Probleem 1:

(a)

$$f(x) \;=\; (1+x^3)^{1/2} \quad \Rightarrow \quad f'(x) \;=\; \frac{3}{2}\frac{x^2}{(1+x^3)^{1/2}}$$

Dus $f'(1) = \frac{3}{2\sqrt{2}}$ en $f'(3) = \frac{27}{2\sqrt{28}}$. Volgens die Euler-Maclaurin foutafskatting,

$$I - T_n \;\approx\; -\frac{h^2}{12}\left(\frac{27}{2\sqrt{28}} - \frac{3}{2\sqrt{2}}\right) \;=\; -0.1242166684\, h^2.$$

Ons vereis dat die absolute fout nie groter as 10^{-4} moet wees nie, dus

$$|I - T_n| \;\leq\; 10^{-4} \quad \Rightarrow \quad h \;\leq\; 0.02837 \quad \Rightarrow \quad n \;=\; \frac{b-a}{h} \;\geq\; 70.5.$$

Gebruik dus 71 intervalle.

(b)

Aanvaar $I - T_n = Ch^2$, dus $I - T_{2n} = C(\frac{h}{2})^2 = C\frac{h^2}{4}$. Elimineer I deur die twee vergelykings van mekaar af te trek. Dan $T_{2n} - T_n = \frac{3}{4}Ch^2$. Dus

$$C \;=\; \frac{4}{3}\left(\frac{T_{2n} - T_n}{h^2}\right).$$

Met $n = 4$ is $h = \frac{1}{2}$, $T_4 = 6.26094238308067$ en $T_8 = 6.23771877158169$. Dus

$$C \;=\; \frac{4}{3}(T_8 - T_4)\cdot 4 \;=\; -0.1238592613.$$

[LW: Dit is baie naby aan die skatting van deel (a).]

Ons vereis weer dat die absolute fout nie groter as 10^{-4} moet wees nie, dus

$$|Ch^2| \;\leq\; 10^{-4} \quad \Rightarrow \quad h \;\leq\; 0.02841 \quad \Rightarrow \quad n \;=\; \frac{b-a}{h} \;\geq\; 70.4.$$

Gebruik dus 71 intervalle.

(c)

Die volgende stukkie kode bevestig dat 71 intervalle inderdaad die gevraagde akkuraatheid lewer:

```
>> f = inline('sqrt((1+x.^3))');
>> for n = 68:72,
      T = trap(f,1,3,n); disp([n abs(T-6.229959387883636)])
   end

  68   1.0745e-004
  69   1.0436e-004
  70   1.0140e-004
  71   9.8564e-005
  72   9.5846e-005
```

Die 71 intervalle van die metodes van dele (a) en (b) is dus baie nader aan die kol as die 95 intervalle van Huiswerk 10, Probleem 3.

(d)

Aanvaar $I - S_n = Ch^4$, dus $I - S_{2n} = C(\frac{h}{2})^4 = C\frac{h^4}{16}$. Elinimeer I deur die twee vergelykings van mekaar af te trek. Dan $S_{2n} - S_n = \frac{15}{16}Ch^4$. Dus

$$C = \frac{16}{15}\left(\frac{S_{2n} - S_n}{h^4}\right).$$

Met $n = 4$ is $h = \frac{1}{2}$, $S_4 = 6.23030381335719$ en $S_8 = 6.22997756774869$. Dus

$$C = \frac{16}{15}(S_8 - S_4) \cdot 16 = -0.00556792505173.$$

Ons vereis nou dat die absolute fout nie groter as 10^{-10} moet wees nie, dus

$$|Ch^4| \leq 10^{-10} \quad \Rightarrow \quad h \leq 0.01158 \quad \Rightarrow \quad n = \frac{b-a}{h} \geq 172.7.$$

Gebruik dus 174 intervalle. (Nie 173 nie, die Simpsonreël moet 'n ewe aantal intervalle hê!)

Die volgende kode wys dat 162 intervalle inderdaad genoeg is:

```
>> for n = 160:2:174
       S = simpson(f,1,3,n); disp([n abs(S-6.229959387883636)])
   end

  160   1.0147e-010
  162   9.6549e-011
  164   9.1926e-011
  166   8.7575e-011
  168   8.3477e-011
  170   7.9618e-011
  172   7.5980e-011
  174   7.2544e-011
```

Dit is bietjie minder as ons geskatte 174 intervalle, maar ons skatting was nie te ver van die kol nie.

[As S_8 en S_{16} instede van S_4 en S_8 gebruik word, word $C = -0.0047$, en dan word $n = 166$ intervalle verkry, wat nader aan die kol is.]

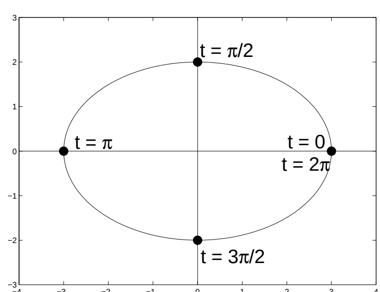
Probleem 2:

(a)

Stel $x = 3\cos t$ en $y = 2\sin t$ in die linkerkant van die vergelyking,

$$\begin{aligned} 4x^2 + 9y^2 &= 4 \cdot 9\cos^2 t + 9 \cdot 4\sin^2 t \\ &= 36(\cos^2 t + \sin^2 t) \;=\; 36, \end{aligned}$$

en dit is die regterkant.



(b)

Herleiding van die formule vir booglengte (interessantheidshalwe):

$$\begin{aligned}
\Delta s^2 &= \Delta x^2 + \Delta y^2 \\
\left(\frac{\Delta s}{\Delta t}\right)^2 &= \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 \\
\Delta s &= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t
\end{aligned}$$

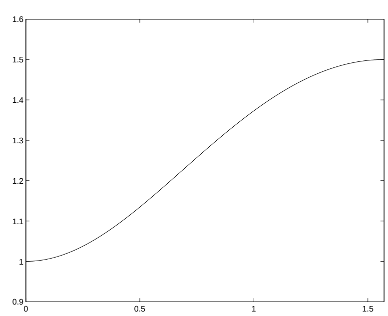
Laat $\Delta t \to 0$ en integreer weerskante,

$$\int_0^s \mathrm{d}s = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Vir hierdie probleem is $x = 3\cos t$ en $y = 2\sin t$ sodat $\frac{\mathrm{d}x}{\mathrm{d}t} = -3\sin t$ en $\frac{\mathrm{d}y}{\mathrm{d}t} = 2\cos t$. Ons bereken die booglengte van een kwart van die ellips se omtrek, en vermenigvuldig met 4 om die omtrek van die hele ellips te kry:

$$\begin{aligned}
s &= 4\int_0^{\frac{\pi}{2}} \sqrt{9\sin^2 t + 4\cos^2 t}\,\mathrm{d}t \\
&= 4\int_0^{\frac{\pi}{2}} \sqrt{9\sin^2 t + 4(1-\sin^2 t)}\,\mathrm{d}t \\
&= 4\int_0^{\frac{\pi}{2}} \sqrt{4 + 5\sin^2 t}\,\mathrm{d}t \\
&= 8\int_0^{\frac{\pi}{2}} \sqrt{1 + \frac{5}{4}\sin^2 t}\,\mathrm{d}t
\end{aligned}$$

(c)



Die helling by $t = 0$ is dieselfde as die helling by $t = \frac{\pi}{2}$, dus sal die Trapesiumreël vinniger as normaalweg konvergeer.

(d)

n	T_n	$\lvert I - T_n \rvert$
2	15.86350275618153	1.9368×10^{-3}
4	15.86543855406580	1.0352×10^{-6}
6	15.86543958840492	8.8567×10^{-10}
8	15.86543958928968	9.1482×10^{-13}
10	15.86543958929059	1.7764×10^{-15}

Konvergensie is inderdaad baie vinniger as wat normaalweg van die Trapesiumreël verwag word. As die aantal intervalle verdubbel word, verwag mens tipies dat die fout met 'n faktor 4 moet verminder. In hierdie probleem kry mens egter 'n faktor $\frac{1.0352\times10^{-6}}{9.1482\times10^{-13}} = 1.1316 \times 10^6$ as die aantal intervalle van 4 na 8 verdubbel word.

Probleem 3:

(a)

Laat $x = \alpha t + \beta$, met $x \in [1,3]$ en $t \in [-1,1]$. D.w.s.

$$\left.\begin{array}{rcl} 1 &=& \alpha(-1)+\beta \\ 3 &=& \alpha(1)+\beta \end{array}\right\} \Longrightarrow \begin{array}{c} \beta = 2 \\ \alpha = 1. \end{array}$$

Dus $x = t+2$ en $\mathrm{d}x = \mathrm{d}t$, sodat

$$I \;=\; \int_1^3 \sqrt{1+x^3}\,\mathrm{d}x \;=\; \int_{-1}^1 \sqrt{1+(t+2)^3}\,\mathrm{d}t.$$

Tweepunt Gauss-reël:

$$\begin{aligned} G_2 &= \sqrt{1+\left(-\frac{1}{\sqrt{3}}+2\right)^3} + \sqrt{1+\left(\frac{1}{\sqrt{3}}+2\right)^3} \\ &= 6.226441785 \\ |I - G_2| &= 0.0035. \end{aligned}$$

Driepunt Gauss-reël:

$$\begin{aligned} G_3 &= \frac{5}{9}\sqrt{1+\left(-\sqrt{\frac{3}{5}}+2\right)^3} + \frac{8}{9}\sqrt{1+(0+2)^3} + \frac{5}{9}\sqrt{1+\left(\sqrt{\frac{3}{5}}+2\right)^3} \\ &= 6.22993431873294 \\ |I - G_3| &= 0.000025. \end{aligned}$$

(b)

```
>> I = 6.229959387883646;
>> F = inline('sqrt(1+(2+t).^3)');
>> [t,w]  = gauss(2);
>> G2 = w*F(t)
>> abs(I-G2)

>> [t,w]  = gauss(3);
>> G3 = w*F(t)
>> abs(I-G3)
```

Ons verkry dieselfde waardes vir `G2`, `G3` en die absolute foute as in deel (a).

(c)

```
>> for n = 4:10
       [t,w] = gauss(n);
       G = w*F(t);
       disp([n G abs(G-I)])
   end
```

n	G_n	$\lvert I - G_n \rvert$
4	6.22996851633989	9.1285×10^{-6}
5	6.22995998871568	6.0083×10^{-7}
6	6.22995938095061	6.9330×10^{-9}
7	6.22995938464617	3.2375×10^{-9}
8	6.22995938774417	1.3946×10^{-10}
9	6.22995938789365	1.0011×10^{-11}
10	6.22995938788499	1.3562×10^{-12}

(d)

In Probleem 1(d) is gevind dat Simpson se reël 162 intervalle benodig om die integraal tot 10^{-10} te benader, d.w.s. 163 funksie evaluerings. In Probleem 3(c) is gevind dat 'n 9-punt Gauss-reël 'n absolute fout kleiner as 10^{-10} sal lewer, d.w.s. slegs 9 funksie evaluerings! Die Gauss-reël is dus baie meer effektief.