Probleem 1:

Pas 'n polinoom deur $(1,1),\ (2,\frac{1}{2})$:

$$p_1(x) = \frac{3}{2} - \frac{1}{2}x.$$

Gegee dat $f(x) = \frac{1}{x}$, dus $f'(x) = -\frac{1}{x^2}$ en $f''(x) = \frac{2}{x^3}$. Stel in die foutformule:

$$\begin{aligned}
f(x) - p_1(x) &= \frac{(x-x_0)(x-x_1)}{2} f''(\xi_x) \\
\frac{1}{x} - \left(\frac{3}{2} - \frac{1}{2}x\right) &= \frac{(x-1)(x-2)}{2} \cdot \frac{2}{\xi_x^3} \\
\frac{(x-1)(x-2)}{2x} &= (x-1)(x-2)\frac{1}{\xi_x^3} \\
\xi_x^3 &= 2x \\
\Rightarrow \quad \xi_x &= (2x)^{\frac{1}{3}}.
\end{aligned}$$

As $x \in [1,2]$, volg dit dat $\xi_x \in [2^{\frac{1}{3}}, 4^{\frac{1}{3}}] \approx [1.26, 1.59]$ en dus is ξ_x wel in $(1,2)$.

Probleem 2:

(a)

x	f	
$\frac{\pi}{4}$	1.93973	
		1.48288
$\frac{\pi}{2}$	3.10438	

$$\begin{aligned}
p_1(x) &= 1.93973 + 1.48288(x - \frac{\pi}{4}) \\
p_1(1) &= 2.25796 \ \approx \ f(1)
\end{aligned}$$

Die fout word begrens m.b.v. die formule op p.70 in Burden & Faires ($n = 1$),

$$f(x) - p_1(x) \;=\; \frac{(x-x_0)(x-x_1)}{2} f^{(2)}(\xi_x).$$

Met $x = 1$, $x_0 = \frac{\pi}{4}$, $x_1 = \frac{\pi}{2}$ volg

$$\begin{aligned} f(1) - p_1(1) &= \frac{(1-\frac{\pi}{4})(1-\frac{\pi}{2})}{2} f^{(2)}(\xi) \\ &\approx -0.061247 f^{(2)}(\xi). \end{aligned} \tag{1}$$

Ons begrens nou $f^{(2)}(\xi)$, met $\xi \in [\frac{\pi}{4}, \frac{\pi}{2}]$.

$$\begin{aligned} f'(x) &= e^{\cos x} \\ f''(x) &= -\sin x \; e^{\cos x} \\ f'''(x) &= (\sin^2 x - \cos x) e^{\cos x} \end{aligned}$$

Die kritieke punte van $f''(x)$ kan bepaal word deur $f'''(x) = 0$ te stel (en onthou dat $e^{\cos x} \neq 0$),

$$\begin{aligned} \sin^2 x - \cos x &= 0 \\ \cos^2 x + \cos x - 1 &= 0 \\ \cos x &= \frac{-1 \pm \sqrt{1+4}}{2} = -1.61803 \;\text{ of }\; 0.61803. \end{aligned}$$

Ignoreer eersgenoemde (hoekom?) sodat die kritieke punt gegee word deur

$$x^* = \cos^{-1} 0.61803 = 0.90456.$$

Om $f''(x)$ se ekstreme waardes op $[\frac{\pi}{4}, \frac{\pi}{2}]$ te bepaal vergelyk ons die waarde by die kritieke punt met die waardes by die eindpunte,

$$\begin{aligned} f''(\frac{\pi}{4}) &= -\frac{1}{\sqrt{2}} e^{\frac{1}{\sqrt{2}}} = -1.43409 \\ f''(0.90456) &= -1.94245 \\ f''(\frac{\pi}{2}) &= -1e^0 = -1. \end{aligned}$$

Dus,

$$-1.94245 \leq f''(\xi) \leq -1 \quad \text{as} \quad \xi \in [\frac{\pi}{4}, \frac{\pi}{2}].$$

Stel in (1), dan kry ons

$$0.06124 \leq f(1) - p_1(1) \leq 0.11897.$$

[Die werklike waarde van $f(1) = 2.34157$ sodat $f(1) - p_1(1) = 0.08361$ wat binne die teoretiese foutgrens lê.]

(b)

Uit deel (a) is verkry dat $M_1 = 1.94245$, sodat

$$\max_{\frac{\pi}{4} \leq x \leq \frac{\pi}{2}} |f(x) - p_1(x)| \ \leq \ \frac{1}{8}\left(\frac{\pi}{4}\right)^2 M_1 \ = \ 0.14978.$$

(c)

Duidelik is $f(0) = 0$, sodat

x	f		
$\frac{\pi}{4}$	1.93973		
		1.48288	
$\frac{\pi}{2}$	3.10439		-0.628256
		1.97631	
0	0		

$$\begin{aligned} p_2(x) &= p_1(x) - 0.628256(x - \frac{\pi}{4})(x - \frac{\pi}{2}) \\ p_2(1) &= p_1(1) - 0.628256(1 - \frac{\pi}{4})(1 - \frac{\pi}{2}) \ = \ 2.334918 \end{aligned}$$

Begrens die fout m.b.v. die formule op p.70 van Burden & Faires ($n = 2$),

$$f(x) - p_2(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{3!} f^{(3)}(\xi).$$

Met $x = 1$, $x_0 = \frac{\pi}{4}$, $x_1 = \frac{\pi}{2}$, $x_2 = 0$ volg

$$f(1) - p_2(1) \ = \ -0.020416 f^{(3)}(\xi). \tag{2}$$

Om $f^{(3)}(\xi)$ te begrens stip ons $f'''(x)$ in MATLAB, en sien dat dit 'n streng stygende funksie op $[0, \frac{\pi}{2}]$ is. Dus,

$$\begin{aligned} f(0) \ &\leq \ f^{(3)}(\xi) \ \leq \ f^{(3)}(\frac{\pi}{2}) \\ -e \ &\leq \ f^{(3)}(\xi) \ \leq \ 1. \end{aligned}$$

Stel in (2),

$$-0.020416 \ \leq \ f(1) - p_1(1) \ \leq \ 0.055496.$$

[Die werklike fout is $2.34157 - 2.33492 = 0.00665$ wat binne hierdie grense lê.]

(d)

Uit deel (c) is verkry dat
$$M_2 \;=\; \max_{0\le x\le \frac{\pi}{2}} |f^{(3)}(x)| = e,$$
sodat
$$\max_{\frac{\pi}{4}\le x\le \frac{\pi}{2}} |f(x)-p_2(x)| \;\le\; \frac{1}{9\sqrt{3}}\left(\frac{\pi}{4}\right)^3 e \;=\; 0.084481.$$

Probleem 3:

(a)

Gestel $x \in [x_0, x_1]$. Dan
$$\begin{aligned}
\omega_n(x) &= \underbrace{(x-x_0)}_{\ge 0}\underbrace{(x-x_1)}_{\le 0}\underbrace{(x-x_2)}_{\le 0}\cdots\underbrace{(x-x_n)}_{\le 0}\\
|\omega_n(x)| &= \underbrace{(x-x_0)}_{\le h}\underbrace{(x_1-x)}_{\le h}\underbrace{(x_2-x)}_{\le 2h}\cdots\underbrace{(x_n-x)}_{\le nh}\\
\Rightarrow\quad |\omega_n(x)| &\le n!\,h^{n+1}.
\end{aligned}$$
Gestel nou $x \in [x_1, x_2]$. Dan
$$\begin{aligned}
\omega_n(x) &= \underbrace{(x-x_0)}_{\ge 0}\underbrace{(x-x_1)}_{\ge 0}\underbrace{(x-x_2)}_{\le 0}\cdots\underbrace{(x-x_n)}_{\le 0}\\
|\omega_n(x)| &= \underbrace{(x-x_0)}_{\le 2h}\underbrace{(x-x_1)}_{\le h}\underbrace{(x_2-x)}_{\le 2h}\cdots\underbrace{(x_n-x)}_{\le (n-1)h}\\
|\omega_n(x)| &\le 2(n-1)!\,h^{n+1}.
\end{aligned}$$
In die algemeen, as $x \in [x_\ell, x_{\ell+1}]$ met $\ell = 0, 1, \ldots, n-1$, dan
$$|\omega_n(x)| \;\le\; (\ell+1)!\,(n-\ell)!\,h^{n+1}.$$
Kies $(\ell+1)!\,(n-\ell)!$ 'n maksimum oor alle $\ell = 0, 1, \ldots, n-1$. Die maksimum stem ooreen met $\ell = 0$ en $\ell = n-1$, dit wil sê die maksimum word in die eerste en laaste intervalle bereik. Dus
$$|\omega_n(x)| \;\le\; n!\,h^{n+1}, \quad a \le x \le b.$$

(b)

Uit die foutformule op p.70 in Burden & Faires,

$$\begin{aligned} f(x) - p_n(x) &= \frac{1}{(n+1)!} f^{(n+1)}(\xi)\omega_n(x) \\ |f(x) - p_n(x)| &= \frac{1}{(n+1)!} |f^{(n+1)}(\xi)|\,|\omega_n(x)| \\ &\leq \frac{1}{(n+1)!} |f^{(n+1)}(\xi)| n!\, h^{n+1}. \end{aligned}$$

Dus

$$\max_{x_0 \leq x \leq x_n} |f(x) - p_n(x)| \;\leq\; \frac{1}{n+1} h^{n+1} M_n, \quad \text{met} \;\; M_n = \max_{x_0 \leq x \leq x_n} |f^{(n+1)}(x)|.$$

(c)

$$\begin{aligned} n = 1: &\qquad \max_{x_0 \leq x \leq x_1} |f(x) - p_1(x)| \leq \frac{1}{2} h^2 M_1 \\ n = 2: &\qquad \max_{x_0 \leq x \leq x_2} |f(x) - p_2(x)| \leq \frac{1}{3} h^3 M_2 \end{aligned}$$

Hierdie foutskattings is nie so skerp soos die foutskattings van deel (b) en deel (d) van Probleem 3 nie. Die rede is dat ons die maksimum van $|\omega_n(x)|$ heelwat oorskat het met die werkswyse van (a).

(d)

As $[a, b] = [-1, 1]$ is $h = \frac{2}{n}$ sodat

$$\max_{-1 \leq x \leq 1} |f(x) - p_n(x)| \;\leq\; \frac{1}{(n+1)} \left(\frac{2}{n}\right)^{n+1} M_n.$$

Vergelyk met Chebyshev punte (sien p.361 in Burden & Faires, of p.318 in Kincaid & Cheney):

(blaai om)

n	Gelykverspreide punte $C_n = \frac{1}{n+1}\left(\frac{2}{n}\right)^{n+1}$	Chebyshev punte $C_n = \frac{1}{2^n}\frac{1}{(n+1)!}$
5	6.83×10^{-4}	4.34×10^{-5}
10	1.86×10^{-9}	2.45×10^{-11}
15	6.24×10^{-16}	1.46×10^{-18}

Soos verwag is die C_n van Chebyshev punte kleiner as die van gelykverspreide punte, want Chebyshev punte is optimaal.

Probleem 4:

(a)

Vir gelykverspreide punte kan ons die resultaat van Probleem 3(d) gebruik, nl.

$$\max_{-1\le x\le 1} |f(x)-p_n(x)| \;\le\; \frac{1}{(n+1)}\left(\frac{2}{n}\right)^{n+1} M_n, \quad \text{waar} \quad M_n \;=\; \max_{-1\le x\le 1}\left|f^{(n+1)}(x)\right|.$$

Vir die gegewe funksie is $f^{(n+1)}(x) = \pm\,\pi^{n+1}\sin\pi x$ of $\pm\,\pi^{n+1}\cos\pi x$, afhangende of n ewe of onewe is.

Hoe dit ook al sy, $|f^{(n+1)}(x)| \le \pi^{n+1}$ as $-1 \le x \le 1$, m.a.w. $M_n = \pi^{n+1}$. Dus

$$\max_{-1\le x\le 1} |f(x)-p_n(x)| \;\le\; \frac{1}{(n+1)}\left(\frac{2\pi}{n}\right)^{n+1}.$$

Aangesien die regterkant streef na nul as $n \to \infty$, volg dat $p_n(x) \to f(x)$ by elke x in $[-1,1]$.

(b)

Vir arbitrêr-verspreide punte geld

$$|(x-x_0)(x-x_1)\ldots(x-x_n)| \;\le\; 2^{n+1},$$

en dus uit die foutformule op p.70 volg

$$\max_{-1\le x\le 1} |f(x)-p_n(x)| \;\le\; \frac{1}{(n+1)!}2^{n+1}M_n.$$

Net soos in deel (a) is $M_n = \pi^{n+1}$, en dus

$$\max_{-1\leq x\leq 1} |f(x) - p_n(x)| \ \leq \ \frac{(2\pi)^{n+1}}{(n+1)!}.$$

Die regterkant streef weer na 0 as $n \to \infty$, soos gesien kan word uit

$$\frac{(2\pi)^{n+1}}{(n+1)!} \ = \ \frac{2\pi}{1} \cdot \frac{2\pi}{2} \cdots \frac{2\pi}{n-1} \cdot \frac{2\pi}{n}.$$

Dit volg dus dat $p_n(x) \to f(x)$ by elke x in $[-1, 1]$.