1 Givens Rotations

The Given's rotation technique is an approach to QR factorization that successively produces matrices with zero-valued elements, one-at-a-time, until an upper triangular matrix is reached. The idea is to use a simple 2×2 rotation matrix (which is unitary) placed along the diagonal of an identity matrix and calculated to zero-out one of the elements of the matrix. The elements of the rotation matrix to rotate a vector counter-clockwise by an angle θ is

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

If we have a vector given by $[x_1, x_2]^T$ and we want to zero out the x_2 component, then we want to rotate *clockwise* by an angle θ (or counter-clockwise by an angle of $-\theta$) where

$$\theta = \tan^{-1} \frac{x_2}{x_1}.$$

Thus, the rotation matrix to accomplish this rotatin *clockwise* by $\theta$is:

$$Q_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

where

$$\cos\theta = c = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$$

and

$$\sin\theta = s = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}.$$

Therefore,

$$Q_\theta = \frac{1}{\sqrt{x_1^2 + x_2^2}} \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}.$$

Notice that as desired.

$$Q_\theta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 + sx_2 \\ -sx_1 + cx_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1 + x_2^2}{\sqrt{x_1^2 + x_2^2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{x_1^2 + x_2^2} \\ 0 \end{bmatrix}.$$

If A is $m \times n$, then consider what happens when we place the elements of Q into the sub-matrix defined by the i^{th} row and column and j^{th} row and column ($i < j$). In other words we place this 2×2 matrix along the diagonal at some point:

$$G_{kl} = \begin{cases} \delta_{kl} & k \neq i, l \neq j \\ c & k,l = i, k,l = j \\ s & k = i, l = j \\ -s & k = j, l = i \end{cases} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & & 0 & \cdots & 0 & & 1 \end{bmatrix}.$$

Thus, G is the identity matrix of size $m \times m$ except for replacements

$$\begin{aligned} G_{ii} &= G_{jj} = c \\ G_{ij} &= -G_{ij} = s. \end{aligned}$$

This will be a unitary matrix as long as

$$G^H G = I$$

which means that

$$\sum_l G_{lk} G_{lp} = \delta_{kp}$$

which only requires that

$$c^2 + s^2 = 1$$

and that is true because $\cos^2\theta + \sin^2\theta = 1$ for any angle.

When this matrix is applied to an $m \times n$ matrix we get

$$\begin{aligned} B_{kp} &= \sum_l G_{kl} A_{lp} \\ &= \begin{array}{cc} \sum_l \delta_{kl} A_{lp} & k \neq i, j \\ \sum_l G_{il} A_{lp} & k = i \\ \sum_l G_{jl} A_{lp} & k = j \end{array} = \begin{array}{cc} A_{kp} & k \neq i, j \\ cA_{ip} + sA_{jp} & k = i \\ -sA_{ip} + cA_{jp} & k = j. \end{array} \end{aligned}$$

Thus, the new matrix is changed only in rows i and j. We choose s and c so as to zero out column r in row j by picking

$$s = \frac{A_{jr}}{\sqrt{A_{jr}^2 + A_{ir}^2}} \qquad c = \frac{A_{ir}}{\sqrt{A_{jr}^2 + A_{ir}^2}}.$$

In this way we are guaranteed that

$$B_{jr} = \frac{-A_{jr} A_{ir} + A_{ir} B_{jr}}{\sqrt{A_{jr}^2 + A_{ir}^2}} = 0.$$

Notice multiplication by G produces a zero in the j^{th} row and r^{th} column and only modifies the matrix in rows i and j by taking a linear combination of those two rows. We can obtain an upper triangular matrix by applying a Givens rotation matrix by taking r as the left-most column and j as the bottom-most row and working up and to the right. Then, i must be selected as a row above the current row (*e.g.* $i = j - 1$).

1.1 Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

with

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_1 & s_1 \\ 0 & 0 & -s_1 & c_1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3c_1 + 6s_1 & 2c_1 + 5s_1 & c_1 + 4s_1 \\ -3s_1 + 6c_1 & -2s_1 + 5c_1 & -s_1 + 4c_1 \end{bmatrix}$$

$$\begin{aligned} c_1 &= \frac{3}{\sqrt{3^2 + 6^2}} = \frac{1}{\sqrt{5}} \\ s_1 &= \frac{6}{\sqrt{3^2 + 6^2}} = \frac{2}{\sqrt{5}}. \end{aligned}$$

Thus,

$$G_1A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3\sqrt{5} & \frac{12}{5}\sqrt{5} & \frac{9}{5}\sqrt{5} \\ 0 & \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix}.$$

The next step is

$$\begin{aligned} G_2G_1A &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & s_2 & 0 \\ 0 & -s_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3\sqrt{5} & \frac{12}{5}\sqrt{5} & \frac{9}{5}\sqrt{5} \\ 0 & \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 4c_2 + 3s_2\sqrt{5} & 5c_2 + \frac{12}{5}s_2\sqrt{5} & 6c_2 + \frac{9}{5}s_2\sqrt{5} \\ -4s_2 + 3c_2\sqrt{5} & -5s_2 + \frac{12}{5}c_2\sqrt{5} & -6s_2 + \frac{9}{5}c_2\sqrt{5} \\ 0 & \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix} \end{aligned}$$

with

$$\begin{aligned} c_2 &= \frac{4}{\sqrt{4^2 + \left(3\sqrt{5}\right)^2}} = \frac{4}{\sqrt{61}} \\ s_2 &= \frac{3\sqrt{5}}{\sqrt{4^2 + \left(3\sqrt{5}\right)^2}} = \frac{3\sqrt{5}}{\sqrt{61}}. \end{aligned}$$

Then,

$$G_2G_1A = \begin{bmatrix} 1 & 2 & 3 \\ \sqrt{61} & \frac{56\sqrt{61}}{61} & \frac{51}{61}\sqrt{61} \\ 0 & -\frac{27\sqrt{305}}{305} & -\frac{54\sqrt{305}}{305} \\ 0 & \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix}.$$

Now,

$$G_3 = \begin{bmatrix} c_3 & s_3 & 0 & 0 \\ -s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so that

$$G_3G_2G_1A = \begin{bmatrix} c_3 + s_3\sqrt{61} & 2c_3 + \frac{56}{61}s_3\sqrt{61} & 3c_3 + \frac{51}{61}\sqrt{61} \\ -s_3 + c_3\sqrt{61} & -2s_3 + \frac{56}{61}c_3\sqrt{61} & -3s_3 + \frac{51}{61}c_3\sqrt{61} \\ 0 & -\frac{27}{305}\sqrt{305} & -\frac{54}{305}\sqrt{305} \\ 0 & \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix}$$

where

$$\begin{aligned} c_3 &= \frac{1}{\sqrt{1+61}} = \frac{\sqrt{62}}{62} \\ s_3 &= \frac{\sqrt{61}}{\sqrt{1+61}} = \frac{\sqrt{61 \cdot 62}}{62} = \frac{\sqrt{3782}}{62} \end{aligned}$$

so that

$$G_3G_2G_1A = \begin{bmatrix} \sqrt{62} & \frac{58}{\sqrt{61}} & \frac{54}{\sqrt{61}} \\ 0 & -\frac{66}{\sqrt{61 \cdot 62}} & -\frac{132}{\sqrt{61 \cdot 62}} \\ 0 & -\frac{27}{\sqrt{305}} & -\frac{54}{\sqrt{305}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}.$$

Next, define

$$G_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_4 & s_4 \\ 0 & 0 & -s_4 & c_4 \end{bmatrix}$$

so that

$$G_4G_3G_2G_1A = \begin{bmatrix} \sqrt{62} & \frac{58}{\sqrt{61}} & \frac{54}{\sqrt{61}} \\ 0 & -\frac{66}{\sqrt{61 \cdot 62}} & -\frac{132}{\sqrt{61 \cdot 62}} \\ 0 & -\frac{27}{\sqrt{305}}c_4 + \frac{1}{\sqrt{5}}s_4 & -\frac{54}{\sqrt{305}}c_4 + \frac{2}{\sqrt{5}}s_4 \\ 0 & \frac{27}{\sqrt{305}}s_4 + \frac{1}{\sqrt{5}}c_4 & \frac{54}{\sqrt{305}}s_4 + \frac{2}{\sqrt{5}}c_4 \end{bmatrix}$$

where

$$\begin{aligned} c_4 &= \frac{-\frac{27}{\sqrt{305}}}{\sqrt{\frac{1}{5} + \frac{27^2}{305}}} = -\frac{27}{\sqrt{790}} \\ s_4 &= \frac{\frac{1}{\sqrt{5}}}{\sqrt{\frac{27^2}{305} + \frac{1}{5}}} = \frac{\sqrt{61}}{\sqrt{790}}. \end{aligned}$$

So that

$$G_4G_3G_2G_1A = \begin{bmatrix} \sqrt{61} & \frac{58}{\sqrt{61}} & \frac{54}{\sqrt{61}} \\ 0 & -\frac{66}{\sqrt{61 \cdot 62}} & -\frac{132}{\sqrt{61 \cdot 62}} \\ 0 & \frac{\sqrt{158}}{\sqrt{61}} & \frac{2\sqrt{158}}{\sqrt{61}} \\ 0 & 0 & 0 \end{bmatrix}.$$

With one more Givens rotation matrix we can complete the decompostion: Use

$$G_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_5 & s_5 & 0 \\ 0 & -s_5 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so that

$$G_5G_4G_3G_2G_1A = \begin{bmatrix} \sqrt{62} & \frac{58}{\sqrt{62}} & \frac{54}{\sqrt{62}} \\ 0 & -\frac{66}{\sqrt{61 \cdot 62}}c_5 + \frac{\sqrt{158}}{\sqrt{61}}s_5 & -\frac{132}{\sqrt{61 \cdot 62}}c_5 + \frac{2\sqrt{158}}{\sqrt{61}}s_5 \\ & \frac{66}{\sqrt{61 \cdot 62}}s_5 + \frac{\sqrt{158}}{\sqrt{61}}c_5 & \frac{132}{\sqrt{61 \cdot 62}}s_5 + \frac{2\sqrt{158}}{\sqrt{61}}c_5 \\ 0 & 0 & 0 \end{bmatrix}$$

with

$$\begin{aligned} c_5 &= \frac{\frac{-66}{\sqrt{61 \cdot 62}}}{\sqrt{\frac{66^2}{61 \cdot 62} + \frac{158}{61}}} = -\frac{66}{\sqrt{61 \cdot 62}} \frac{\sqrt{61 \cdot 62}}{\sqrt{66^2 + 158 \cdot 62}} = -\frac{33}{\sqrt{61 \cdot 29 \cdot 2}} \\ s_5 &= \frac{\sqrt{\frac{158}{61}}}{\sqrt{\frac{66^2}{61 \cdot 62} + \frac{158}{61}}} = \frac{\sqrt{158}}{\sqrt{61}} \frac{\sqrt{61 \cdot 62}}{\sqrt{66^2 + 158 \cdot 62}} = \frac{\sqrt{79 \cdot 31}}{\sqrt{61 \cdot 29 \cdot 2}}. \end{aligned}$$

Thus,

$$Q^H A = \begin{bmatrix} \sqrt{62} & \frac{58}{\sqrt{62}} & \frac{54}{\sqrt{62}} \\ 0 & 2\sqrt{\frac{29}{31}} & 4\sqrt{\frac{29}{31}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R.$$

Also,

$$Q^H = G_5G_4G_3G_2G_1 =$$

$$G_2G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & s_2 & 0 \\ 0 & -s_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_1 & s_1 \\ 0 & 0 & -s_1 & c_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & s_2c_1 & s_2s_1 \\ 0 & -s_2 & c_2c_1 & c_2s_1 \\ 0 & 0 & -s_1 & c_1 \end{bmatrix}$$

and

$$G_4G_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_4 & s_4 \\ 0 & 0 & -s_4 & c_4 \end{bmatrix} \begin{bmatrix} c_3 & s_3 & 0 & 0 \\ -s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_3 & s_3 & 0 & 0 \\ -s_3 & c_3 & 0 & 0 \\ 0 & 0 & c_4 & s_4 \\ 0 & 0 & -s_4 & c_4 \end{bmatrix}$$

with

$$G_5G_4G_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_5 & s_5 & 0 \\ 0 & -s_5 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & s_3 & 0 & 0 \\ -s_3 & c_3 & 0 & 0 \\ 0 & 0 & c_4 & s_4 \\ 0 & 0 & -s_4 & c_4 \end{bmatrix} = \begin{bmatrix} c_3 & s_3 & 0 & 0 \\ -c_5s_3 & c_5c_3 & s_5c_4 & s_5c_4 \\ s_5s_3 & -s_5c_3 & c_5c_4 & c_5s_4 \\ 0 & 0 & -s_4 & c_4 \end{bmatrix}$$

so that

$$\begin{aligned} Q^H &= \begin{bmatrix} c_3 & s_3 & 0 & 0 \\ -c_5s_3 & c_5c_3 & s_5c_4 & s_5c_4 \\ s_5s_3 & -s_5c_3 & c_5c_4 & c_5s_4 \\ 0 & 0 & -s_4 & c_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & s_2c_1 & s_2s_1 \\ 0 & -s_2 & c_2c_1 & c_2s_1 \\ 0 & 0 & -s_1 & c_1 \end{bmatrix} \\ &= \begin{bmatrix} c_3 & s_3c_2 & s_3s_2c_1 & s_3s_2s_1 \\ -c_5s_3 & c_5c_3c_2 - s_5c_4s_2 & c_5c_3s_2c_1 + s_5c_4c_2c_1 - s_5c_4s_1 & c_5c_3s_2s_1 + s_5c_4c_2s_1 + s_5c_4c_1 \\ s_5s_3 & -s_5c_3c_2 - c_5c_4s_2 & -s_5c_3s_2c_1 + c_5c_4c_2c_1 - c_5s_4s_1 & -s_5c_3s_2s_1 + c_5c_4c_2s_1 + c_5s_4c_1 \\ 0 & s_4s_2 & -s_4c_2c_1 - c_4s_1 & -s_4c_2s_1 + c_4c_1 \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} c_1, s_1 &= \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \\ c_2, s_2 &= \frac{4}{\sqrt{61}}, \frac{3\sqrt{5}}{\sqrt{61}} \\ c_3, s_3 &= \frac{1}{\sqrt{62}}, \frac{\sqrt{61}}{\sqrt{62}} \\ c_4, s_4 &= -\frac{27}{\sqrt{790}}, \frac{\sqrt{61}}{\sqrt{790}} \\ c_5, s_5 &= -\frac{33}{\sqrt{61 \cdot 29 \cdot 2}}, \frac{\sqrt{79 \cdot 31}}{\sqrt{61 \cdot 29 \cdot 2}} \end{aligned}$$

so that

$$Q = \begin{bmatrix} \frac{1}{\sqrt{62}} & \frac{33}{2\sqrt{29 \cdot 31}} & \frac{79}{2\sqrt{79 \cdot 29}} & 0 \\ \frac{4}{\sqrt{62}} & \frac{39}{2\sqrt{29 \cdot 31}} & -\frac{49}{2\sqrt{79 \cdot 29}} & \frac{3}{\sqrt{79 \cdot 2}} \\ \frac{3}{\sqrt{62}} & -\frac{25}{2\sqrt{29 \cdot 31}} & \frac{21}{2\sqrt{79 \cdot 29}} & \frac{10}{\sqrt{79 \cdot 2}} \\ \frac{6}{\sqrt{62}} & -\frac{19}{2\sqrt{29 \cdot 31}} & \frac{9}{2\sqrt{79 \cdot 29}} & -\frac{7}{\sqrt{79 \cdot 2}} \end{bmatrix}$$

Thus, the Givens rotation method gives us the factorization:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \\ 6 & 5 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{62}} & \frac{33}{2\sqrt{29 \cdot 31}} & \frac{79}{2\sqrt{79 \cdot 29}} & 0 \\ \frac{4}{\sqrt{62}} & \frac{39}{2\sqrt{29 \cdot 31}} & -\frac{49}{2\sqrt{79 \cdot 29}} & \frac{3}{\sqrt{79 \cdot 2}} \\ \frac{3}{\sqrt{62}} & -\frac{25}{2\sqrt{29 \cdot 31}} & \frac{21}{2\sqrt{79 \cdot 29}} & \frac{10}{\sqrt{79 \cdot 2}} \\ \frac{6}{\sqrt{62}} & -\frac{19}{2\sqrt{29 \cdot 31}} & \frac{9}{2\sqrt{79 \cdot 29}} & -\frac{7}{\sqrt{79 \cdot 2}} \end{bmatrix} \begin{bmatrix} \sqrt{62} & \frac{58}{\sqrt{62}} & \frac{54}{\sqrt{62}} \\ 0 & 2\sqrt{\frac{29}{31}} & 4\sqrt{\frac{29}{31}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2 CORDIC rotations

The Givens metehod for QR decomposition can be approximated using high-speed CORDIC rotations. CORDIC rotations are a method for approximating rotation by an aribtrary angle by using a sequence of special angles whose tangents are powers of 2. Consider, the fundamental step in Givens rotation which is the two-dimensional rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where θ is chosen so that $y' = 0$. Thus,

$$\tan\theta = -\frac{y}{x}.$$

This can be re-written as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, fundamental operation is computed repeatedly in the Givens rotation. In order to make it fast we consider that rotation by an angle can be accomplished by repeatedly rotating by smaller angles. In particular, we choose the smaller angles to have values of $\tan\theta$ that are powers of 2. In other words, let $\rho_i = \pm 1$, then

$$\theta = \sum_{i=0}^{\infty} \rho_i \theta_i$$

where $\tan\theta_i = \frac{1}{2^i} = 2^{-i}$. The approximation is truncated at some level (say $i_{\max} = 10$). Thus, we can write

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \prod_{i=0}^{\infty} \cos\theta_i \prod_{i=0}^{\infty} \begin{bmatrix} 1 & -\rho_i 2^{-i} \\ \rho_i 2^{-i} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In practice, the micro-rotations are truncated at some maximum value and the constant

$$\kappa = \prod_{i=0}^{i_{\max}} \cos\theta_i$$

is pre-computed and used only if the magnitude of the result is important.

The recursive-algorithm is thus,

$$\begin{aligned}
\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} &= \kappa \begin{bmatrix} x \\ y \end{bmatrix} \\
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \rho_0 2^0 \begin{bmatrix} -y_0 \\ x_0 \end{bmatrix} \\
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \rho_1 2^{-1} \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} \\
&\vdots \\
\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} x_{i_{\max}} \\ y_{i_{\max}} \end{bmatrix} + \rho_{i_{\max}} 2^{-i_{\max}} \begin{bmatrix} -y_{i_{\max}} \\ x_{i_{\max}} \end{bmatrix}.
\end{aligned}$$

Notice this algorithm only requires additions and shifts to accomplish. The only thing left to determine is ρ_i. We can determine this without ever computing the angle of rotation, by looking at the current value of $[x_i, y_i]^T$. If this vector is in quadrant I or quadrant III then we need a negative rotation. If this vector is in quadrant II or quadrant IV then we need a positive rotation. Thus, we choose the sign of the micro-rotation as

$$\rho_i = -\operatorname{sign}(x_i) \operatorname{sign}(y_i).$$

The following table provides the nice angles that allow CORDIC rotation. Notice that except for κ you don't actually need to know any of these angles (all you need is $\tan\theta_i = 2^{-i}$) in order to proceed with a CORDIC algorithm.

i	$\tan\theta_i$	θ_i(degrees)	κ
0	1	45	0.70711
1	$\frac{1}{2}$	26.5605	0.63245
2	$\frac{1}{4}$	14.0362	0.61357
3	$\frac{1}{8}$	7.12502	0.60883
4	$\frac{1}{16}$	3.5763	0.60765
5	$\frac{1}{32}$	1.7899	0.60735
6	$\frac{1}{64}$	08952	0.60728
7	$\frac{1}{128}$	0.4476	0.60726
8	$\frac{1}{256}$	0.2238	0.60725
9	$\frac{1}{512}$	0.1119	0.60725
10	$\frac{1}{1024}$	0.05595	0.60725