Homework E.16 - Solution

From HW E.7, the state space equations for the ballbeam system are given by

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & -18.1923 & 0 & 0 \\ -9.8000 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 2.6519 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x.$$

Step 1. The controllability matrix is therefore

$$\mathcal{C} = [B, AB, A^2B, A^3B] = \begin{pmatrix} 0 & 2.6519 & 0 & 0 \\ 0 & 0 & 0 & -25.9890 \\ 2.6519 & 0 & 0 & 0 \\ 0 & 0 & -25.9890 & 0 \end{pmatrix}.$$

The determinant is $\det(\mathcal{C}) = -4750 \neq 0$, therefore the system is controllable.

Step 2. The open loop characteristic polynomial

$$\Delta_{ol}(s) = \det(sI - A) = s^4 - 178.2842$$

which implies that

$$\mathbf{a} = (0, 0, 0, -178.2842)$$
$$\mathcal{A} = I.$$

Step 3. When $\omega_\theta = 23.0115$, $\zeta_\theta = 0.707$, $\omega_z = 1.3982$, $\zeta_z = 0.707$, the desired closed loop polynomial

$$\Delta_{cl}^d(s) = (s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2)(s^2 + 2\zeta_z \omega_{n_z} s + \omega_{n_z}^2)$$
$$= s^4 + 34.5s^3 + 595.8s^2 + 1110.5s + 1035.2$$

which implies that

$$\boldsymbol{\alpha} = (34.5595.81110.51035.2).$$

Step 4. The gains are therefore given as

$$\begin{aligned} K &= (\boldsymbol{\alpha} - \mathbf{a})\mathcal{A}^{-1}\mathcal{C}^{-1} \\ &= \begin{pmatrix} 224.6715 & -46.6927 & 13.0152 & -42.7304 \end{pmatrix} \end{aligned}$$

To compute the reference gain $k_r = -1/C_{out}(A - BK)^{-1}B$, we need to use C_{out}, the output matrix matching the reference input, which since the desired reference input is z_d and since $x = (\theta, z, \dot{\theta}, \dot{z})^\top$, we have $C_{out} = (0, 1, 0, 0)$, which gives

$$\begin{aligned} k_r &= \frac{-1}{C_{out}(A - BK)^{-1}B} \\ &= -39.8327. \end{aligned}$$

Alternatively, we could have used the following Matlab script

```
% state space design
A = [...
    0, 0, 1, 0;...
    0, 0, 0, 1;...
    0, -P.m1*P.g/((P.m2*P.L^2)/3+P.m1*(P.L/2)^2), 0, 0;...
    -P.g, 0, 0, 0;...
];
B = [0; 0; P.L/(P.m2*P.L^2/3+P.m1*P.L^2/4); 0; ];
C = [...
    1, 0, 0, 0;...
    0, 1, 0, 0;...
    ];

% gains for pole locations
wn_z    = 2*0.6991;
zeta_z  = 0.707;
wn_th   = 23.0115;
zeta_th = 0.707;

ol_char_poly = charpoly(A);
des_char_poly = conv([1,2*zeta_z*wn_z,wn_z^2],...
                     [1,2*zeta_th*wn_th,wn_th^2]);
des_poles = roots(des_char_poly);

% is the system controllable?
if rank(ctrb(A,B))≠4, disp('System Not Controllable'); end
```

```
P.K = place(A,B,des_poles);
Cout = [0, 1, 0, 0];
P.kr = -1/(Cout*inv(A-B*P.K)*B);
```

The Matlab code for the controller is given by

```
function F=ballbeam_ctrl(in,P)
    z_d   = in(1);
    z     = in(2);
    theta = in(3);
    t     = in(4);

     % use a digital differentiator to find zdot and thetadot
    persistent zdot
    persistent z_d1
    persistent thetadot
    persistent theta_d1
    % reset persistent variables at start of simulation
    if t<P.Ts,
        zdot        = 0;
        z_d1        = 0;
        thetadot    = 0;
        theta_d1    = 0;
    end
    zdot = (2*P.tau-P.Ts)/(2*P.tau+P.Ts)*zdot...
        + 2/(2*P.tau+P.Ts)*(z-z_d1);
    thetadot = (2*P.tau-P.Ts)/(2*P.tau+P.Ts)*thetadot...
        + 2/(2*P.tau+P.Ts)*(theta-theta_d1);
    z_d1 = z;
    theta_d1 = theta;

    % construct the state
    x = [theta; z; thetadot; zdot];
    % equilibrium force
    Fe = 0.5*P.m2*P.g;% + P.m1*P.g*z/P.L;
    % compute the state feedback controller
    F_tilde = -P.K*x + P.kr*z_d;
    F = sat( Fe + F_tilde, P.Fmax);
end
```