ECEn 483 / ME 431 Case Studies

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Contents

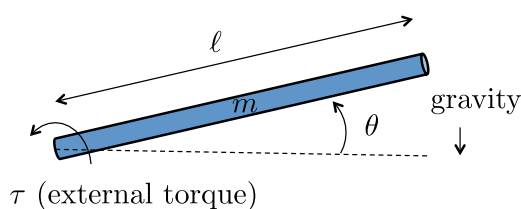# Design Problem 1

Single Link Robot Arm

Figure 1.1: Single Link Robot Arm

Figure 1.1 shows the single link robot arm system. The robot arm has mass m and length ℓ. The angle of the robot is given by θ as measured from level. The angular speed of the arm is $\dot{\theta}$. There is an applied torque τ at the joint, and assuming a damping torque that opposes the rotation of the joint of magnitude $-b\dot{\theta}$.

Assume the following physical constants: $m = 0.5$ kg, $\ell = 0.3$ m, $g = 9.8$ m/s^2, $b = 0.01$ Nms.

Homework I.1 Using the configuration variable θ, write an expression for the kinetic energy of the system.

Homework I.2 Create a simulink animation of the single link robot arm. The input should be a slider for θ.

Homework I.3

    (a) Find the potential energy for the system.

(b) Define the generalized coordinates.

(c) Find the generalized forces.

(d) Derive the equations of motion using the Euler-Lagrange equations.

Homework I.4 Modify the simulink model created in homework I.2 by creating an s-function that implements the equations of motion. The input to the s-function should be a slider for torque. The output should go to the animation developed in homework I.2.

Homework I.5

(a) Find the equilibria of the system.

(b) Linearize the system about the equilibria.

Homework I.6 Find the transfer function of the system from the torque τ to the angle θ.

Homework I.7 Defining the states as $x = (\tilde{\theta}, \dot{\tilde{\theta}})^\top$, the input as $u = \tilde{\tau}$, and the measured output as $y = \tilde{\theta}$, find the linear state space equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework I.8 Suppose that the size of the input torque is limited to $\tau_{\max} = 1$ Nm.

(a) Suppose that a PD controller is used to stabilize the system and regulate θ to θ^d. Draw the (linearized) closed loop block diagram using the transfer function found in problem I.6. Derive the actual transfer function from $\tilde{\theta}^d$ to $\tilde{\theta}$.

(b) Considering the equilibrium torque τ_e, derive a bound $\tau_{\max}$ on the linearized torque $\tilde{\tau}$ such that $|\tilde{\tau}| \leq \tau_{\max}$ when the maximum possible equilibrium torque is used.

(c) Select the proportional gain k_p so that $\tilde{\tau}$ just saturates when a step of size $A_{th} = 50$ degrees is placed on $\tilde{\theta}_d$.

(d) If the desired transfer function is given by

$$\tilde{\theta}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\tilde{\theta}^d(s),$$

find the natural frequency ω_n and the proportional gain k_d so that the actual transfer function equals the desired transfer function, where ζ is a design parameter.

(e) What are the closed loop poles?

Homework I.9 Implement the PD controller designed in Homework I.8 on the single link robot arm in Simulink. Note that the input torque from the PD controller is given by

$$\tilde{\tau} = k_p(\theta^d - \theta) - k_d\dot{\theta}.$$

You should select the input torque as

$$\tau = \tau_e + \tilde{\tau},$$

where τ_e is the equilibrium torque found in Homework I.5.

Homework I.10

(a) With PD control, what is the system type? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

(b) Consider the case where a constant disturbance acts at the input to the plant (for example gravity in this case). What is the steady state error to a constant input disturbance when the integrator is not present, and when it is present?

(c) Add an integrator to the PD control developed in Homework I.9. Draw the block diagram of the closed loop system with PID control. Find the transfer function for the closed loop system with the integrator added.

(d) Put the characteristic equation in Evan's form and use the Matlab SISO tool to plot the root locus verses the integrator gain k_i.

Homework I.11 The objective of this problem is to implement the PID controller using Matlab code using only measured outputs of the system.

(a) Change the parameter file param.m to specify actual parameters and the parameters known to the controller. Change the values for mass and length known to the controller by 5% to simulate imperfect knowledge of the plant.

(b) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$. The controller should only assume knowledge of the angle θ and the desired angle θ^d.

(c) Implement the PID controller designed in Problems I.8 and I.10 using an m-function called arm_ctrl.m. Use the dirty derivative gain of $\tau = 0.05$.

Homework I.12 Draw by hand the Bode plot of the single link robot arm from torque $\tilde{\tau}$ to angle $\tilde{\theta}$ given that the equilibrium angle is $\theta_e = 0$. Use the Matlab `bode` command and compare your results.

Homework I.13 For the single link robot arm, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, (2) the plant under PD control, and (3) the plant under PID control, using the control gains calculated in Homework 11.

(a) To what percent error can the closed loop system under PID control track the desired input if all of the frequency content of $\theta^d(t)$ is below $\omega = 0.4$ radians per second?

(b) If the desired input is $\theta^d(t) = 5t$ for $t \geq 0$, what will be the steady state tracking error to this input?

(c) If all of the frequency content of the input disturbance $d_i(t)$ is below $\omega = 0.01$ radians per second, what percentage of the input disturbance shows up in the output θ?

(d) If all of the frequency content of the noise $n(t)$ is greater than $\omega = 100$ radians per second, what percentage of the noise shows up in the output signal θ?

Homework I.14 For this homework assignment we will use loopshaping to improve the PD controllers developed in Homework 9. Download the simulink files on the web that are associated with this homework. When you open the Simulink file, notice the presence of the LTI system with parameter C, and also the prefilter with parameter F.

(a) At the Matlab prompt, type >> C = 1 and F=1 to set the additional compensator ant prefilter to one. Run the simulation to establish baseline performance.

(b) Note in `param.m` the line `L=series(G,C_pd)`, where $L(s)$ is the loop gain. Load the `sisotool` and enter L for the system plant under the tab `System Data`.

(c) The objective will be to (1) improve tracking and disturbance rejection by a factor of 10 for reference signals and disturbances below 0.007 radians/sec, (2) improve noise attention by a factor of 10 for frequencies above 200 radians/sec. In `sisotool` add constraints on the open loop Bode plot that reflect these objectives.

(d) Add a phase lag filter to satisfy the disturbance rejection and tracking objective, and add one to two poles to meet the noise rejection objective. If needed, add a phase lead filter to keep the phase margin around 60 degrees.

(e) Export $C(s)$, and run the combined controller in Simulink.

(f) Note that despite having a good phase margin, there is still significant overshoot, due in part to the windup effect in the phase lag filter. This can be mitigated by adding a prefilter, that essentially modifies the hard step input into the system. In sisotool, add the a plot for the closed loop Bode response. Add a loop pass filter for $F(s)$ as a prefilter to flatten the closed loop Bode response.

Homework I.15 The objective of this problem is to implement state feedback controller using the full state. Start with the simulation files developed in Homework I.11.

(a) Modify arm_dynamics.m so that the outputs are $(\theta, x)^\top$, where $x = (\theta, \dot\theta)^\top$. In this problem we will use x instead of θ but in the next problem we will again remove x as an output of the plant and reconstruct the state using an observer.

(b) Rearrange the block diagram so that the state x is an input to the controller.

(c) Select the closed loop poles as the roots of the equations $s^2 + 2\zeta\omega_n + \omega_n^2 = 0$ where ω_n, and ζ were found in Homework I.11.

(d) Add the state space matrices A, B, C, D derived in Homework I.7 to your param file.

(e) Verify that the state space system is controllable by checking that $\text{rank}(\mathcal{C}) = n$.

(f) Using the Matlab `place` command, find the feedback gain K such that the eigenvalues of $(A - BK)$ are equal to desired closed loop poles. Find the reference gain k_r so that the DC-gain from θ^c to θ is equal to one. Note that $K = (k_p, k_d)$ where k_p and k_d are the proportional and derivative gains found in Homework I.11. Why?

(g) Implement the state feedback scheme in Simulink and tune the closed loop poles to get good response.

Hints:

- For comparison, using exact parameters, I get the following state space model:

$$A = \begin{pmatrix} 0 & 1.0000 \\ 0 & -0.667 \end{pmatrix}$$
$$B = \begin{pmatrix} 0 \\ 66.667 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

- We also need to input the equilibrium torque as

$$\tau_e = \frac{mg\ell}{2} \cos(\theta^d)$$
$$\tau = \tau_e - Kx + k_r\theta^d.$$

Homework I.16

(a) Modify the state feedback solution developed in Homework 15 to add an integrator with anti-windup.

(b) Change the parameters known to the controller by $\pm 5\%$.

(c) Tune the integrator to get good tracking performance.

Homework I.17 The objective of this problem is to replace the state x in Homework 16 with the estimated state $\hat{x}$ produced by an observer.

(a) Modify the simulink diagram from Homework 16 so that y is also an input to the controller, as well as x, and so that $\hat{x}$ is also an output of the controller. Add a scope to plot the estimation error $x - \hat{x}$.

(b) Verify that the state space system is observable by checking that $\text{rank}(\mathcal{O}) = n$.

(c) In the control block, add an observer to estimate the state $\hat{x}$. Do not yet use $\hat{x}$ in the state feedback equations. Run the controller using the actual state x and tune the observer so that the estimation error is small. Make sure the $\text{eig}(A - BK - LC)$ are in the left half plane.

(d) Replace x in the controller with the estimated state $\hat{x}$ and tune the controller and observer if necessary to obtain good performance.

(e) Modify the simulink diagram to add an input disturbance of 0.2 Newton-meters. Add a state to the observer to estimate the disturbance. Replace the integrator with an estimate of the disturbance and compare the performance.

Hints:

- *Note that the observer is estimating the linearized state $\tilde{x} = x - x_e$. Therefore the observer update equation will be*

$$\dot{\hat{\tilde{x}}} = A\hat{\tilde{x}} + B\tilde{u} + L(\tilde{y} - C\hat{\tilde{x}}),$$

but sense $\dot{\tilde{x}} = \dot{x} - \dot{x}_e = \dot{x}$, and since $y_e = Cx_e$, this is equivalent to

$$\dot{\hat{x}} = A(\hat{x} - x_e) + B(u - u_e) + L(y - C\hat{x}).$$
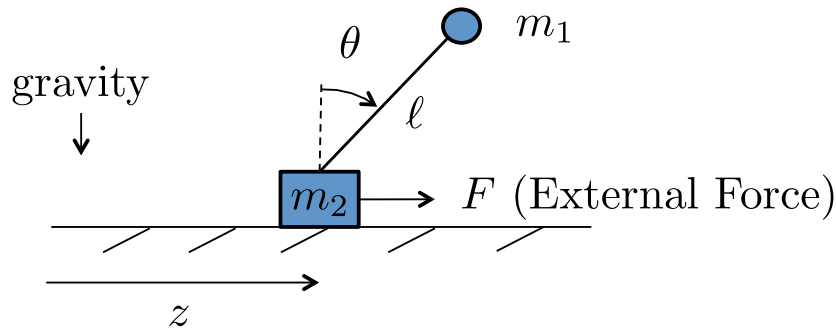
Design Problem 2

Pendulum on a Cart

Figure 2.1: Pendulum on a cart.

Figure 2.1 shows the pendulum on a cart system. The position of the cart measured from the origin is z and the linear speed of the cart is $\dot{z}$. The angle of the pendulum from straight up is given by θ and the angular velocity is $\dot{\theta}$. The pendulum arm is assumed to be massless, and m_1 is assumed to be a point mass. Gravity acts in the down direction. The only applied force is F which acts in the direction of z. The cart slides on a frictionless surface but air friction produces a damping force equal to $-b\dot{z}$.

Assume the following physical constants: $m_1 = 0.25$ kg, $m_2 = 1.0$ kg, $\ell = 0.5$ m, $g = 9.8$ m/s^2, $b = 0.05$ Ns.

Homework II.1 Using the configuration variables z and θ, write an expression for the kinetic energy of the system.

Homework II.2 Create a simulink animation of the pendulum on a cart system. The inputs should be sliders for z and θ.

Homework II.3

(a) Find the potential energy for the system.

(b) Define the generalized coordinates.

(c) Find the generalized forces.

(d) Derive the equations of motion using the Euler-Lagrange equations.

Homework II.4 Modify the simulink model created in homework II.2 by creating an s-function that implements the equations of motion. The input to the s-function should be a slider for force. The output should go to the animation developed in homework II.2.

Homework II.5

(a) Find the equilibria of the system.

(b) Linearize the system about the equilibria.

Homework II.6

(a) Using the Laplace transform, convert from time domain to the s-domain.

(b) Find the transfer functions that define the cascade connected from force F to the cart position z to the pendulum angle θ.

(c) Find explicit expressions for the disturbances that have been ignored.

Homework II.7 Defining the states as $x = (z, \tilde{\theta}, \dot{z}, \dot{\tilde{\theta}})^\top$, the input as $u = F$, and the measured output as $y = (z, \tilde{\theta})^\top$, find the linear state space equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework II.8 Suppose that the size of the input force on the cart is limited to $F_{\max} = 5$ N. The objective of this problem is to design separate PD controllers for the inner loop (force to angle) and the outer loop (angle to position).

(a) Using the transfer functions derived inn problem II.6, and using a PD controller for the inner loop to regulate $\tilde{\theta}$ to $\tilde{\theta}^d$, and a PD controller for the outer loop to regulate $\tilde{z}$ to ^{d}, draw the block diagram of the complete closed loop system.

(b) Derive the actual transfer function of the inner loop from $\tilde{\theta}^d$ to $\tilde{\theta}$.

(c) Considering the equilibrium force F_e, derive a bound $F_{\max}$ on the linearized force $\tilde{F}$ such that $\left|\tilde{F}\right| \leq F_{\max}$ when the maximum possible equilibrium force is used.

(d) Select the proportional gain k_{p_θ} so that $\tilde{F}$ just saturates when a step of size A_{th} is placed on $\tilde{\theta}_d$.

(e) If the desired transfer function is given by

$$\tilde{\theta}(s) = \frac{\omega_{n_\theta}^2}{s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2} \tilde{\theta}^d(s),$$

find the natural frequency ω_{n_θ} and the derivative gain k_{d_θ} so that the actual transfer function equals the desired transfer function, where ζ_θ is a design parameter.

(f) What are the closed loop poles of the inner loop?

(g) Assuming that the inner loop is working properly and that $\tilde{\theta} \approx \tilde{\theta}^d$, draw the block diagram for the outer loop and derive the actual transfer function from $\tilde{z}^d$ to $\tilde{z}$.

(h) Select the proportional gain k_{p_z} so that $\tilde{\theta}^d$ just saturates at A_{th} when a step of size $A_z = 2$ meters is placed on $\tilde{z}^d$.

(i) If the desired transfer function is given by

$$\tilde{z}(s) = \frac{\omega_{n_z}^2}{s^2 + 2\zeta_z \omega_{n_z} s + \omega_{n_z}^2} \tilde{z}^d(s),$$

find the natural frequency ω_{n_z} and the derivative gain k_{d_z} so that the actual transfer function equals the desired transfer function, where ζ_z is a design parameter.

(j) What are the closed loop poles of the outer loop?

Homework II.9 Implement the PD controllers for the inner and outer loops designed in Homework II.8 on the inverted in Simulink.

Homework II.10

(a) Consider the inner loop with PD control, what is the system type? Characterize the steady state error when $\tilde{\theta}^d$ is a step, a ramp, and a parabola. How does this change if you add an integrator?

(b) Consider the outer loop with PD control, what is the system type? Characterize the steady state error when $\tilde{z}^d$ is a step, a ramp, and a parabola. How does this change if you add an integrator?

Homework II.11 The objective of this problem is to implement the PID controller using Matlab code using only measured outputs of the system.

(a) Change the parameter file param.m to specify actual parameters and the parameters known to the controller. Change the values for m_1, m_2, ℓ and b known to the controller by 5% to simulate imperfect knowledge of the plant.

(b) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$. The controller should only assume knowledge of the position z and the angle θ, as well as the desired position z^d.

(e) Implement the nested PID loops designed in Problems II.8 and II.10 using an m-function called pendulum_ctrl.m. Use the dirty derivative gain of $\tau = 0.05$.

Homework II.12

(a) Draw by hand the Bode plot of the inner loop transfer function from force F to angle θ for the inverted pendulum. Use the Matlab `bode` command and compare your results.

(b) Draw by hand the Bode plot of the outer loop transfer function from force F to angle θ for the inverted pendulum. Use the Matlab `bode` command and compare your results.

Homework II.13 For the inner loop of the inverted pendulum, use the Matlab `bode` command to create a graph that simultaneously displays

the Bode plots for (1) the plant, (2) the plant under PD control, and (3) the plant under PID control, using the control gains calculated in Homework 11.

(a) To what percent error can the closed loop system under PID control track the desired input if all of the frequency content of $\theta^d(t)$ is below $\omega = 1.0$ radians per second?

(b) If the commaned input to the inner loop $\theta^d(t)$ is a step of size 2.0 radians what will be the steady state tracking error under PD control?

(c) If all of the frequency content of the noise $n(t)$ is greater than $\omega = 200$ radians per second, what percentage of the noise shows up in the output signal θ?

For the outer loop of the inverted pendulum, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PD control, using the control gains calculated in Homework 11.

(d) If the frequency content of an output disturbance is contained below 0.1 radian/sec, what percentage of the output disturbance will be contained in the output?

(e) If the reference signal $y^d(t)$ is a ramp of magnitude 2 meters/sec, what is the tracking error.

Homework II.14 For this homework assignment we will use loopshaping to improve the PD controllers developed in Homework 9. Download the simulink files on the web that are associated with this homework. We will focus on the outer loop. When you open the Simulink file, notice the presence of the LTI system with parameter C.

(a) At the Matlab prompt, type `>> C = 1` to set the additional compensator originally to one. Run the simulation to establish baseline performance.

(b) Note in `param.m` the line `L=series(P_out,C_out_pd)`, where $L(s)$ is the loop gain for the out loop. Load the `sisotool` and enter L for the system plant under the tab `System Data`.

(c) The objective will be to (1) improve disturbance rejection and tracking by a factor of 10 for input signals and disturbance below 0.001 radians/sec, (2) improve noise attention by a factor of 10 for frequencies above 20 radians/sec. In `sisotool` add constraints that reflect these objectives.

(d) Add a phase lag filter to satisfy the disturbance rejection and tracking objective, and add one to two poles to meet the noise rejection objective. The phase margin in this case is interpreted different. The phase above 0 degrees corresponds to the normal phase margin. This should remain around 60 degrees. To understand why, you will need to understand the Nyquist criteria.

(e) Export $C(s)$, and run the combined controller in Simulink.

Homework II.15 The objective of this problem is to implement state feedback controller using the full state. Start with the simulation files developed in Homework II.11.

(a) Modify pendulum_dynamics.m so that the outputs are $(z, \theta, x)^\top$, where $x = (z, \theta, \dot{z}, \dot{\theta})^\top$. In this problem we will use x instead of z and θ, but in the next problem we will again remove x as an output of the plant and reconstruct the state using an observer.

(b) Rearrange the block diagram so that the state x is an input to the controller.

(c) Using the values for ω_{n_z}, ζ_z, ω_{n_θ}, and ζ_θ selected in Homework 11, find the desired closed loop poles.

(d) Add the state space matrices A, B, C, D derived in Homework 7 to your param file.

(e) Verify that the state space system is controllable by checking that $\text{rank}(\mathcal{C}) = n$.

(f) Using the Matlab `place` command, find the feedback gain K such that the eigenvalues of $(A - BK)$ are equal to desired closed loop poles. Find the reference gain k_r so that the DC-gain from z^c to z is equal to one.

(g) Implement the state feedback scheme in Simulink and tune the closed loop poles to get good response. You should be able to get much faster response using state space methods.

Hints:

- There are several mistakes in previous solutions. For comparison, using exact parameters, I get the following state space model:

$$A = \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & -2.4500 & -0.0500 & 0 \\ 0 & 24.5000 & 0.1000 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Be careful in computing $k_r = -1/C(A-BK)^{-1}B$. In this formula, C is the output matrix matching the reference input. Therefore, for this problem $C = (1, 0, 0, 0)$.

Homework II.16

(a) Modify the state feedback solution developed in Homework 15 to add an integrator with anti-windup to the position feedback.

(b) Change the parameters known to the controller by $\pm 5\%$.

(c) Add a constant input disturbance of 0.5 Newtons to the input of the plant and verify that when the integrator gain is set to zero, there is a steady state tracking error.

(c) Tune the integrator to get zero steady state tracking performance.

Hints:

- The integrator gain will be negative.

- To get fast integrator action, try turning off the integrator during transition, i.e., when $|\dot{y}|$ is greater than some limit.

Homework II.17 The objective of this problem is to replace the state in Homework 16 with the estimated state $\hat{x}$ produced by an observer.

(a) Modify the simulink diagram from Homework 16 so that y is also an input to the controller, as well as x, and so that $\hat{x}$ is also an output of the controller. Add a scope to plot the estimation error $x - \hat{x}$.

(b) Verify that the state space system is observable by checking that $\text{rank}(\mathcal{O}) = n$.

(c) In the control block, add an observer to estimate the state $\hat{x}$. Do not yet use $\hat{x}$ in the state feedback equations. Run the controller using the actual state x and tune the observer so that the estimation error is small. Make sure the $\text{eig}(A - BK - LC)$ are in the left half plane.

(d) Replace x in the controller with the estimated state $\hat{x}$ and tune the controller and observer if necessary to obtain good performance.

(e) Modify the simulink diagram to add an input disturbance of 0.2 Newton-meters. Add a state to the observer to estimate the input disturbance . Replace the integrator with an estimate of the disturbance and compare the performance.

Hints:

- *Note the eigenvalues of the controller: $\text{eig}(A - BK - LC)$. If any of the eigenvalues of the observer are in the RHP, then tune the desired poles if $\text{eig}(A - LC)$ until all of $\text{eig}(A - BK - LC)$ are in the LHP.*

Design Problem 3

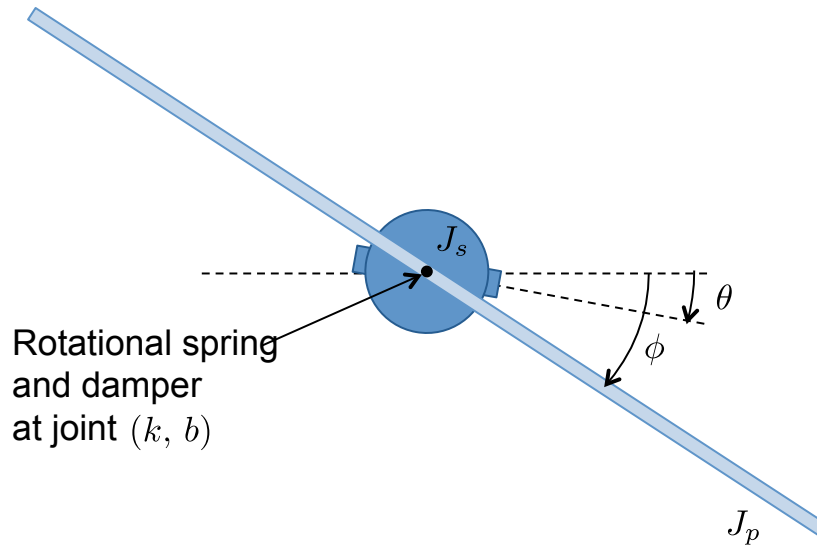# Satellite Attitude Control

Figure 3.1: Satellite with flexible solar panels.

Figure 3.1 shows a simplified version of a satellite with flexible solar panels. We will model the flexible panels using a rigid sheet with moment of inertia J_p connected to the main satellite body by a torsional compliant element with spring constant k and damping constant b. The moment of inertia of the satellite J_s. The angle of satellite body from the inertial reference is θ and the angle of the panel from the inertial reference is denoted as ϕ. Thrusters are used to command an external torque of τ about the axis of the satellite body.

Assume the following physical constants: $J_s = 5$ kg m^2, $J_p = 1$ kg m^2, $k = 0.15$ N m, $b = 0.05$ Nms.

Homework III.1 Using the configuration variables θ and ϕ, write an expression for the kinetic energy of the system.

Homework III.2 Create a simulink animation of the satellite system. The inputs should be sliders for θ and ϕ.

Homework III.3

(a) Find the potential energy for the system.

(b) Define the generalized coordinates.

(c) Find the generalized forces.

(d) Derive the equations of motion using the Euler-Lagrange equations.

(e) Express the equations of motion as four first-order differential equations with states $x = (\theta, \phi, \dot{\theta}, \dot{\phi})^\top$.

Homework III.4 Modify the simulink model created in homework III.2 by creating an s-function that implements the equations of motion. The input to the s-function should be a slider for torque. The output should go to the animation developed in homework III.2.

Homework III.5 The system is linear and it doesn't make sense to linearize.

Homework III.6

(a) Using the Laplace transform, convert from time domain to the s-domain.

(b) Find the transfer functions that define the cascade connected from the torque τ to the angle θ, and the transfer function from θ to ϕ.

(c) Find explicit expressions for the disturbances that have been ignored.

Homework III.7 Suppose that a star tracker is used to measure θ and a strain gage is used to approximate $\phi - \theta$. Defining the states as

$x = (\theta, \phi, \dot{\theta}, \dot{\phi})^\top$, the input as $u = \tau$, and the measured output as $y = (\theta, \phi - \theta)^\top$, find the linear state space equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework III.8 Suppose that the size of the input torque on the satellite body is limited to $\tau_{\max} = 5$ Nm. Design a PD controller for the inner loop and a proportional (P) controller for the outer loop.

(a) Using the transfer functions derived inn problem III.6, and using a PD controller for the inner loop to regulate θ to θ^d, and a P controller for the outer loop to regulate ϕ to ϕ^d, draw the block diagram of the complete closed loop system. Note that only a P controller is needed on the outer loop because the outer loop is a first order system.

(b) Using $\tau_{\max}$ find the proportional gain k_{p_θ} so that the torque τ is equal to $\tau_{\max}$ the moment after a step size of A_θ is placed on θ^d.

(c) Derive the actual transfer function of the inner loop from θ^d to θ.

(d) If the desired transfer function is given by

$$\theta(s) = \frac{\omega_{n_\theta}^2}{s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2} \theta^d(s),$$

find the natural frequency ω_{n_θ} and the derivative gain k_{d_θ} so that the actual transfer function equals the desired transfer function, where ζ_θ is a design parameter.

(e) Find the DC gain of the inner loop.

(f) Assuming that the inner loop is working properly, replace the inner loop system with its DC gain and draw the block diagram for the outer loop and derive the actual transfer function from ϕ^d to ϕ.

(g) Select the proportional gain k_{p_ϕ} so that θ^d just saturates at A_θ when a step of size $A_\phi = 30$ degrees is placed on ϕ^d.

(h) Find the DC gain of the outer loop. What does it mean that this value does not equal to one?

(i) What are the closed loop poles of the inner and the outer loops?

Homework III.9 Implement the PD and P controllers for the inner and outer loops designed in Homework III.8 on the satellite system in Simulink.

Homework III.10

(a) With PD control for the inner loop, what is the system type of the inner loop? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

(b) Recall from Problem 6, that in deriving the transfer functions that we obtained the expression

$$\Theta(s) = \left(\frac{1/J_s}{s^2 + b/J_s s + k/J_s}\right) \tau(s) + \left(\frac{b/J_s s + k/J_s}{s^2 + b/J_s s + k/J_s}\right) \Psi(s),$$

where we considered

$$d(s) = \left(\frac{b/J_s s + k/J_s}{s^2 + b/J_s s + k/J_s}\right) \Psi(s)$$

as an output disturbance on the inner loop. What is the response of the inner loop to a step on ψ with and without an integrator in the controller?

(c) With the integrator included, put the characteristic equation of the inner loop in Evan's form and use the Matlab SISO tool to plot the root locus verses the integrator gain k_{i_θ}.

(d) With PD control for the outer loop, what is the system type of the outer loop? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

(e) With the integrator, put the characteristic equation of the outer loop in Evan's form and use the Matlab SISO tool to plot the root locus verses the integrator gain k_{i_ψ}.

Homework III.11 The objective of this problem is to implement the PID controller using Matlab code using only measured outputs of the system.

(a) Change the parameter file param.m to specify actual parameters and the parameters known to the controller. Change the values for J_s, J_p, k and b known to the controller by 5%, to simulate imperfect knowledge of the plant.

(b) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$. Assume that the controller only has knowledge of the angles ϕ and θ as well as the desired angle ϕ^d.

(c) Implement the nested PID loops designed in Problems III.8 and III.10 using an m-function called satellite_ctrl.m. Use the dirty derivative gain of $\tau = 0.05$.

Homework III.12

(a) Draw by hand the Bode plot of the inner loop transfer function from force F to angle θ for the inverted pendulum. Use the Matlab `bode` command and compare your results.

(b) Draw by hand the Bode plot of the outer loop transfer function from force F to angle θ for the inverted pendulum. Use the Matlab `bode` command and compare your results.

Homework III.13 For the inner loop of the satellite attitude control, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, (2) the plant under PD control, and (3) the plant under PID control, using the control gains calculated in Homework 11.

(a) To what percent error can the closed loop system under PD control track a step in $\theta^d(t)$?

(b) If the input disturbance has frequency content below 0.02 radians per second, what percentage of the input disturbance appears in the output.

For the outer loop of the satellite attitude control, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, (2) the plant under P control, and (3) the plant under PI control, using the control gains calculated in Homework 11.

(c) Under PID control, what is the percent tracking error of the outer loop for reference signals $\phi^d(t)$ with frequency content below 0.01 radians per second?

(d) If all of the frequency content of the noise $n(t)$ is greater than $\omega = 1$ radians per second, what percentage of the noise shows up in the output signal θ, using PID control?

Homework III.14 For this homework assignment we will use loopshaping to improve the PD controllers developed in Homework 9. Download the simulink files on the web that are associated with this homework. We will focus on the outer loop. When you open the Simulink file, notice the presence of the LTI system with parameter C.

(a) At the Matlab prompt, type `>> C = 1` to set the additional compensator originally to one. Run the simulation to establish baseline performance.

(b) Note in `param.m` the line `L=series(P_out,C_out_pd)`, where $L(s)$ is the loop gain for the out loop. Load the `sisotool` and enter L for the system plant under the tab `System Data`.

(c) The objective will be to (1) improve disturbance rejection and tracking by a factor of 10 for input signals and disturbance below 0.003 radians/sec, (2) improve noise attention by a factor of 10 for frequencies above 10 radians/sec. In `sisotool` add constraints that reflect these objectives.

(d) Add a PI or phase lag filter to satisfy the disturbance rejection and tracking objective, and add one to two poles to meet the noise rejection objective. Take care to maintain roughly the same cross over frequency and adequate phase margin.

(e) Export $C(s)$, and run the combined controller in Simulink.

Homework III.15 The objective of this problem is to implement state feedback controller using the full state. Start with the simulation files developed in Homework 11. We first note that the there was a mistake in the derivation of the dynamics in the solution to problem 3 where $k(\phi - \theta)$ was not copied to the last line of the solutions. This has propagated through all of the solutions. The correct dynamics should have

been

$$\begin{pmatrix} J_s & 0 \\ 0 & J_p \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \tau - b(\dot{\theta} - \dot{\phi}) - k(\theta - \phi) \\ -b(\dot{\phi} - \dot{\theta}) - k(\phi - \theta) \end{pmatrix}.$$

Therefore, letting $x = (\theta, \phi, \dot{\theta}, \dot{\phi})^\top$, $u = \tau$, and $y = (\theta, \phi - \theta)^\top$, the state space equations should have been

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_s} & -\frac{k}{J_s} & \frac{b}{J_s} & -\frac{b}{J_s} \\ \frac{k}{J_p} & -\frac{k}{J_p} & \frac{b}{J_p} & -\frac{b}{J_p} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{J_s} \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} x.$$

You will need to modify the `satellite_dynamics.m` to reflect the correct dynamics.

(a) Modify satellite_dynamics.m so that the outputs are $(\theta, \phi, x)^\top$, where $x = (\theta, \phi, \dot{\theta}, \dot{\phi})^\top$. In this problem we will use x instead of θ and ϕ, but in the next problem we will again remove x as an output of the plant and reconstruct the state using an observer.

(b) Rearrange the block diagram so that the state x is an input to the controller.

(c) Select the closed loop poles as the roots of the equations $s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2 = 0$ and $s^2 + 2\zeta_\phi \omega_{n_\phi} s + \omega_{n_\phi}^2 = 0$. Where ω_{n_θ}, ω_{n_θ}, ζ_ϕ, and ζ_ϕ are tuning parameters. Start with the numbers used in problem 11.

(d) Add the state space matrices A, B, C, D to your param file.

(e) Verify that the state space system is controllable by checking that $\text{rank}(\mathcal{C}) = n$.

(f) Using the Matlab `place` command, find the feedback gain K such that the eigenvalues of $(A - BK)$ are equal to desired closed loop poles. Find the reference gain k_r so that the DC-gain from ϕ^d to ϕ is equal to one. In this case, the C matrix used to find k_r is $C = (0, 1, 0, 0)$. Why?

(g) Implement the state feedback scheme in Simulink and tune the closed loop poles to get good response.

Homework III.16

   (a) Modify the state feedback solution developed in Homework 15 to
         add an integrator with anti-windup to the feedback loop for ϕ.

   (b) Change the parameters known to the controller by $\pm 5\%$.

   (c) Add a constant input disturbance of 0.5 Newtons-meters to the
         input of the plant and verify that when the integrator gain is set
         to zero, there is a steady state tracking error.

   (c) Tune the integrator to get zero steady state tracking performance.

   Hints:

   • To get fast integrator action, try turning off the integrator during
     transition, i.e., when $|\dot{\theta}_1|$ is greater than some limit.

Homework III.17 The objective of this problem is to replace the state in
   Homework 16 with the estimated state $\hat{x}$ produced by an observer.

   (a) Modify the simulink diagram from Homework 16 so that y is also
         an input to the controller, as well as x, and so that $\hat{x}$ is also an
         output of the controller. Add a scope to plot the estimation error
         $x - \hat{x}$.

   (b) Verify that the state space system is observable by checking that
         $\text{rank}(\mathcal{O}) = n$.

   (c) In the control block, add an observer to estimate the state $\hat{x}$. Do
         not yet use $\hat{x}$ in the state feedback equations. Run the controller
         using the actual state x and tune the observer so that the estima-
         tion error is small. Make sure the $\text{eig}(A - BK - LC)$ are in the
         left half plane.

   (d) Replace x in the controller with the estimated state $\hat{x}$ and tune the
         controller and observer if necessary to obtain good performance.

   (e) Modify the simulink diagram to add an input disturbance of 0.5 Newton-
         meters. Add a state to the observer to estimate the input distur-
         bance . Replace the integrator with an estimate of the disturbance
         and compare the performance.

Design Problem 4
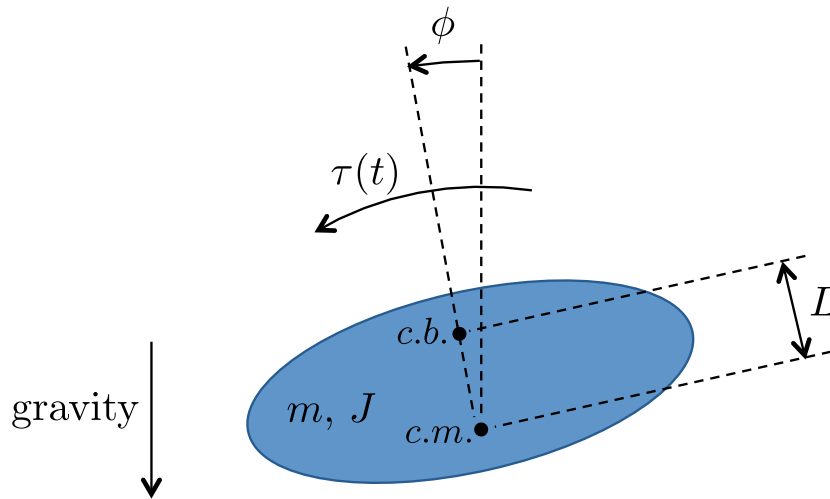
UUV Roll Control

Figure 4.1: Front view of an unmanned underwater vehicle

Figure 4.1 shows a schematic depiction of the front view of an unmanned underwater vehicle (UUV). We are interested in the rolling motion of the vehicle. Torque from thrusters causes the UUV to roll about its center of mass (c.m.). A gravity-buoyancy couple create a moment that restores the UUV to its nominal horizontal position. The surrounding water opposes the rolling motion by exerting a viscous damping torque proportional to the angular speed of the UUV's rolling motion. The damping constant is given by b. The UUV has mass m and mass moment of inertia J. The center of

mass and the center of buoyancy are separated by a distance L. The roll angle of the UUV is given by ϕ and the roll rate is given by $\Omega = \dot{\phi}$.

Assume the following physical constants: $J = 45$kg-m^2, $m = 200$ kg, $L = 0.03$ m, $b = 5$ N-m-s.

Homework IV.1

Homework IV.2 Create a Simulink animation of the UUV system. The input should be a slider for ϕ. Turn in a screen capture of the animation.

Homework IV.3

(a) Apply Newton's 2nd Law for a rotating body to derive the equation of motion for the UUV system.

(b) Convert the resulting second-order ODE describing the motion to a system of two first-order ODE's with the states $x = (\Omega, \phi)^\top$.

Homework IV.4 Modify the Simulink model created in homework IV.2 by creating an s-function that implements the equations of motion. The input to the s-function should be a slider for thruster torque. The output should go to the animation developed in homework IV.2.

Homework IV.5

(a) Find the equilibria of the system. (Hint: For this system, it would be most typical and appropriate to consider zero applied torque as an equilibrium condition.)

(b) Linearize the system about the equilibria.

Homework IV.6

(a) Using the Laplace transform, convert from time domain to the s-domain.

(b) Find the transfer function from the force τ to the UUV angular position ϕ.

(c) Draw the associated block diagram.

Homework IV.7 Defining the states as $x = (\phi, \dot{\phi})^\top$, the input as $u = \tau$, and the measured output as $y = \phi$, find the linear state space equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework IV.8 Suppose that the size of the input force is limited to $\tau_{\max} = 50$ N-m.

(a) In this problem, you will use a PD controller to stabilize the system and regulate ϕ to ϕ^d. Assume that the proportional gain acts on the roll error ($\phi^d - \phi$, in the forward path) and that the derivative gain acts on the roll rate ($-\dot{\phi}$, in the feedback path). Draw the (linearized) closed loop block diagram using the transfer function found in problem IV.6. Derive the actual transfer function from ϕ^d to ϕ.

(b) Select the proportional gain k_p so that τ just saturates when a step of size 30 deg (0.524 rad) is placed on ϕ^d.

(c) If the desired closed-loop characteristic equation for the UUV is given by

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0,$$

find the natural frequency ω_n and the proportional gain k_d so that the actual characteristic equation equals the desired characteristic equation. For this problem, choose $\zeta = 0.7$.

(e) What are the closed-loop poles?

Homework IV.9 Implement the PD controller designed in Homework IV.8 on the UUV system in Simulink.

Homework IV.10 In this problem, we will examine the effect of integral control on system type.

(a) With PD control, what is the system type in response to reference inputs? Characterize the steady-state error when the reference input is a step and a ramp. How does this change if you add an integral feedback term in parallel with the proportional feedback term?

(b) Consider the case where a constant disturbance torque acts at the input to the plant (for example, an inaccurate knowledge of the separation distance L between the centers of mass and buoyancy.). What is the steady-state error to a constant input disturbance torque when the integral feedback term is not present, and when it is present?

(c) Add an integral feedback term to the PD control developed in Homework IV.9. Draw the block diagram of the closed-loop system with PID control. Find the transfer function for the closed-loop system with the integral feedback term added.

(d) Put the characteristic equation in Evan's form and use the Matlab rlocus function or rltool to plot the root locus verses the integral feedback gain k_i. Choose a value of the gain k_i is sufficiently large to drive the error to zero as quickly as possible without significantly degrading the transient response of the system. Submit a root locus plot supporting your decision. (This is a design trade-off that may yield mixed results.)

(e) Implement the PID controller you design on your Simulink case study. Submit your Simulink block diagram and plot of simulation result for a 30-degree step input.

Homework IV.11 In this problem, you will use root-locus methods to design a lead-lag compensator. The desired performance specifications are rise time less than 0.9 s, settling time less than 3 s, percent overshoot less than 10 percent, and steady-state error less than 10 percent in response to a step input of 30 degrees (0.524 rad). Assume that we have turbo charged our thrusters so that we can produce a peak torque of 250 N-m on the UUV.

(a) Use root locus methods to design a lead compensator that satisfies the transient response characteristics (rise time, settling time, percent overshoot). Submit a plot illustrating your root locus design. Specify the gain, the lead zero, and lead pole values.

(b) Implement your lead controller in your Simulink simulation of the UUV. Submit a plot showing the UUV response to a 30-degree step input. How do your transient response characteristics match up with your expectation? What is the steady-state error to the 30-degree step input?

(c) Use root locus methods to design a lag compensator to complement your lead compensator so that steady-state error is reduced to within the desired range. Submit a plot of the root locus illustrating your lead-lag design. Specify your lag zero and pole values.

(d) Implement your lead-lag controller in your Simulink simulation of the UUV. Submit a plot showing the UUV response to a 30-degree step input. Are your design specifications met?

(e) Try adding a model-based feedforward control term that predicts the amount of torque required to overcome the gravity-buoyancy torque based on the desired roll angle and and adds that torque to the torque generated by the feedback controller.

Homework IV.12 The objective of this problem is to implement the PID controller using Matlab code using only measured outputs of the system.

(a) Change the parameter file param.m to specify actual parameters and the parameters known to the controller. Change the values of m, b, and L known to the controller by 5% to simulate imperfect knowledge of the plant.

(b) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$. The controller should only assume knowledge of the angular position ϕ and the desired angular position ϕ^d.

(c) Implement the PID controller designed in Problems IV.8 and IV.11 using an m-function called uuv_ctrl.m. Use the dirty derivative gain of $\tau = 0.05$. Tune the integrators to so that there is no steady-state error.

Homework IV.13 For this homework assignment, you will use frequency-response-based design techniques to design a dynamics compensator for the UUV system. Here are the specifications: zero steady-state error to a step input in the roll angle command, rise time of approximately 1 s (set your crossover frequency at about 2 rad/s), and a percent overshoot to a step command of less than 10 percent (phase margin of at least 60 deg).

(a) The steady-state error specification requires the use of an integrator. Let $D(s) = 1/s$ and plot the open-loop frequency response for $D(s)G(s)$. What is the phase margin of the system at the desired crossover frequency (2 rad/s) with this compensation?

(b) Because the phase is so low at the desired crossover frequency, we need to do something to change the phase at frequencies near the cross over and above. The integrator is important at lower frequencies well below the crossover frequency. We can improve the phase at crossover by adding a zero to our compensation D(s). Set the zero frequency to be one decade below the crossover frequency. Note that the combination of a zero and an integrator is equivalent to PI control. Plot the frequency response of the new open-loop system. Notice that the phase margin is no longer negative. It should be about zero at 2 rad/s. This becomes a good starting point for the design of a lead compensator.

(c) Design a lead compensator to add to the existing $D(s)$ to improve the phase lead at the crossover frequency. Specify the pole and zero locations of your lead design. Apply your lead compensator (with a gain of 1) and plot the frequency response of the open-loop system. Set the gain of the compensator so that the magnitude plot crosses over unity magnitude at 2 rad/s. Plot the closed-loop response of the system.

(d) The PI/lead compensator does a pretty good job, but perhaps not as well as we would like. The lightly-damped mode of the open-loop system (as shown by the resonance in the frequency response) leads to marginal performance if it is not addressed. We can address this by notching out the resonance using a notch filter. Starting with just the open-loop plant, notch out the resonant mode with a notch filter with a zero frequency corresponding to the lightly damped mode of the system $\omega_z = \sqrt{\frac{mgL}{J}}$. Set the zero damping of the filter to $\zeta_z = 0.05$. (This is easy to do using SISO-tool in Matlab). Apply this filter as $D(s)$ and plot the open-loop frequency response. This should have eliminated the resonance in the magnitude plot. Add the integrator to your compensation and plot the frequency response once again. What is the phase margin at the desired cross-over frequency?

(e) Design a lead compensator to get the desired lead ratio at the crossover frequency. Depending on the amount of phase to be added by the lead, a double lead may be required. Complete your design and plot the frequency response to verify that the crossover

and phase margin specifications are met. Plot the closed-loop step response – you should observe significantly improved performance.

(f) Simulate the response of the UUV system in Simulink using the compensator devised in part (e). Submit plots showing the design steps you took in parts (a)-(e). Submit a plot of your Simulink block diagram as well as the response of your Simulink model.

Homework IV.14 The objective of this problem is to implement a state feedback controller using the full state. You will use this controller to first regulate all of the states to zero given non-zero initial conditions on the states. As a final step you will introduce a reference input and control the system to track the reference input. Start with the simulation files developed in Homework IV.12. You will replace the transfer-function based controller blocks with a Matlab function-based controller block (like the autopilot block you've used with the most recent labs.)

(a) Create a controller function block that takes in the commanded roll angle ϕ^c, the state vector $x = (\phi, \dot{\phi})^\top$, and the parameter variable P. The controller function should output the applied thruster torque $T = u = -Kx$. You will calculate the control gains below. The commanded roll angle ϕ^c will be set to zero initially.

(b) Select closed-loop pole locations to give a closed-loop step response with a rise time of about 1 s and percent overshoot of about 5 percent.

(c) Using the state space matrices F, G, H, and J derived in Homework IV.7, first verify that the state space system is controllable by verifying that the controllability matrix is full rank: $\text{rank}(\mathcal{C}) = n$.

(d) Next, using the Matlab `place` command, find the feedback gain K such that the eigenvalues of $(F - GK)$ are equal to desired closed loop poles. Note that $K \approx (k_p, k_d)$ where k_p and k_d are the proportional and derivative gains found in Homework IV.8. Why?

(e) Implement the state feedback scheme $T = u = -Kx$ in Simulink and plot the closed-loop response to the initial condition $x_0 = [0.5, 0]^\top$. Submit a printout of your Simulink block diagram and your response plots.

(f) Calculate the reference input gain $\bar{N}$ using the methods discussed in class. Introduce the reference input into your control block so that the control torque is $T = u = -Kx + \bar{N}r$. Set the initial conditions on the states to zero and plot the response of the system to a step input of 30 deg in ϕ. Submit a printout of your Simulink block diagram and your response plots.

Homework IV.15 The objective of this problem is to replace the state in Homework IV.13 with the estimated state $\hat{x}$ produced by an estimator.

(a) Modify the Simulink diagram from Homework IV.13 so that thruster torque, and the estimated state $\hat{x}$ are outputs of the controller and use a scope to plot the states and estimated states. From the states coming out of the dynamics block, select the roll angle as the measured output of the system. Create a controller block with inputs roll angle command, roll angle output, and time t from a Clock Simulink block. Set the sample period of the controller block to be $T_s = 0.01$ s.

(b) Verify that the state space system is observable by checking that the observability matrix is full rank: $\text{rank}(\mathcal{O}) = n$.

(c) In the control block, add an observer (estimator) to estimate the state $\hat{x}$. Do not yet use $\hat{x}$ in the state feedback equations. Run the controller using the actual state x and tune the observer so that the estimation error is small. Make sure the eigenvalues $\text{eig}(F - GK - LH)$ are in the left half plane.

(d) Replace x in the controller with the estimated state $\hat{x}$ and tune the controller and observer if necessary to obtain good performance.

(e) Test your system by giving the commanded roll angle a series of step changes. Plot the values of the states and estimated states using Scope blocks. Submit screen shots of your system response and block diagrams. Submit a printout of your controller block code.
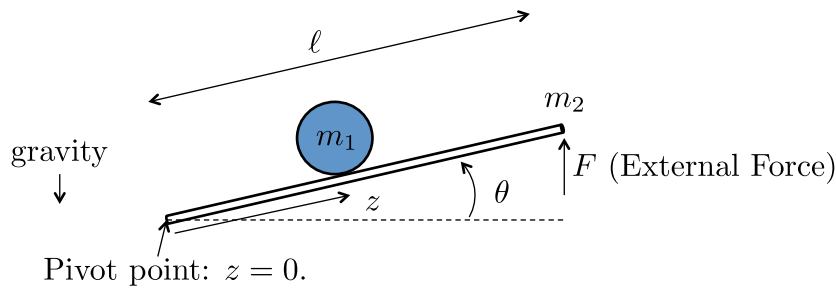
Design Problem 5

Ball on Beam

Figure 5.1: Ball on Beam Problem

Figure 5.1 shows the ball and beam system. The position of the ball measured from the pivot point is z and the speed of the ball along the direction of the beam is $\dot{z}$. The angle of the beam from level is θ and the angular speed of the beam is $\dot{\theta}$. Gravity acts in the down direction. The mass of the ball is m_1 and the mass of the beam is m_2. The length of the beam is ℓ. An external force is applied at the end of the beam as shown in Figure 5.1.

Use the following physical parameters: $m_1 = 0.35$ kg, $m_2 = 2$ kg, $\ell = 0.5$ m, $g = 9.8$ m/s^2.

Homework V.1 Using the configuration variable z and θ, write an expression for the kinetic energy of the system.

Homework V.2 Create a Simulink animation of the ball on beam. The inputs should be sliders for z and θ. Turn in a screen capture of the animation.

Homework V.3

(a) Find the potential energy for the system.

(b) Define the generalized coordinates.

(c) Find the generalized forces.

(d) Derive the equations of motion using the Euler-Lagrange equations.

Homework V.4 Modify the Simulink model created in homework V.2 by creating an s-function that implements the equations of motion. The input to the s-function should be a slider for force. The output should go to the animation developed in homework V.2.

Homework V.5

(a) Find the equilibria of the system.

(b) Linearize the system about the equilibria.

Homework V.6

(a) Using the Laplace transform, convert from time domain to the s-domain.

(b) Find the transfer functions from the input force $\tilde{F}$ to the beam angle $\tilde{\theta}$ and the ball position $\tilde{z}$. Find the transfer function from $\tilde{\theta}$ to $\tilde{z}$.

(c) From small deviations of the ball position from equilibrium (small $\tilde{z}$), the moment term $m_1 g \tilde{z}$ can be neglected in the equation governing the rotational dynamics of the beam. With this simplifying assumption, find an alternate expression for the transfer function from $\tilde{F}$ to $\tilde{\theta}$.

(d) Draw the associated block diagram for the simplified system.

Homework V.7 Defining the states as $x = (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$, the input as $u = F$, and the measured output as $y = (\tilde{z}, \tilde{\theta})^\top$, find the linear state space equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework V.8 Suppose that the size of the input force on the beam is limited to $F_{\max} = 15$ N. Design separate PD controllers for the inner loop and the outer loop.

(a) Using the transfer functions derived in problem V.6, and using a PD controller for the inner loop to regulate $\tilde{\theta}$ to $\tilde{\theta}^d$, and a PD controller for the outer loop to regulate z to z^d, draw the block diagram of the complete closed loop system.

(b) Derive the actual transfer function of the inner loop from $\tilde{\theta}^d$ to $\tilde{\theta}$.

(c) Considering the equilibrium force F_e, derive a bound $F_{\max}$ on the linearized force $\tilde{F}$ such that $\left|\tilde{F}\right| \le F_{\max}$ when the maximum possible equilibrium force is used.

(d) Select the proportional gain k_{p_θ} so that $\tilde{F}$ just saturates when a step of size A_{th} is placed on $\tilde{\theta}_d$.

(e) If the desired transfer function is given by

$$\tilde{\theta}(s) = \frac{\omega_{n_\theta}^2}{s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2}\tilde{\theta}^d(s),$$

find the natural frequency ω_{n_θ} and the derivative gain k_{d_θ} so that the actual transfer function equals the desired transfer function, where ζ_θ is a design parameter.

(f) What are the closed loop poles of the inner loop?

(g) Assuming that the inner loop is working properly and that $\tilde{\theta} \approx \tilde{\theta}^d$, draw the block diagram for the outer loop and derive the actual transfer function from $\tilde{z}^d$ to $\tilde{z}$.

(h) Select the proportional gain k_{p_z} so that $\tilde{\theta}^d$ just saturates at A_{th} when a step of size $A_z = 0.25$ meters is placed on $\tilde{z}^d$.

(i) If the desired transfer function is given by

$$\tilde{z}(s) = \frac{\omega_{n_z}^2}{s^2 + 2\zeta_z \omega_{n_z} s + \omega_{n_z}^2}\tilde{z}^d(s),$$

find the natural frequency ω_{n_z} and the derivative gain k_{d_z} so that the actual transfer function equals the desired transfer function, where ζ_z is a design parameter.

   (j) What are the closed loop poles of the outer loop?

Homework V.9 Implement the PD controllers for the inner and outer loops
   designed in Homework V.8 on the ball and beam in Simulink.

Homework V.10 (a) Consider the ball position outer loop with PD con-
   trol. What is the steady-state error when $\tilde{z}^d$ is a step, a ramp, and
   a parabola. What is the system type with respect to the reference
   input $\tilde{z}^d$? How does this change if you add an integrator?

   (b) Add an integrator to the outer loop PD control developed in Home-
   work V.8. Draw the block diagram of the closed loop system with
   PID control. Find the transfer function $E(s)/\tilde{Z}(s)$ for the closed
   loop system with the integrator added. What is the system type
   with respect to $\tilde{z}^d$ with PID control?

   (c) Consider the case where a constant disturbance acts at the input
   to the outer loop plant (for example an inaccurate knowledge of
   the equilibrium force in this case). What is the steady-state error
   to a constant input disturbance when the integrator is not present,
   and when it is present?

   (d) With PID control, put the closed-loop characteristic equation in
   Evan's form and use the Matlab SISO tool to plot the root locus
   verses the integrator gain k_i. Use values for k_p and k_d from the ball
   position control design of problem V.8. Choose k_i so that there
   is zero steady-state error in response to a disturbance without
   significantly degrading the transient response.

   Homework V.11 Root locus design.

Homework V.12 The objective of this problem is to implement the PID
   controller using Matlab code using only measured outputs of the sys-
   tem.

   (a) Modify ballbeam_dynamics.m so that the outputs are z and θ.
   (b) Modify drawBallbeam.m so that it does not expect $\dot{z}$ and $\dot{\theta}$ as
   inputs.
   (c) Change the parameter file param.m to specify actual parameters
   and the parameters known to the controller. Change the values
   for m_1, m_2, and ℓ known to the controller by 5% to simulate
   imperfect knowledge of the plant.

(d) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$.

(e) Implement the nested PID loops in Problems V.8 using an m-function called ballbeam_ctrl.m. Use the dirty derivative gain of $\tau = 0.05$. Tune the integrator to so that there is no steady state error.

(f) Note that the integrator gain will need to be negative which will cause problems for the anti-windup scheme that we have been implementing. Remove the old anti-windup scheme and implement a new scheme where the integrator only winds up when $|\dot{z}|$ is small.

Homework V.13 For this homework assignment, you will design lead controllers using frequency-response-based methods to position the ball in the ball/beam system. For these designs, you will only have access to the ball position measurement z and the beam angle measurement θ. Use the transfer functions for $\tilde{\theta}(s)/\tilde{F}(s)$ and $\tilde{z}(s)/\tilde{\theta}(s)$ derived in problem V.6 and used in problem V.8 to create your designs.

Inner loop:

(a) Replace the PD controller designed in problem V.8 for the inner loop controlling the beam angle with a lead controller. Design the inner loop to have a bandwidth of 20 rad/s and closed-loop damping ratio of 0.7.

(b) Consider the weight of the beam and the weight of the ball to be disturbance forces to the angle positioning system. When the ball is at $z = \ell$, a disturbance force of 13.2 N results. With the controller implemented in part (a), calculate the steady-state angle positioning error for a constant disturbance of 13.2 N. The disturbance force sums with the force produced by the controller. Supplement the controller of part (a) with a lag controller to boost the low-frequency gain of the system so that the positioning error is less than 1 degree = 0.0175 rad. Clearly specify the final transfer function of your inner-loop compensator.

Outer loop:

(c) Assuming that the inner-loop closed-loop transfer function is approximately 1 over its bandwidth, design a lead controller

for the outer loop to have a bandwidth of 2 rad/s and a closed-loop damping ratio of 0.6. Clearly specify the transfer function of your outer-loop compensator.

(d) Implement your frequency-response-based designs in your Simulink simulation. Submit plots showing the Simulink block diagram and the step response of your system. Hint: You will need to use a feedforward command to overcome the weight of ball and beam. This will work best if your feedforward command is based on the measured position of the ball (not true feedforward!). However, the lag compensator used to beef up the inner loop makes it possible to use the commanded ball position successfully in the feedforward command.

Homework V.14 The objective of this problem is to implement state feedback controller using the full state. In this problem you will develop a regulator that controls all of the states to be zero. Start with the simulation files developed in Homework V.12. You will replace the transfer-function based controller blocks with a Matlab function-based controller block. This problem is a little challenging because we will regulate the system to the equilibrium state. Let equilibrium position of the ball be $z_e = L/2 = 0.25$ m. The equilibrium values for the other states of the system (beam angle, ball velocity, beam angular velocity) will be zero.

(a) Create a controller function block that takes in the state vector $\tilde{x} = (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$, and the parameter variable P. The controller function should output the force applied to the beam $F = u = -K\tilde{x}$. You will calculate the control gains below.

(b) Using the state space matrices F, G, H, and J derived in Homework V.7, first verify that the state space system is controllable by verifying that the controllability matrix is full rank: $\text{rank}(\mathcal{C}) = n$.

(c) Next, using the Matlab `place` command, find the feedback gain K so that the eigenvalues of $(F - GK)$ are placed in the following locations: $s = -4.5 \pm 4.5j, -6 \pm 1.5j$. These locations were chosen to give acceptably fast, well-damped response of the ball position.

(d) Implement the state feedback scheme $F = u = -K\tilde{x}$ in Simulink and plot the closed-loop response to the initial condition $\tilde{x}_0 =$

$(0.1, 0, 0, 0)^\top$. Submit a printout of your Simulink block diagram and your response plots.

Hints:

- Make sure that the states are in the right order: $\tilde{x} = (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^\top$.

- The control was designed using the system linearized around an equilibrium state. Thus, the controller utilizes the equilibrium state $F = u = -K\tilde{x}$ rather than the true state. You will need to take this into account as you implement your controller. You will need to use the equilibrium force as a feedforward term. To regulate the system to the equilibrium state, you will use the following approach:

$$F_e = \frac{m_1 g z_e}{L} + \frac{1}{2} m_2 g$$
$$x_e = (z_e, 0, 0, 0)^\top$$
$$F = F_e - K\tilde{x} = F_e - K(x - x_e)$$

- For comparison, here is the state-space model linearized about the equilibrium state x_e.

$$F = \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & -9.8100 & 0 & 0 \\ -18.2108 & 0 & 0 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2.6519 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

$$J = 0.$$

Homework V.15 The objective of this problem is to replace the state in Homework V.13 with the estimated state $\hat{x}$ produced by an estimator. We will also introduce a reference input.

(a) Modify the Simulink diagram from Homework V.13 so that beam force and the estimated state $\hat{x}$ are outputs of the controller and use a Scope to plot the states and estimated states. From the states coming out of the dynamics block, select the ball position as the measured output of the system. Create a controller block with inputs ball position command, ball position output, and time t from a Clock Simulink block. Set the sample period of the controller block to be $T_s = 0.001$ s.

(b) Verify that the state space system is observable by checking that the observability matrix is full rank: rank($\mathcal{O}$) = n.

(c) In the control block, add an observer (estimator) to estimate the state $\hat{x}$. Do not yet use $\hat{x}$ in the state feedback equations. Run the controller using the actual state x and tune the observer so that the estimation error is small. Make sure the eigenvalues eig($F - GK - LH$) are in the left half plane.

(d) Replace the state x in the controller with the estimated state $\hat{x}$ and tune the controller and observer if necessary to obtain good performance.

(e) Test your system by giving the commanded ball position a series of step changes. Plot the values of the states and estimated states using Scope blocks. Submit screen shots of your system response and block diagrams. Submit a printout of your controller block code.

Hints:

- Recall that the observer is estimating $\tilde{x}$. Therefore the observer update equation will be

$$\dot{\tilde{x}} = F\tilde{x} + G\tilde{u} + L(\tilde{y} - H\tilde{x}),$$

but since $\dot{\tilde{x}} = \dot{\hat{x}} - \dot{x}_e = \dot{\hat{x}}$, and since $y_e = Hx_e$, this is equivalent to

$$\dot{\hat{x}} = F(\hat{x} - x_e) + G(u - u_e) + L(y - H\hat{x}).$$
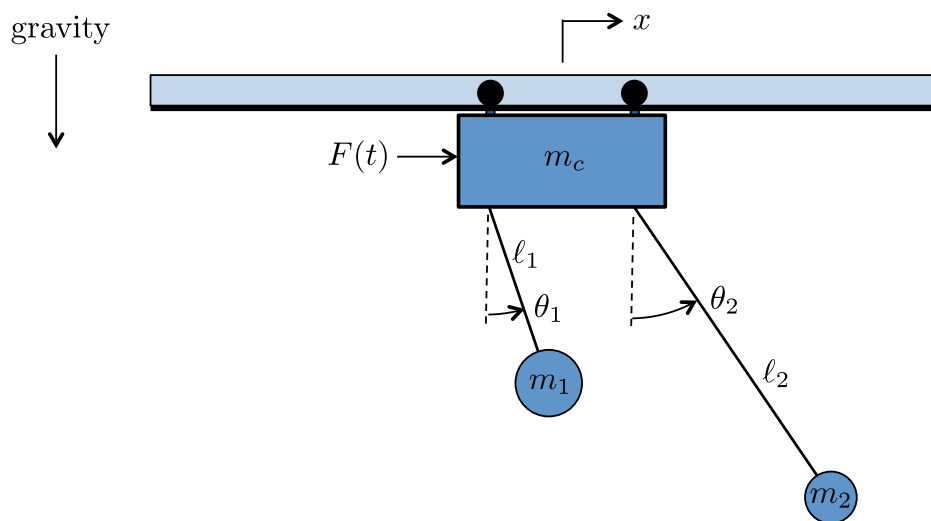
Design Problem 6

Gantry Crane

Figure 6.1: Gantry Crane

Figure 6.1 shows a gantry crane system with two slung loads. The crane car is suspended from a track and has mass m_c. The rolling friction on the track is proportional the the velocity of car with friction constant b. The crane carries two loads. Load 1 has mass m_1 and is suspended by a cable of length ℓ_1. Load 2 has mass m_2 and is suspended by a cable of length ℓ_2. The crane car is propelled along the track by a force $F(t)$ and its position is given by x. The angles of the loads are given by θ_1 and θ_2 respectively.

Use the following physical parameters: $m_c = 1500$ kg, $m_1 = 500$ kg,

$m_2 = 200$ kg, $b = 200$ N-s/m, $\ell_1 = 10$ m, $\ell_2 = 24$ m, $g = 9.81$ m/s^2.

Homework VI.1 Using the configuration variable x, θ_1, and θ_2, write an expression for the kinetic energy of the system.

Homework VI.2 Create a Simulink animation of the gantry crane system. The inputs should be sliders for x, θ_1 and θ_2. Turn in a screen capture of the animation.

Homework VI.3

   (a) Find the potential energy for the system.

   (b) Define the generalized coordinates.

   (c) Find the generalized forces.

   (d) Derive the equations of motion using the Euler-Lagrange equations.

Homework VI.4 Modify the Simulink model created in homework VI.2 by creating an s-function that implements the equations of motion. The input to the s-function should be a slider for F. The output should go to the animation developed in homework VI.2.

Homework VI.5

   (a) Find the equilibria of the system.

   (b) Linearize the system about the equilibria.

Homework VI.6

   (a) Simplify the equations of motion for the system by neglecting friction on the cart and the dynamic effects of pendulum load 2. This converts our model to a single pendulum system with no friction. The order of the system is reduced from six to four. Using the Laplace transform, convert the simplified equations of motion from time domain to the s-domain.

   (b) Find transfer functions from force to θ_1 and from θ_1 to car position x.

   (c) Draw a block diagram of the open-loop system showing the transfer function cascade with F as the input, θ_1 as the intermediate state, and x as the output of the cascade.

Homework VI.7 Defining the states as $x = (x, \theta_1, \theta_2, \dot{x}, \dot{\theta}_1, \dot{\theta}_2)^\top$, the input as $u = F$, and the measured output as $y = (x, \theta_1, \theta_2)^\top$, find the linear state space equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework VI.8 To make this problem more reasonable, let's consider the gantry crane with a single pendulum load. Let $m_2 = \ell_2 = b = 0$. For simplicity in notation, let $m_1 = m$ and $\ell_1 = \ell$. We will use successive loop closure with the inner loop controlling the pendulum angle θ and the outer loop controlling the car position x. We will use PD control for both feedback loops. For the inner loop, place the proportional feedback term in the forward path and the derivative feedback term in the feedback path. For the outer loop, place the derivative feedback gain block ($k_{d_x} s$) in the forward path in parallel with the proportional gain. This will make the analysis a little simpler.

(a) Using the transfer functions developed in problem VI.6, draw the block diagram for the complete closed-loop system showing the feedback loops for θ and x.

(b) Derive the transfer function for the inner loop from θ^d to θ.

(c) Design the inner loop controller so that a rise time of 1 second and a percent overshoot of 50 percent is obtained. If the desired transfer function is given by

$$\theta(s) = \frac{\omega_{n_\theta}^2}{s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2} \theta^d(s),$$

find the proportional gain k_{p_θ} and the derivative gain k_{d_θ} so that the actual transfer function equals the desired transfer function.

(d) What are the closed-loop poles of the pendulum angle loop?

(e) What is the dc-gain of the θ/θ^d inner loop? We will call this $K_{\theta_{DC}}$.

(f) Assuming that the inner loop is working properly and that $\theta \approx K_{\theta_{DC}} \theta^d$, draw the block diagram for the outer loop and derive the transfer function from x^d to x.

(g) For the simplified outer loop, use root locus methods to design the PD controller. Place the closed-loop poles for the outer loop at locations corresponding to a natural frequency between 0.4 and 0.5 rad/s and a damping ratio between 0.3 and 0.4. What are the values of k_{p_x} and k_{d_x} that you selected?

(h) What are the closed-loop poles of the outer loop? What is the natural frequency of the complex pair of closed-loop roots?

Homework VI.9 Implement the PD controllers for the angle and position loops designed in Homework VI.8 on the gantry crane system in Simulink. Note that the derivative of the positions (θ and x) are available for feedback as outputs of your dynamics block. You will need to simplify your dynamics block to take out the dynamics for θ_2. (Keep a copy of the 6th-order equations – you'll use them later.)

TWM: Continue updating here...

Homework VI.10

(a) Considering the outer cart position loop with PD control, what is the steady-state error when the desired cart position input is a step, a ramp, and a parabola? What is the type of this system with respect to the desired cart position input? How does this change if you add an integrator?

(b) Find the response of the outer cart position loop to a unit step input disturbance in pendulum angle, both when the outer loop controller does not have an integrator and when it does.

(c) With PID control implemented, put the characteristic equation of the outer loop in Evan's form and use the Matlab SISO tool to plot the root locus verses the integrator gain k_i.

Homework VI.11 The objective of this problem is to implement the PID controller using Matlab code using only measured outputs of the system.

(a) Change the parameter file param.m to specify actual parameters and the parameters known to the controller. Change the values for m_1, m_2, m_c, ℓ_1, ℓ_2, and b known to the controller by $\pm 5\%$ to simulate imperfect knowledge of the plant.

(b) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$. The controller should only assume knowledge of the position z and the angles θ_1, and θ_2, as well as the desired position z^d.

(c) Implement the nested PID loops designed in Problems VI.8 and VI.10 using an m-function called gantry_ctrl.m. Use the dirty derivative gain of $\tau = 0.05$.

Homework VI.12 The objective of this problem is to implement the PD controllers designed in problem VI.8 in a Matlab function using only the cart position and pendulum angle outputs of the system.

(a) Change the parameter file param.m to specify actual parameters and the parameters known to the controller. Change the values for m, m_c, ℓ, and b known to the controller by $\pm 5\%$ to simulate imperfect knowledge of the plant.

(b) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$. The controller should only assume knowledge of the position z and the angle θ, as well as the desired position z^d.

(c) Implement the nested PD loops designed in Problems VI.8 using an m-function called gantry_ctrl.m. Use a dirty derivative gain of $\tau = 0.05$.

Homework VI.13 For this homework assignment, we will use the frequency response design methodology to design controllers for the gantry crane system with the single slung load. For these designs, you will only have access to the cart position measurement x and the load angle measurement θ. Use the transfer functions for $\tilde{\theta}(s)/\tilde{F}(s)$ and $\tilde{x}(s)/\tilde{\theta}(s)$ derived and used in problem VI.8 to create your designs.

Angle loop:

(a) Replace the PD controller designed in problem VI.8 for the inner loop controlling the pendulum angle with a lead controller. Design the inner loop to have a bandwidth of 3 rad/s and a phase margin of 25 deg. Clearly specify the transfer function of your inner-loop compensator.

Position loop:

(b) Calculate the DC gain of the inner loop $K_{\theta_{DC}}$. Assuming that the inner-loop closed-loop transfer function is approximately $K_{\theta_{DC}}$ over its bandwidth, design a lead controller for the outer loop to have a bandwidth of 0.5 rad/s and a damping ratio of 0.4. Clearly specify the transfer function of your outer-loop compensator.

(c) Implement your frequency-response-based designs in your Simulink simulation. Submit plots showing the Simulink block diagram and the step response of your system.

Homework VI.14 The objective of this problem is to implement a state feedback controller using the full state. You will use this controller to first regulate all of the states to zero given non-zero initial conditions on the states. As a final step you will introduce a reference input and control the system to track the reference input. Start with the single pendulum system of Homework VI.13. You will replace the transfer-function based controller blocks with a Matlab function-based controller block (like the autopilot block you've used with the most recent labs.)

(a) Create a controller function block that takes in the commanded cart position z^c, the state vector $x = (\dot{z}, \dot{\theta}, z, \theta)^\top$ and the parameter variable P. The controller function should output the force on the cart $F = u = -Kx$. You will calculate the control gains below. The commanded cart position z^c will be set to zero initially.

(b) Using the state space matrices F, G, H, and J derived in Homework IV.7, first verify that the state space system is controllable by verifying that the controllability matrix is full rank: $\text{rank}(\mathcal{C}) = n$.

(c) Next, using the Matlab `place` command, find the feedback gain K such that the eigenvalues of $(F - GK)$ are placed in the following locations: $s = -0.6, -1.5, -0.55 \pm 0.75j$. The complex pole pair corresponds to the pendulum mode with increased damping. The real poles correspond to the closed-loop poles of the cart.

(d) Implement the state feedback scheme $F = u = -Kx$ in Simulink and plot the closed-loop response to the initial condition $x_0 = [0, 0, -5, 0]^\top$. Submit a printout of your Simulink block diagram and your response plots.

(e) Calculate the reference input gain $\bar{N}$ using the methods discussed in class. Introduce the reference input into your control block so that the control force is $F = u = -Kx + \bar{N}r$. Set the initial conditions on the states to zero and plot the response of the system to a step input of 10 m in z^c. Submit a printout of your Simulink block diagram and your response plots.

Hints:

- For comparison, I get the following state space model for the single-pendulum crane system:

$$F = \begin{pmatrix} -1.3333 & 0 & 0 & 3.2700 \\ 0.1333 & 0 & 0 & -1.3080 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 6.6667e - 04 \\ -6.6667e - 05 \\ 0 \\ 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

$$J = 0.$$

Better yet, in symbolic form:

$$F = \begin{pmatrix} -\frac{b}{m_c} & 0 & 0 & \frac{mg}{m_c} \\ \frac{b}{m_c L} & 0 & 0 & -\frac{(m_c+m)g}{m_c L} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} \frac{1}{m_c} \\ -\frac{1}{m_c L} \\ 0 \\ 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

$$J = 0.$$

Homework VI.15 The objective of this problem is to replace the state in Homework VI.13 with the estimated state $\hat{x}$ produced by an estimator.

(a) Modify the Simulink diagram from Homework VI.13 so that cart force and the estimated state $\hat{x}$ are outputs of the controller and use a Scope to plot the states and estimated states. From the states coming out of the dynamics block, select the cart position as the measured output of the system. Create a controller block with inputs cart position command, cart position output, and time t from a Clock Simulink block. Set the sample period of the controller block to be $T_s = 0.001$ s.

(b) Verify that the state space system is observable by checking that the observability matrix is full rank: $\mathrm{rank}(\mathcal{O}) = n$.

(c) In the control block, add an observer (estimator) to estimate the state $\hat{x}$. Do not yet use $\hat{x}$ in the state feedback equations. Run the controller using the actual state x and tune the observer so that the estimation error is small. Make sure the eigenvalues $\mathrm{eig}(F - GK - LH)$ are in the left half plane.

(d) Replace the state x in the controller with the estimated state $\hat{x}$ and tune the controller and observer if necessary to obtain good performance.

(e) Test your system by giving the commanded cart position a series of step changes. Plot the values of the states and estimated states using Scope blocks. Submit screen shots of your system response and block diagrams. Submit a printout of your controller block code.