Benchmarking NLopt and state-of-art algorithms for Continuous Global Optimization via Hybrid IACO$_{\mathbb{R}}$

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Abstract

This paper presents a comparative analysis of the performance of the Incremental Ant Colony algorithm for continuous optimization ($IACO_{\mathbb{R}}$), with different algorithms provided in the NLopt library. The key objective is to understand how the various algorithms in the NLopt library perform in combination with the Multi Trajectory Local Search (Mtsls1) technique. A hybrid approach has been introduced in the local search strategy by the use of a parameter which allows for probabilistic selection between Mtsls1 and a NLopt algorithm. In case of stagnation, the algorithm switch is made based on the algorithm being used in the previous iteration. The paper presents an exhaustive comparison on the performance of these approaches on Soft Computing (SOCO) and Congress on Evolutionary Computation (CEC) 2014 benchmarks. For both benchmarks, we conclude that the best performing algorithm is a hybrid variant of Mtsls1 with BFGS for local search.

Keywords. ACO, Global optimization, $IACO_{\mathbb{R}}$, $IACO_{\mathbb{R}}$-Local Search, Mtsls1, NLopt, BFGS, Hybrid $IACO_{\mathbb{R}}$

1. Introduction

The NLopt (Non-Linear Optimization) library (v2.4.2) [24] is a rich collection of optimization routines and algorithms, which provides a platform-independent interface for their use for global and local optimization. The library has been widely used for practical implementations of optimization algorithms as well as for benchmarking new algorithms.

The work in this paper is based on the $IACO_{\mathbb{R}}$-LS algorithm proposed by Liao, Dorigo et al. [34] This algorithm introduced the local search procedure in the original $IACO_{\mathbb{R}}$ technique, specifically Mtsls1 by Tseng et al.[60] for local search. The $IACO_{\mathbb{R}}$ was an extension of the $ACO_{\mathbb{R}}$ algorithm for continuous optimization with the added advantage of a variable size solution archive. The premise of our work lies in improving the local search strategy adopted by $IACO_{\mathbb{R}}$-LS, by allowing algorithms other than Mtsls1 to be used for local search.

We present a comparison of using various algorithms from the NLopt library for local search procedure in the $IACO_{\mathbb{R}}$-LS algorithm. In order to introduce a hybrid approach for local search, we use a parameter that probabilistically determines whether to use the Mtsls1 algorithm or the NLopt library algorithm. In case of stagnation, we switch between Mtsls1 or the NLopt algorithm based on the algorithm being used in the previous iteration. The objective is to rigorously analyze the effect of using various optimization algorithms in the local search procedure for $IACO_{\mathbb{R}}$-LS, as well as provide results on benchmark functions to enable a naive researcher to choose an algorithm easily. To the best of our knowledge, available works in literature have not provided exhaustive comparisons using optimization algorithm libraries on ant colony based approaches, other than [50]. However, surveys on state-of-art in multi-objective evolutionary algorithms [75], differential evolution [9] and real-parameter evolutionary multimodal optimization [8] have appeared in literature.

The rest of the paper is organized as follows. Section 2 discusses our hybrid approach which allows using Mtsls1 alongwith an NLopt library algorithm for local search phase of $IACO_{\mathbb{R}}$-LS. This is followed by a discussion on the NLopt library in Section 3. We present our results and a discussion in Section 4, followed by the conclusions in Section 5.

2. Hybrid Local Search using Mtsls1 and NLopt algorithms

We begin by introducing the Mtsls1 algorithm, and the motivation to develop a hybrid approach for local search. This is followed by a description of our algorithm which uses the hybrid local search using Mtsls1 and the algorithms from the NLopt library.

The Multi-Trajectory Local Search, or Mtsls1 algorithm [60] exploits the search space across multiple paths. The approach has evolved to many variants, notable among which are the self-adaptive evolution by Zhao et al. [74], multi-objective optimization [61] and dynamic search trajectories by Snyman et al. [55].

Mtsls1 searches along one dimension, and optimum value of one dimension is used as starting point for the next dimension. At each dimension, Mtsls1 tries to move by a step size s along one dimension, and evaluates the change in the function value. If the function value decreases, then new point is used for optimization along the next dimension, If the function value increases, then algorithm goes back to the starting point and moves by a factor of the step size, $0.5 * s$ towards negative direction and evaluates the function. Again the function value is compared and based on minimum value of the function, the optimum point is provided.

We propose an hybrid local-search approach which incorporates the non-gradient based Mtsls1, alongwith an algorithm from the NLopt library as part of the $IACO_{\mathbb{R}} - LS$ technique. Our approach offers a choice between selecting either of the two, based on a probabilistically determined choice. Ths is indicated by the algorithm parameter $P(nlopt)$. In case this probabilistic choice fails to provide any improvement after a specific number of iterations, we switch the algorithm being used based on the algorithm used in the previous iteration. The parameters $ctr_{localsearch}$ and $thresh_{localsearch}$ have been used in our algorithm implement this, as we select a different local search algorithm when $ctr_{localsearch}$ crosses $thresh_{localsearch}$. This ensures that our local-search approach does not stagnate, and also gives our approach an "adaptive" flavor.

All algorithms from NLopt library are used as part of the hybrid local search approach. The Nlopt algorithms meant for global optimization are allowed as many function evaluations as set for global search, but for local search, maximum allowed function evaluations in a single local search call is set to 160. It may be noted here that the method for approximating the gradient is directly linked to algorithm's ability to escape local minima. Solomon [53] opines that *"if, however, the gradient is estimated by independent trials with a distance along each axis, the difference between both classes of algorithms almost vanishes."* Hence, our computations of gradient are based on approximating the derivative using central differences. By using this method, the maximum number of function evaluations would effectively be ($2*n$ *(for gradient)*$+1$ *(function evaluation)*)$*160$) for local search, for an n-dimensional problem. Our approach is illustrated in Algorithm 1; note that hybrid local-search approach is incorporated at Steps (10)-(28), for additional details reader may refer to [60].

Algorithm 1 Hybrid $IACO_{\mathbb{R}}$ Algorithm

procedure HYBRID $IACO_{\mathbb{R}}$(Probability (p), constant parameter (ζ), initial archive size (α), growth (γ), maximum archive size (α_{max}), function tolerance (τ), maximum failure ($Fail_{max}$), maximum stagnation iterations ($Siter_{max}$), dimensions (N), termination criteria (Tc), probability of switching to NLopt algorithm ($P(nlopt)$), maximum local search iterations ($thresh_{localsearch}$))
    Initialize α solutions
    Evaluate initial solutions
    while (Tc not satisfied) **do**
        if ($Fail_{(i,best)} < Fail_{max}$) **then**
            Local search from Sol_{best}
        else if ($Fail_{(i,random)} < Fail_{max}$) **then**
            Local search from Sol_{random}
        end if
        if $P(nlopt) == 0$ **then** ▷ Local Search
            Use Mtsls1
        else if $P(nlopt) == 1$ **then**
            Use NLopt algo
        else
            if $ctr_{localsearch} < thresh_{localsearch}$ **then**
                if $rand() < P(nlopt)$ **then**
                    Use NLopt algo
                else
                    Use Mtsls1
                end if
            else
                if Last iteration used Mtsls1 **then**
                    Use NLopt algo
                else
                    Use Mtsls1
                end if
            end if
        end if
        if No improvement in solution **then**
            Increment $Fail_i$
        end if

▷ Continued...

if $rand() < p$ **then** ▷ Generate new solution
    Sample best Gaussian for each dimension
    if New soln is better **then** ▷ Exploitation
        Substitute new solution for Sol_{best}
    end if
else
    for all $j \in [1 : \alpha]$ **do** ▷ Exploration
        Sample Gaussian along each dimension j
        if new solution is better **then**
            Substitute new solution for Sol_j
        end if
    end for
end if
if $Iter_{curr}$ is a multiple of γ and $\alpha < \alpha_{max}$ **then** ▷ Archive Growth
    Initialize new solutions using $S_{new} = S_{new} + rand() \cdot (S_{best} - S_{new})$
    Add new solution to archive
    Increment α
end if
if $ctr_{globalsearch}{==}Siter_{max}$ **then** ▷ Restart
    Re-initialize solution set without S_{best}
end if
end while
end procedure

3. The NLopt Library

The NLopt library optimization algorithms are partitioned into four categories as shown in Figure 1; algorithms in each category are listed in Table 1. For the sake of brevity, each algorithm has been assigned a numeric identifier in Table 1 (Col. *"ID"*) which is used to refer to them in the subsequent sections of this paper.

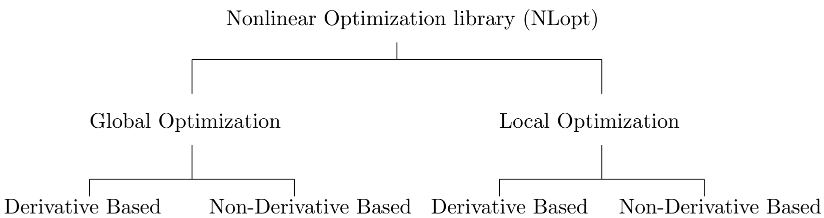


Figure 1: Categories of algorithms in the NLopt library

The global search algorithms can be categorized into derivative and non-derivative based algorithms. All global optimization algorithms require bound constraints to be specified on the optimization parameters [60]. The DIviding RECTangles (DIRECT) algorithm proposed by Jones et al. [25, 16] is based on dividing the search space into hyperrectangles and searching simultaneously at the global and local level. We consider the DIRECT (A0), unscaled DIRECT (A3) and original DIRECT (A6) versions of the algorithm. A locally biased variant of the DIRECT approach proposed by Gablonsky et al. [17] is the DIRECT-L, which is expected to perform well on functions with a single global minima and few local minima. We consider the

Table 1: NLopt algorithms

Summary of Nlopt Algorithms			
S. No.	ID	Algorithm	Code
Global Search Algorithms (Non Derivative Based)			
1	A0	DIRECT	NLOPT_GN_DIRECT
2	A1	DIRECT-L	NLOPT_GN_DIRECT_L
3	A2	Randomized DIRECT-L	NLOPT_GN_DIRECT_L_RAND
4	A3	Unscaled DIRECT	NLOPT_GN_DIRECT_NOSCAL
5	A4	Unscaled DIRECT-L	NLOPT_GN_DIRECT_L_NOSCAL
6	A5	Unscaled Randomized DIRECT-L	NLOPT_GN_DIRECT_L_RAND_NOSCAL
7	A6	Original DIRECT version	NLOPT_GN_ORIG_DIRECT
8	A7	Original DIRECT-L version	NLOPT_GN_ORIG_DIRECT_L
9	A19	Controlled random search (CRS2) with local mutation	NLOPT_GN_CRS2_LM
10	A20	Multi-level single-linkage (MLSL), random	NLOPT_GN_MLSL
11	A22	Multi-level single-linkage (MLSL), quasi-random	NLOPT_GN_MLSL_LDS
12	A35	ISRES evolutionary constrained optimization	NLOPT_GN_ISRES
13	A36	Augmented Lagrangian method	NLOPT_AUGLAG
14	A37	Augmented Lagrangian method for equality constraints	NLOPT_AUGLAG_EQ
15	A38	Multi-level single-linkage (MLSL), random	NLOPT_G_MLSL
16	A39	Multi-level single-linkage (MLSL), quasi-random	NLOPT_G_MLSL_LDS
17	A42	ESCH evolutionary strategy	NLOPT_GN_ESCH
Global Search Algorithms (Derivative Based)			
18	A8	Stochastic Global Optimization (StoGO)	NLOPT_GD_STOGO
19	A9	Stochastic Global Optimization (StoGO), random	NLOPT_GD_STOGO_RAND
20	A21	Multi-level single-linkage (MLSL), random	NLOPT_GD_MLSL
21	A23	Multi-level single-linkage (MLSL) quasi-random	NLOPT_GD_MLSL_LDS
Local Search Algorithms (Non Derivative Based)			
22	A12	Principal-axis, PRAXIS	NLOPT_LN_PRAXIS
23	A25	COBYLA	NLOPT_LN_COBYLA
24	A26	NEWUOA unconstrained optimization via quadratic models	NLOPT_LN_NEWUOA
25	*A27**	*Bound-constrained optimization via NEWUOA-based quadratic models*	*NLOPT_LN_NEWUOA_BOUND*
26	A28	Nelder-Mead simplex algorithm	NLOPT_LN_NELDERMEAD
27	A29	Sbplx variant of Nelder-Mead	NLOPT_LN_SBPLX
28	A30	Augmented Lagrangian method	NLOPT_LN_AUGLAG
29	A32	Augmented Lagrangian method for equality constraints	NLOPT_LN_AUGLAG_EQ
30	A34	BOBYQA bound-constrained optimization via quadratic models	NLOPT_LN_BOBYQA
Local Search Algorithms (Derivative Based)			
31	*A10***	*Original L-BFGS by Nocedal et al.*	*NLOPT_LD_LBFGS_NOCEDAL*
32	A11	Limited-memory BFGS	NLOPT_LD_LBFGS
33	A13	Limited-memory variable-metric, rank 1	NLOPT_LD_VAR1
34	A14	Limited-memory variable-metric, rank 2	NLOPT_LD_VAR2
35	A15	Truncated Newton	NLOPT_LD_TNEWTON
36	A16	Truncated Newton with restarting	NLOPT_LD_TNEWTON_RESTART
37	A17	Preconditioned truncated Newton	NLOPT_LD_TNEWTON_PRECOND
38	A18	Preconditioned truncated Newton with restarting	NLOPT_LD_TNEWTON_PRECOND_RESTART
39	A24	Method of Moving Asymptotes (MMA)	NLOPT_LD_MMA
40	A31	Augmented Lagrangian method	NLOPT_LD_AUGLAG
41	A33	Augmented Lagrangian method for equality constraints	NLOPT_LD_AUGLAG_EQ
42	A40	Sequential Quadratic Programming (SQP)	NLOPT_LD_SLSQP
43	A41	CCSA with simple quadratic approximations	NLOPT_LD_CCSAQ
** - This algorithm has not been considered in this study as runtime errors were encountered during execution.* *** - The original algorithm is not part of NLopt library, after minor modification it has been made part of A11.*			

DIRECT-L (A1), randomized DIRECT-L (A2), unscaled DIRECT-L (A4), unscaled randomized DIRECT-L (A5) and the original DIRECT-L (A7) versions of this algorithm in our study.

The Controlled Random Search (CRS) algorithm with local mutation by Kaelo et al. [26] is similar to the idea behind genetic algorithms, where an initial population evolves across generations to converge to the minima. This algorithm uses an evolution strategy similar the Nelder Mead algorithm [30]. The version of the CRS algorithm provided in the NLopt library supports bound constraints and starts with an initial population size of $10 * (n + 1)$ for an n dimensional problem. We use the CRS2 with local mutation (A19) for our study.

The Multi-Level Single Linkage (MLSL) algorithm by Kan and Timmer [27] is based on selecting multiple start points initially at random, and then using clustering heuristics to traverse the search space effeciently without redundancy. The algorithm configuration in the NLopt library allows for sampling four random points by default, with default function and variable tolerances set to 10^{-15} and 10^{-7} respectively. We have used the non-derivative-based random MLSL (A20) and quasi-random MLSL (A22) for global search, their derivative-independent versions (A38 and A39), and derivative-based versions (A21 and A23) respectively.

The Improved Stochastic Ranking Evolution Strategy (ISRES) algorithm by Runarsson and Yao [52] supports optimization with both linear and nonlinear constraints (A35). The algorithm uses a mutation rule with log-normal step size and an update rule similar to the Nelder Mead method. The default configuration for the initial population size in the NLopt library is $20*(n+1)$ for an n-dimensional function. Another evolutionary algorithm available in the library and used in our study is the ESCH algorithm by Santos et al. [54] which supports only linear bound constraints (A42). The STOchastic Global Optimization algorithm (StoGO) by Madsen et al. [37, 73, 21] is a derivative-based global search algorithm which supports only bound constraints. We consider the original StoGO (A8) and its randomized variant (A9).

In the category of non-gradient based local search methods, the Constrained Optimization BY Linear Approximation (COBYLA) by Powell [44] supports non-linear equality and inequality constraints (A25). It is based on linearly approximating the objective function using a simplex of $(n+1)$ points for an n-dimensional problem. Another version of this algorithm which supports bound constraints is Bound Optimization BY Quadratic Approximation, referred to as BOBYQA [48] (A34). Its enhanced version is NEWUOA [45, 47, 46] (A26), with support for constrained and unconstrained problems. The PRincipal AXIS algorithm (PRAXIS) by Brent [3] primarily supports unconstrained optimization (A12). Other algorithms available in this category in the NLopt library that have been used in this study include the well known Nelder-Mead Simplex method [41] (A28), and its subplex variant by Rowan [51] (A29).

We now present a brief summary of the derivative based algorithms for local search. The Broyden-Fletcher-Goldfarb-Shanno (BFGS) [68] algorithm (A11) has been a classical optimization algorithm belonging to the class of approximate Newton methods. Along with several variants [66, 63, 35, 32], BFGS has been widely used in the domain of unconstrained optimization. The Method of Moving Asymptotes (MMA) algorithm (A24) by Svanberg [57] is based on locally approximating the gradient of the objective function and a quadratic penalty term. It is an enhanced version of the original Conservative Complex Separable Approximation (CCSA) algorithm [57] (A41), which provides for pre-conditioning of the Hessian in the version available as part of the NLopt library. The Sequential Quadratic Programming (SQP) by Kraft [28, 29] (A40) is available for both linear and non-linear equality and inequality constraints. The preconditioned truncated Newton method [11] allows for using gradient information from previous iterations, which provides for faster convergence with the trade-off for greater memory requirement. We have used the truncated Newton method (A15), with restart (A16), pre-conditioned Newton method (A17) and pre-conditioned with restart (A18) for our study.

A limited memory variable metric algorithm by Vlček and Lukšan [62] is available with rank-1 (A13) and rank-2 (A14) methods in the NLopt library. Also, an Augmented Lagrangian algorithm by Conn et al. [7, 2] is available for all categories including gradient/non-gradient and global/local search. This algorithm combines the objective and associated constraints into a single function with a penalty term. This is solved separately as another problem without non-linear constraints, to finally converge to the desired solution. Variants of this algorithm to consider penalty function for only equality constraints is also available in the library. We have used the Augmented Lagrangian method (A30), a version with equality constraint support (A32), and the corresponding derivative based versions A31 and A33 respectively. We now present the results of our comparative analysis using these algorithms alongwith Mtsls1 for local search in the following section.

4. Study and Discussion

We evaluate the performance of our approach by comparing its performance with the algorithms featured in SOCO and CEC 2014 benchmarks. The results are categorized into three subsections, presenting the results on the SOCO, CEC 2014 benchmarks, and a comparison of the standalone performance of the NLopt algorithms with our hybrid approach. For our study, we use $P(nlopt)$ as 0.6, and parameters for the NLopt algorithms as $x_{tol_{rel}} = x_{tol_{abs}} = 1e-7$, $f_{tol_{rel}} = f_{tol_{abs}} = 1e-15$ (for description of these parameters, the reader may refer to [24]). Parameters specific to algorithms have been used with their default configuration, while ranking parameters for performance have been elucidated in [6],

4.1. Results on SOCO benchmarks

We used the 50-dimensional versions of the 19 benchmark functions suite shown in Table 2. Functions $F1 - F6$ were originally proposed for the special session on large scale global optimization organized for the IEEE 2008 Congress on Evolutionary Computation (CEC 2008) [59], Functions $F7 - F11$ were proposed at the ISDA 2009 Conference. Functions $F12 - F19$ are hybrid functions that combine two functions belonging to $F1 - F11$. Some properties of the benchmark functions are listed in Table 2. The detailed description of these functions is available in [22, 36].

We applied the termination conditions used for SOCO, that is, the maximum number of function evaluations was $5000 \times D$, where D denotes the number of dimensions in which the function is considered. All the investigated algorithms were run 25 times on each function. We report error values defined as $f(x) - f(x^*)$, where x is a candidate solution and x^* is the optimal solution. Error values lower than 10^{-14} (this value is referred to as 0-threshold) are approximated to 0. Our analysis is based on either the whole solution quality distribution, or on the median and average errors. For the evaluation of our $IACO_{\mathbb{R}}$-Hybrid approach, we use the algorithm parameters as indicated in Table 2 of [34].

The average and median errors obtained on the benchmark functions have been shown in Tables 3 and 4 respectively. The algorithms from the NLopt library (used for hybridization within $IACO_{\mathbb{R}}$-Mtsls1 framework) have been indicated as the rows of the table using numeric identifiers provided in Table 1, while the SOCO benchmark functions are provided as the columns. These provide a comprehensive analysis of the performance of these algorithms on the functions; cases where the error is zero have been indicated in boldface. It may be noted here that only the Differential Evolution algorithm (DE) [56], the co-variance matrix adaptation evolution strategy with increasing population size (G-CMA-ES) [1] and the real-coded CHC algorithm (CHC) [15] have been considered as baseline algorithms for performance evaluation on SOCO benchmarks in [22, 36]. Further, the source code for implementation of IACO$_{\mathbb{R}}$ is available online at http://iridia.ulb.ac.be/supp/IridiaSupp2011-008/.

Table 2: SOCO Benchmark functions

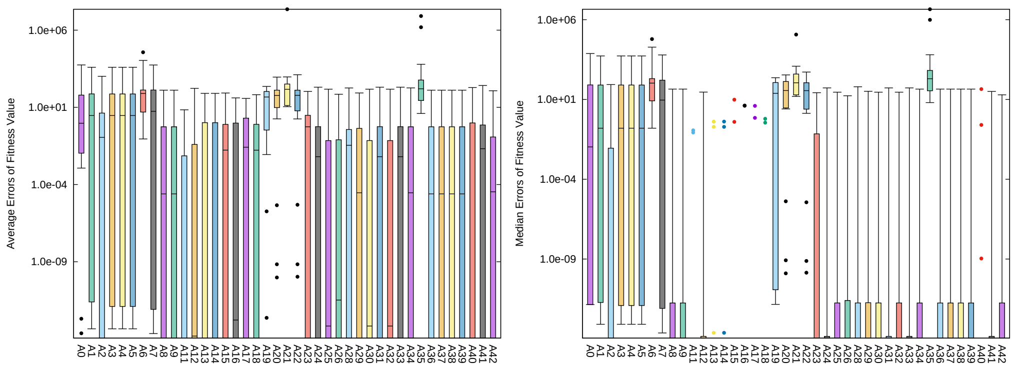
ID	Name	Analytical Form	Uni(U)/ Multi(M) Modal	Sep.	Rotated	Easily optimized dimension-wise
$F1$	Shift Sphere	$\sum_{i=1}^{D} z_i^2 + f_{bias}, z = x - o$	U	Y	N	Y
$F2$	Shifted Schwefel 2.21	$max_i\{\|z_i\|, 1 \leq i \leq D\} + f_{bias}, z = x - o$	U	N	N	N
$F3$	Shifted Rosenbrock	$\sum_{i=1}^{D-1}(100(z_i^2 + z_{i+1})^2 + (z_i - 1)^2)) + f_{bias}, z = x - o$	M	N	N	Y
$F4$	Shift. Rastrigin	$\sum_{i=1}^{D}(z_i^2 - 10cos(2\pi z_i) + 10) + f_{bias}, z = x - o$	M	Y	N	Y
$F5$	Shift. Griewank	$\sum_{i=1}^{D} \frac{z_i^2}{4000} - \prod_{i=1}^{D} cos(\frac{z_i}{\sqrt{i}}) + 1 + f_{bias}, z = x - o$	M	N	N	N
$F6$	Shift. Ackley	$-20e^{-0.2\sqrt{\frac{1}{D}\sum_{i=1}^{D} z_i^2}} - e^{\frac{1}{D}\sum_{i=1}^{D} cos(2\pi z_i)} +$ $20 + e + f_{bias}, z = x - o$	M	Y	N	Y
$F7$	Shift. Schwefel 2.22	$\sum_{i=1}^{D} \|z_i\| + \prod_{i=1}^{D} \|z_i\|, z = x - o$	U	Y	N	Y
$F8$	Shift Schwefel 1.2	$\sum_{i=1}^{D}(\sum_{j=1}^{i} z_j)^2, z = x - o$	U	N	N	N
$F9$	Shift. Extended F_{10}	$\sum_{i=1}^{D-1} f_{10}(z_i, z_{i+1}) + f_{10}(z_D, z_1), z = x - o$ where $f_{10} = (x^2 + y^2)^{0.25}(sin^2(50(x^2 + y^2)^{0.1}) + 1)$	U	N	N	Y
$F10$	Shift. Bohachevsky	$\sum_{i=1}^{D} z_i^2 + 2z_{i+1}^2 - 0.3cos(3\pi z_i) - 0.4cos(4\pi z_{i+1}) +$ $0.7, z = x - o$	U	N	N	N
$F11$	Shift. Schafler	$\sum_{i=1}^{D-1}(z_i^2 + z_{i+1}^2)^{0.25}(sin^2(50(z_i^2 + z_{i+1}^2)^{0.1}) + 1), z = x - o$	U	N	N	Y
$F12$	Hybrid Function	$F9$ + 0.25 $F1$	M	N	N	N
$F13$	Hybrid Function	$F9$ + 0.25 $F3$	M	N	N	N
$F14$	Hybrid Function	$F9$ + 0.25 $F4$	M	N	N	N
$F15$	Hybrid Function	$F10$ + 0.25 $F7$	M	N	N	N
$F16$	Hybrid Function	$F9$ + 0.5 $F1$	M	N	N	N
$F17$	Hybrid Function	$F9$ + 0.75 $F3$	M	N	N	N
$F18$	Hybrid Function	$F9$ + 0.75 $F4$	M	N	N	N
$F19$	Hybrid Function	$F10$ + 0.75 $F7$	M	N	N	N

Table 3: Average error for NLopt algorithms for 50D

ALGOs	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11	F12	F13	F14	F15	F16	F17	F18	F19
A0	2.04E-13	6.23E+01	2.15E+02	2.64E+01	2.33E-14	1.43E+00	3.00E-02	1.95E+01	6.69E-01	1.51E-01	5.69E-02	1.17E-03	3.69E+02	7.07E+01	9.34E-01	1.08E-02	5.58E+03	3.91E+01	**0.00E+00**
A1	5.94E-14	2.51E-12	1.61E+02	7.44E+01	5.62E-03	8.90E-02	2.09E-13	1.76E+03	7.76E+00	1.89E-01	4.47E+01	6.05E-14	1.47E+02	5.23E+01	1.49E+00	4.54E-14	3.96E+03	5.92E+00	2.96E+00
A2	**0.00E+00**	5.11E+01	1.38E+02	4.26E+00	5.42E-03	1.41E-01	**0.00E+00**	1.80E-01	1.13E-01	**0.00E+00**	5.64E-02	**0.00E+00**	7.32E+00	2.26E+00	**0.00E+00**	**0.00E+00**	1.02E+03	4.93E-01	**0.00E+00**
A3	5.94E-14	1.28E-12	1.61E+02	7.39E+01	5.62E-03	8.90E-02	2.20E-13	1.76E+03	7.76E+00	1.89E-01	4.47E+01	6.05E-14	1.47E+02	5.43E+01	1.49E+00	4.54E-14	3.96E+03	6.15E+00	2.96E+00
A4	5.94E-14	1.28E-12	1.61E+02	7.39E+01	5.62E-03	8.90E-02	2.20E-13	1.76E+03	7.76E+00	1.89E-01	4.47E+01	6.05E-14	1.47E+02	5.43E+01	1.49E+00	4.54E-14	3.96E+03	6.15E+00	2.96E+00
A5	5.94E-14	1.28E-12	1.61E+02	7.39E+01	5.62E-03	8.90E-02	2.20E-13	1.76E+03	7.76E+00	1.89E-01	4.47E+01	6.05E-14	1.47E+02	5.43E+01	1.49E+00	4.54E-14	3.96E+03	6.15E+00	2.96E+00
A6	4.96E+00	3.69E+01	1.10E+04	1.14E+02	6.81E-01	8.90E-02	2.42E+00	5.51E+03	1.26E+02	4.69E-01	1.32E+02	1.30E+02	3.68E+04	7.88E+01	7.07E+00	8.53E+01	4.65E+03	3.91E+01	5.61E+00
A7	**0.00E+00**	8.27E-13	2.11E+02	1.13E+02	5.62E-03	8.90E-02	2.23E-14	5.51E+03	1.26E+02	2.98E-01	1.32E+02	**0.00E+00**	1.94E+02	7.87E+01	3.17E+00	**0.00E+00**	4.58E+03	3.91E+01	5.53E+00
A8	**0.00E+00**	3.57E-12	6.34E+01	7.56E-01	**0.00E+00**	6.89E-14	**0.00E+00**	2.48E-05	**0.00E+00**	**0.00E+00**	1.05E-02	4.25E-02	3.05E+00	5.50E-01	**0.00E+00**	5.36E-03	1.30E+02	4.64E-02	**0.00E+00**
A9	**0.00E+00**	3.57E-12	6.34E+01	7.56E-01	**0.00E+00**	6.89E-14	**0.00E+00**	2.48E-05	**0.00E+00**	**0.00E+00**	1.05E-02	4.25E-02	3.05E+00	5.50E-01	**0.00E+00**	5.36E-03	1.30E+02	4.64E-02	**0.00E+00**
A11	**0.00E+00**	**0.00E+00**	1.13E-11	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	7.27E-03	1.86E+00	4.37E-01	**0.00E+00**	**0.00E+00**	6.89E+00	1.63E-01	**0.00E+00**
A12	**0.00E+00**	1.53E-14	8.20E+01	3.98E-02	**0.00E+00**	1.14E-14	**0.00E+00**	3.07E-09	**0.00E+00**	**0.00E+00**	3.75E-02	**0.00E+00**	3.38E+00	3.90E-01	**0.00E+00**	6.64E-03	1.70E+02	2.62E-03	**0.00E+00**
A13	**0.00E+00**	**0.00E+00**	8.11E-14	1.75E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	7.82E-02	5.15E-03	1.49E+00	1.12E+00	**0.00E+00**	4.29E-04	8.01E+01	1.02E+00	**0.00E+00**
A14	**0.00E+00**	**0.00E+00**	8.11E-14	1.75E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	7.82E-02	5.15E-03	1.49E+00	1.12E+00	**0.00E+00**	4.29E-04	8.01E+01	1.02E+00	**0.00E+00**
A15	**0.00E+00**	**0.00E+00**	3.19E-01	5.97E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.33E-01	**0.00E+00**	7.84E-01	1.71E-02	6.37E+00	2.85E+00	**0.00E+00**	1.51E-01	8.49E+01	2.57E+00	**0.00E+00**
A16	**0.00E+00**	**0.00E+00**	9.57E-01	1.68E-13	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.88E-02	**0.00E+00**	6.32E-04	6.00E-02	6.81E+00	1.69E+00	**0.00E+00**	1.84E-01	4.10E+01	1.30E+00	**0.00E+00**
A17	**0.00E+00**	**0.00E+00**	4.78E-01	7.96E-02	3.95E-04	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.97E+00	**0.00E+00**	1.96E+00	2.60E-02	8.10E+00	1.75E+00	**0.00E+00**	2.00E+00	3.78E+01	3.10E+00	**0.00E+00**
A18	**0.00E+00**	**0.00E+00**	7.97E-01	1.59E+00	6.57E-13	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.35E-01	**0.00E+00**	6.95E-01	6.98E-02	2.47E+00	1.42E+00	**0.00E+00**	1.68E-02	6.57E+01	3.52E-01	**0.00E+00**
A19	2.34E-13	1.72E+01	1.53E+02	5.66E+01	8.77E-03	1.37E-01	1.83E-06	2.95E+01	1.03E+02	2.24E+00	1.06E+02	4.71E+01	1.06E+02	6.97E+01	3.37E-01	1.35E+02	2.25E+02	4.67E+01	2.21E+00
A20	9.60E-11	2.56E+01	1.73E+02	9.99E+01	6.87E-10	1.80E+00	9.59E+00	4.58E-06	2.42E+02	1.09E+01	2.55E+02	5.88E+01	8.80E+01	1.06E+02	1.39E+01	1.30E+02	9.07E+02	7.73E+01	1.39E+01
A21	**0.00E+00**	3.75E+01	2.07E+02	3.98E+02	**0.00E+00**	1.10E+01	2.25E+07	3.06E+01	2.76E+02	1.18E+01	2.80E+02	7.64E+01	1.03E+02	3.22E+02	9.38E+02	1.66E+02	4.23E+02	1.48E+02	1.31E+01
A22	1.07E-10	2.61E+01	1.52E+02	1.04E+02	6.82E-10	1.78E+00	5.94E+00	4.90E-06	2.47E+02	1.08E+01	2.50E+02	5.97E+01	8.86E+01	8.88E+01	1.47E+01	1.30E+02	1.30E+03	7.11E+01	1.39E+01
A23	**0.00E+00**	3.70E-03	1.07E+02	3.03E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	5.52E-01	2.37E+00	**0.00E+00**	1.93E+00	4.39E-01	1.90E+01	3.32E+00	2.62E-13	1.58E+00	2.06E+01	2.69E+00	**0.00E+00**
A24	**0.00E+00**	6.37E-03	1.81E+02	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	5.60E-01	**0.00E+00**	**0.00E+00**	5.92E-02	1.54E-01	9.38E+00	1.92E+00	**0.00E+00**	1.08E-01	2.03E+02	2.39E-01	**0.00E+00**
A25	**0.00E+00**	3.68E-12	7.69E+01	4.78E-01	**0.00E+00**	6.91E-14	**0.00E+00**	3.34E-05	**0.00E+00**	**0.00E+00**	5.87E-02	7.04E-02	2.10E+00	9.85E-03	**0.00E+00**	**0.00E+00**	1.50E+02	**0.00E+00**	**0.00E+00**
A26	**0.00E+00**	3.31E-12	7.07E+01	7.96E-02	**0.00E+00**	6.82E-14	**0.00E+00**	3.37E-05	3.47E-02	**0.00E+00**	1.05E-02	8.72E-03	2.74E+00	2.88E-01	**0.00E+00**	**0.00E+00**	2.31E+01	**0.00E+00**	**0.00E+00**
A28	**0.00E+00**	3.40E-12	1.35E+02	6.77E-01	**0.00E+00**	7.25E-14	**0.00E+00**	5.09E-05	1.18E-01	**0.00E+00**	3.47E-02	9.32E-02	2.71E+00	3.66E-01	**0.00E+00**	1.82E-01	1.80E+02	1.59E-01	**0.00E+00**
A29	**0.00E+00**	3.54E-12	4.35E+01	6.37E-01	**0.00E+00**	6.79E-14	**0.00E+00**	2.93E-05	2.08E-03	**0.00E+00**	4.64E-02	7.51E-02	3.25E+00	4.33E-01	**0.00E+00**	**0.00E+00**	8.55E+01	3.18E-01	**0.00E+00**
A30	**0.00E+00**	3.68E-12	7.69E+01	4.78E-01	**0.00E+00**	6.91E-14	**0.00E+00**	3.35E-05	**0.00E+00**	**0.00E+00**	5.87E-02	7.04E-02	2.10E+00	9.85E-03	**0.00E+00**	**0.00E+00**	1.50E+02	**0.00E+00**	**0.00E+00**
A31	**0.00E+00**	6.37E-03	1.81E+02	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	5.60E-01	**0.00E+00**	**0.00E+00**	5.92E-02	1.54E-01	9.38E+00	1.92E+00	**0.00E+00**	1.08E-01	2.03E+02	2.39E-01	**0.00E+00**
A32	**0.00E+00**	3.68E-12	7.69E+01	4.78E-01	**0.00E+00**	6.91E-14	**0.00E+00**	3.35E-05	**0.00E+00**	**0.00E+00**	5.87E-02	7.04E-02	2.10E+00	9.85E-03	**0.00E+00**	**0.00E+00**	1.50E+02	**0.00E+00**	**0.00E+00**
A33	**0.00E+00**	6.37E-03	1.81E+02	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	5.60E-01	**0.00E+00**	**0.00E+00**	5.92E-02	1.54E-01	9.38E+00	1.92E+00	**0.00E+00**	1.08E-01	2.03E+02	2.39E-01	**0.00E+00**
A34	**0.00E+00**	3.57E-12	1.01E+02	7.56E-01	**0.00E+00**	6.89E-14	**0.00E+00**	2.94E-05	**0.00E+00**	**0.00E+00**	1.05E-02	2.60E-03	3.06E+00	5.50E-01	**0.00E+00**	5.36E-03	1.81E+02	4.64E-02	**0.00E+00**
A35	6.74E+02	1.17E+01	8.15E+06	2.44E+02	7.24E+00	4.01E+00	1.61E+01	6.14E+03	1.59E+02	7.39E+01	1.60E+02	2.92E+02	1.54E+06	1.67E+02	2.77E+01	2.10E+02	5.89E+02	7.42E+01	3.97E+01
A36	**0.00E+00**	3.57E-12	6.34E+01	7.56E-01	**0.00E+00**	6.89E-14	**0.00E+00**	2.48E-05	**0.00E+00**	**0.00E+00**	1.05E-02	4.25E-02	3.05E+00	5.50E-01	**0.00E+00**	5.36E-03	1.30E+02	4.64E-02	**0.00E+00**
A37	**0.00E+00**	3.57E-12	6.34E+01	7.56E-01	**0.00E+00**	6.89E-14	**0.00E+00**	2.48E-05	**0.00E+00**	**0.00E+00**	1.05E-02	4.25E-02	3.05E+00	5.50E-01	**0.00E+00**	5.36E-03	1.30E+02	4.64E-02	**0.00E+00**
A38	**0.00E+00**	3.57E-12	6.34E+01	7.56E-01	**0.00E+00**	6.89E-14	**0.00E+00**	2.48E-05	**0.00E+00**	**0.00E+00**	1.05E-02	4.25E-02	3.05E+00	5.50E-01	**0.00E+00**	5.36E-03	1.30E+02	4.64E-02	**0.00E+00**
A39	**0.00E+00**	3.57E-12	6.34E+01	7.56E-01	**0.00E+00**	6.89E-14	**0.00E+00**	2.48E-05	**0.00E+00**	**0.00E+00**	1.05E-02	4.25E-02	3.05E+00	5.50E-01	**0.00E+00**	5.36E-03	1.30E+02	4.64E-02	**0.00E+00**
A40	**0.00E+00**	**0.00E+00**	4.33E+00	9.66E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	1.18E-01	**0.00E+00**	9.78E-01	**0.00E+00**	2.35E+00	1.24E+00	**0.00E+00**	**0.00E+00**	1.90E+02	**0.00E+00**	**0.00E+00**
A41	**0.00E+00**	9.23E-03	4.88E+01	3.98E-02	**0.00E+00**	**0.00E+00**	**0.00E+00**	7.20E-01	5.63E+00	**0.00E+00**	2.10E-02	**0.00E+00**	1.71E+01	1.54E-01	**0.00E+00**	5.57E-01	2.60E+02	8.07E-02	**0.00E+00**
A42	**0.00E+00**	4.48E-12	6.94E+01	1.59E-01	**0.00E+00**	6.92E-14	**0.00E+00**	3.40E-05	**0.00E+00**	**0.00E+00**	5.69E-02	1.29E-02	2.49E+00	4.80E-02	**0.00E+00**	5.36E-03	1.19E+02	1.21E-01	**0.00E+00**

Table 4: Median error for NLopt algorithms for 50D

ALGOs	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11	F12	F13	F14	F15	F16	F17	F18	F19
A0	**0.00E+00**	9.66E+01	8.19E+01	1.49E+01	**0.00E+00**	2.56E+00	3.61E-11	9.73E+00	9.34E-05	**0.00E+00**	1.49E-08	9.22E-04	5.32E+02	8.87E+01	1.41E-12	1.07E-02	7.61E+03	4.38E+01	**0.00E+00**
A1	**0.00E+00**	2.40E-12	1.11E+02	8.06E+01	7.40E-03	1.59E-01	3.74E-13	5.90E+02	1.26E+00	1.90E-12	2.67E+01	1.08E-13	1.53E+02	5.55E+01	2.18E-09	8.11E-14	5.35E+03	3.98E+00	1.16E-07
A2	**0.00E+00**	7.66E+01	3.10E+01	**0.00E+00**	**0.00E+00**	7.51E-14	**0.00E+00**	8.91E-03	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.77E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	8.59E+01	**0.00E+00**	**0.00E+00**
A3	**0.00E+00**	1.22E-12	1.11E+02	8.06E+01	7.40E-03	1.59E-01	3.92E-13	5.90E+02	1.26E+00	1.90E-12	2.67E+01	1.08E-13	1.53E+02	5.55E+01	2.18E-09	8.11E-14	5.35E+03	3.98E+00	1.16E-07
A4	**0.00E+00**	1.22E-12	1.11E+02	8.06E+01	7.40E-03	1.59E-01	3.92E-13	5.90E+02	1.26E+00	1.90E-12	2.67E+01	1.08E-13	1.53E+02	5.55E+01	2.18E-09	8.11E-14	5.35E+03	3.98E+00	1.16E-07
A5	**0.00E+00**	1.22E-12	1.11E+02	8.06E+01	7.40E-03	1.59E-01	3.92E-13	5.90E+02	1.26E+00	1.90E-12	2.67E+01	1.08E-13	1.53E+02	5.55E+01	2.18E-09	8.11E-14	5.35E+03	3.98E+00	1.16E-07
A6	8.26E+00	5.68E+01	1.91E+04	1.26E+02	1.06E+00	1.59E-01	4.04E+00	5.99E+03	1.62E+02	7.54E-01	1.63E+02	2.02E+02	6.13E+04	1.06E+02	1.04E+01	1.20E+02	6.31E+03	4.38E+01	9.32E+00
A7	**0.00E+00**	8.10E-13	2.55E+02	1.25E+02	7.40E-03	1.59E-01	2.32E-14	5.99E+03	1.62E+02	4.70E-01	1.63E+02	**0.00E+00**	2.59E+02	1.06E+02	4.67E+00	**0.00E+00**	6.22E+03	4.38E+01	9.19E+00
A8	**0.00E+00**	1.75E-12	1.88E+01	**0.00E+00**	**0.00E+00**	7.51E-14	**0.00E+00**	2.16E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.06E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.48E+01	**0.00E+00**	**0.00E+00**
A9	**0.00E+00**	1.75E-12	1.88E+01	**0.00E+00**	**0.00E+00**	7.51E-14	**0.00E+00**	2.16E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.06E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.48E+01	**0.00E+00**	**0.00E+00**
A11	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	1.17E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	8.55E-02	**0.00E+00**	**0.00E+00**
A12	**0.00E+00**	1.42E-14	1.73E+01	**0.00E+00**	**0.00E+00**	1.11E-14	**0.00E+00**	1.98E-09	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.02E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.91E+01	**0.00E+00**	**0.00E+00**
A13	**0.00E+00**	**0.00E+00**	2.38E-14	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	1.92E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.12E-01	**0.00E+00**	**0.00E+00**
A14	**0.00E+00**	**0.00E+00**	2.38E-14	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	1.92E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.12E-01	**0.00E+00**	**0.00E+00**
A15	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.95E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	9.71E+00	**0.00E+00**	**0.00E+00**
A16	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.05E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.19E+00	**0.00E+00**	**0.00E+00**
A17	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.99E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	7.04E-01	**0.00E+00**	**0.00E+00**
A18	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.52E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	6.19E-01	**0.00E+00**	**0.00E+00**
A19	**0.00E+00**	2.35E+01	9.07E+01	1.89E+01	1.45E-12	1.44E-12	1.19E-11	2.42E+01	1.15E+02	**0.00E+00**	1.19E+02	3.73E+01	1.20E+02	5.47E+01	2.86E-08	2.02E+02	2.32E+02	4.98E+01	5.50E-11
A20	1.27E-10	3.64E+01	4.08E+00	1.08E+02	8.32E-10	2.94E+00	2.55E+00	4.14E-06	3.28E+02	1.62E+01	3.47E+02	7.34E+01	9.12E+01	1.34E+02	1.76E+01	1.81E+02	3.26E+02	9.68E+01	2.11E+01
A21	**0.00E+00**	4.08E+01	6.79E+01	5.45E+02	**0.00E+00**	1.95E+01	1.17E+05	2.49E+01	3.94E+02	1.83E+01	3.86E+02	1.04E+02	1.12E+02	4.51E+02	1.22E+03	2.37E+02	3.89E+02	1.74E+02	1.56E+01
A22	1.39E-10	3.60E+01	9.27E+00	1.13E+02	7.58E-10	2.48E+00	1.35E+00	3.58E-06	3.28E+02	1.62E+01	3.47E+02	7.61E+01	9.44E+01	9.46E+01	1.82E+01	1.81E+02	5.25E+02	8.91E+01	1.90E+01
A23	**0.00E+00**	3.81E-12	2.67E+01	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	1.72E-01	7.05E-02	**0.00E+00**	1.07E-02	**0.00E+00**	1.83E+01	7.37E-08	**0.00E+00**	**0.00E+00**	2.26E+00	**0.00E+00**	**0.00E+00**
A24	**0.00E+00**	1.42E-14	5.35E+01	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	1.06E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	8.55E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	5.12E+00	**0.00E+00**	**0.00E+00**
A25	**0.00E+00**	1.78E-12	1.88E+01	**0.00E+00**	**0.00E+00**	6.79E-14	**0.00E+00**	2.73E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.41E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.87E+01	**0.00E+00**	**0.00E+00**
A26	**0.00E+00**	2.52E-12	1.72E+01	**0.00E+00**	**0.00E+00**	6.79E-14	**0.00E+00**	2.60E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.36E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	1.91E+00	**0.00E+00**	**0.00E+00**
A28	**0.00E+00**	1.78E-12	6.27E+01	**0.00E+00**	**0.00E+00**	7.51E-14	**0.00E+00**	3.75E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	7.51E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.94E+01	**0.00E+00**	**0.00E+00**
A29	**0.00E+00**	1.82E-12	1.72E+01	**0.00E+00**	**0.00E+00**	6.79E-14	**0.00E+00**	2.14E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	1.06E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.26E+01	**0.00E+00**	**0.00E+00**
A30	**0.00E+00**	1.78E-12	1.88E+01	**0.00E+00**	**0.00E+00**	6.79E-14	**0.00E+00**	2.74E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.41E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.87E+01	**0.00E+00**	**0.00E+00**
A31	**0.00E+00**	1.42E-14	5.35E+01	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	1.06E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	8.55E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	5.12E+00	**0.00E+00**	**0.00E+00**
A32	**0.00E+00**	1.78E-12	1.88E+01	**0.00E+00**	**0.00E+00**	6.79E-14	**0.00E+00**	2.74E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.41E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.87E+01	**0.00E+00**	**0.00E+00**
A33	**0.00E+00**	1.42E-14	5.35E+01	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	1.06E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	8.55E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	5.12E+00	**0.00E+00**	**0.00E+00**
A34	**0.00E+00**	1.75E-12	3.07E+01	**0.00E+00**	**0.00E+00**	7.51E-14	**0.00E+00**	1.99E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.06E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.48E+01	**0.00E+00**	**0.00E+00**
A35	8.97E+02	1.47E+01	4.53E+06	3.12E+02	8.89E+00	6.35E+00	2.44E+01	6.37E+03	2.00E+02	9.82E+01	1.93E+02	3.88E+02	9.83E+05	2.45E+02	3.54E+01	2.89E+02	6.61E+02	9.09E+01	5.76E+01
A36	**0.00E+00**	1.75E-12	1.88E+01	**0.00E+00**	**0.00E+00**	7.51E-14	**0.00E+00**	2.16E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.06E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.48E+01	**0.00E+00**	**0.00E+00**
A37	**0.00E+00**	1.75E-12	1.88E+01	**0.00E+00**	**0.00E+00**	7.51E-14	**0.00E+00**	2.16E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.06E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.48E+01	**0.00E+00**	**0.00E+00**
A38	**0.00E+00**	1.75E-12	1.88E+01	**0.00E+00**	**0.00E+00**	7.51E-14	**0.00E+00**	2.16E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.06E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.48E+01	**0.00E+00**	**0.00E+00**
A39	**0.00E+00**	1.75E-12	1.88E+01	**0.00E+00**	**0.00E+00**	7.51E-14	**0.00E+00**	2.16E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.06E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.48E+01	**0.00E+00**	**0.00E+00**
A40	**0.00E+00**	**0.00E+00**	1.08E-09	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.54E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	4.50E+01	**0.00E+00**	**0.00E+00**
A41	**0.00E+00**	1.42E-14	2.46E+01	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	2.09E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	8.52E+00	**0.00E+00**	**0.00E+00**	**0.00E+00**	3.21E+01	**0.00E+00**	**0.00E+00**
A42	**0.00E+00**	1.78E-12	1.88E+01	**0.00E+00**	**0.00E+00**	7.51E-14	**0.00E+00**	3.15E-05	**0.00E+00**	**0.00E+00**	**0.00E+00**	**0.00E+00**	5.60E-01	**0.00E+00**	**0.00E+00**	**0.00E+00**	1.97E+01	**0.00E+00**	**0.00E+00**



(a) Average error

(b) Median error

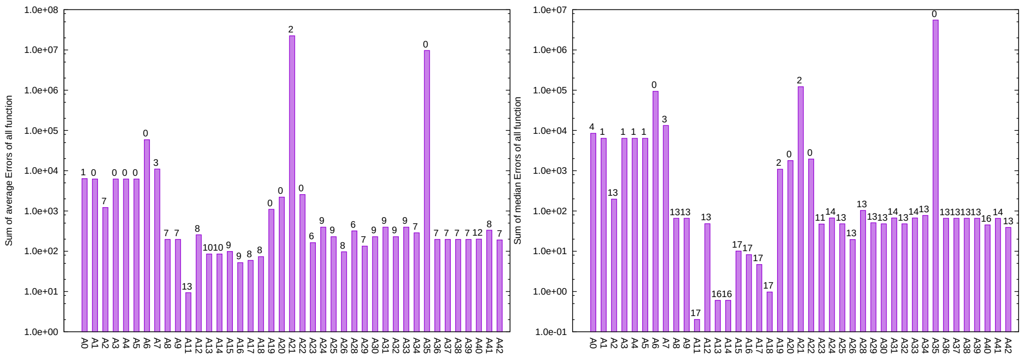
Figure 2: Box plots for average and median errors obtained on the SOCO benchmark functions.

Further, the box plots for the average and median errors are shown in Figures 2a and 2b respectively. The horizontal axis represents the algorithms from the NLopt library referred by the numeric identifiers as given in Table 1, and the vertical axis represents the mean and median errors in the two cases. It may be noted that all plots indicating mean/median errors use a logarithmic scale. One can infer from the plots that in terms of average error, the choice of algorithms A6, A19, A20, A21, A22 and A35 provides better performance, whereas in case of median error, the algorithms A11, A13, A14, A15, A16, A17, A18 and A40 demonstrate superior performance.

The sum of average and median errors are shown in Figures (3a) and (3b) respectively. The horizontal axis indicates the algorithm from the NLopt library, while the vertical axis shows the sum of the corresponding average and median errors. The number at the top of each bar in the graph indicates the number of times the global minima were obtained on the SOCO benchmark functions. In terms of average error, algorithms A11 and A40 are able to find the global minima for 13 and 12 functions out of the 19 benchmarks, respectively. In case of median error, algorithms A11, A15, A16, A17 and A18 are able to find global minima for 17 out of the 19 benchmark functions. None of the reference algorithms have achieved zero median error on as many functions.

Figure 4 aims to provide a ranking of the algorithms from the NLopt library on our approach. The ranking is done on the basis of the sum of average and median errors obtained on the benchmark functions, which is indicated on the vertical axis. The bars represent the value of sum of average and median errors for the algorithms on the horizontal axis. The rank of the algorithm is indicated above the bars. From our analysis, the top three algorithms for our approach are A11 (limited memory BFGS), A16 (Truncated Newton method with restart) and A17 (Preconditioned Truncated Newton), all of which belong to the class of gradient-based methods for local search. The worst performing algorithm from our analysis is A21 (MLSL-random) which belongs to the category of gradient-based global search algorithms of the NLopt library.

Finally, we also present a comparison of the performance of the best 3 NLopt algorithms (A11, A16 and A17) on the SOCO benchmark functions vis-à-vis reference algorithms available in literature. These include the Differential Evolution algorithm (DE) [56] and its variants, the co-variance matrix adaptation evolution strategy with increasing population size (G-CMA-ES) [1], the real-coded CHC algorithm (CHC) [15], Shuffle Or Update Parallel Differential Evolu-



(a) Sum of average errors.

(b) Sum of median errors.

Figure 3: Sum of average and median errors on SOCO benchmarks for the NLopt algorithms. The number above the bar represents the number of times optima were found.

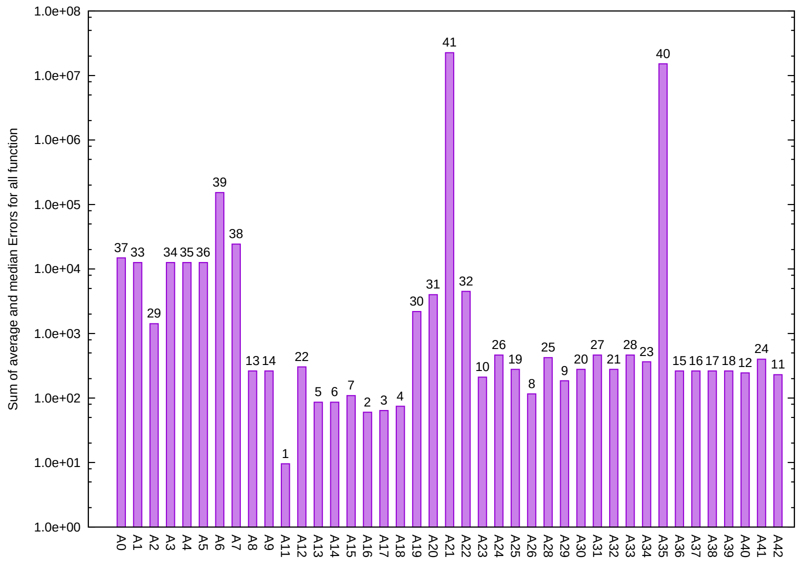
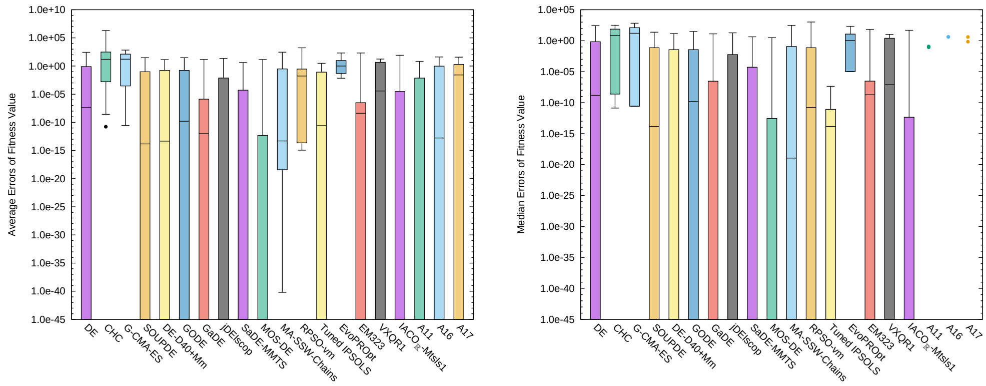


Figure 4: Ranking of all algorithms based upon the sum of average and median errors on SOCO benchmarks

tion (SOUPDE) [65], $DE - D^40 + M^m$ [18], Generalized Opposition-based Differential Evolution (GODE) [64], Generalized Adaptive Differential Evolution (GADE) [70], jDElscop [4], Self-adaptive Differential Evolution with Multi-Trajectory Search (SaDE-MMTS) [74], MOS-DE [31], MA-SSW-Chains [40], Restart Particle Swarm Optimization with Velocity Modulation (RPSO-VM) [19], Tuned IPSOLS [10] EVO-PROpt [12], EM323 [20] and VXQR [42] among others.

The box plots of average and median error when comparing with $IACO_{\mathbb{R}}$-Mtsls1 are shown in Figure 5. It can be inferred that the error obtained using NLopt algorithms are much lower than the reference algorithms. A ranking of the best 3 NLopt algorithms with the reference algorithms is shown in Figure 6. The best performing algorithm was found to be the Tuned-IPSOLS, whereas CHC performs the worst.



(a) Average Error (b) Median Error

Figure 5: Plots comparing average and median error of SOCO functions on reference algorithms and best 3 NLopt algorithms.

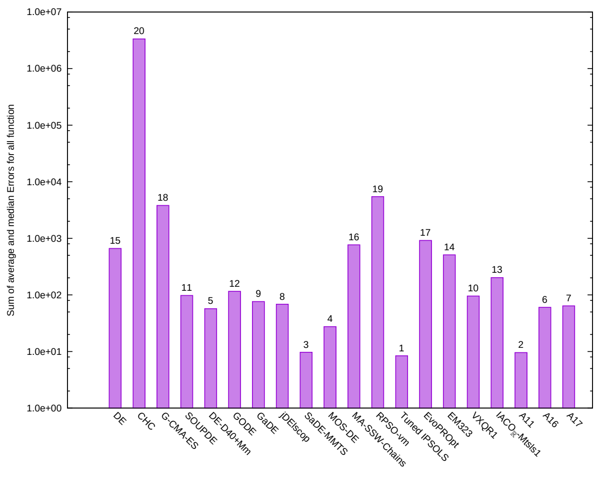


Figure 6: Ranking of best 3 NLopt and reference algorithms on SOCO benchmarks.

4.2. Results on CEC 2014 benchmarks

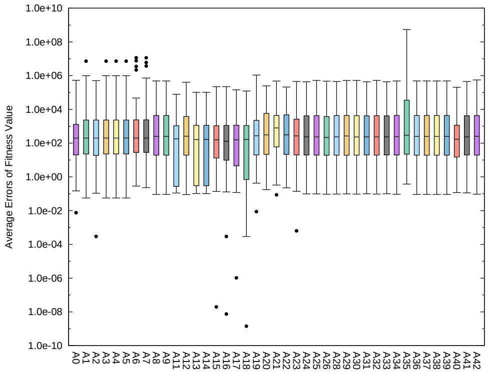
We now present the results on the CEC 2014 benchmarks [33], where we have considered the 50-dimensional versions of the functions. The maximum number of function evaluations allowed was set to $10000 \times D$, where D represents the number of dimensions in which the function is considered. The search range is $[-100, 100]^D$, and uniform random initialization within the search space has been done. The algorithm was run 51 times on each function; error values were defined as $f(x) - f(x^*)$, where x is a candidate solution and x^* is the optimal solution. Error values lower than 10^{-8} are approximated to 0. A summary of the benchmark functions is presented in Table 5.

Table 5: CEC 2014 Benchmark Functions [6]

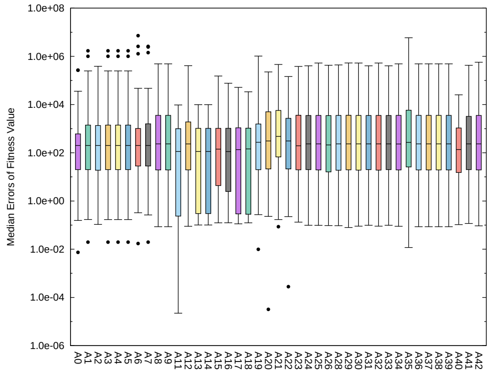
Category	**S. No.**	**Function**
Unimodal Functions	1	Rotated High Conditioned Elliptic Function
	2	Rotated Bent Cigar Function
	3	Rotated Discus Function
Multimodal Functions	4	Shifted and Rotated Rosenbrocks Function
	5	Shifted and Rotated Ackleys Function
	6	Shifted and Rotated Weierstrass Function
	7	Shifted and Rotated Griewanks Function
	8	Shifted Rastrigins Function
	9	Shifted and Rotated Rastrigins Function
	10	Shifted Schwefels Function
	11	Shifted and Rotated Schwefels Function
	12	Shifted and Rotated Katsuura Function
	13	Shifted and Rotated HappyCat Function
	14	Shifted and Rotated HGBat Function
	15	Shifted and Rotated Expanded Griewanks plus Rosenbrocks Function
	16	Shifted and Rotated Expanded Scaffers F6 Function
Hybrid Function - 1	17	Hybrid Function 1 (N=3)
	18	Hybrid Function 2 (N=3)
	19	Hybrid Function 3 (N=4)
	20	Hybrid Function 4 (N=4)
	21	Hybrid Function 5 (N=5)
	22	Hybrid Function 6 (N=5)
Composition Functions	23	Composition Function 1 (N=5)
	24	Composition Function 2 (N=3)
	25	Composition Function 3 (N=3)
	26	Composition Function 4 (N=5)
	27	Composition Function 5 (N=5)
	28	Composition Function 6 (N=5)
	29	Composition Function 7 (N=3)
	30	Composition Function 8 (N=3)

The average and median errors of the NLopt algorithms on the CEC benchmarks are shown in Figure 7. The box plots showing average error are shown in Figure 7a, while median error is shown in Figure 7b. It may be noted here that the plots are for the 50-dimensional versions of the functions. Further, the sum of average and median errors are shown in Figures 7c and 7d respectively.

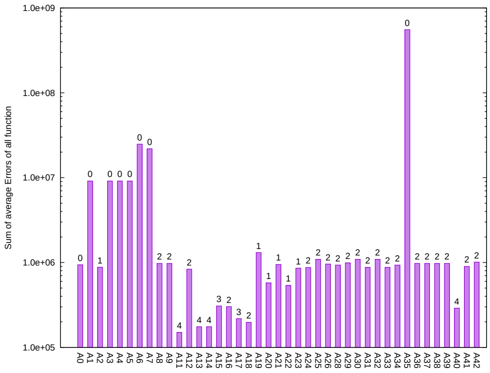
The ranking of the algorithms on the CEC 2014 benchmarks is shown in Figure 8. One can observe that A11 (limited memory BFGS) performs the best while A35 (ISRES evolutionary constrained optimization) performs the worst.



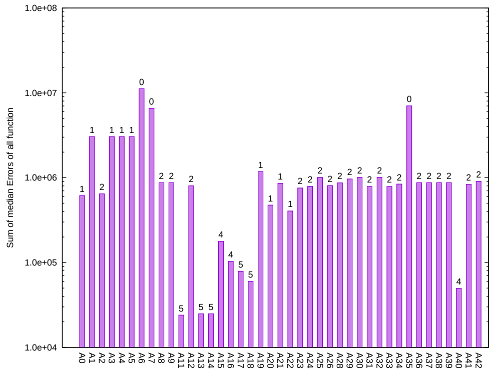
(a) Average Error



(b) Median Error



(c) Average Error Ranking



(d) Median Error Ranking

Figure 7: Plots showing average and median error of NLopt algorithms with their ranking on CEC 2014 benchmarks.

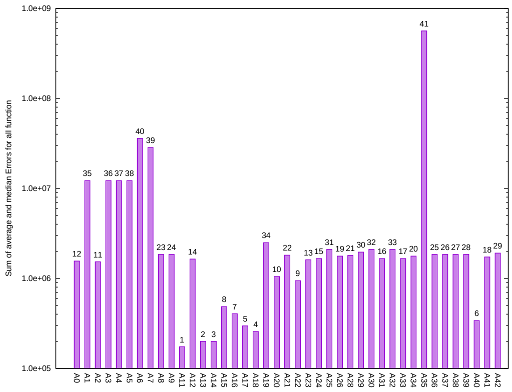
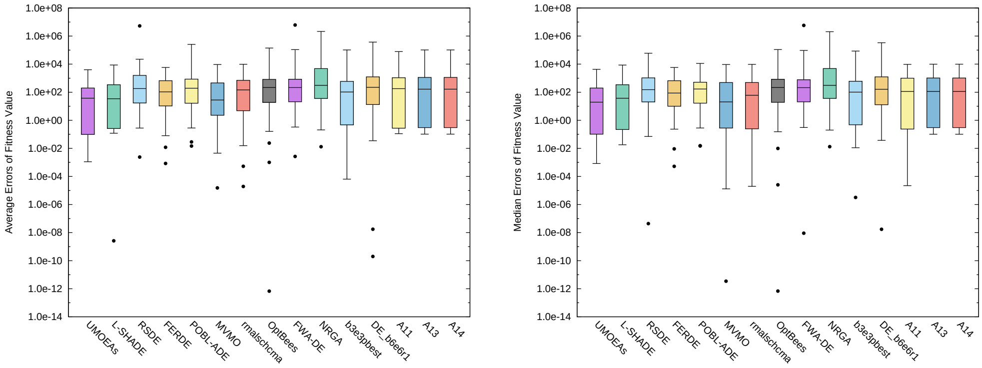


Figure 8: Ranking of the NLopt algorithms on the CEC 2014 benchmarks.

We also provide a comparison of the top 3 NLopt algorithms (A11, A13 and A14) on the CEC 2014 benchmarks with other reference algorithms. These include the United Multi-Operator Evolutionary Algorithms (UMOEA) [13], Success-History based Adaptive Differential Evolution using linear population size reduction (L-SHADE) [58], Differential Evolution with Replacement Strategy (RSDE) [69], Memetic Differential Evolution Based on Fitness Euclidean-Distance Ratio (FERDE) [49], Partial Opposition-Based Adaptive Differential Evolution (POBL-ADE) [23], Mean-Variance Mapping Optimization (MVMO) [14], rmalschcma [39], Opt Bees [38], Fireworks Algorithm with Differential Mutation (FWA-DE) [72], Non-uniform Real-coded Genetic Algorithm (NRGA) [71], b3e3pbest [5] and DE_b6e6rl [43].

The box plots for average and median errors are shown in Figure 9, specifically average error in Figure (9a) and median error in Figure (9b) respectively. The range of average errors of the reference algorithms are relatively lower than the top 3 NLopt algorithms except on UMOEA, and also median error except MVMO and rmalschcma.



(a) Average Error

(b) Median Error

Figure 9: Plots showing average and median error of best 3 NLopt and reference algorithms on CEC 2014 benchmarks.

A relative ranking of these algorithms is shown in Figure 10. One can observe that the best performing algorithm is UMOEA, while the worst performing algorithm is FWA-DE.

4.3. Comparison of Hybrid approach with standalone algorithms used for Local Search

We also provide a comparison of the standalone performance of the algorithms in Table 6. The comparison is provided in terms of Wilcoxon Signed-Rank test [67], which is a measure of the extent of statistical deviations in the results obtained using a particular approach. A p-value less than 0.05 indicates that the results of our approach have a significant statistical difference with the results obtained using the algorithms being compared, whereas p-values greater than 0.05 indicate non-significant statistical difference. The columns (Wp) and (Wn) denote the sum of signed ranks. The column (N) indicates the number of instances for which there is a difference in the result between two algorithms.

It can be observed from Table 6 that on the SOCO benchmark functions, the NLopt algorithms give significant statistical difference on **all but one** (A34) in terms of average error, and on **all but two** (A6 and A7) in terms of median error. On the CEC 2014 benchmarks, there is significant statistical difference on all but **8** and **3** out of **42** algorithms in terms of average and median error respectively.

Table 6: Wilcoxon signed rank test between hybrid and standalone approach

	SOCO								**CEC2014**							
	Average				*Median*				*Average*				*Median*			
	Wp	*Wn*	*n*	*p*	*Wp*	*Wn*	*n*	*p*	*Wp*	*Wn*	*n*	*p*	*Wp*	*Wn*	*n*	*p*
A0	7	183	19	**3.98E-04**	0	78	12	**4.88E-04**	26	250	23	**6.58E-04**	0	190	19	**1.32E-04**
A1	4	186	19	**2.50E-04**	0	91	13	**2.44E-04**	32	268	24	**7.48E-04**	0	153	17	**2.93E-04**
A2	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	26	274	24	**3.96E-04**	0	231	21	**5.96E-05**
A3	3	187	19	**2.14E-04**	0	91	13	**2.44E-04**	32	268	24	**7.48E-04**	0	153	17	**2.93E-04**
A4	3	187	19	**2.14E-04**	0	91	13	**2.44E-04**	32	268	24	**7.48E-04**	0	153	17	**2.93E-04**
A5	3	187	19	**2.14E-04**	0	91	13	**2.44E-04**	32	268	24	**7.48E-04**	0	153	17	**2.93E-04**
A6	0	190	19	**1.32E-04**	0	0	0	1.00E+00	33	243	23	**1.41E-03**	0	105	14	**1.22E-04**
A7	2	134	16	**6.43E-04**	0	0	0	1.00E+00	33	243	23	**1.41E-03**	0	91	13	**2.44E-04**
A8	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	61	404	30	**4.20E-04**	62	403	30	**4.53E-04**
A9	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	61	404	30	**4.20E-04**	62	403	30	**4.53E-04**
A11	1	135	16	**5.31E-04**	0	91	13	**2.44E-04**	109	216	25	1.50E-01	68	208	23	**3.33E-02**
A12	1	152	17	**3.52E-04**	1	135	16	**5.31E-04**	71	307	27	**4.58E-03**	72	306	27	**4.94E-03**
A13	0	153	17	**2.93E-04**	2	134	16	**6.43E-04**	109	242	26	9.12E-02	29	247	23	**9.16E-04**
A14	0	153	17	**2.93E-04**	2	134	16	**6.43E-04**	109	242	26	9.12E-02	29	247	23	**9.16E-04**
A15	0	136	16	**4.38E-04**	0	120	15	**6.10E-05**	93	313	28	**1.23E-02**	32	374	28	**9.86E-05**
A16	0	136	16	**4.38E-04**	0	120	15	**6.10E-05**	91	315	28	**1.08E-02**	34	344	27	**1.96E-04**
A17	3	150	17	**5.03E-04**	0	120	15	**6.10E-05**	93	285	27	**2.11E-02**	36	315	26	**3.96E-04**
A18	1	152	17	**3.52E-04**	0	105	14	**1.22E-04**	92	286	27	**1.98E-02**	29	271	24	**5.46E-04**
A19	15	175	19	**1.28E-03**	16	174	19	**1.48E-03**	147	288	29	1.27E-01	98	253	26	5.05E-02
A20	17	173	19	**1.70E-03**	7	183	19	**3.98E-04**	71	254	25	**1.38E-02**	62	238	24	**1.19E-02**
A21	0	153	17	**2.93E-04**	0	153	17	**2.92E-04**	53	325	27	**1.08E-03**	31	320	26	**2.42E-04**
A22	18	172	19	**1.94E-03**	13	177	19	**9.67E-04**	95	230	25	6.73E-02	78	247	25	**2.30E-02**
A23	0	153	17	**2.93E-04**	2	151	17	**4.21E-04**	66	340	28	**1.81E-03**	57	294	26	**2.62E-03**
A24	0	153	17	**2.93E-04**	0	153	17	**2.93E-04**	52	354	28	**5.85E-04**	44	307	26	**8.38E-04**
A25	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	99	336	29	**1.04E-02**	99	336	29	**1.04E-02**
A26	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	240	195	29	6.27E-01	229	236	30	9.34E-01
A28	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	83	382	30	**2.11E-03**	86	379	30	**2.58E-03**
A29	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	56	409	30	**2.83E-04**	88	377	30	**2.96E-03**
A30	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	99	336	29	**1.04E-02**	99	336	29	**1.04E-02**
A31	0	153	17	**2.93E-04**	0	153	17	**2.93E-04**	38	340	27	**2.86E-04**	33	318	26	**2.96E-04**
A32	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	99	336	29	**1.04E-02**	99	336	29	**1.04E-02**
A33	0	153	17	**2.93E-04**	0	153	17	**2.93E-04**	38	340	27	**2.86E-04**	33	318	26	**2.96E-04**
A34	38	115	17	6.84E-02	0	153	17	**2.93E-04**	253	182	29	4.43E-01	229	149	27	3.37E-01
A35	0	190	19	**1.32E-04**	9	181	19	**5.39E-04**	2	433	29	**3.17E-06**	0	435	29	**2.56E-06**
A36	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	61	404	30	**4.20E-04**	62	403	30	**4.53E-04**
A37	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	61	404	30	**4.20E-04**	62	403	30	**4.53E-04**
A38	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	61	404	30	**4.20E-04**	62	403	30	**4.53E-04**
A39	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	61	404	30	**4.20E-04**	62	403	30	**4.53E-04**
A40	0	136	16	**4.38E-04**	0	136	16	**4.38E-04**	99	226	25	8.75E-02	62	214	23	**2.08E-02**
A41	0	153	17	**2.93E-04**	0	153	17	**2.93E-04**	71	335	28	**2.65E-03**	38	313	26	**4.79E-04**
A42	0	190	19	**1.32E-04**	0	190	19	**1.32E-04**	37	428	30	**5.79E-05**	37	428	30	**5.79E-05**

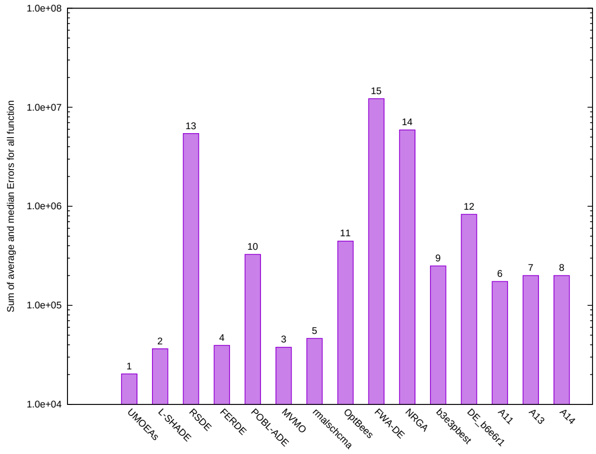


Figure 10: Ranking of the top 3 NLopt and reference algorithms on the CEC 2014 benchmarks.

5. Conclusions

This paper presented an exhaustive analysis of using optimization algorithms from the NLopt library in combination with the Mtsls1 algorithm within the $IACO_{\mathbb{R}}$-Mtsls1 framework for continuous global optimization. The results on SOCO and CEC 2014 benchmark functions present a ready reference on the performance of these approaches and would be of help to a researcher in deciding on a choice among these algorithms. The nature of functions on which these algorithms perform better can also be inferred from the results. A relative ranking of these approaches has also been provided based on the total error obtained in using them, which would provide a measure of the versatility of the algorithms.

The results of our analysis have been summarized in Table 7. On both the benchmark function sets, the hybrid $IACO_{\mathbb{R}}$-Mtsls1 with gradient-based local search performs better (A11, A16, A17 for SOCO and A11, A13, A14 for CEC). The best results are obtained using hybridization with BFGS on both benchmarks. On SOCO benchmarks, the hybrid approach outperforms the original $IACO_{\mathbb{R}}$-Mtsls1, as it has been able to achieve zero median error on **17** out of **19** functions, which has not been achieved by any other algorithm considered. We believe that the analysis presented in this paper would be of use to the research community at large.

Table 7: Summary of algorithm rankings.

Function	Ranking	NLopt	State-of-Art
SOCO	Best	A11	Tuned IPSOLS
	Worst	A21	CHC
CEC 2014	Best	A11	UMOEA
	Worst	A35	FWA-DE

A11: Limited Memory BFGS
A21: Multi-Level Single Linkage, random
A35: ISRES Evolutionary Constrained Optimization
Tuned IPSOLS: Incremental Particle Swarm for Large Scale Optimization
UMOEA: united Multi Operator Evolutionary Algorithms
FWA-DE: FireWorks Algorithm with Differential Evolution

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