V.5 a) The equations of motion are

$$m_1\ddot{y} - m_1 y\dot{\theta}^2 + m_1 g\sin\theta = 0$$

$$\left(\frac{m_2 l^2}{3} + m_1 y^2\right)\ddot{\theta} + 2m_1 y\dot{y}\dot{\theta} + m_1 g y\cos\theta + m_2 g\frac{l}{2}\cos\theta = lF\cos\theta$$

The equilibria are when $\dot{y} = \ddot{y} = \dot{\theta} = \ddot{\theta} = 0$ or

$$\begin{bmatrix} m_1 g\sin\theta_e \\ m_1 g y_e\cos\theta_e + m_2 g\frac{l}{2}\cos\theta_e \end{bmatrix} = \begin{bmatrix} 0 \\ l F_e \cos\theta_e \end{bmatrix}$$

$m_1 g\sin\theta_e = 0 \Rightarrow \theta_e = 0$ since other configurations don't make sense

$$m_1 g y_e + m_2 g\frac{l}{2} = lF_e$$

and so $F_e = m_1 g\, y_e/l + \frac{1}{2}m_2 g$ (1.1)

(y_e, θ_e, F_e) s.t. $\theta_e = 0$ and (y_e, F_e) satisfy (1.1)

b) Let $y = y_e + \tilde{y}$, $\theta = \theta_e + \tilde{\theta}$, $\tilde{F} = F_e + \tilde{F}$

$$y\dot{\theta}^2 \simeq y_e\dot{\theta}_e^2 + \frac{\partial}{\partial y}(y\dot{\theta}^2)\Big|_e \tilde{y} + \frac{\partial}{\partial\dot{\theta}}(y\dot{\theta}^2)\Big|_e \tilde{\dot{\theta}}$$

$$= y_e\dot{\theta}_e^2 + \dot{\theta}_e^2\tilde{y} + 2y_e\dot{\theta}_e\tilde{\dot{\theta}} = 0$$

$\sin\theta \simeq \tilde{\theta}$ (as shown in class

$$y\dot{y}\dot{\theta} \simeq y_e\dot{y}_e\dot{\theta}_e + \frac{\partial}{\partial y}(y\dot{y}\dot{\theta})\Big|_e\tilde{y} + \frac{\partial}{\partial\dot{y}}(y\dot{y}\dot{\theta})\Big|_e\tilde{\dot{y}} + \frac{\partial}{\partial\dot{\theta}}(y\dot{y}\dot{\theta})\Big|_e\tilde{\dot{\theta}}$$

$$= y_e\dot{y}_e\dot{\theta}_e + \dot{y}_e\dot{\theta}_e\tilde{y} + y_e\dot{\theta}_e\tilde{\dot{y}} + y_e\dot{y}_e\tilde{\dot{\theta}} = 0$$

$$y\cos\theta \simeq y_e\cos\theta_e + \frac{\partial}{\partial y}(y\cos\theta)\Big|_e\tilde{y} + \frac{\partial}{\partial\theta}(y\cos\theta)\Big|_e\tilde{\theta}$$

$$= y_e\cos\theta_e + \cos\theta_e\tilde{y} - y_e\sin\theta_e\tilde{\theta}$$

$$F\cos\theta \approx F_e\cos\theta_e + \frac{\partial}{\partial F}(F\cos\theta)\Big|_e \tilde{F} + \frac{\partial}{\partial\theta}(F\cos\theta)\Big|_e \tilde{\theta} = F_e\cos\theta_e + \cos\theta_e\tilde{F} - F_e\sin\theta_e\tilde{\theta}$$ ②

$\cos\theta \approx 1$ (as shown in class)

$$y^2\ddot{\theta} \approx y_e^2\ddot{\theta}_e + \frac{\partial\, y^2\ddot{\theta}}{\partial y}\Big|_e \tilde{y} + \frac{\partial}{\partial\ddot{\theta}}(y^2\ddot{\theta})\Big|_e \ddot{\tilde{\theta}} = y_e^2\ddot{\theta}_e + 2y_e\ddot{\theta}_e\tilde{y} + y_e^2\ddot{\tilde{\theta}}$$
$$= y_e^2\ddot{\tilde{\theta}}$$

$\therefore$ the linearized equations are

$$m_1\ddot{y}_1 + m_1 g\tilde{\theta} = 0$$

$$\left(\frac{m_2\ell^2}{3} + m_1 y_e^2\right)\ddot{\tilde{\theta}} + m_1 g(y_e + \tilde{y}) + m_2 g\frac{\ell}{2} = \ell(F_e\cos\theta_e + \tilde{F})$$

using (1.1) gives

$$\boxed{\begin{aligned} &m_1\ddot{\tilde{y}} + m_1 g\tilde{\theta} = 0 \\ &\left(\frac{m_2\ell^2}{3} + m_1 y_e^2\right)\ddot{\tilde{\theta}} + m_1 g\tilde{y} = \ell\tilde{F}\end{aligned}}$$