TW324 **Tutoriaaltoets #4** **2004**

NAAM: Weideman - Antwoorde US NR: ____________

INSTRUKSIES: Vyf probleme, 20 punte, 50 minute. Toon alle berekenings by Probleme 4 en 5.

INSTRUCTIONS: *Five problems, 20 marks, 50 minutes. Show all work on Problems 4 and 5.*

WENKE (Gebruik enige plek sonder bewys)

HINTS *(Use anywhere without proof)*

Simpson

$$S_n = \tfrac{1}{3}h\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 4f(x_{n-1}) + f(x_n)\Big)$$

Euler-Maclaurin

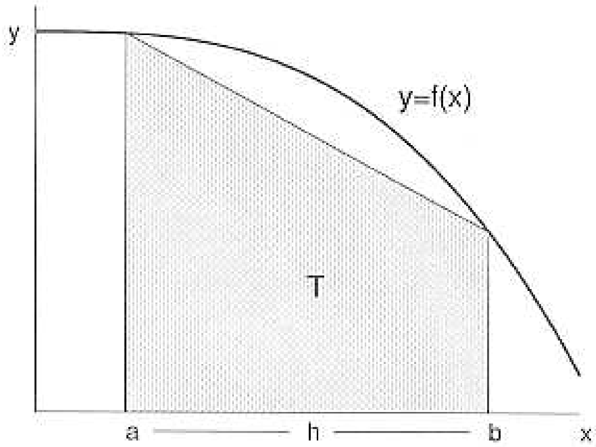
$$I(f) - T_n(f) = -\frac{h^2}{12}\Big(f'(b) - f'(a)\Big) - \frac{h^4}{720}\Big(f'''(b) - f'''(a)\Big) + \ldots + \text{resterm}/\textit{remainder term}$$

Gauss

$$\int_{-1}^{1} f(t)\,dt \approx f\Big(-\frac{1}{\sqrt{3}}\Big) + f\Big(\frac{1}{\sqrt{3}}\Big), \qquad \int_{-1}^{1} f(t)\,dt \approx \frac{5}{9}f\Big(-\sqrt{\frac{3}{5}}\Big) + \frac{8}{9}f\Big(0\Big) + \frac{5}{9}f\Big(\sqrt{\frac{3}{5}}\Big)$$

P1	P2	P3	P4	P5	T
3	3	1	4	9	20

Probleem 1 (3 punte)



Laat

Let

$$I = \int_a^b f(x)\,dx$$

en gestel hierdie integraal word met die basiese trapesiumreël benader soos in die figuur. Laat die staplengte $h = b - a$ wees, T die oppervlakte van die trapesium, en E die fout in die benadering, soos gedefinieer deur

and suppose this integral is estimated with the basic trapezoidal rule as shown in the figure. Let the step length be $h = b - a$, T the area of the trapezoid, and E the error in the approximation, defined by

$$E = I - T.$$

(a) Omsirkel die korrekte formule vir T. *Circle the correct formula for T.*

(A) $T = 2h\Big(f(a) + f(b)\Big)$ (B) $T = h\Big(f(a) + f(b)\Big)$ (C) $T = \frac{1}{2}h\Big(f(a) + f(b)\Big)$

(D) $T = h\,f\left(\frac{a+b}{2}\right)$ (E) $T = \frac{1}{3}h\Big(f(a) + 4f(\frac{1}{2}(a+b)) + f(b)\Big)$

teorie.

(b) Omsirkel die korrekte formule vir E. (Hier is ξ 'n getal in $[a, b]$.) *Circle the correct formula for E. (Here ξ is a number in $[a, b]$.)*

(A) $E = \frac{1}{12}h^3 f''(\xi)$ (B) $E = -\frac{1}{12}h^3 f''(\xi)$ (C) $E = -\frac{1}{12}h^3 f'(\xi)$

(D) $E = \frac{1}{12}h^3 f'''(\xi)$ (E) $E = -\frac{1}{12}h^3 f'''(\xi)$

teorie.

(c) Omsirkel die korrekte formule vir die basiese trapesiumreël met endpunt korreksie. *Circle the basic trapezoidal rule with endpoint correction.*

(A) $T - \frac{h^2}{12}\Big(f'(b) - f'(a)\Big)$ (B) $T + \frac{h^2}{12}\Big(f'(b) - f'(a)\Big)$ (C) $T - \frac{h^4}{720}\Big(f'''(b) - f'''(a)\Big)$

(D) $T - \frac{h^2}{3}\Big(f'(b) - f'(a)\Big)$ (E) $T + \frac{h^2}{3}\Big(f''(b) - f''(a)\Big)$

Tel eerste term van EM formule by.

Probleem 2 (3 *punte*)

Beskou die integraal

Consider the integral

$$I = \int_1^2 \frac{dx}{\sqrt{8+x^3}}.$$

Hieronder stel S_n die saamgestelde Simpson-reël benadering tot I voor, as n intervalle van uniforme lengte gebruik word.

Below, S_n represents the composite Simpson rule approximation to I, when n intervals of uniform length are used.

(a) Bereken S_2 (tot vier desimale syfers).

Compute S_2 (to four decimal places).

(A) 0.2852 (B) 0.2873 (C) 0.2898 (D) 0.2921 (E) 0.2949

(b) Gegee dat $S_4 = 0.29488299$ en $S_8 = 0.29488223$ (tot agt desimale syfers), wat is jou beste skatting vir die absolute fout in S_8?

Given that $S_4 = 0.29488299$ and $S_8 = 0.29488223$ (to eight decimal places), what is your best estimate for the absolute error in S_8?

$\frac{1}{15}|S_8 - S_4|$

(A) 5×10^{-8} (B) 5×10^{-7} (C) 5×10^{-6} (D) 5×10^{-5} (E) 5×10^{-4}

(c) Pas Richardson ekstrapolasie toe op S_4 en S_8 om 'n beter benadering tot I te bereken.

Apply Richardson extrapolation to S_4 and S_8 to derive a better approximation to I.

(A) 0.29488304 (B) 0.29488218 (C) 0.29488203 (D) 0.29488175 (E) 0.29488168

$S_8 + \frac{1}{15}(S_8 - S_4)$

Probleem 3 (1 *punt*)

Wat is die polinoomgraad van die negepunt Gauss-reël?

What is the polynomial degree of the nine point Gauss rule?

(A) 9 (B) 10 (C) 17 (D) 18 (E) 19

Hermite interpolasie op 9 punte $\Rightarrow$ pol. is van graad $2 \times 9 - 1 = 17$.

Probleem 4 (*4 punte*)

Beskou die integraal van Probleem 2, naamlik

Consider the integral of Problem 2, namely

$$I = \int_1^2 \frac{dx}{\sqrt{8+x^3}}.$$

Bereken 'n benadering tot I met die 2-punt Gauss-reël.

Compute an approximation to I with the 2-point Gauss rule.

Laat $x = at + b$, met t in $[-1, 1]$.

$$\left.\begin{aligned} 1 &= a(-1) + b \\ 2 &= a(+1) + b \end{aligned}\right\} \Rightarrow b = 3/2,\ a = 1/2$$

Dus $x = \frac{1}{2}(t+3)$, $dx = \frac{1}{2}\,dt$

$$I = \frac{1}{2}\int_{-1}^{1} \frac{dt}{\sqrt{8 + \frac{1}{8}(t+3)^3}}$$

$$\approx \frac{1}{2}\left[\frac{1}{\sqrt{8+\frac{1}{8}(-1/\sqrt{3}+3)^3}} + \frac{1}{\sqrt{8+\frac{1}{8}(1/\sqrt{3}+3)^3}}\right]$$

$$= 0.294878\ldots \longrightarrow$$

Probleem 5 ($3+3+3=9$ *punte*)

Gestel 'n numeriese integrasiereël konvergeer by benadering volgens

Suppose a rule for numerical integration converges approximately according to

$$I - I_n = C\,h^p.$$

Hier is I die werklike waarde van die integraal, I_n die benaderde waarde bereken met n intervalle, en h die (uniforme) staplengte. C en p is konstantes.

Here I is the actual value of the integral, I_n the approximate value computed with n subintervals, and h the (uniform) step size. C and p are constants.

(a) Gestel benaderings I_n, I_{2n} en I_{4n} is beskikbaar. Toon aan dat

Suppose approximations I_n, I_{2n} and I_{4n} are available. Show that

$$p = \frac{\log((I_{2n} - I_n)/(I_{4n} - I_{2n}))}{\log 2}.$$

(1) $I - I_n = Ch^p$, $\quad I - I_{2n} = C\left(\frac{h}{2}\right)^p = Ch^p/2^p$ (2)

$I - I_{4n} = C\left(\frac{h}{4}\right)^p = Ch^p/2^{2p}$ (3)

(1) - (2) $\quad I_{2n} - I_n = Ch^p\left(1 - \frac{1}{2^p}\right)$ (4)

(3) - (2) $\quad I_{4n} - I_{2n} = Ch^p\left(\frac{1}{2^p} - \frac{1}{2^{2p}}\right) = Ch^p\,\frac{1}{2^p}\left(1 - \frac{1}{2^p}\right)$ (5)

(4) ÷ (5) $\quad \dfrac{I_{2n} - I_n}{I_{4n} - I_{2n}} = 2^p$

$\ln(\text{ditto}) = p\ln 2$

$\Rightarrow \quad p = \dfrac{\ln\left(\dfrac{I_{2n} - I_n}{I_{4n} - I_{2n}}\right)}{\ln 2} \longrightarrow$

(b) Laat $a = 1$, en beskou

Let $a = 1$ and consider

$$I = \int_1^2 \sqrt{ax^3 - 1}\, dx.$$

Benadering met die trapesiumreël lewer

Approximation with the trapezoidal rule yields

$$T_{32} = 1.51411876635361, \quad T_{64} = 1.51526934665329, \quad T_{128} = 1.51568991773747.$$

Die formule van deel (a) lewer

The formula of part (a) yields

$$p \approx 1.45.$$

Is hierdie orde van konvergensie tipies van die trapesiumreël? Indien nie, verduidelik die oorsprong van die nie-tipiese gedrag volledig.

Is this order of convergence typical of the trapezoidal rule? If not, explain the origin of the non-typical behaviour in detail.

Tipiese orde van konvergensie is $p = 2$ vir trapesiumreël, dus stadiger konvergensie as normaalweg.

Verklaring: $f(x) = \sqrt{x^3 - 1}$

$$f'(x) = \frac{3x^2}{2\sqrt{x^3-1}}$$ ongedefinieerd as $x = 1$.

Die integrand het dus nie twee kontinue afgeleides op $[1,2]$ nie, en die standaard foutformule op p. 70, nl. $I - T_n = -\frac{1}{12} h^2 f''(\xi)$, is nie toepaslik nie.

(c) Herhaal deel (b) as $a = 15/8$, met data

Repeat part (b) if $a = 15/8$, with data

$$T_{32} = 2.32145943528963, \; T_{64} = 2.32145938452611, \; T_{128} = 2.32145938133835 \quad \Longrightarrow \quad p \approx 3.99.$$

$p \approx 4 \Rightarrow$ konvergensie is vinniger as normaalweg.

Verklaring: $f(x) = \sqrt{\tfrac{15}{8}x^3 - 1}$

$$f'(x) = \frac{\tfrac{45}{8}x^2}{2\sqrt{\tfrac{15}{8}x^3 - 1}}$$

$$\left.\begin{aligned} f'(1) &= \frac{45}{16}\cdot\frac{1}{\sqrt{7/8}} \\ f'(2) &= \frac{45}{16}\cdot\frac{4}{\sqrt{14}} \end{aligned}\right\} \Rightarrow$$ $f'(1) = f'(2)$ sodat 1^e term in EM formule verdwyn en die fout dus v orde h^4 i.p.v. h^2 is. →