Second Edition

DIFFERENTIAL EQUATIONS

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4.5 THE TACOMA NARROWS BRIDGE

On July 1, 1940, the $6 million Tacoma Narrows Bridge opened for traffic. On November 7, 1940, during a windstorm, the bridge broke apart and collapsed. During its short stand, the structure, a suspension bridge more than a mile long, became known as "Galloping Gertie" because the roadbed oscillated dramatically in the wind. The collapse of the bridge proved to be a scandal in more ways than one, including the fact that because the insurance premiums had been embezzled, the bridge was uninsured.*

The roadbed of a suspension bridge hangs from vertical cables that are attached to cables strung between towers (see Figure 4.30 for a schematic picture). If we think of the vertical cables as long springs, then it is tempting to model the oscillations of the roadbed with a harmonic oscillator equation. We can think of the wind as somehow providing periodic forcing. It is very tempting to say, "Aha, the collapse must be due to resonance."

It turns out that things are not quite so simple. We know that to cause dramatic effects, the forcing frequency of a forced harmonic oscillator must be very close to its natural frequency. The wind seldom behaves in such a nice way for very long, and it would be very bad luck indeed if the oscillations caused by the wind happened to have a frequency almost exactly the same as the natural frequency of the bridge.

Recent research on the dynamics of suspension bridges (by two mathematicians, A. C. Lazer and P. J. McKenna♣) indicates that the linear harmonic oscillator does not make an accurate model of the movement of a suspension bridge. The vertical cables do act like springs when they are stretched. That is, when the roadbed is below its rest position, the cables pull up. However, when the roadbed is significantly above its rest position, the cables are slack, so they do not push down. Hence the roadbed feels less

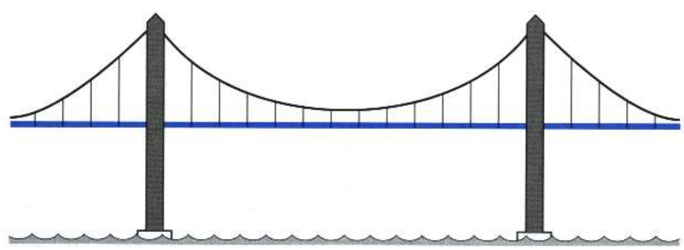


Figure 4.30
Schematic of a suspension bridge.

*The story of the bridge and its collapse can be found in Martin Braun, *Differential Equations and Their Applications*, Springer-Verlag, 1993, p. 173, and Matthys Levy and Mario Salvadori, *Why Buildings Fall Down: How Structures Fail*, W. W. Norton Co., 1992, p. 109.

♣See "Large-amplitude Periodic Oscillations in Suspension Bridges: Some New Connections with Nonlinear Analysis" by A. C. Lazer and P. J. McKenna, in *SIAM Review*, Vol. 32, No. 4, 1990, pp. 537–578.

force trying to pull it back into the rest position when it is pushed up than when it is pulled down (see Figure 4.31).

In this section, we study a model for a system with these properties. The system of equations we study was developed by Lazer and McKenna from more complicated models of oscillations of suspension bridges. This system gives considerable insight into the possible behaviors of suspension bridges and even hints at how they can be made safer.

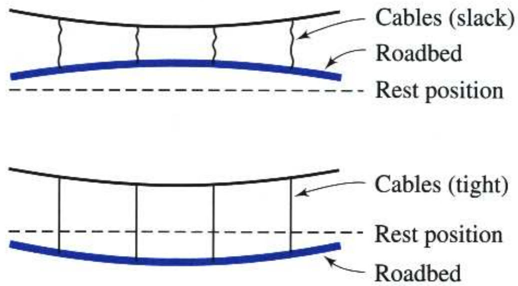


Figure 4.31
Close-up of the vertical cable when the roadbed is above and below the rest position.

Derivation of the Equations

The model we consider for the motion of the bridge uses only one variable to describe the position of the bridge. We assume that the bridge oscillates up and down as in Figure 4.31. We let $y(t)$ (measured in feet or meters) denote the vertical position of the center of the bridge, with $y = 0$ corresponding to the position where the cables are taut but not stretched. We let $y < 0$ correspond to the position in which the cables are stretched and $y > 0$ correspond to the position in which the cables are slack (see Figure 4.32). Of course, using one variable to study the motion of the bridge ignores many possible motions, and we comment on other models at the end of this section.

To develop a model for $y(t)$, we consider the forces that act on the center of the bridge. Gravity provides a constant force in the negative direction of y. We also assume that the cables provide a force that pulls the bridge up when $y < 0$ and that is proportional to y. On the other hand, when $y > 0$, the cable provides no force. When $y \neq 0$, there is also a restoring force that pulls y back toward $y = 0$ due to the stretching of the roadbed. Finally, there will also be some damping, which is assumed to be proportional to dy/dt. We choose units so that the mass of the bridge is 1.

Based on these assumptions, the equation developed by Lazer and McKenna to model a suspension bridge on a calm day (no wind) is

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y + c(y) = -g.$$

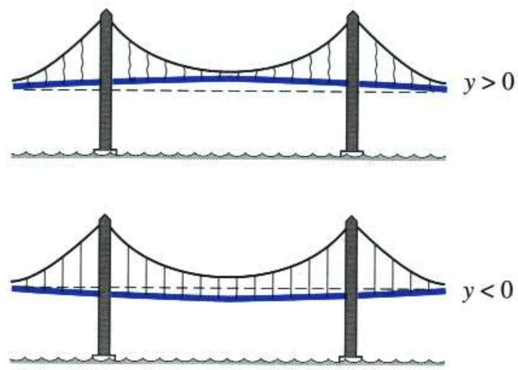


Figure 4.32
Positions of the bridge corresponding to $y > 0$ and $y < 0$.

The first term is the vertical acceleration. The second term, $\alpha(dy/dt)$, arises from the damping. Since suspension bridges are relatively flexible structures, we assume that α is small. The term βy accounts for the force provided by the material of the bridge pulling the bridge back toward $y = 0$. The function $c(y)$ accounts for the pull of the cable when $y < 0$ (and the lack thereof when $y \geq 0$), and therefore it is given by

$$c(y) = \begin{cases} \gamma y, & \text{if } y < 0; \\ 0, & \text{if } y \geq 0. \end{cases}$$

The constant g represents the force due to gravity.

We can convert this to a system in the usual way, obtaining

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -\beta y - c(y) - \alpha v - g.$$

This is an autonomous system. Simplifying the right-hand side of this system by combining the $-\beta y$ term with the terms in $c(y)$, we obtain

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -h(y) - \alpha v - g,$$

where $h(y)$ is the piecewise-defined function

$$h(y) = \begin{cases} ay, & \text{if } y < 0; \\ by, & \text{if } y \geq 0, \end{cases}$$

and $a = \beta + \gamma$ and $b = \beta$.

To study this example numerically, we choose particular values of the parameters. (These values are not motivated by any particular bridge.) Following Lazer and McKenna, we take $a = 17$, $b = 13$, $\alpha = 0.01$, and $g = 10$. So the system we study is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -h(y) - 0.01v - 10,$$

where

$$h(y) = \begin{cases} 17y, & \text{if } y < 0; \\ 13y, & \text{if } y \geq 0. \end{cases}$$

We can easily compute that this system has only one equilibrium point, which is given by $(y, v) = (-10/17, 0)$. The y-coordinate of this equilibrium point is negative because gravity forces the bridge to sag a little, stretching the cables. Numerical results indicate that solutions spiral toward the equilibrium point very slowly. This is what we expect because there is a small amount of damping present. The direction field and a typical solution curve are shown in Figure 4.33.

The behavior of the solutions indicate that the bridge oscillates around the equilibrium position. The amplitude of the oscillation dies out slowly due to the small amount of damping. Because the forces controlling the motion change abruptly along $y = 0$, solutions also change direction when they cross from the left half-plane to the right half-plane.

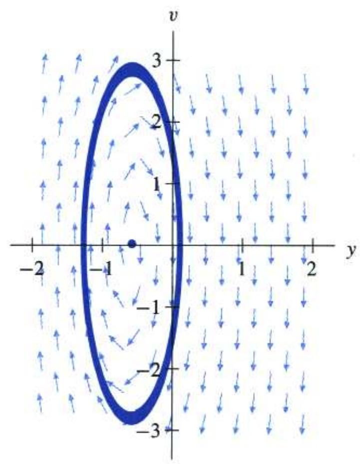


Figure 4.33
Direction field and typical solution for the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -h(y) - 0.01v - 10.$$

There is a spiral sink at $(y, v) = (-10/17, 0)$. Solution curves spiral toward the equilibrium point very slowly.

The effect of wind

To add the effect of wind into the model, we add an extra term to the right-hand side of the equation. The effect of the wind is very difficult to quantify. Not only are there gusts of more or less random duration and strength, but also the way in which the wind interacts with the bridge can be very complicated. Even if we assume that the wind has constant speed and direction, the effect on the bridge need not be constant. As air moves past the bridge, swirls or vortices (like those at the end of an oar in water) form above and below the roadbed. When they become large enough, these vortices "break off," causing the bridge to rebound. Hence even a constant wind can give a periodic push to the bridge.

Despite these complications, we assume, for simplicity, that the wind provides a forcing term of the form $\lambda \sin \mu t$. Since it is unlikely that turbulent winds will give a forcing term with constant amplitude λ or constant frequency $\mu/(2\pi)$, we will look for behavior of solutions that persist for a range of λ- and μ-values.

The system with forcing is given by

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -h(y) - 0.01v - 10 + \lambda \sin \mu t.\end{aligned}$$

This is a fairly simple model for the complicated behavior for a bridge moving in the wind, but we see below that even this simple model has solutions that behave in a surprising way.

Behavior of Solutions

We have seen that a linear system with damping and sinusoidal forcing has one periodic solution to which every other orbit tends as time increases, the steady-state solution. In other words, no matter what the initial conditions, the long-term behavior of the system will be the same. The amplitude and frequency of this periodic solution are determined by the amplitude and frequency of the forcing term (see Section 4.2).

The behavior of the system

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -h(y) - 0.01v - 10 + \lambda \sin \mu t\end{aligned}$$

is quite different. We now describe the results of a numerical study of these solutions of this system carried out by Glover, Lazer, and McKenna.*

If, for example, we choose $\mu = 4$ and λ very small ($\lambda < 0.05$), then every solution tends toward a periodic solution with small magnitude near $y = -10/17$ (see Figure 4.34). For this periodic solution, $y(t)$ is negative for all t. Since this solution

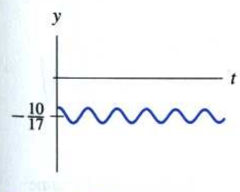


Figure 4.34
Solution of the system with small forcing.

*See "Existence and Stability of Large-scale Nonlinear Oscillations in Suspension Bridges" by J. Glover, A. C. Lazer, and P. J. McKenna, *ZAMP*, Vol. 40, 1989, pp. 171–200.

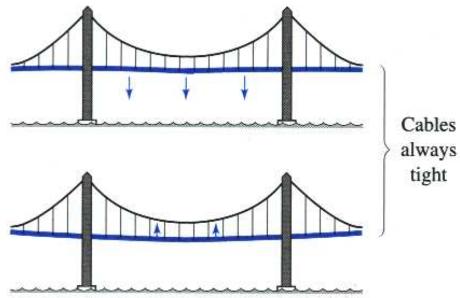


Figure 4.35
Schematic of the bridge oscillating in light winds.

never crosses $y = 0$, it behaves just like the solution of the forced linear system

$$\begin{aligned}\frac{dy}{dt} &= v\\ \frac{dv}{dt} &= -17y - 0.01v - 10 + \lambda \sin 4t.\end{aligned}$$

In terms of the behavior of the bridge, this means that in light winds, we expect the bridge to oscillate with small amplitude. Gravity keeps the bridge sagging downward and the cables are always stretched somewhat (see Figure 4.35). In this range, modeling the cables as linear springs is reasonable.

As λ increases, a new phenomenon is observed. Initial conditions near $(y, v) = (-10/17, 0)$ still yield solutions that oscillate with small amplitude (see Figure 4.36). However, if $y(0) = -10/17$ but $v(0)$ is large, solutions can behave differently. There is another periodic solution that oscillates around $y = -10/17$, but with (relatively) large amplitude (see Figure 4.37).

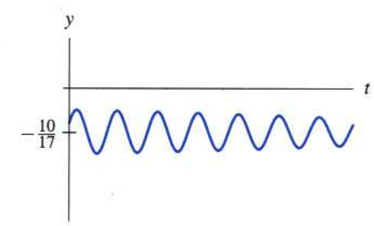


Figure 4.36
Solution of the forced system with larger forcing than in Figure 4.34 and initial conditions near the equilibrium.

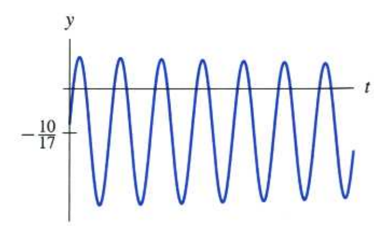


Figure 4.37
Solution of the forced system with the same large forcing as in Figure 4.36 but with initial conditions farther from the equilibrium.

This has dramatic implications for the behavior of the bridge. If the initial displacement is small, then we expect to see small oscillations as before. However, if a gust of wind gives the bridge a kick large enough to cause it to rise above $y = 0$, then the cables will go slack and the linear model will no longer be accurate (see Figure 4.38). In this situation the bridge can start oscillating with much larger amplitude, and these oscillations *do not die out*. So, in a moderate wind ($\lambda > 0.06$), a single strong gust could suddenly cause the bridge to begin oscillating with much larger amplitude, perhaps with devastating consequences.

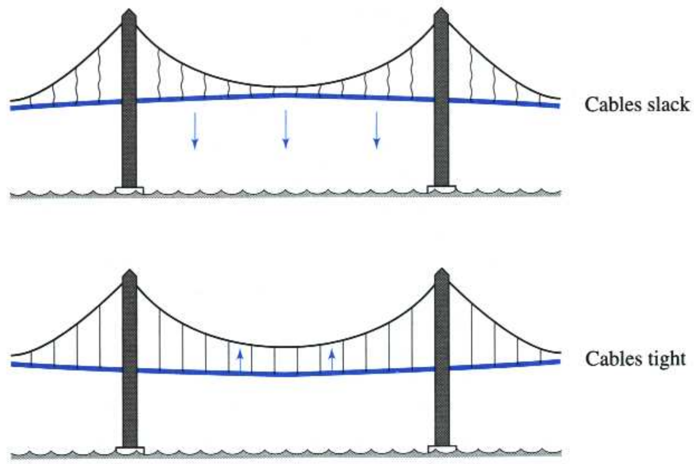


Figure 4.38
Schematic of large-amplitude oscillations of the bridge.

Varying the parameters

As mentioned above, because the effects of the wind are not particularly regular, we should investigate the behavior of solutions as λ and μ are varied. It turns out that the large-amplitude periodic solution persists for a fairly large range of λ and μ. This means that even in winds with uneven velocity and direction, the sudden jump in behavior to a persistent oscillation with large amplitude is possible.

Does This Explain the Tacoma Narrows Bridge Disaster?

As with any simple model of a complicated system, a note of caution is in order. To construct this model, we have made a number of simplifying assumptions. These include, but are not limited to, assuming that the bridge oscillates in one piece. The bridge can oscillate in two or more sections (see Figure 4.39). To include this in our model, we would have to include a new independent variable for the position along the bridge. The resulting model is a partial differential equation.

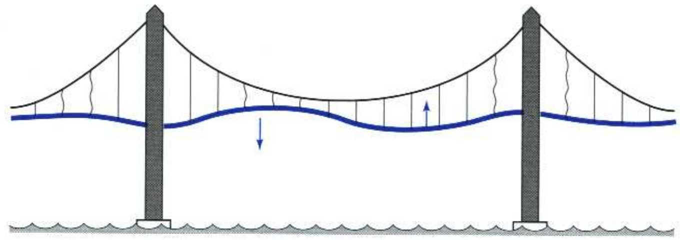


Figure 4.39
More complicated forms of oscillation of a suspension bridge.

Another factor we have ignored is that the roadbed of the bridge has width as well as length. The final collapse of the Tacoma Narrows Bridge was preceded by violent twisting motions of the roadbed, alternately stretching and loosening the cables on either side of the road. Analysis of a model including the width gives considerable insight into the final moments before the bridge's collapse (see the paper by Lazer and McKenna cited on page 421).

This being said, the simple model discussed above still helps a great deal in understanding the behavior of the bridge. If this simple system can feature the surprising appearance of large-amplitude periodic solutions, then it is not at all unreasonable to expect that more complicated and more accurate models will also exhibit this behavior. So this model does what it is supposed to do: It tells us what to look for when studying the behavior of a flexible suspension bridge.

Joseph McKenna (1948–), born in Dublin, Ireland, received his Ph.D. in mathematics at the University of Michigan. For most of his professional life he has been involved in research in differential equations, applying them to diverse problems in soil physics, fluid dynamics, optics, biology, and most recently on flexing in bridges and ships.

His work with A. C. Lazer, described in this section, has received considerable attention in both the mathematics and engineering communities. Their research directly contradicts the long-standing view that resonance phenomena caused the collapse of the Tacoma Narrows Bridge. They have also suggested several alternative types of differential equations that govern the motion of such suspension bridges, including the equation described in this section as well as a more complicated model involving nonlinear equations somewhat like those we will discuss in the next chapter.

McKenna is currently Professor of Mathematics at the University of Connecticut.

EXERCISES FOR SECTION 4.5

For Exercises 1–3, recall that our simple model of a suspension bridge is

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y + c(y) = -g,$$

where α is the coefficient of damping, β is a parameter corresponding to the stiffness of the roadbed, the function $c(y)$ accounts for the pull of the cables, and g is the gravitational constant. For each of the following modifications of bridge design,

(a) discuss which parameters are changed, and

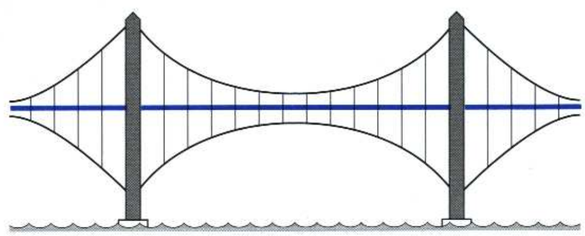
(b) discuss how you expect a change in the parameter values to affect the solutions. (For example, does the modification make the system look more or less like a linear system?)

1. The "stiffness" of the roadbed is increased, for example, by reinforcing the concrete or adding extra material that makes it harder for the roadbed to bend.

2. The coefficient of damping is increased.

3. The strength of the cables is increased.

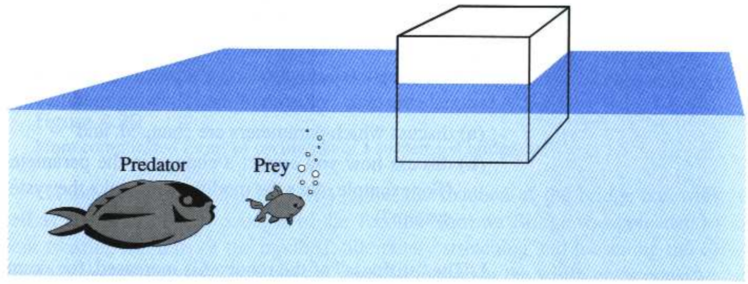
4. The figure below is a schematic for an alternate suspension bridge design called the Lazer-McKenna light flexible long span suspension bridge. Why does this design avoid the problems of the standard suspension bridge design? Discuss this in a paragraph and give model equations similar to those in the text for this design.



A schematic of the Lazer-McKenna light flexible long span suspension bridge.

In Exercises 5–8, we consider another application of the ideas in this section. Lazer and McKenna observe that the equations they study may also be used to model the up-and-down motion of an object floating in water, which can rise completely out of the water. This has serious implications for the behavior of a ship in heavy seas.

Suppose we have a cube made of a light substance floating in water. Gravity always pulls the cube downward. The cube floats at an equilibrium level at which the mass of the water displaced equals the mass of the cube. If the cube is higher or lower than the equilibrium level, then there is a restoring force proportional to the size of the displacement. We assume that the bottom and top of the cube stay parallel to the surface of the water at all times and that the system has a small amount of damping.



Cube floating in water.

5. Write a differential equation model for the up-and-down motion of the cube, assuming that it always stays in contact with the water and is never completely submerged.

6. Write a differential equation for the up-and-down motion of the cube, assuming that it always stays in contact with the water, but that it can be completely submerged.

7. Write a differential equation model for the up-and-down motion of the cube, assuming that it will never be completely submerged but can rise completely out of the water by some distance.

8. **(a)** Adjust each of the models in Exercises 5–7 to include the effect of waves on the motion of the cube (assuming the top and bottom remain parallel to the average water level).

(b) Discuss the implications of the behavior of solutions of this system considered in the text for the motion of the cube.