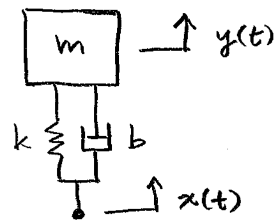
# Transfer Functions

What is a transfer function?

- Mathematical representation of the input/output dynamics of a system
- Laplace domain    input $\longrightarrow \boxed{G(s)} \longrightarrow$ output
- Describes how input to a system is affected by system to produce output response
- An algebraic expression (instead of an ODE)
  - easy to manipulate
- Can quickly determine natural response from roots of denominator polynomial
  - eigenvalues
  - characteristic roots
  - poles
- Gives insights into frequency response of system.
- Defined for linear, time-invariant systems — do not exist for nonlinear systems
- Response to forcing input — IC's not considered

# Finding Transfer Functions Using the Laplace Transform:

**Example:** Simple Suspension System




EOM

$$m\ddot{y} + b(\dot{y}-\dot{x}) + k(y-x) = 0$$

1) Take Laplace transform — Assume zero IC's:

$$ms^2 Y(s) + bsY(s) - bsX(s) + kY(s) - kX(s) = 0$$

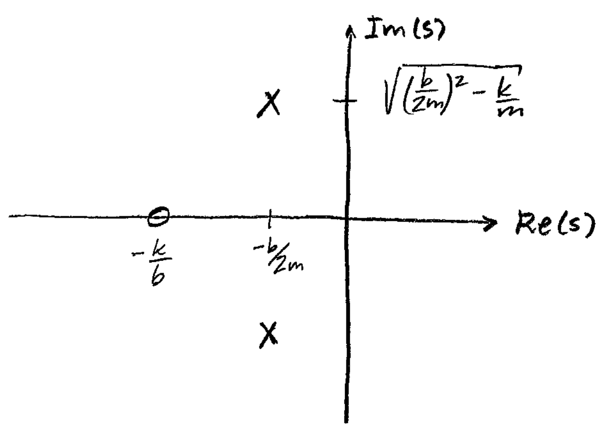
$$(ms^2 + bs + k)Y(s) - (bs+k)X(s) = 0$$

2) Manipulate algebraically to obtain desired transfer fcn:

$$\frac{Y(s)}{X(s)} = \frac{bs+k}{ms^2+bs+k}$$

TF zero at $s = -k/b$

TF poles at $s = -\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}\ ,\ -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$



# Finding Transfer Functions:

## CS I: Single Link Robot Arm

Find TF from $\tau$ to $\theta$:

Take Laplace x-form w/ zero IC's:

$$J\ddot{\tilde{\theta}} + b\dot{\tilde{\theta}} - \underbrace{\left(mg\frac{l}{2}\sin\theta_0\right)}_{c}\tilde{\theta} = \tau(t)$$

$$Js^2\Theta(s) + bs\,\Theta(s) - c\,\Theta(s) = T(s)$$

$$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + bs - c} = \frac{1}{Js^2 + bs - mg\frac{l}{2}\sin\theta_0}$$

## CS II: Inverted Pendulum

$$(m_1+m_2)\ddot{y} + m_1 l\ddot{\theta} + b\dot{y} = F(t)$$
$$m_1 l\ddot{y} + m_1 l^2\ddot{\theta} - m_1 g l\theta = 0$$

$$\begin{cases} (m_1+m_2)s^2 Y(s) + m_1 l s^2\Theta(s) + bs\,Y(s) = F(s) \\ m_1 l s^2 Y(s) + m_1 l^2 s^2\Theta(s) - m_1 g l\,\Theta(s) = 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} (m_1+m_2)s^2+bs & m_1 l s^2 \\ m_1 l s^2 & (m_1 l^2 s^2 - mgl) \end{bmatrix}}_{A}\begin{bmatrix} Y(s) \\ \Theta(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

Want $\dfrac{Y(s)}{F(s)}$, $\dfrac{\Theta(s)}{F(s)}$

Use Cramer's Rule

$$Y(s) = \frac{1}{\det A} \cdot \det \begin{bmatrix} F(s) & m_1 l s^2 \\ 0 & m_1 l^2 s^2 - m_1 g l \end{bmatrix}$$

$$= \frac{(m_1 l^2 s^2 - m_1 g l)\, F(s)}{[(m_1+m_2)s^2 + bs](m_1 l^2 s^2 - mgl) - m_1^2 l^2 s^4}$$

$$\boxed{\frac{Y(s)}{F(s)} = \frac{(m_1 l^2 s^2 - mgl)}{[(m_1+m_2)s^2 + bs](m_1 l^2 s^2 - mgl) - m_1^2 l^2 s^4}}$$

$$\Theta(s) = \frac{1}{\det A} \cdot \det \begin{bmatrix} (m_1+m_2)s^2 + bs & F(s) \\ m_1 l s^2 & 0 \end{bmatrix}$$

$$\boxed{\frac{\Theta(s)}{F(s)} = \frac{m_1 l s^2}{[(m_1+m_2)s^2 + bs](m_1 l^2 s^2 - mgl) - m_1^2 l^2 s^4}}$$

What about $\frac{Y(s)}{\Theta(s)}$ ?

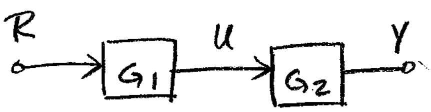
$$\frac{Y(s)}{\Theta(s)} = \frac{Y}{F} \cdot \frac{F}{\Theta} = \frac{(m_1 l^2 s^2 - m_1 g l)}{\det A} \cdot \frac{\det A}{m_1 l s^2}$$

$$\frac{Y(s)}{\Theta(s)} = \frac{m_1 l^2 s^2 - m_1 g l}{m_1 l s^2} = \underline{\underline{\frac{l s^2 - g}{s^2}}}$$

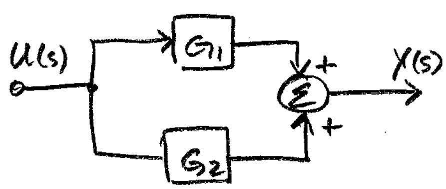
## Block Diagrams

- To obtain the transfer function for a complete system, we first find the transfer functions for the components of the system (actuator, plant, controller, sensors, etc) and then solve the algebraic relations to find the overall system transfer function.
- In many control systems, the dynamics of the components do not interact except for the output of one component being the input to another component. In these cases, it is easy to draw a component block diagram to represent the relationships between the components.
- The transfer function for each component is placed in a box and the input-output relations between components are indicated with lines and arrows.

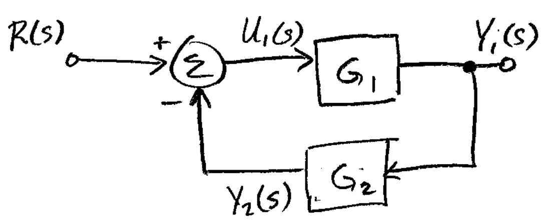
Some Examples:




$$\frac{Y}{R} = \frac{U}{R} \cdot \frac{Y}{U} = \underline{G_1 G_2}$$




$$Y = G_1 U + G_2 U \Rightarrow \underline{\frac{Y}{U} = G_1 + G_2}$$




$$U_1(s) = R(s) - Y_2(s)$$
$$Y_2(s) = G_2 Y_1(s)$$
$$Y_1(s) = G_1 U_1(s)$$

$$\frac{Y_1}{G_1} = R - G_2 Y_1$$

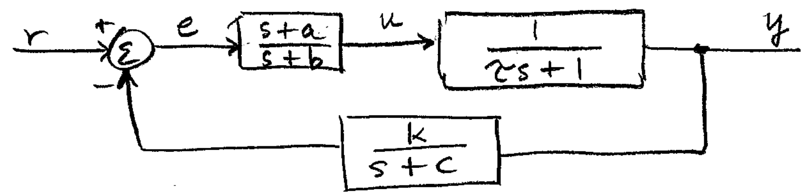
$$\left(\frac{1}{G_1} + G_2\right) Y = R$$

$$\frac{Y}{R} = \frac{1}{\frac{1}{G_1} + G_2}$$

Rule: The gain of a single-loop negative feedback system is given by the forward gain divided by the sum of 1 plus the loop gain

$$\boxed{\frac{Y}{R} = \frac{G_1}{1 + G_1 G_2}}$$

Example




$$\frac{Y}{R} = \frac{G_1}{1 + G_1 G_2}$$

Here $G_1 = \dfrac{s+a}{(s+b)(\tau s+1)}$

$G_2 = \dfrac{k}{s+c}$

$$\frac{Y}{R} = \frac{\dfrac{s+a}{(s+b)(\tau s+1)}}{1 + \dfrac{k(s+a)}{(s+b)(\tau s+1)(s+c)}} = \frac{\dfrac{s+a}{\cancel{(s+b)(\tau s+1)}}}{1 + \dfrac{k(s+a)}{\cancel{(s+b)(\tau s+1)(s+c)}}} \cdot \frac{(s+b)\cancel{(\tau s+1)}\cancel{(s+c)}}{(s+b)(\tau s+1)(s+c)}$$

$$\boxed{\frac{Y}{R} = \frac{(s+a)(s+c)}{(s+b)(\tau s+1)(s+c) + k(s+a)}}$$

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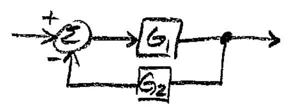
Alternative Expression :

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Then do example above.

$$\frac{Y}{R} = \frac{G_1}{1+G_1 G_2} = \frac{\dfrac{N_{G1}}{D_{G1}}}{1 + \dfrac{N_{G1}}{D_{G1}}\dfrac{N_{G2}}{D_{G2}}} \cdot \frac{D_{G1} D_{G2}}{D_{G1} D_{G2}}$$

$$\boxed{\frac{Y}{R} = \frac{N_{G1} D_{G2}}{D_{G1} D_{G2} + N_{G1} N_{G2}}}$$

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Q : What about case where feedback is positive ?

If $N_{G2} = D_{G2} = 1$,

$$\Rightarrow \frac{Y}{R} = \frac{N_{G1}}{D_{G1} + N_{G1}}$$