Probleem 1:

$$x' = -x + y \qquad x(0) = 1$$

$$y' = x + y \qquad y(0) = 1.$$

met $\underline{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ is die lineêre stelsel equivalent aan

$$\frac{d}{dt}\underline{u} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\underline{u} \quad , \quad \underline{u}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Diagonaliseer $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

Die eiewaardes van A is $\lambda = \pm\sqrt{2}$.

<u>Ooreenstemmende eievektore:</u>

Vir

$\lambda = -\sqrt{2}$,

soek ons alle $\underline{x} \in \mathbb{R}^2$ sodat

$$\begin{bmatrix} -1+\sqrt{2} & 1 \\ 1 & 1+\sqrt{2} \end{bmatrix}\underline{x} = \underline{0}$$

$$\begin{bmatrix} -1+\sqrt{2} & 1 \\ 1-\sqrt{2} & (1+\sqrt{2})(1-\sqrt{2}) \end{bmatrix}$$ (Maal ry 2 met $1-\sqrt{2}$)

$$\rightarrow \begin{bmatrix} -1+\sqrt{2} & 1 \\ 1-\sqrt{2} & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1+\sqrt{2} & 1 \\ 0 & 0 \end{bmatrix}$$

dus $x_2 = s$ (vrye veranderlike)

$$x_1 = \frac{-s}{-1+\sqrt{2}} = \frac{s}{1-\sqrt{2}} \times \frac{1+\sqrt{2}}{1+\sqrt{2}}$$

$$= -s(1+\sqrt{2})$$

$\Rightarrow$ 'n eievektor is $\begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix}$

Vir $\lambda = \sqrt{2}$ is 'n eievektor

$$\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}$$ (toon self)

Dus $\qquad \dfrac{d\underline{u}}{dt} = P D P^{-1} \underline{u}$

met $\qquad P = \begin{bmatrix} -1-\sqrt{2} & -1+\sqrt{2} \\ 1 & 1 \end{bmatrix}$ en $D = \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

$\Rightarrow \qquad P^{-1} \dfrac{d\underline{u}}{dt} = D P^{-1} \underline{u}$

P^{-1} onafhanklik van t, dus

$$\frac{d}{dt}\left(P^{-1}\underline{u}\right) = D\left(P^{-1}\underline{u}\right)$$

Definieer nuwe vektor $\underline{w} = P^{-1}\underline{u}$ en los dan op vir $\underline{w}$ in

$$\frac{d}{dt}\underline{w} = D\underline{w}$$

$$\frac{d}{dt}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

oftewel

$$\left.\begin{aligned} \frac{dw_1}{dt} &= -\sqrt{2}\, w_1 \\ \frac{dw_2}{dt} &= \sqrt{2}\, w_2 \end{aligned}\right\}$$

Herken hierdie as eksponsiële verval

$$\Rightarrow \quad w_1 = c_1 e^{-\sqrt{2}t}$$

$$w_2 = c_2 e^{+\sqrt{2}t}$$

$$\Rightarrow \quad \underline{w} = \begin{bmatrix} c_1 e^{-\sqrt{2}t} \\ c_2 e^{\sqrt{2}t} \end{bmatrix}.$$

Nou dat ons al hierdie moeite gedoen het om $\underline{w}$ te kry, is $\underline{u}$ gelukkig net 'n matriks vermenigvuldiging $\underline{w} = P^{-1}\underline{u}$

$$\Rightarrow \quad \underline{u} = P\underline{w} = \begin{bmatrix} -1-\sqrt{2} & -1+\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-\sqrt{2}t} \\ c_2 e^{\sqrt{2}t} \end{bmatrix}$$

$$= \begin{bmatrix} -c_1(1+\sqrt{2})e^{-\sqrt{2}t} + c_2(-1+\sqrt{2})e^{\sqrt{2}t} \\ c_1 e^{-\sqrt{2}t} + c_2 e^{\sqrt{2}t} \end{bmatrix}$$

Dit bly oor om vir c_1 en c_2 op te los.

Om dit te doen, herskryf ons die vergelyking hierbo as

$$\underline{u} = \begin{bmatrix} -(1+\sqrt{2})e^{-\sqrt{2}t} & (-1+\sqrt{2})e^{\sqrt{2}t} \\ e^{-\sqrt{2}t} & e^{\sqrt{2}t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad t$$

en gebruik die aanvangswaardes $x(0) = y(0) = 1$

om vir c_1 en c_2 op te los as volg:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -(1+\sqrt{2}) & (-1+\sqrt{2}) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-2\sqrt{2}} \begin{bmatrix} 1 & (1-\sqrt{2}) \\ -1 & -(1+\sqrt{2}) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} \sqrt{2}-1 \\ -\sqrt{2}-1 \end{bmatrix}$$

$$\Rightarrow \quad x = u_1 = -\frac{1}{2}(1-\sqrt{2})(1+\sqrt{2})e^{-\sqrt{2}t} + \frac{1}{2}(1+\sqrt{2})(-1+\sqrt{2})e^{\sqrt{2}t}$$

$$= +\frac{1}{2}e^{-\sqrt{2}t} + \frac{1}{2}e^{\sqrt{2}t} = \cosh\sqrt{2}t$$

Soos antwoord verkry deur Laplace in toets

en

$$y = u_2 = -\frac{1}{2}(\sqrt{2}-1)e^{-\sqrt{2}t} + \frac{1}{2}(\sqrt{2}+1)e^{\sqrt{2}t}$$

Probleem 2:

(a) Here $m = 2$ slugs and $k = 64/.5 = 128$ lb-s/ft. Therefore, the equation of motion (1) is

$$2y'' + cy' + 128y = 0. \tag{4}$$

The characteristic equation is

$$2r^2 + cr + 128 = 0,$$

which has roots

$$r = \frac{-c \pm \sqrt{c^2 - 8 \cdot 128}}{4}.$$

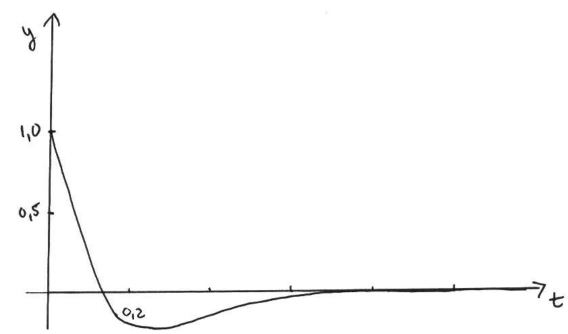
Therefore, the damping is critical if

$$c = \sqrt{8 \cdot 128} = 32 \text{ lb-s/ft}.$$

Imposing the initial conditions $y(0) = 1$ and $y'(0) = -20$ in (5) and (6) yields $1 = c_1$ and $-20 = -8 + c_2$. Hence, the solution of this initial value problem is

$$y = e^{-8t}(1 - 12t).$$

Therefore, the object moves downward through equilibrium just once, and then approaches equilibrium from below as $t \to \infty$.



(b) ligte demping: $c = 4 < 32$ ($c = 32$ is in (a) bepaal as die kritiek waarde vir die dempingskonstante c), of $c^2 - 4mk = 16 - 4 \times 2 \times 128 < 0$, dus ligte demping.

With $c = 4$ the equation of motion (4) becomes

$$y'' + 2y' + 64y = 0 \tag{7}$$

after canceling the common factor 2. The characteristic equation

$$r^2 + 2r + 64 = 0$$

has complex conjugate roots

$$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 64}}{2} = -1 \pm 3\sqrt{7}i.$$

Therefore, the motion is underdamped and the general solution of (7) is

$$y = e^{-t}\left(c_1 \cos 3\sqrt{7}t + c_2 \sin 3\sqrt{7}t\right).$$

Differentiating this yields

$$y' = -y + 3\sqrt{7}e^{-t}\left(-c_1 \sin 3\sqrt{7}t + c_2 \cos 3\sqrt{7}t\right).$$

Imposing the initial conditions $y(0) = 1.5$ and $y'(0) = -3$ in the last two equations yields $1.5 = c_1$ and $-3 = -1.5 + 3\sqrt{7}c_2$. Hence, the solution of the initial value problem is

$$y = e^{-t}\left(\frac{3}{2} \cos 3\sqrt{7}t - \frac{1}{2\sqrt{7}} \sin 3\sqrt{7}t\right). \tag{8}$$

The amplitude of the function in parentheses is

$$R = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2\sqrt{7}}\right)^2} = \sqrt{\frac{9}{4} + \frac{1}{4 \cdot 7}} = \sqrt{\frac{64}{4 \cdot 7}} = \frac{4}{\sqrt{7}}.$$

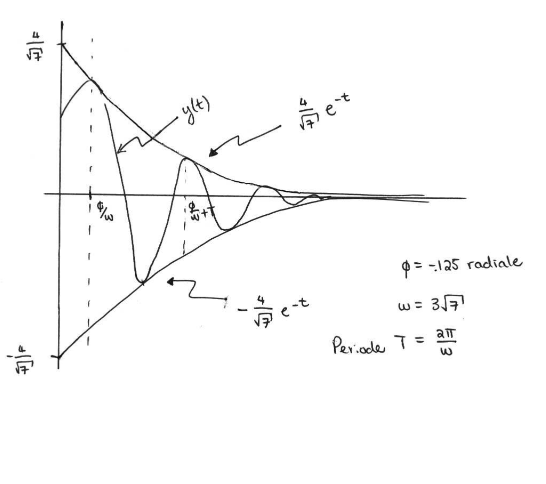
Therefore, we can rewrite (8) as

$$y = \frac{4}{\sqrt{7}} e^{-t} \cos\left(3\sqrt{7}t - \phi\right),$$

where

$$\cos \phi = \frac{3}{2R} = \frac{3\sqrt{7}}{8} \quad \text{and} \quad \sin \phi = -\frac{1}{2\sqrt{7}R} = -\frac{1}{8}.$$

Therefore $\phi \cong -.125$ radians.



(c) swaar demping: $c = 40 > 32$ ($c = 32$ is in (a) bepaal as die kritiek waarde vir die dempingskonstante c), of $c^2 - 4mk = 40^2 - 4 \times 2 \times 128 > 0$, dus swaar demping.

With $c = 40$ the equation of motion (4) reduces to

$$y'' + 20y' + 64y = 0 \tag{9}$$

after canceling the common factor 2. The characteristic equation

$$r^2 + 20r + 64 = (r + 16)(r + 4) = 0$$

has the roots $r_1 = -4$ and $r_2 = -16$. Therefore, the general solution of (9) is

$$y = c_1 e^{-4t} + c_2 e^{-16t}. \tag{10}$$

Differentiating this yields

$$y' = -4e^{-4t} - 16c_2 e^{-16t}.$$

The last two equations and the initial conditions $y(0) = 1$ and $y'(0) = 1$ imply that

$$\begin{aligned} c_1 + \quad c_2 &= 1 \\ -4c_1 - 16c_2 &= 1. \end{aligned}$$

The solution of this system is $c_1 = 17/12, c_2 = -5/12$. Substituting these into (10) yields

$$y = \frac{17}{12}e^{-4t} - \frac{5}{12}e^{-16t}$$

as the solution of the given initial value problem.