INSTRUKSIES:

(a) 8 probleme, 90 minute, 40 punte

(b) Alle probleme is van die uitskryf-tipe. Toon alle bewerkings. 'n Korrekte antwoord verdien geen punte sonder die nodige verduideliking nie.

(c) Beantwoord alle probleme in die toetsboek wat voorsien word.

(d) Die formules hieronder mag enige plek in die toets sonder bewys gebruik word.

INSTRUCTIONS:

8 problems, 90 minutes, 40 marks

All solutions are to be written out. Show all work. A correct answer earns no marks without the proper justification.

Solve all problems in the test book that will be provided.

The formulas below may be used without proof anywhere in the test.

$$\begin{aligned}
\text{Secant:}\quad x_{n+1} &= x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})},\\
\text{Newton:}\quad x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}\\
\text{Newton Fout}/\textit{Error}:\quad x_{n+1} - p &= \frac{f''(\xi_n)}{2f'(x_n)}(x_n - p)^2,\\
\text{Nuttige identiteit}/\textit{Useful identity}:\quad u^3 - v^3 &= (u - v)(u^2 + uv + v^2)
\end{aligned}$$

Probleem 1 (2 punte)

Beskou

Consider

$$p(x) = 1 - 2x + 3x^2 - 4x^3 + 5x^4.$$

Bereken $p(2)$ met Horner se algoritme en handberekening (d.w.s. sonder sakrekenaar). Die doel van die oefening is om te demonstreer dat jy Horner se algoritme verstaan, so alle tussenstappe moet getoon word.

Compute $p(2)$ with Horner's algorithm and hand calculation (i.e., no calculator). The object of the exercise is to demonstrate that you understand Horner's algorithm, so all intermediate steps need to be shown.

Probleem 2 (2 punte)

Beskou die limiet

Consider the limit

$$\lim_{x \to 0} \left(1 + \sin x\right)^{2/x}.$$

'n Vriend (wat nie L'Hospital se reël ken nie) gebruik die volgende MATLAB kode in 'n poging om die limiet numeries te bereken. Verwag jy dat hierdie kode die limietwaarde tot volle akkuraatheid sal bereken? Verduidelik.

A friend (who does not know L'Hospital's rule) uses the following MATLAB *code in an attempt to compute this limit numerically. Do you expect this code to compute the value of the limit to full precision? Explain.*

```
for k = [1:20]
  x = 10^(-k);
  y = (1+sin(x))^(2/x)
end
```

Probleem 3 (3 punte)

Gestel die halveringsmetode word gebruik vir die oplos van 'n sekere vergelyking $f(x) = 0$, met aanvanklike interval $[a, b] = [-2, 2]$. Wat is die minimum aantal stappe wat uitgevoer moet word om 'n absolute fout nie groter nie as 10^{-10} te waarborg?

Suppose the bisection method is used for the solution of a certain equation $f(x) = 0$, with initial interval $[a, b] = [-2, 2]$. What is the minimum number of steps that has to be executed to guarantee an absolute error no larger than 10^{-10}?

Probleem 4 $(2 + 2 + 3 + 3 = 10$ punte)

Cardano se formule vir die oplos van kubiese vergelykings lui soos volg: As p en q reëel is, met $p^3+q^2 > 0$, dan het die kubiese vergelyking

Cardano's formula for the solution of cubic equations reads as follows: If p and q are real, with $p^3 + q^2 > 0$, then the cubic equation

$$x^3 + 3px + 2q = 0$$

'n reële wortel, x_1, wat gegee word deur

has a real root, x_1, given by

$$x_1 = u - v,$$

waar

where

$$u^3 = \sqrt{p^3 + q^2} - q, \qquad v^3 = \sqrt{p^3 + q^2} + q.$$

Cardano so formule, soos die kwadratiese formule, is onderhewig aan kansellasiefoute soos in die volgende twee MATLAB implementerings gesien kan word. Vir elk van (a) en (b) bestudeer die kode, en verduidelik dan presies waar die kansellasie plaasvind.

Cardano's formula, like the quadratic formula, is subject to cancellation errors as can be seen in the following two MATLAB *implementations. For each of (a) and (b) study the code, and then explain precisely where the cancellation occurs.*

(a) $p = 0.01$, $q = -2004$:

```
>> p  = 0.01; q = -2004;
>> u  = (sqrt(p^3+q^2)-q)^(1/3);
>> v  = (sqrt(p^3+q^2)+q)^(1/3);
>> x1 = u-v

x1 =  15.88395666559116   (Eksak/Exact: 15.88395660496455)
```

(b) $p = 2004$, $q = 0.01$:

```
>> p  = 2004; q = 0.01;
>> u  = (sqrt(p^3+q^2)-q)^(1/3);
>> v  = (sqrt(p^3+q^2)+q)^(1/3);
>> x1 = u-v

x1 = -3.326679973270075e-06   (Eksak/Exact: -3.326679973386554e-06)
```

(c) Verbeter die implementering van Cardano se formule in deel (a) hierbo.

Improve the implementation of Cardano's formula in part (a) above.

(d) Verbeter die implementering van Cardano se formule in deel (b) hierbo.

Improve the implementation of Cardano's formula in part (b) above.

Probleem 5 ($2+4+2=8$ *punte*)

Beskou die vergelyking

Consider the equation

$$x \ln x - 1 = 0.$$

(a) Stel vas, grafies of andersinds, hoeveel reële wortels hierdie vergelyking het.

Determine, graphically or otherwise, how many real roots this equation has.

(b) Met aanvanklike skattings $x_0 = 1$, $x_1 = 2$, gebruik twee stappe van die secant metode om 'n wortel van die vergelyking te benader, d.w.s., bereken x_2 en x_3. Bied jou berekeninge só aan dat dit duidelik is dat net een funksie-evaluering per iterasie nodig is.

With initial guesses $x_0 = 1$, $x_1 = 2$, use two steps of the secant method to approximate a root of this equation, i.e., compute x_2 and x_3. Present your computations such that it is clear that only one function evaluation per iteration is necessary.

(c) Waar of Vals? As twee stappe van die Regula-Falsi metode op hierdie vergelyking toegepas word, met aanvanklike interval $[1, 2]$, sal presies dieselfde benaderings as met die secant metode van deel (b) verkry word. Motiveer jou antwoord.

True or False? If two steps of the Regula-Falsi method are applied to this equation, with initial interval $[1, 2]$, exactly the same approximations as with the secant method of part (b) will be obtained. Justify your answer.

Probleem 6 (5 *punte*)

Aldrie die volgende funksies het $p = 3^{1/3}$ as 'n wortel (aanvaar sonder bewys)

Each of these three functions has $p = 3^{1/3}$ as root (assume without proof)

$$f(x) = x^3 - 3, \qquad g(x) = x^2 - \frac{3}{x}, \qquad h(x) = x^6 - 6x^3 + 9.$$

Wanneer Newton se metode op hierdie funksies toegepas word, is die spoed van konvergensie egter heel verskillend, soos waargeneem in die tabel hieronder (bereken met MATLAB). Watter van hierdie resultate verteenwoordig die tipiese spoed van konvergensie van Newton se metode? Verklaar ook die oorsaak van die vinniger/stadiger konvergensie in die nie-tipiese gevalle.

When Newton's method is applied to these functions, however, the rate of convergence is very different, as observed in the table below (computed with MATLAB). Which of these results represent the typical convergence of Newton's method? In those cases that are non-typical, explain the cause of the quicker/slower convergence.

$f(x) = x^3 - 3$		$g(x) = x^2 - 3/x$		$h(x) = x^6 - 6x^3 + 9$	
x_n	$\lvert x_n - p\rvert$	x_n	$\lvert x_n - p\rvert$	x_n	$\lvert x_n - p\rvert$
1.47111111111111	2.8862e-02	1.40000000000000	4.2250e-02	1.43043456828063	1.1815e-02
1.44281209824934	5.6253e-04	1.44222431668238	2.5254e-05	1.43639099799476	5.8586e-03
1.44224978959900	2.1929e-07	1.44224957030740	5.1070e-15	1.43933224800129	2.9173e-03
1.44224957030744	3.3307e-14	1.44224957030741	2.2204e-16	1.44079386765107	1.4557e-03

Probleem 7 (5 *punte*)

Laat a 'n positiewe konstante wees. Aanvaar sonder bewys dat met geskikte aanvangsvoorwaardes, die volgende iterasie na $\sqrt{a}$ sal konvergeer.

Let a be a positive constant. Assume without proof that with suitable initial conditions, the following iteration will converge to $\sqrt{a}$.

$$x_{n+1} = \frac{1}{2a}\left(3ax_n - x_n^3\right)$$

Bepaal die orde van konvergensie, α, en die foutkonstante, C, soos gedefinieer deur

Determine the order of convergence, α, and the error constant, C, as defined by

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = C \qquad (e_n = x_n - \sqrt{a})$$

Probleem 8 ($3 + 2 = 5$ *punte*)

Die iterasie van Probleem 7 kom inderwaarheid van 'n toepassing van Newton se metode op die volgende vergelyking (aanvaar sonder bewys)

The iteration of Problem 7 comes in fact from an application of Newton's method to the following equation (assume without proof)

$$\frac{a}{x^2} - 1 = 0.$$

Die grafiek van die funksie links word hieronder getoon.

The graph of the function on the left is shown below.

(a) Vind die grootste interval (b, c) (sien figuur), met die eienskap dat enige aanvanklike skatting x_0 in (b, c) sal lei tot konvergensie na $\sqrt{a}$.

Find the largest interval (b, c) (see figure), with the property that each initial guess x_0 in (b, c) will lead to convergence to $\sqrt{a}$.

(b) Is daar ook aanvanklike skattings buite (b, c) wat sal lei tot konvergensie na $\sqrt{a}$? Motiveer jou antwoord grafies.

Are there also initial guesses outside (b, c) that will lead to convergence to $\sqrt{a}$? Justify your answer graphically.