State Variable Form :

When applying first principles (Newton's 2nd Law), the resulting ODE's that describe the system behavior are typically 2nd order or 1st order ODE's

Newton's 2nd $\Rightarrow$ 2nd order ODE's

- position x
- velocity $\dot{x}$
- acceleration $\ddot{x}$

} forces / torques are function of these

In order to solve numerically (ie. get them into a Simulink s-function) we need to express them in "state variable" form :

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

In general, n states
m inputs

$\mathbf{x}$ is $n \times 1$ vector

$\mathbf{u}$ is $m \times 1$ vector

f is a vector function

or

$$\left.\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
\dot{x}_2 &= f_2(x_1, x_2, \ldots x_n, u_1, u_2, \ldots, u_m) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots x_n, u_1, u_2, \ldots, u_m)
\end{aligned}\right\} \text{a system of } n \text{ 1st order ODE's}$$

Steps :

1. Starting with derived EOM, solve each equation for the highest-order derivative term
2. Terms on left are state derivatives
   Terms on right are states, inputs, or physical parameters
3. For each state identified, need an equation expressing state derivative as a function of states, inputs, parameters

In other words, need

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \quad \text{for each state}$$

Example : Case Study I : Single link robot arm

EOM $$J_o \ddot{\theta} + b\dot{\theta} + mg\frac{l}{2}\cos\theta = \tau(t)$$

2nd order system — we expect that in state variable form, it will be represented by a system of 2 1st-order ODE's

Solve for highest-order derivative term :

$$\boxed{\underbrace{\ddot{\theta}}_{\text{derivative of state}} = -\frac{b}{J_o}\underbrace{\dot{\theta}}_{\text{state}} - \frac{mgl}{2J_o}\cos\underbrace{\theta}_{\text{state}} + \frac{1}{J_o}\underbrace{\tau(t)}_{\text{input}}}$$

$$x = \begin{bmatrix} \dot{\theta} \\ \theta \end{bmatrix} \qquad u = \tau(t)$$

$$\dot{x} = \begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \end{bmatrix}$$

one of our state equations — 1st order ?

need 2nd ODE : $\dot{\theta} = \ ?$ $\Rightarrow$ $\boxed{\underbrace{\dot{\theta}}_{\text{derivative of position}} = \underbrace{\dot{\theta}}_{\text{velocity}}}$

derivative of 2nd state = 1st state Confusing !

Let's try a substitution : $\dot{\theta} = \Omega \leftarrow$ angular velocity

$$\rightarrow \ddot{\theta} = \dot{\Omega}$$

$$\Rightarrow \boxed{\underbrace{\dot{\Omega}}_{\text{derivative of state}} = -\frac{b}{J_o}\Omega - \frac{mgl}{2J_o}\cos\theta + \frac{1}{J_o}\underbrace{\tau(t)}_{\text{input}}} \leftarrow \text{1st EOM}$$

states

$$x = \begin{bmatrix} \Omega \\ \theta \end{bmatrix} \qquad 2^{nd} \text{ EOM}: \quad \boxed{\dot{\theta} = \Omega}$$

derivative of 2^{nd} state $=$ 1^{st} state

Example : Case Study V - Ball Beam

$$\frac{7}{5}\ddot{z} - z\dot{\theta}^2 + g\sin\theta = 0$$

$$\left(\frac{1}{3}m_2 l^2 + m_1 z^2\right)\ddot{\theta} + 2m_1 z\dot{z}\dot{\theta} + g\cos\theta\left(m_2\frac{l}{2} + m_1 z\right) = F(t)\, l\cos\theta$$

Let's do velocity variable substitutions to avoid confusion:

Let $\boxed{\dot{z} = v}$ and $\boxed{\dot{\theta} = \Omega}$

$$\Rightarrow \frac{7}{5}\dot{v} - z\Omega^2 + g\sin\theta = 0 \qquad (1)$$

$$\left(\frac{1}{3}m_2 l^2 + m_1 z^2\right)\dot{\Omega} + 2m_1 z v\Omega + g\cos\theta\left(m_2\frac{l}{2} + m_1 z\right) = F(t)\, l\cos\theta \qquad (2)$$

Solving (1) & (2) for highest - order derivative terms:

$$\boxed{\dot{v} = \frac{5}{7}z\Omega^2 - \frac{5}{7}g\sin\theta}$$

$$\boxed{\dot{\Omega} = \frac{-2m_1 z v\Omega - g\cos\theta\left(m_2\frac{l}{2} + m_1 z\right) + F(t)\, l\cos\theta}{\left(\frac{1}{3}m_2 l^2 + m_1 z^2\right)}}$$

$$x = \begin{bmatrix} z \\ \theta \\ v \\ \Omega \end{bmatrix} \qquad u = F(t)$$

<u>Linearization</u>:

Given a nonlinear system of ODE's

$$\dot{x} = f(x, u),$$

we can linearize about an equilibrium point (x_0, u_0) using the Taylor Series Approximation:

$$\dot{x} = f(x,u) = f(x_0,u_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0}(x-x_0) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x_0\\u_0}}(u-u_0) + \text{H.O.T.}$$

At equilibrium, $f(x_0,u_0) = 0$ and ignoring H.O.T we get:

$$\dot{x} \approx \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0}(x-x_0) + \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0}(u-u_0)$$

Define $\tilde{x} \triangleq x - x_0$ and $\tilde{u} \triangleq u - u_0$

Note $\dot{\tilde{x}} = \dot{x} - \dot{x}_0$ but x_0 is constant

$\longrightarrow \dot{\tilde{x}} = \dot{x}$ $\qquad \longrightarrow \dot{x}_0 = 0$

Substituting,

$$\boxed{\dot{\tilde{x}} \approx \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0}\tilde{x} + \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0}\tilde{u}}$$

which is linear in the perturbations $\tilde{x}$, $\tilde{u}$.

This is state-space form (partially)

CSI : Single-link robot arm, <u>Linearization.</u>

$$J_o \ddot{\theta} + b\dot{\theta} + mg\frac{l}{2}\cos\theta = \tau(t)$$

At equilibrium, $\dot{\theta} = \ddot{\theta} = 0$, $\theta = \theta_o$, $\tau = \tau_o$

$$\longrightarrow \quad \underline{mg\frac{l}{2}\cos\theta_o = \tau_o}$$

$\longrightarrow$ <u>Specify θ_o, calculate τ_o</u>

For our state, take $x = \begin{bmatrix} \Omega \\ \theta \end{bmatrix}$ where $\dot{\theta} = \Omega$

$$J_o\dot{\Omega} + b\Omega + mg\frac{l}{2}\cos\theta = \tau(t)$$

<u>or</u>

$$\dot{\Omega} = -\frac{b}{J_o}\Omega - \frac{mgl}{2J_o}\cos\theta + \frac{1}{J_o}\tau(t) = f_1$$

and

$$\dot{\theta} = \Omega = f_2$$

$$\left.\frac{\partial f_1}{\partial \Omega}\right|_{\Omega_o,\theta_o,\tau_o} = -\frac{b}{J_o} \qquad \left.\frac{\partial f_1}{\partial \theta}\right|_{\Omega_o,\theta_o,\tau_o} = \frac{mgl}{2J_o}\sin\theta_o$$

$$\left.\frac{\partial f_2}{\partial \Omega}\right|_{\Omega_o,\theta_o,\tau_o} = 1 \qquad \left.\frac{\partial f_2}{\partial \theta}\right|_{\Omega_o,\theta_o,\tau_o} = 0$$

$$\left.\frac{\partial f_1}{\partial \tau}\right|_{\Omega_o,\theta_o,\tau_o} = \frac{1}{J_o} \qquad \left.\frac{\partial f_2}{\partial \tau}\right|_{\Omega_o,\theta_o,\tau_o} = 0$$

From this, we can write:

$$\boxed{\begin{bmatrix} \dot{\tilde{\Omega}} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} -b/J_o & \frac{mgl}{2J_o}\sin\theta_o \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \tilde{\Omega} \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} 1/J_o \\ 0 \end{bmatrix}\tilde{\tau}(t)}$$

We can also linearize nonlinearities term-by-term.

Referring to EoM:

$$J_0 \dot{\Omega} + b\Omega + mg\frac{l}{2}\cos\theta = \tau(t)$$

What are nonlinear terms? $\cos\theta$ only

let $a(\theta) = \cos\theta$

Taylor Series approx abt $\theta = \theta_0$

$$\rightarrow \quad a \approx a(\theta_0) + \left.\frac{\partial a}{\partial \theta}\right|_{\theta=\theta_0} (\theta - \theta_0)$$

$$\approx \cos\theta_0 - (\sin\theta_0)\tilde{\theta}$$

Subst.

$$J_0 \dot{\Omega} + b\Omega + mg\frac{l}{2}\left[\cos\theta_0 - (\sin\theta_0)\tilde{\theta}\right] = \tau(t)$$

Recall that: $\tilde{\tau} = \tau - \tau_0$

$$\rightarrow \quad \tau = \tilde{\tau} + \tau_0$$

$$\underline{\tau = \tilde{\tau} + mg\frac{l}{2}\cos\theta_0}$$

Subst.

$$J_0 \dot{\Omega} + b\Omega + \cancel{mg\frac{l}{2}\cos\theta_0} - mg\frac{l}{2}\sin\theta_0\,\tilde{\theta} = \tilde{\tau} + \cancel{mg\frac{l}{2}\cos\theta_0}$$

$$\boxed{J_0 \dot{\Omega} + b\Omega - mg\frac{l}{2}\sin\theta_0\,\tilde{\theta} = \tilde{\tau}(t)}$$

Why do this term-by-term?

Case Study II provides an example

CS II: (Inverted Pendulum) Linearization

EOM

$$\begin{bmatrix} m_1 + m_2 & m_1 l \cos\theta \\ m_1 l \cos\theta & m_1 l_1^2 \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} m_1 l \dot{\theta}^2 \sin\theta - b\dot{y} + F(t) \\ m_1 g l \sin\theta \end{bmatrix}$$

Solving for $\begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix}$,

$$\begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} m_1 + m_2 & m_1 l \cos\theta \\ m_1 l \cos\theta & m_1 l_1^2 \end{bmatrix}^{-1} \begin{bmatrix} m_1 l \dot{\theta}^2 \sin\theta - b\dot{y} + F(t) \\ m_1 g l \sin\theta \end{bmatrix}}$$

- This gets messy.
- Easier to linearize before taking inverse and multiplying
- Linearize term by term
- First, find equilibrium

Equilibrium:

Let $\ddot{y} = \ddot{\theta} = \dot{y} = \dot{\theta} = 0$

$$\Rightarrow \begin{bmatrix} F(t) \\ m_1 g l \sin\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (y_0, \theta_0, F_0) = (\underset{\text{can be anything}}{\text{free}}, \underset{\text{up or down}}{k\pi}, 0) \Rightarrow (\theta_0, F_0) = (0, 0)$$

Linearize about (θ_0, F_0):

for $\theta_0 = 0$

$$\cos\theta \approx \cos\theta_0 + \frac{\partial}{\partial\theta}\cos\theta\Big|_{\theta_0}\tilde{\theta} = \cos\theta_0 - \sin\theta_0\,\tilde{\theta} = \underline{\underline{1}}$$

$$\sin\theta \approx \sin\theta_0 + \frac{\partial}{\partial\theta}\sin\theta\Big|_{\theta_0}\tilde{\theta} = \sin\theta_0 + \cos\theta_0\,\tilde{\theta} = \underline{\underline{\tilde{\theta}}}$$

(cont.)

Also,

$$\dot\theta^2 \sin\theta \approx \dot\theta_0^2 \sin\theta_0 + \frac{\partial}{\partial\theta}(\dot\theta^2 \sin\theta)\Big|_{\substack{\theta_0\\ \dot\theta_0}} \tilde\theta + \frac{\partial}{\partial\dot\theta}(\dot\theta^2 \sin\theta)\Big|_{\substack{\theta_0\\ \dot\theta_0}} \dot{\tilde\theta}$$

$$\approx \dot\theta_0^2 \cancelto{0}{\sin\theta_0} + \dot\theta_0^2 \cancelto{0}{\cos}\,\theta_0 \tilde\theta + 2\dot\theta_0 \cancelto{0}{\sin\theta_0}\, \dot{\tilde\theta}$$

$$\underline{\underline{\approx 0}}$$

Subst. gives:

$$\begin{bmatrix} m_1+m_2 & m_1 l \\ m_1 l & m_1 l^2 \end{bmatrix} \begin{bmatrix} \ddot y \\ \ddot\theta \end{bmatrix} = \begin{bmatrix} -b\dot y + F(t) \\ m_1 g l \tilde\theta \end{bmatrix}$$

Convert to $\tilde\theta$, $\tilde F$ notation

$$\tilde\theta = \theta - \theta_0 \qquad \tilde F = F - F_0$$
$$\tilde\theta = \theta \qquad \tilde F = F$$

$$\begin{bmatrix} m_1+m_2 & m_1 l \\ m_1 l & m_1 l^2 \end{bmatrix} \begin{bmatrix} \ddot y \\ \ddot{\tilde\theta} \end{bmatrix} = \begin{bmatrix} -b\dot y + \tilde F(t) \\ m_1 g l \tilde\theta \end{bmatrix}$$

State-Space Form

- Depending on how they are derived, equations of motion (ODE's) can take a variety of forms

- For analysis, two forms are particularly useful
  - State-space form
  - transfer function form

  (Both of these forms require the equations to be linear, constant-coefficient ODE's)

State-space form looks like this:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

For an n^{th}-order SISO system,

$x, \dot{x}$: state, state derivative vectors $(n \times 1)$

y, u : output, input (scalar)

A : system matrix $(n \times n)$

B : input matrix $(n \times 1)$

C : output matrix $(1 \times n)$

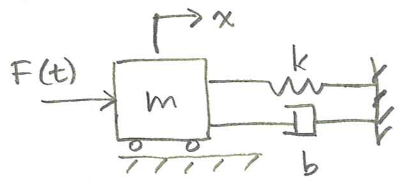
D : direct transmission matrix (scalar)

- State-space form can be used for control design and analysis

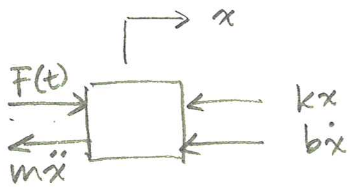
- Can be used to specify EOM to Matlab for simulation

Simple Simulations in Matlab

Example : Mass-spring system



FBD



EOM

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

Two Approaches

State Space

- Solve for highest-order derivative term

$$\ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x + \frac{1}{m}F(t)$$

- Terms on right are composed of states, inputs, physical parameters

  states : $\dot{x}$, x

  inputs : $F(t)$

  parameters : m, b, k

- Want EOM in this form

$$\dot{z} = Az + Bu$$
$$y = Cz + Du$$

$$z = \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \qquad u = F(t)$$
$$y = x$$

Transfer Function

- Take Laplace transform (assume zero IC's)

$$(ms^2 + bs + k)\,X(s) = F(s)$$

- Solve for $\frac{\text{Output}}{\text{Input}}$

- $\dfrac{X(s)}{F(s)} = \dfrac{1}{ms^2 + bs + k}$

State Space (cont.)

- A little easier to understand if we use

$$z = \begin{bmatrix} v \\ x \end{bmatrix} \quad \text{where} \quad v = \dot{x}$$
$$\dot{v} = \ddot{x}$$

- Substituting ...

$$\dot{v} = -\frac{b}{m} v - \frac{k}{m} x + \frac{1}{m} F(t)$$

$$\dot{x} = v$$

$$\begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -\frac{b}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} F(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} + 0 \; F(t)$$