess of the airplane's fixed cost. Under assumption A7, firms may react quickly to competitors' actions by varying prices and schedules.

If Assumption A7 is valid, then each air carrier chooses its schedule and prices to optimize against the others' choices of schedules and prices. As each carrier can change its choices, a reasonable "solution" to this game is a vector of schedules and prices for routes, one for each airline, such that each airline's decision is an optimal response to the others'. Technically, this is the Nash equilibrium for the game where airlines choose schedules and prices simultaneously.

The first assumption is necessary to study issues of free entry and is a weak assumption. The second assumption is fundamental to this research. The third assumption is the most problematic. If airplanes, gates, landing rights and takeoff slots can be bought and sold or leased at low transaction costs, the assumption holds.

However, note that gates, landing rights, and takeoff slots are not currently traded at many hubs, so that transaction costs are likely to be high. If airlines experience substantial costs for changing schedules, but low cost for changing prices, each air carrier would forecast its profit before setting its schedule. When no firm can, a priori, be predicted to lead the others in setting its schedule, it is reasonable to assume that airlines choose schedules simultaneously, and then prices simultaneously. A reasonable solution to this game would be a vector of schedules, one for each airline, such that each airline's schedule is an optimal response to the others', anticipating how schedules affect prices. Anticipated prices would be the Nash equilibrium prices with the fixed schedules. Technically, this is the subgame perfect Nash equilibrium to a two stage game where, first, airlines simultaneously choose schedules, then, simultaneously choose prices for routes. See LEDERER[12] for an analysis of this game.

Two simple games are analyzed. For each, two firms compete to serve two cities. Although hub and two spoke city examples may seem uninteresting, the analysis may be extended to examples with many spoke cities. Aside from the two competing firms, there is an additional source of transportation, which we call the "alternate provider." The alternate provider corresponds to a mode with a fixed schedule and fixed prices. We interpret the alternate provider as automobile transport. The alternate provider is low priced but has long route duration. For the examples, the airlines have the same per hour flying cost, fixed cost for airplanes, and airplane capacity. This assumption may be generalized to asymmetric costs and fleets.

In the following examples, equilibria were found by optimizing one airline's schedule and prices against an initially chosen set of schedules and prices for a competing airline. Then, the competitor's schedules and prices were optimized against the airline's new choices. This procedure was repeated until each airline's choice was optimal.

Assumptions about model parameters for the first example are the same as those in the Sample Problem, except that the airplane capacity is 100 seats. Data about the alternate provider, airline costs, route duration and consumer density are: the alternate provider's routes are every two hours, its price is $50 and its route duration is 20 hours. The fixed cost per plane (F) is $5000, the cost per flying hour (V) is $4000, and the consumer density per hour is 170 for all origin-destinations. Airplanes can be dispatched on even numbered hours. Flights to/from the hub take 1 hour. Therefore a routing takes 3 hours. Consumer density is constant throughout the day. The alternate provider is dispatched each even numbered hour.

In equilibrium both airlines have full schedules and operate at capacity with four airplanes: (d_f, λ_f) (flight loading and shadow price on flight seats) equals (100, $129.7) for all flights, and (p_r, d_r, w_r) (route price, route demand and average schedule delay) equals ($298.2, 100 and 1.2) for all routes. The solution is symmetric for the two markets. Each airline serves approximately 29% of the market and the alternate provider serves the balance. Each airline's profit is approximately $503,700. The airlines' price in terms of ticket cost plus duration is $340.2 ($298.2 + 3 hours ($14/hour)) versus $330 ($50.0 + 20 hours ($14/hour)) for the alternate provider.

This is the only equilibrium to the game with two airlines. One airline cannot keep the other out of the market because there is too much demand for just one film.

The second example is a two-city problem where α, consumer disutility for incorrect takeoff time, consumer disutility for duration and number of periods are the same as for the first example, except that airplane capacity is 70 seats. Data about the alternate provider, airline costs, route duration and consumer density are: the alternate provider's routes are every 2 hours, its price is $80 and its route duration is 7 hours. The fixed cost per plane (F) is $5000, the cost per flying hour (V) is $4000, and the consumer density per hour is 70 for all origin-destinations. Again, airplanes can be dispatched every two hours, and a routing takes three hours. In this example, the consumer density has been greatly reduced.

There are two equilibria for this game. In the first, only one airline operates offering flights at even numbered hours and using four airplanes: $(d_f, \lambda_f) = (70, 40.2)$ for all flights, and $(p_r, d_r, w_r) = (135.4, 70, 1.2)$ for all routes. The airline carries exactly half the passengers. The airline's price in terms of ticket cost plus duration is $177.4 ($135.4 + 3 hours ($14/hour)) versus $227 ($80.0 + 7 hours

($14/hour)) for the alternate provider. The airline's profit is $15,472.

In the second equilibrium two airlines serve, one offering routes beginning at times 2, 6, 10, 14, 18 and 22 and the other offering routes two hours later. Each operates two airplanes: $(d_f, \lambda_f) = (70, 49.3)$ for all flights, and $(p_r, d_r, w_r) = (135, 70, 1.2)$ for all routes. All flights are at capacity. The airlines choose not to offer flights at the same time. Equilibrium prices are $135, slightly lower than in the other solution. Now, each airline carries exactly 25% of the total market, with the alternate provider serving half the market. Each airline's profit is $7,400; the total profit for both airlines is slightly less than the airline profit in the other equilibrium.

A number of observations flow from these examples. First, airplanes were always at capacity, indicating that revenue increases as route prices fall, until capacity constraints are binding. Apparently, the value of extra demand attracted by lowering a route price exceeds the revenue lost from existing customers. This is because of the sensitivity of passengers to fares and demonstrates the value to firms of reducing fares.

Second, there are many equilibria to the competitive scheduling and pricing problem. For both examples, there are equilibria with more than two firms serving the market. For the first competitive example, the solution uses eight airplanes and an equilibrium exists with up to eight airlines competing. In alternate equilibrium solutions, each airline provides at least one plane which is scheduled to complete a tour of the cities.

Third, the more firms that actually serve the markets, the lower the fares. What is somewhat surprising in the second example is that fares are not very different when two firms serve the markets compared with when only one firm serves them. If assumption A7 and parameter estimates are valid, this may show that domination of a hub by a single airline does not imply that there is a "wealth transfer" from customers to an airline. Although an airline that dominates a hub may have operational advantages such as the ability to offer more frequent routings than non-hub competitors, it may not be able to exploit these advantages through much higher prices. This result seems contrary to some empirical studies of airline competition such as those by BORENSTEIN[3] and Government Accounting Office.[9] These studies seem to show that airlines operating out of hubs that they dominate enjoy higher profits (as measured by revenue per seat mile) than other airlines. However, note two differences in our work from these studies. The first difference is we assume via A7 that com-

petitors can enter markets and compete; this may be unrealistic because of shortages of gates and landing slots at dominated hubs. Without A7 it may be true that an airline dominating a hub will get higher fares and earn higher profits. The second difference is that the empirical studies focus on routings originating or ending at the hub. We assume that all passengers originate and end at spoke cities. Therefore, the empirical studies focus on airfares that we do not study.

Fourth, competing firms may not wish to bunch flights if ideal departure time densities are not peaked. In the second example, competing firms choose to offset departure times rather than offer flights at the same time. Similar behavior is observed in the Second Series of Test Problems. The differentiation between airlines induced by the logit function causes airlines not to desire to bunch schedules. This observation contrasts with GREENHUT, NORMAN and HUNG'S[10] (pp. 297–300) that more competition causes more bunching of flights. Their model assumed that customers purchased tickets from the airline with the least cost; demand was not based on the logit functions assumed in this paper. Changing assumptions of consumer choice seems to change airlines' bunching behavior.

Fifth, as noted in Section 3, the interrelationship between demand in different markets drives the structure of the flight schedule. For example, in the Sample Problem, bunching of routes for cities $(1, 2)$ and $(2, 1)$ was driven by demand peaks. The peaks caused flights between the hub and cities 1 and 2 to be scheduled primarily in the morning and afternoon, and these flights caused routings between cities 1 (or 2) and other cities to be peaked as well. Understanding the complicated nature of route interrelationships is important when analyzing competition.

Clearly much more work needs to be done to help understand airline competition. Many more examples need to be analyzed; crucial assumptions, such as A7, need to be relaxed; and more complicated models allowing non-hub competition are needed. However, this section has shown that the heuristic may help to study the effect of competition on flights and schedules.

4. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

THE IDEAS of this paper would be especially valuable if the heuristic could solve realistically sized problems, as opposed to the rather small ones studied. To answer this we need to know how much larger a realistically sized problem is, how computation time increases with the size of the problem, and to what extent one could take advantage of parallel computational techniques to solve this problem. Let us address each of these in turn.

Observation of the current airline industry's hub-and-spoke systems suggests that to solve a real problem we would need to be able to handle a 50-city problem, with perhaps 24 periods (flights scheduled at most every hour), 2 classes of customers, and several sizes of planes. The number of origin-destination pairs as a function of the number of cities is $n(n - 1)$. The number of planes in the initial solution and the number of possible flights grows roughly linearly with the number of cities and linearly with the number of periods. This is because initially in the heuristic every possible flight is scheduled. With two classes of customers, say, business and non-business, demand on routes for each class would be considered separately, doubling the number of origin-destination pairs. The important point to realize is that once the capacity constraints have been relaxed and a Lagrange multiplier has been picked, the problem separates by origin-destination pair. Thus, it would be possible to solve the $n(n - 1)$ relatively small non-linear unconstrained optimizations in parallel. Although machines with several thousand parallel processors are not yet commercially available, such systems could be built today.

With such a parallel computer, we see that solution to our realistically sized problem is possible since the number of variables in the inner, unconstrained optimizations would be approximately twice as many (because of 24 periods rather than 12) and the number of major iterations (involving removal of flight pairs or removal of a tour of a plane) would only grow linearly with the number of cities. Thus the application of our heuristic to a large problem may be possible. Given the magnitude of the potential gain from better scheduling for a large airline and given the exponential increase in computing power that has occurred for several decades, the custom building of such a parallel machine is likely to be economically and technically feasible.

The difficulty of a fleet with several sizes of aircraft can be handled relatively easily. Each city would be assigned a plane size. Cities could be clustered based on the size of plane that would serve them. The tour elimination procedure would simply work on each of these clusters individually rather than on the network as a whole. Thus this complication could be handled heuristically without substantially affecting the computational complexity of the problem.

Many extensions of this work are possible. On the practical side, a refined price optimization routine would be useful (since our code for price optimiza-

tion was slow), as would adjustments to the code to allow parallel computation. As for the heuristic, faster and better procedures for estimating the impact of a change, e.g., the deletion of a flight, would allow for more thorough searches. Additional insights about competition between airlines would result from solving a series of well chosen examples. Finally such models are only as good as the data available for them. The estimation of the parameters of the problem, e.g., α, schedule delay disutility, and the consumer's demand density, is crucial. Such analysis is well beyond the scope of this paper but is requisite work before this model could be applied.

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