NAAM/*Name:* Oplossings US NR. ____________

INSTRUKSIES:

(a) Twee uur plus vyftien minute, toeboek, agt probleme, 11 bladsye.

(b) Daar is 52 punte op die vraestel, maar 50 is volpunte.

(c) Alle probleme moet uitgeskryf word. Berekenings moet getoon word en alle stappe gemotiveer word. 'n Korrekte antwoord verdien nie volpunte sonder die nodige verduideliking nie.

(d) Beantwoord alle probleme in die toetsboek wat voorsien word.

(e) **Let Wel:** Die wenke hieronder kan enige plek in die vraestel **sonder bewys** gebruik word.

(f) Moenie omblaai voordat u aangesê word om dit te doen nie.

INSTRUCTIONS:

(a) Two hours plus fifteen minutes, closed book, eight problems, 11 pages.

(b) There are 52 marks in the paper, but 50 is full marks.

(c) All problems are to be written out. Calculations have to be shown and all steps must be justified. A correct answer does not earn full marks without the necessary explanation.

(d) Solve all problems in the test book that is provided.

(e) **Note:** *The hints below may be used* **without proof** *anywhere in the paper.*

(f) Do not turn the page until you are told to do so.

Wenke/Hints:

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots, \quad |x|<1, \qquad \sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \ldots,$$

$$\arctan x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \ldots, \quad |x|<1, \qquad \cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \ldots.$$

$$\int \frac{dx}{x^2+1} = \arctan x + C, \qquad \int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + C$$

$$\frac{dN}{dt} = bN - sN^2, \quad N(0)=\alpha \quad \Longrightarrow \quad N = \frac{b\alpha}{s\alpha + (b-s\alpha)e^{-bt}}$$

$$\frac{dy}{dx} + p(x)\,y = q(x) \quad \Longrightarrow \quad \text{Int. Faktor} = e^{\int p(x)\,dx}$$

$$\text{Newton:} \quad f(x)=0 \quad \Longrightarrow \quad x_{n+1} = x_n - f(x_n)/f'(x_n)$$

Vraag 1 (2+2+4+2=10 punte)

Beskou die Malthus model vir bevolkingsgroei, naamlik

Question 1 *(2+2+4+2=10 marks)*

Consider the Malthus model for population growth, namely

$$\frac{dy}{dt} = k\,y, \qquad y(0) = \alpha.$$

(a) Los die DV op en bepaal gevolglik 'n uitdrukking vir die verdubbelingstyd, sê T, van die bevolking.

(a) Solve the DE and hence determine an expression for the doubling time, say T, of the population.

(b) Gestel die Malthus model word soos volg gewysig

(b) Suppose the Malthus model is modified to

$$\frac{dy}{dt} = k\left(1 + \cos\left(\tfrac{2\pi}{P}t\right)\right)y, \qquad y(0) = \alpha.$$

Bespreek die moontlike aannames wat tot 'n model soos hierdie aanleiding kon gee. Sê spesifiek wat die fisiese betekenis van die konstante P kan wees.

Discuss the possible assumptions that may have lead to a model such as this. Say, in particular, what the physical meaning of the constant P may be.

(c) Toon aan dat die model van deel (b) se verdubbelingstyd, T, bepaal word deur

(c) Show that the doubling time, T, of the model in part (b) is determined by

$$T + \tfrac{P}{2\pi}\sin\left(\tfrac{2\pi}{P}T\right) = \tfrac{\ln 2}{k}.$$

(d) Toon aan dat vir groot waardes van P die verdubbelingstyd van die gewysigde model ongeveer die helfte van die verdubbelingstyd van die oorspronlike Malthus model is.

(d) Show that, for large values of P, the doubling time of the modified model is about one half of the doubling time of the original Malthus model.

(a) $\dfrac{dy}{dt} = ky$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln y = kt + c$$

<u>beginwaarde:</u> $y(0) = \alpha$

$$\ln\alpha = c$$

$$\Rightarrow \ln y = kt + \ln\alpha$$

$$\boxed{y = \alpha e^{kt}}$$

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Verdubbelingstyd T as

$y(T) = 2\alpha$:

$$\ln(2\alpha) = kT + \ln\alpha$$

$$kT = \ln\left(\frac{2\alpha}{\alpha}\right)$$

$$\boxed{T = \frac{\ln 2}{k}}$$

(b) $\cos \frac{2\pi}{p} t$ is periodies, met periode $= p$

$\Rightarrow$ $\frac{dy}{dt} \propto$ een of ander periodiese faktor soos bv.

seisoene jare, temp van dag teenoor temp van nag.

p is dan die periode wees, bv aantal dae in jaar.

(c)

$$\int \frac{1}{y}\, dy = \int k\left(1+\cos\left(\frac{2\pi}{p}t\right)\right) dt$$

$$\ln y = kt + \frac{p}{2\pi}\cdot \sin\left(\frac{2\pi}{p}t\right) + C$$

beginwaarde: $y(0) = \alpha$

$$C = \ln \alpha$$

$$\Rightarrow \quad \ln y = kt + k.\frac{p}{2\pi} \sin\left(\frac{2\pi}{p}t\right) + \ln\alpha$$

Verdubbelingstyd T as $y(T) = 2\alpha$

$$\ln 2\alpha - \ln\alpha = kT + \cancel{k}\frac{p}{2\pi} \sin\left(\frac{2\pi}{p}T\right)$$

$$\frac{\ln 2}{k} = T + \frac{\quad}{2\pi} \sin\left(\frac{2\pi}{p}T\right)$$

(d)

$$\lim_{p\to\infty}\left(\frac{\ln 2}{k}\right) = \lim_{p\to\infty}\left(T + \frac{p}{2\pi}\sin\left(\frac{2\pi}{p}T\right)\right)$$

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

$$\frac{\ln 2}{k} = T + \left(\lim_{p\to\infty} \frac{\sin \frac{2\pi}{p}T}{\frac{2\pi}{p}T}\right)T$$

$$\frac{\ln 2}{k} = 2T \quad \Rightarrow \quad \boxed{T = \frac{1}{2}\left(\frac{\ln 2}{k}\right)}$$

Vraag 2 (4 punte)

Beskou weer die gewysigde Malthus model van Probleem 1, en in die besonder die vergelyking vir die verdubbelingstyd in Prob 1(c). Met die keuse van parameters $k = 1$, $P = 2\pi$, word die vergelyking

Question 2 *(4 marks)*

Consider again the modified Malthus model of Problem 1, and specifically the equation of the doubling time in Prob 1(c). With the parameter choice $k = 1$, $P = 2\pi$, the equation becomes

$$T + \sin T = \ln 2.$$

Los hierdie vergelyking op met Newton se metode en aanvanklike skatting $T = 0$. Staak die iterasie sodra twee opeenvolgende skattings vir T binne 0.005 vanmekaar is.

Solve this equation using Newton's method with initial guess $T = 0$. Stop the iteration as soon as two consecutive approximations of T are within 0.005 of each other.

$$f(x) = x + \sin x - \ln 2$$

$$f'(x) = 1 + \cos x$$

Newton:
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

n	x_n	$f(x_n)$	$f'(x_n)$
0	0	-0.693147	2
1	0.346574	-0.006896	1.940542
2	0.350127	-0.152113×10^{-6}	1.939329
3	0.350128		

$$\Rightarrow \boxed{T \approx 0.350128}$$

Vraag 3 (4 punte)

'n Bevolking van 100 selle word geplaas in 'n medium waarvan die dra-kapasiteit 1000 selle is. Na 7 uur is daar 200 selle teenwoordig. Aanvaar dat die bevolking volgens die logistiese model groei, en skat die grootte daarvan na 14 uur.

Question 3 *(4 marks)*

A population of 100 cells is placed in a medium with carrying capacity 1000 cells. After 7 hours 200 cells are present. Assume that the population grows according to the logistic model, and estimate its size after 14 hours.

$\alpha = 100$ selle $\qquad \frac{b}{s} = 1000$ selle

Laat $N(t)$ die aantal selle op tyd t wees.

$N(7) = 200$

$N(14) = ?$

Gegee:

$$N(t) = \frac{b\alpha}{s\alpha + (b - \alpha s)e^{-bt}}$$

$$= \frac{\frac{b}{s}\alpha}{\alpha + \left(\frac{b}{s} - \alpha\right)e^{-bt}}$$

$$= \frac{1000 \times 100}{100 + (1000 - 100)e^{-bt}}$$

$$= \frac{1000}{1 + 9e^{-bt}}$$

Stel nou $N(7) = 200$ in om vir b op te los:

$$200 = \frac{1000}{1 + 9e^{-b(7)}}$$

$$1 + 9e^{-7b} = 5$$

$$-7b = \ln\frac{4}{9} \qquad \boxed{-b = \frac{1}{7}\ln\frac{4}{9}}$$

$$N(t) = \frac{1000}{1 + 9e^{\frac{t}{7}\ln 4/9}}$$

$$N(t) = \frac{1000}{1 + 9\cdot\left(\frac{4}{9}\right)^{t/7}}$$

$$N(14) = \frac{1000}{1 + 9e^{2\ln 4/9}}$$

$$= 360$$

Wat 'n sinvolle antwoord is omdat ons verwag dat $N(7) < N(14) < b/s$. Ook dit het 7 uur geneem voordat die 100 selle 200 geword het, dan in die volgende 7 uur het die selle amper weer verdubbel.

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Vraag 4 (4 punte)

'n Groot dam huisves 'n skool visse wat volgens die logistiese model groei, terwyl die vis ook gehengel word teen 'n konstante tempo. Die DV wat die grootte van die skool modelleer is

Question 4 *(4 marks)*

A large dam accommodates a school of fish that grows according to the logistic model, and which is also subject to angling at a constant rate. The DE that models the size of the school is

$$\frac{dN}{dt} = 4N - N^2 - 8, \qquad N(0) = 10,$$

waar N die aantal vis op tyd t voorstel (N in duisende, t in jaar). Toon aan dat die skool in bietjie meer as 'n jaar sal uitsterf.

where N is the number of fish at time t (N in thousands, t in years). Show that the school will die out in a little over a year.

$$\frac{b^2}{4s} = \frac{16}{4} = 4 < E = 8 \quad \Rightarrow \text{ groot jag}$$

(kwadratiese uitdrukking het nie reële wortels nie sal kwadraats voltooiing moet doen.

$$\int \frac{1}{N^2 - 4N + 8}\, dN = -\int dt$$

Beskou eers:

$$\int \frac{1}{N^2-4N+8}\, dN = \int \frac{1}{(N-2)^2 + 4}\, dN$$

$$= \frac{1}{4} \int \frac{1}{\left(\frac{N-2}{2}\right)^2 + 1}\, dN$$

$$= \frac{1}{4} \times 2 \arctan\left(\frac{N-2}{2}\right) = \frac{1}{2} \arctan\left(\frac{N-2}{2}\right)$$

$$\Rightarrow \quad \frac{1}{2} \arctan\left(\frac{N-2}{2}\right) = -t + C$$

$$N(0) = 10 \quad \Rightarrow \quad \frac{1}{2} \arctan(4) = C$$

$$\Rightarrow \quad \arctan\left(\frac{N-2}{2}\right) = -2t + \arctan(4) \qquad (*)$$

$$\boxed{N(t) = 2\tan(-2t + \arctan(4)) + 2}$$

$N = 0 \qquad t = ?$

Stel $N=0$ in $(*)$ en los op vir t

$$\arctan(-1) = -2t + \arctan(4)$$

$$t = \frac{1}{2}\left(\arctan(4) - \arctan(-1)\right)$$

$$= \frac{1}{2}\left(\arctan(4) + \frac{\pi}{4}\right) \approx 1.0556$$

Visse sterf uit na 1.0556 jaar.

Vraag 5 (4+4=8 punte)

Beskou die probleem van 'n klip wat vertikaal opwaarts gegooi word met aanvanklike snelheid v_0. As aangeneem word dat die lugweerstand eweredig aan die kwadraat van die snelheid is, word die opwaartse fase van die vlug gemodelleer deur

Question 5 *(4+4=8 marks)*

Consider the problem of a stone that is thrown vertically upwards with initial velocity v_0. If it is assumed that air resistance is proportional to the square of the velocity, the upward phase of the flight is modelled by

$$\frac{dv}{dt} = -g - k\,v^2, \qquad v(0) = v_0.$$

Hier is v die snelheid, t die tyd, k 'n konstante, en g die swaartekragversnelling. (Opwaarts positief.)

(a) Los hierdie aanvangswaardeprobleem op, en herlei sodoende 'n uitdrukking vir v as 'n funksie van t.

(b) Valideer u antwoord van deel (a) deur 'n dimensionele analise. Dit wil sê, toon aan dat u uitdrukking vir v wel die eenhede van 'n snelheid het.

Here v is the velocity, t the time, k a constant, and g the acceleration due to gravity. (Upwards positive.)

(a) Solve the initial value problem, and thus derive an expression for v as a function of t.

(b) Validate your result of part (a) by a dimensional analysis. That is, show that your expression for v indeed has the units of a velocity.

(a) $$\int \frac{1}{g+kv^2}\,dv = -\int dt$$

$$\int \frac{1}{1+\frac{k}{g}v^2}\,dv = -gt + K$$

Stel $a^2 = \frac{k}{g} \Rightarrow a = \sqrt{\frac{k}{g}}$

Beskou dan eers:

$$\int \frac{1}{1+\frac{k}{g}v^2}\,dv = \int \frac{1}{1+(av)^2}\,dv$$

$$= \frac{1}{a}\arctan(av)$$

$$\Rightarrow \arctan(av) = -gat + C$$

Beginwaarde: $v(0) = v_0$

$$\Rightarrow C = \arctan(av_0)$$

$$\arctan(av) = \arctan(av_0) - gat$$

$$\boxed{\arctan\left(\sqrt{\frac{k}{g}}\,v\right) = \arctan\left(\sqrt{\frac{k}{g}}\,v_0\right) - \sqrt{kg}\,t}$$

implisiete Formule

$$\sqrt{\frac{k}{g}}\,v = \tan\left(\arctan\sqrt{\frac{k}{g}}\,v_0 - \sqrt{kg}\,t\right)$$

$$\boxed{v(t) = \sqrt{\frac{g}{k}}\,\tan\left(\arctan\sqrt{\frac{k}{g}}\,v_0 - \sqrt{kg}\,t\right)}$$

explisiete formule

(b)

$$[g] = \frac{m}{s^2} \qquad \Rightarrow \qquad [k] = \frac{1}{m}$$

$$[v] = \frac{m}{s}$$

sodat DV sin maak.

$$\left[\sqrt{\frac{k}{g}}\, v_0\right] = \sqrt{\frac{1}{m}\cdot\frac{s^2}{m}}\ \frac{m}{s} = \text{dimensieloos.}$$

$\Rightarrow$ argument van arctan is dimensieloos.

$$\left[\sqrt{kg}\; t\right] = \sqrt{\frac{1}{m}\cdot\frac{m}{s^2}}\ s = \text{dimensieloos}$$

$\Rightarrow$ argument van tan is die som van twee dim.loos terme.

$$\left[\sqrt{\frac{g}{k}}\right] = \sqrt{\frac{m}{s^2}\cdot\frac{m}{1}} = \frac{m}{s} \quad \rightarrow \quad \text{dimensie van snelheid}$$

$\Rightarrow$ korrekte dimensies.

Vraag 6 (5+3=8 punte)

Beskou weer die situasie van Probleem 5, maar aanvaar nou dat lugweerstand eweredig aan die snelheid is (en nie die kwadraat van die snelheid nie), d.w.s.,

Question 6 *(5+3=8 marks)*

Consider again the situation of Problem 5, but assume that air resistance is now proportional to velocity (and not the square of velocity), i.e.,

$$\frac{dv}{dt} = -g - k\,v, \qquad v(0) = v_0.$$

(a) Aanvaar sonder bewys dat integrasie van hierdie vergelyking lei na

(a) Assume without proof that integration of this equation leads to

$$v = -\tfrac{g}{k} + \left(v_0 + \tfrac{g}{k}\right)e^{-kt}. \qquad (*)$$

Herlei nou 'n formule vir die maksimum hoogte wat die klip bereik. Vereenvoudig u antwoord sover moontlik.

Now derive a formula for the maximum heigth that the stone reaches. Simplify your answer as far as possible.

(b) Aanvaar sonder bewys dat as lugweerstand buite rekening gelaat word, dan bereik so 'n klip 'n maksimumhoogte

(b) Assume without proof that if air resitance is ignored, such a stone will reach a maximum height

$$H = \frac{v_0^2}{2g}.$$

Bevestig nou dat u antwoord van deel (a) na hierdie formule reduseer in die limiet $k \longrightarrow 0$.

Now confirm that your answer of part (a) reduces to this formula in the limit $k \longrightarrow 0$.

(a)

By maks hoogte is $v=0$.

Stel $v=0$ in vgl $(*)$ en los op vir $t=T$:

$$0 = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kT}$$

$$e^{-kT} = \frac{g/k}{v_0 + g/k}$$

$$-kT = \ln\left(\frac{g}{kv_0 + g}\right)$$

$$T = \frac{1}{k}\ln\left(\frac{kv_0 + g}{g}\right)$$

$$= \frac{1}{k}\ln\left(1 + \frac{k}{g}v_0\right)$$

· Integreer $(*)$ mbt tyd

$$\frac{dy}{dt} = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}$$

$$y = -\frac{g}{k}t - \frac{1}{k}\left(v_0 + \frac{g}{k}\right)e^{-kt} + C$$

$y(0) = 0 \Rightarrow$

$$C = \frac{1}{k}\left(v_0 + \frac{g}{k}\right)$$

$\Rightarrow$

$$y(t) = \frac{1}{k}\left[\left(v_0 + \frac{g}{k}\right) - gt - \left(v_0 + \frac{g}{k}\right)e^{-kt}\right]$$

$$= \frac{1}{k}\left[\left(v_0 + \frac{g}{k}\right)\left(1 - e^{-kt}\right) - gt\right]$$

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Maks hoogte H as $t = T$:

$$H = y(T) = \frac{1}{k}\left[\left(v_0 + \frac{g}{k}\right)\left(1 - \frac{g}{kv_0 + g}\right) - \frac{g}{k}\ln\left(1 + \frac{k}{g}v_0\right)\right]$$

Waar ons die vereenvoudiging gebruik het.

$$= \frac{1}{k}\left[v_0 - \frac{g}{k}\ln\left(1 + \frac{k}{g}v_0\right)\right]$$

$$\boxed{\left(v_0 + \frac{g}{k}\right)\left(1 - \frac{g}{kv_0 + g}\right) = \left(\frac{v_0 k + g}{k}\right)\left(\frac{kv_0}{kv_0 + g}\right) = v_0}$$

(b) Gebruik die Taylor reeks

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \qquad |x| < 1.$$

$$H = \frac{1}{k}\left[v_0 - \frac{g}{k}\left(\frac{k}{g}v_0 - \frac{1}{2}\left(\frac{k}{g}v_0\right)^2 + \frac{1}{3}\left(\frac{k}{g}v_0\right)^3 - \dots\right]\right.$$

$$= \frac{1}{k}\left[\frac{1}{2}\frac{k}{g}v_0^2 - \frac{1}{3}\frac{k^2}{g^2}v_0^3 + \dots\right]$$

$$= \frac{1}{2}\frac{v_0^2}{g} - \frac{1}{3}\frac{k}{g^2}v_0^2 + \dots \quad \text{(hoër magte van } k\text{)}$$

$$\longrightarrow \frac{1}{2}\frac{v_0^2}{g} = \text{maks hoogte as } k \to 0.$$

maks hoogte van model sonder lugweerstand

Vraag 7 (2+2+2+4=10 punte)

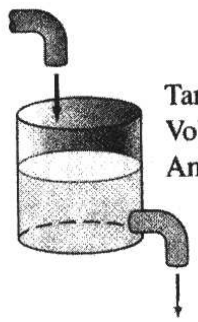
(a) Die tenk in die eerste figuur het 'n volume $100\,\ell$. Aanvanklik is dit gevul met water waarin $8\,kg$ sout opgelos is. Suiwer water vloei die tenk nou binne teen $5\,\ell/min$ en die mensgsel, wat goed geroer word, vloei na buite teen dieselfde tempo. Laat $x = x(t)$ die massa sout in die tenk wees op tyd t. Herlei, met die nodige verduideliking, 'n DV wat die tempo van toename/afname van x beskryf.

(b) Los die DV van deel (a) op, en toon sodoende aan dat $x = Ce^{-\frac{1}{20}t}$ waar u die konstante C moet bepaal.

Question 7 *(2+2+2+4=10 marks)*

(a) The tank in the first figure has volume $100\,\ell$. Initially it is filled with water in which $8\,kg$ salt is dissolved. Pure water now flows into the tank at a rate $5\,\ell/min$, and the mixture, which is stirred well, flows out at the same rate. Let $x = x(t)$ be the mass of salt in the tank at time t. Derive, with the necessary explanation, a DE that describes the rate of increase/decrease in x.

(b) Solve the DE of part (a), and hence show that $x = Ce^{-\frac{1}{20}t}$ where you need to determine the constant C.



(a)

$$\begin{pmatrix}\text{tempo van}\\ \text{toename/afname}\end{pmatrix} = \begin{pmatrix}\text{tempo van}\\ \text{sout wat}\\ \text{in kom}\end{pmatrix} - \begin{pmatrix}\text{tempo van}\\ \text{sout wat}\\ \text{uit gaan}\end{pmatrix}$$

$$= \frac{0\text{ kg}}{\ell} \times \frac{5\,\ell}{\text{min}} - \frac{x\text{ kg}}{100\,\ell} \times \frac{5\,\ell}{\text{min}}$$

$$\frac{dx}{dt} = -\frac{1}{20}x$$

$$x(0) = 8$$

(b)

$$\int \frac{1}{x}\,dx = -\frac{1}{20}\int dt$$

$$\ln x = -\frac{1}{20}t + C$$

$$x = Ce^{-\frac{1}{20}t}$$

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Beginwaarde: $x(0)=8$

$\Rightarrow C = 8$

$$\boxed{x(t) = 8e^{-\frac{1}{20}t}}$$

(c) Gestel nou die uitvloeistroom van die tenk hierbo word die invloeistroom vir 'n tweede tenk, soos in die tweede figuur. Die tweede tenk het volume 200 ℓ, en is ook aanvanklik gevul met water waarin 8 kg sout opgelos is. Die uitvloeitempo uit die tweede tenk is ook 5 ℓ/min. Laat $y = y(t)$ die massa sout in die tweede tenk wees op tyd t. Toon aan dat

(c) Suppose the exit stream of the tank above becomes the inflow stream of a second tank, as in the second figure. The second tank has volume 200 ℓ, and is also initially filled with water in which 8 kg salt is dissolved. The outflow rate of the second tank is also 5 ℓ/min. Let $y = y(t)$ be the mass of salt in the second tank at time t. Show that

$$\frac{dy}{dt} = \frac{x}{20} - \frac{y}{40}.$$

(d) Los die DV van probleem (c) op, en bepaal die maksimum hoeveelheid sout wat daar op enige stadium in die tweede tenk is. *Wenk:* Gebruik die resultaat van deel (b) en stel dit in die regterkant van die DV in (c).

(d) Solve the DE of part (c), and determine the maximum amount of salt in the second tank at any stage. Hint: Use the result of part (b) and substitute it into the right-hand side of the DE in (c).

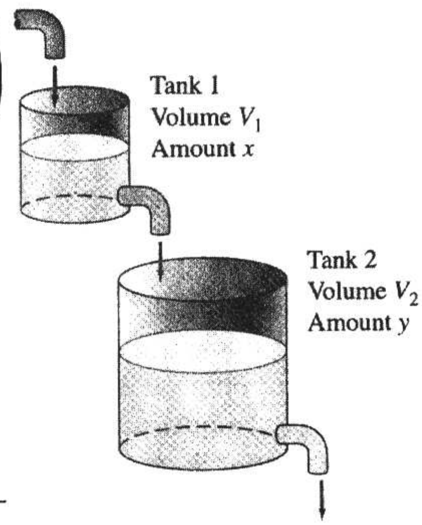
(c)

$$\begin{pmatrix}\text{tempo van}\\ \text{toename/afname}\end{pmatrix} = \begin{pmatrix}\text{tempo}\\ \text{wat sout}\\ \text{in kom}\end{pmatrix} - \begin{pmatrix}\text{tempo}\\ \text{wat sout}\\ \text{uit gaan}\end{pmatrix}$$

$$= \frac{x\,kg}{100\,\ell}\cdot\frac{5\ell}{min} - \frac{y}{200\,\ell}\cdot\frac{5\ell}{min}$$

$$\frac{dy}{dt} = \frac{x}{20} - \frac{y}{40}$$

$$y(0) = 8$$



(d)

$$\frac{dy}{dt} = \frac{1}{20}\cdot 8e^{-\frac{1}{20}t} - \frac{1}{40}y$$

lineêre diff. vgl.

$$\frac{dy}{dt} + \frac{1}{40}y = \frac{2}{5}e^{-\frac{1}{20}t}$$

Integrasie faktor:

$$I(x) = e^{\int 1/40\, dt} = e^{t/40}$$

$$e^{t/40}\frac{dy}{dt} + e^{t/40}\frac{1}{40}y = \frac{2}{5}e^{-\frac{1}{40}t}$$

$$\frac{d}{dt}\left(e^{t/40}y\right) = \frac{2}{5}e^{-1/40t}$$

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$$e^{t/40}\, y = \frac{2}{5}\int e^{-1/40\, t}\, dt$$

$$= \frac{2}{5}(-40)\, e^{-1/40 t} + K.$$

$$y = -16 e^{-t/20} + K e^{-t/40}$$

$y(0)=8 \Rightarrow$

$$8 = -16 + K \qquad \Rightarrow \qquad K = 24$$

$$\boxed{y(t) = 24 e^{-t/40} - 16 e^{-t/20}}$$

$\begin{pmatrix} \text{MAKS hoeveelheid} \\ \text{in tenk 2} \end{pmatrix}$ waar $\dfrac{dy}{dt} = 0$

$$\Rightarrow \quad 24 \cdot \left(-\frac{1}{40}\right) e^{-t/40} - 16\left(-\frac{1}{20}\right) e^{-t/20} = 0$$

$$-\frac{3}{5} e^{-t/40} + \frac{4}{5} e^{-t/20} = 0$$

$$3 e^{-t/40} = 4 e^{-t/20}$$

$$e^{t/40} = \frac{4}{3}$$

$$\frac{t}{40} = \ln 4/3$$

$$t_{maks} = t_m = 40 \ln(4/3) \approx 11.5 \text{ minute}$$

$$y(t_{maks}) = 24 e^{-\ln 4/3} - 16 e^{-2\ln 4/3} = 24\left(\tfrac{3}{4}\right) - 16\left(\tfrac{9}{16}\right) = 9 \text{ kg}$$

Vraag 8 (4 punte)

Volgens een van Kepler se wette word die baan van 'n planeet om die son gegee deur die formule

Question 8 *(4 marks)*

According to one of Kepler's laws the orbit of a planet around our sun is given by

$$r = \frac{p}{1+\epsilon\cos\theta},$$

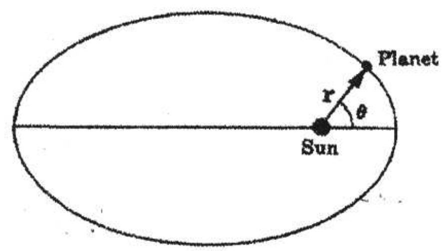
waar r en θ in die figuur gedefinieer word, en p en ϵ konstantes is.
Versonderstel 'n groot aantal eksperimentele datawaardes van die vorm

where r and θ are defined in the figure, and p and ϵ are constants.
Suppose you are given, for a specific planet, a large number of experimental data values

$$(\theta_j, r_j), \qquad j = 1, 2, \ldots, n,$$

word aan u verskaf. Bespreek hoe u te werk sal gaan om goeie skattings vir die waardes van p en ϵ te bereken. *Wenk:* Herskryf die formule as $r(1+\epsilon\cos\theta) = p$.

Discuss how you will go about computing good estimates for the values of p and ϵ. Hint: Rewrite the formula as $r(1+\epsilon\cos\theta) = p$.



$$r(1+\varepsilon\cos\theta) = p$$

Ons wil vir ε en p oplos.

$$r + r\varepsilon\cos\theta = p$$

$$r\varepsilon\cos\theta - p = -r$$

Stel data in:

$$r_1\varepsilon\cos\theta_1 - p = -r_1$$

$$r_2\varepsilon\cos\theta_2 - p = -r_2$$

$$r_3\varepsilon\cos\theta_3 - p = -r_3$$

$$\vdots$$

Ons moet vir ε en p op los.

Herskryf dit as 'n lineêre stelsel

$$\underbrace{\begin{bmatrix} r_1 \cos\theta_1 & -1 \\ r_2 \cos\theta_2 & -1 \\ r_3 \cos\theta_3 & -1 \\ \vdots & \\ r_n \cos\theta_n & -1 \end{bmatrix}}_{A} \begin{bmatrix} \varepsilon \\ \rho \end{bmatrix} = \underbrace{\begin{bmatrix} -r_1 \\ -r_2 \\ -r_3 \\ \vdots \\ -r_n \end{bmatrix}}_{b}$$

Die kleinste-kwadrate oplossing is dan gegee $(A^T A)^{-1} (A^T \underline{b})$.