Supplementary Material: A Spatially-Lifted SCvx NMPC for Executing SIPP-Generated Time-Corridor Trajectories on an Articulated Loader

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1 Proof of Lemma 1 (Well-Posedness)

Recall from the main paper that

$$f(z, w, s) = f_a(z, s) + G(z, s)w$$

with

$$f_a(z, s) = \begin{bmatrix} (1 - d_f \kappa_{\mathrm{ref}}) \tan e_\theta \\ \dfrac{(1 - d_f \kappa_{\mathrm{ref}}) \sin \gamma}{(L_f \cos \gamma + L_r) \cos e_\theta} - \kappa_{\mathrm{ref}}(s) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$G(z, s) = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{L_r}{L_f \cos \gamma + L_r} & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Assumption 1 ensures that all denominators are bounded away from zero on Z_s, and that Z_s is compact. We also define

$$C_w := (1 + \omega_{\max} + a_{\max})\, w_{2,\max},$$

so that $\|w\| \leq C_w$ for all $w \in \mathcal{W}$.

We first show that f_a and G have uniformly bounded Jacobians with respect to z on Z_s. Let $D(\gamma) = L_f \cos \gamma + L_r$; by Assumption 1, $D(\gamma) \geq D_{\min} > 0$ for all $z \in Z_s$. For the first component

$$f_{a,1} = (1 - d_f \kappa_{\mathrm{ref}}) \tan e_\theta,$$

we have

$$\frac{\partial f_{a,1}}{\partial d_f} = -\kappa_{\text{ref}} \tan e_\theta, \qquad \frac{\partial f_{a,1}}{\partial e_\theta} = (1 - d_f \kappa_{\text{ref}}) \sec^2 e_\theta.$$

On Z_s we have $|d_f| \le \kappa_{\max}^{-1}$, $|\kappa_{\text{ref}}(s)| \le \kappa_{\max}$, and $|e_\theta| \le \frac{\pi}{2} - \varepsilon_\theta$, so $\cos e_\theta \ge c_\theta := \cos(\frac{\pi}{2} - \varepsilon_\theta) > 0$. Hence $\tan e_\theta$ and $\sec e_\theta$ are bounded, and therefore there exist constants $C_{1d}, C_{1e} > 0$ such that

$$\left|\tfrac{\partial f_{a,1}}{\partial d_f}\right| \le C_{1d}, \qquad \left|\tfrac{\partial f_{a,1}}{\partial e_\theta}\right| \le C_{1e}$$

uniformly on $Z_s \times [0, s_f]$.

For the second component,

$$f_{a,2} = \frac{(1 - d_f \kappa_{\text{ref}}) \sin \gamma}{D(\gamma) \cos e_\theta} - \kappa_{\text{ref}}(s),$$

the partial derivatives have the same structure: they are rational functions of (d_f, e_θ, γ) with denominators involving $D(\gamma)$ and $\cos e_\theta$, both bounded away from zero on Z_s. As a representative example,

$$\left|\frac{\partial f_{a,2}}{\partial e_\theta}\right| = \left|\frac{(1 - d_f \kappa_{\text{ref}}) \sin \gamma \tan e_\theta}{D(\gamma) \cos e_\theta}\right| \le \frac{C_\kappa}{D_{\min} c_\theta} C_t,$$

for suitable constants $C_\kappa, C_t > 0$ depending on the bounds on $d_f, \kappa_{\text{ref}}(s), \gamma, e_\theta$. All entries of $\partial f_a / \partial z$ are thus uniformly bounded on Z_s by continuity and compactness. Hence there exists $L_a > 0$ such that

$$\left\|\tfrac{\partial f_a}{\partial z}(z, s)\right\| \le L_a, \qquad \forall (z, s) \in Z_s \times [0, s_f]. \tag{1}$$

For the input matrix $G(z, s)$, the only z-dependent entry is

$$G_{21}(z, s) = \frac{L_r}{D(\gamma)}.$$

We compute

$$\frac{\partial G_{21}}{\partial \gamma} = \frac{L_r L_f \sin \gamma}{D(\gamma)^2},$$

which is bounded on Z_s since $|\sin \gamma| \le 1$ and $D(\gamma) \ge D_{\min}$. All other entries of G are constant. Thus there exists $L_G > 0$ such that, for each column g_i of G,

$$\left\|\tfrac{\partial g_i}{\partial z}(z, s)\right\| \le L_G, \quad i = 1, 2, 3, \ \forall (z, s) \in Z_s \times [0, s_f]. \tag{2}$$

Using these bounds, we now derive a global Lipschitz constant for f on Z_s. For any $z_1, z_2 \in Z_s$, $w \in \mathcal{W}$, and $s \in [0, s_f]$, the mean value theorem together with (1) gives

$$\|f_a(z_1, s) - f_a(z_2, s)\| \le L_a \|z_1 - z_2\|.$$

Similarly, from (2),

$$\|G(z_1, s)w - G(z_2, s)w\| \leq \|G(z_1, s) - G(z_2, s)\| \, \|w\| \leq L_G \|z_1 - z_2\| \, \|w\|.$$

The definition of $\mathcal{W}$ implies

$$\|w\| \leq |w_1| + |w_2| + |w_3| \leq (\omega_{\max} + 1 + a_{\max}) w_{2,\max} = C_w,$$

and therefore

$$\|f(z_1, w, s) - f(z_2, w, s)\| \leq (L_a + L_G C_w) \, \|z_1 - z_2\|.$$

Thus f is globally Lipschitz in z on Z_s with constant

$$L := L_a + L_G C_w. \tag{3}$$

It remains to verify the Carathéodory conditions for the spatial ODE

$$\frac{dz}{ds} = f(z(s), w(s), s), \qquad z(0) = z_0. \tag{4}$$

For each fixed (w, s), $f(\cdot, w, s)$ is continuous in z on Z_s, because f_a and G are smooth in z. For each fixed (z, w), the map $s \mapsto f(z, w, s)$ is measurable since $s \mapsto \kappa_{\text{ref}}(s)$ is measurable by assumption and compositions of measurable and continuous maps are measurable. Finally, because $Z_s \times \mathcal{W} \times [0, s_f]$ is compact and f is continuous, there exists $m_{\max} > 0$ such that $\|f(z, w, s)\| \leq m_{\max}$ for all (z, w, s), and we may choose $m(s) \equiv m_{\max}$ as an integrable majorant on $[0, s_f]$.

Thus f satisfies the Carathéodory conditions (see, e.g., [2, 3]), and for any $z_0 \in Z_s$ and measurable essentially bounded input $w(\cdot) \in L^\infty([0, s_f]; \mathcal{W})$ the ODE (4) admits a unique absolutely continuous solution on $[0, s_f]$ that depends continuously on $(z_0, w(\cdot))$ in the $C^0 \times L^\infty$ topology. This proves Lemma 1. □

2 Proof of Theorem 2 (Recursive Feasibility)

Theorem 2 states that, under the Lipschitz constant L from Lemma 1, the horizon condition $L N_p \Delta s \leq 1$, and a bounded trust-region radius $\rho_k \leq \rho_{\max}$, one can construct at each MPC step $k + 1$ a shift–append candidate with state deviation and slack bounds of the form (27)–(29).

We first construct a candidate trajectory at step $k + 1$ by shifting the optimal solution of the previous step. Let $\{(z^\star_{k,j}, w^\star_{k,j})\}_{j=0}^{N_p - 1}$ be the optimal solution of the SCvx subproblem (23) at step k, and denote this nominal trajectory by $(\bar{z}_j, \bar{w}_j)$. At step $k + 1$ we define the candidate controls by reusing the shifted nominal controls and appending a terminal feedback law: we set $\tilde{z}_0 = z_{k+1}$ (the current measured/estimated state), $\tilde{w}_j = \bar{w}_{j+1}$ for

$j = 0, \dots, N_p - 2$, and $\tilde{w}_{N_p-1} = K_f \tilde{x}_{N_p-1}$, where $\tilde{x}_{N_p-1}$ is the tracking-error component of $\tilde{z}_{N_p-1}$ and K_f is the terminal feedback gain from Assumption 2. The candidate states $\tilde{z}_j$ are generated by simulating the nonlinear spatial dynamics with step size Δs,

$$\tilde{z}_{j+1} = \tilde{z}_j + \Delta s\, f(\tilde{z}_j, \tilde{w}_j, s_j), \qquad j = 0, \dots, N_p - 1.$$

Let $\bar{z}_j$ denote the nominal states from step k, which satisfy

$$\bar{z}_{j+1} = \bar{z}_j + \Delta s\, f(\bar{z}_j, \bar{w}_j, s_j) + \mathcal{O}(\Delta s^2),$$

where the $\mathcal{O}(\Delta s^2)$ term accounts for discretization error of the Euler scheme. We define the deviation between the candidate and the shifted nominal as

$$\delta z_j := \tilde{z}_j - \bar{z}_{j+1}, \qquad j = 0, \dots, N_p - 1.$$

Using the Lipschitz continuity of f with respect to z (Lemma 1) and a bound C_G on its sensitivity to w (derived from boundedness of G on Z_s), one obtains the recursive inequality

$$\|\delta z_{j+1}\| \le (1 + L\Delta s)\|\delta z_j\| + C_G \rho_{\max} \Delta s, \tag{5}$$

for $j = 0, \dots, N_p - 2$, where the discretization error has been absorbed into the constant C_G (since Δs is fixed). Unfolding (5) yields

$$\|\delta z_j\| \le (1 + L\Delta s)^j \|\delta z_0\| + C_G \rho_{\max} \Delta s \sum_{i=0}^{j-1} (1 + L\Delta s)^i.$$

Using $(1 + L\Delta s)^j \le e^{Lj\Delta s}$ and the geometric-sum bound

$$\sum_{i=0}^{j-1} (1 + L\Delta s)^i \le \frac{e^{Lj\Delta s} - 1}{L\Delta s},$$

we obtain

$$\|\delta z_j\| \le e^{Lj\Delta s} \|\delta z_0\| + \frac{C_G \rho_{\max}}{L} (e^{Lj\Delta s} - 1). \tag{6}$$

The horizon condition $L N_p \Delta s \le 1$ implies $Lj\Delta s \le 1$, hence $e^{Lj\Delta s} \le e$ for all $j \le N_p$, and

$$\|\delta z_j\| \le e \|\delta z_0\| + \frac{e - 1}{L} C_G \rho_{\max}.$$

Under perfect state measurement one may take $\delta z_0 = 0$; more generally, any bounded δz_0 can be absorbed into a constant $C_z > 0$, yielding

$$\|\delta z_j\| \le C_z \rho_{\max} N_p \Delta s, \tag{7}$$

which is precisely the deviation bound (27) in the main paper.

We now bound the consistency slack. Recall

$$\psi(z) = \frac{1 - d_f \kappa_{\text{ref}}(s)}{v_f \cos e_\theta}$$

and the linearized consistency constraint

$$w_{2,j} = \psi(\bar{z}_j) + \nabla\psi(\bar{z}_j)^\top (z_j - \bar{z}_j) + \nu_j. \tag{8}$$

For the candidate trajectory we define the slack

$$\tilde{\nu}_j := \psi(\tilde{z}_j) - \psi(\bar{z}_j) - \nabla\psi(\bar{z}_j)^\top (\tilde{z}_j - \bar{z}_j), \tag{9}$$

which is exactly the second-order Taylor remainder of ψ along the segment joining $\bar{z}_j$ and $\tilde{z}_j$. By Taylor's theorem with integral remainder, there exists a point ξ_j on that segment such that

$$\tilde{\nu}_j = \tfrac{1}{2}(\tilde{z}_j - \bar{z}_j)^\top \nabla^2\psi(\xi_j)(\tilde{z}_j - \bar{z}_j).$$

On Z_s, the function ψ is twice continuously differentiable and its Hessian is bounded: there exists $H_\psi > 0$ such that $\|\nabla^2\psi(z)\| \le H_\psi$ for all $z \in Z_s$. Therefore,

$$\|\tilde{\nu}_j\|_\infty \le \tfrac{1}{2} H_\psi \|\tilde{z}_j - \bar{z}_j\|^2.$$

Combining this with the deviation bound (7) gives

$$\|\tilde{\nu}_j\|_\infty \le \tfrac{1}{2} H_\psi (C_z \rho_{\max} N_p \Delta s)^2 =: \epsilon_\nu, \tag{10}$$

which coincides with (28) in the main paper. Since $\tilde{\nu}_j$ is a feasible choice for the slack in (8), the optimal slack $\nu_j^\star$ of the SCvx subproblem must satisfy $\|\nu_j^\star\|_\infty \le \epsilon_\nu$.

We proceed similarly for the time-corridor slack. Let δt_j be the deviation of the time state between candidate and shifted nominal, i.e. the last component of δz_j. From the time-state update and the consistency constraint we obtain a recursion of the form

$$\delta t_{j+1} = \delta t_j + \Delta s(\tilde{w}_{2,j} - \bar{w}_{2,j+1}),$$

where the difference $\tilde{w}_{2,j} - \bar{w}_{2,j+1}$ can be bounded by the Lipschitz continuity of ψ and the bound on $\tilde{\nu}_j$. More precisely, there exists $C_\psi > 0$ such that

$$|\tilde{w}_{2,j} - \bar{w}_{2,j+1}| \le C_\psi \|\delta z_j\| + \|\tilde{\nu}_j\| \le \tilde{C} \rho_{\max} N_p \Delta s,$$

for some $\tilde{C} > 0$ depending on C_z and H_ψ. Iterating this recursion yields

$$\begin{aligned} |\delta t_j| &\le \sum_{i=0}^{j-1} |\tilde{w}_{2,i} - \bar{w}_{2,i+1}| \Delta s \\ &\le j\, \tilde{C} \rho_{\max} N_p \Delta s^2 \le N_p \tilde{C} \rho_{\max} N_p \Delta s^2. \end{aligned}$$

Thus there exists $C_\sigma > 0$ such that

$$|\delta t_j| \le C_\sigma \rho_{\max} N_p^2 \Delta s^2 =: \epsilon_\sigma, \tag{11}$$

which corresponds to (29).

The candidate SIPP slacks are defined by

$$\tilde{\sigma}_j^- := \max\{0, t_{\min}(s_j) - \tilde{t}_j\}, \quad \tilde{\sigma}_j^+ := \max\{0, \tilde{t}_j - t_{\max}(s_j)\},$$

so that $\|\tilde{\sigma}_j^\pm\|_\infty \le |\delta t_j|$ up to a nominal feasibility margin. Since these $\tilde{\sigma}_j^\pm$ are feasible choices for the slacks in the SCvx subproblem, the optimal slacks $\sigma_j^{\pm,\star}$ satisfy $\|\sigma_j^{\pm,\star}\|_\infty \le \epsilon_\sigma$.

Together with the state deviation bound (7), these estimates show that the shift–append candidate at MPC step $k+1$ satisfies (27)–(29), and hence the SCvx subproblem is feasible with optimally bounded slacks. This proves Theorem 2. □

3 Proof of Theorem 3 (Local Practical Stability)

Theorem 3 asserts that, under Assumptions 1 and 2 and the slack bounds of Theorem 2, the MPC value function V_k satisfies

$$V_{k+1} - V_k \le -\alpha_3(\|x_k\|) + \beta\,\epsilon,$$

for suitable class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and a constant $\beta > 0$, where $\epsilon = \max\{\epsilon_\nu, \epsilon_\sigma\}$. This implies local practical stability of the origin.

We begin by recalling the terminal ingredients. Let $(A_{\text{lin}}, B_{\text{lin}})$ denote the linearization of the error dynamics around the terminal reference along the spatial grid. Choose a stabilizing feedback gain K_f and a positive definite matrix P_f that solve the discrete-time algebraic Riccati equation for the pair $(A_{\text{lin}}, B_{\text{lin}})$ with stage weight $Q_f \succ 0$. Define the closed-loop matrix $A_{\text{cl}} = A_{\text{lin}} + B_{\text{lin}} K_f$ and the quadratic Lyapunov function

$$V(x) = x^\top P_f x.$$

For the linearized dynamics $x^+ = A_{\text{cl}} x$, the Riccati equation yields

$$V(x^+) - V(x) = x^\top (A_{\text{cl}}^\top P_f A_{\text{cl}} - P_f) x = -x^\top Q_f x.$$

For the nonlinear closed-loop dynamics $x^+ = F(x)$ under $w = K_f x$, we write

$$x^+ = A_{\text{cl}} x + \phi(x),$$

where $\phi(x)$ collects higher-order terms. Since F is C^2 and $F(0) = 0$, there exists $c_\phi > 0$ and a neighborhood of the origin such that $\|\phi(x)\| \le c_\phi \|x\|^2$. Then

$$\begin{aligned} V(x^+) - V(x) &= (A_{\text{cl}} x + \phi(x))^\top P_f (A_{\text{cl}} x + \phi(x)) - x^\top P_f x \\ &= -x^\top Q_f x + 2x^\top A_{\text{cl}}^\top P_f \phi(x) + \phi(x)^\top P_f \phi(x). \end{aligned}$$

Using bounds $\|A_{\rm cl}^\top P_f\| \le c_A$ and $\|P_f\| \le c_P$, we obtain

$$|2x^\top A_{\rm cl}^\top P_f \phi(x)| \le 2c_A c_\phi \|x\|^3, \qquad |\phi(x)^\top P_f \phi(x)| \le c_P c_\phi^2 \|x\|^4.$$

Hence, for $\|x\| \le r$,

$$V(x^+) - V(x) \le -\lambda_{\min}(Q_f)\|x\|^2 + 2c_A c_\phi \|x\|^3 + c_P c_\phi^2 \|x\|^4.$$

By choosing $r > 0$ such that

$$2c_A c_\phi r + c_P c_\phi^2 r^2 \le \tfrac{1}{2}\lambda_{\min}(Q_f),$$

we ensure that, whenever $\|x\| \le r$,

$$V(x^+) - V(x) \le -\tfrac{1}{2}\lambda_{\min}(Q_f)\|x\|^2.$$

The terminal set

$$X_f = \{x : x^\top P_f x \le \rho_f\}$$

is then chosen so that $X_f \subseteq \{x : \|x\| \le r\}$ and $X_f \subseteq Z_s$, and such that all state and input constraints are satisfied under $w = K_f x$. This construction verifies Assumption 2 and provides an explicit local invariance margin around the origin.

We now turn to the MPC value function. Let V_k denote the optimal cost of the nonlinear MPC problem at step k, including stage cost, terminal cost, and the penalties on slacks. Let $(z^\star, w^\star)$ be an optimal solution at step k, and $(\tilde{z}, \tilde{w})$ the shift–append candidate at step $k+1$ constructed in Theorem 2. By optimality of V_{k+1},

$$V_{k+1} \le J_{\rm nl}(\tilde{z}, \tilde{w}),$$

where $J_{\rm nl}$ denotes the true nonlinear cost evaluated along $(\tilde{z}, \tilde{w})$.

The difference $V_{k+1} - V_k$ is decomposed into a nominal decrease plus error terms arising from linearization and slacks. The nominal decrease comes from the stage costs and the terminal Lyapunov property inside X_f, and yields a bound of the form

$$J_{\rm nl}(\tilde{z}, \tilde{w}) - J_{\rm nl}(z^\star, w^\star) \le -\underline{\alpha}\,\|x_k\|^2 + \beta_{\rm term}\|x_k\|^2,$$

for some $\underline{\alpha}, \beta_{\rm term} > 0$ depending on Q, R, Q_f, P_f and the size of X_f. The linearization error between the nonlinear cost and the linearized cost used by SCvx can be bounded using the Lipschitz continuity of ℓ and Φ in z together with the state deviation bound (27) from Theorem 2. There exist constants $L_\ell, L_\Phi > 0$ such that

$$|\ell(\tilde{z}_j, \tilde{w}_j) - \ell(\bar{z}_j, \bar{w}_j)| \le L_\ell \|\delta z_j\|, \quad |\Phi(\tilde{x}_{N_p}) - \Phi(\bar{x}_{N_p})| \le L_\Phi \|\delta z_{N_p}\|.$$

Using (27) we obtain

$$|\Delta_{\mathrm{lin}}| := \left|J_{\mathrm{nl}}(\tilde{z}, \tilde{w}) - J_{\mathrm{lin}}(\tilde{z}, \tilde{w})\right| \leq \beta_{\mathrm{lin}} \rho_{\max} N_p^2 \Delta s,$$

for some constant $\beta_{\mathrm{lin}} > 0$ depending on L_ℓ, L_Φ and C_z.

The contribution of the slacks is directly controlled by the bounds (28)–(29). Let $\epsilon = \max\{\epsilon_\nu, \epsilon_\sigma\}$ and define

$$\beta_{\mathrm{slack}} := N_p(\rho_\nu + 2\rho_\sigma),$$

where ρ_ν and ρ_σ are the penalties in the SCvx cost. Since the optimal slacks satisfy $\|\nu_j^\star\|_\infty \leq \epsilon_\nu$ and $\|\sigma_j^{\pm,\star}\|_\infty \leq \epsilon_\sigma$, we have

$$\sum_{j=0}^{N_p-1} \Big(\rho_\nu \|\nu_j^\star\|_1 + \rho_\sigma(\|\sigma_j^{+,\star}\|_1 + \|\sigma_j^{-,\star}\|_1)\Big) \leq \beta_{\mathrm{slack}} \epsilon.$$

Collecting all contributions, the value function satisfies

$$V_{k+1} - V_k \leq -\alpha_3(\|x_k\|) + (\beta_{\mathrm{lin}} + \beta_{\mathrm{slack}})\, \epsilon,$$

for a suitable class-$\mathcal{K}_\infty$ function α_3 and $\beta := \beta_{\mathrm{lin}} + \beta_{\mathrm{slack}}$. This is exactly inequality (30) in the main paper:

$$V_{k+1} - V_k \leq -\alpha_3(\|x_k\|) + \beta\, \epsilon.$$

Standard Lyapunov arguments for discrete-time systems (see, e.g., [2,4]) then imply that the error state x_k converges to a neighborhood of the origin whose radius is proportional to $\sqrt{\epsilon}$. This establishes local practical stability and completes the proof of Theorem 3. □

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References

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