NAAM: Oplossings (Gedeeltlik) US NR. ____________

INSTRUKSIES:

(a) 40 minute, 2 probleme, 27 punte, toeboek.

(b) Toon al jou bewerkings en motiveer alle stappe. Korrekte antwoorde verdien nie volpunte sonder die nodige verduideliking nie.

(c) **Let Wel:** Die wenke hieronder kan enige plek in die vraestel **sonder bewys** gebruik word.

(d) Moenie omblaai voordat u aangesê word om dit te doen nie.

INSTRUCTIONS:

(a) 40 minutes, 2 problems, 27 marks, closed book.

(b) Calculations are to be shown and all steps must be justified. Correct answers do not earn full marks without the necessary explanation.

(c) **Note:** *The hints below may be used* **without proof** *anywhere in the paper.*

(d) Do not turn the page until you are told to do so.

Wenke/Hints:

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots, \quad |x|<1, \qquad \sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \dots,$$

$$\arctan x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \dots, \quad |x|<1, \qquad \cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \dots.$$

$$\int \frac{dx}{x^2+1} = \arctan x + C, \qquad \int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + C$$

$$\frac{dN}{dt} = bN - sN^2, \quad N(0)=\alpha \quad \Longrightarrow \quad N = \frac{b\alpha}{s\alpha + (b-s\alpha)e^{-bt}}$$

$$\frac{dy}{dx} + p(x)\,y = q(x) \quad \Longrightarrow \quad \text{Int. Faktor} = e^{\int p(x)\,dx}$$

$$\text{Newton:} \quad f(x)=0 \quad \Longrightarrow \quad x_{n+1} = x_n - f(x_n)/f'(x_n)$$

Vraag 1 (5+5+5=15 punte)

Beskou die probleem van 'n projektiel wat teen 'n hoek θ en aanvanklike snelheid v_0, (sien figuur) gelaseer word. As aangeneem word dat die lugweerstand eweredig aan die die snelheid van die projektiel is, word die vlug van die projektiel gemodelleer deur

Question 1 *(5+5+5=15 marks)*

Consider the problem of a projectile that is lanched at an angle of θ and initial velocity v_0 (see figure). If it is assumed that air resistance is proportional to the velocity of the projectile, then the flight of the projectile is modelled by

$$m\frac{d\mathbf{v}}{dt} = -mg\mathbf{j} - k\,\mathbf{v}, \qquad v(0) = v_0.$$

Hier is $\mathbf{v}$ die snelheid, t die tyd, k 'n konstante, m die massa van die projektiel en g die swaartekragversnelling. (Opwaarts positief.)

(a) Los hierdie aanvangswaardeprobleem op, en herlei sodoende uitdrukkings vir die horisontale en vertikale snelhede, as funksies van t, van die projektiel .

(b) Bereken die tyd T wat verloop voordat die projektiel op sy maksimum hoogte H is.

Here $\mathbf{v}$ is the velocity, t the time, k a constant, m the mass of the projectile and g the acceleration due to gravity. (Upwards positive.)

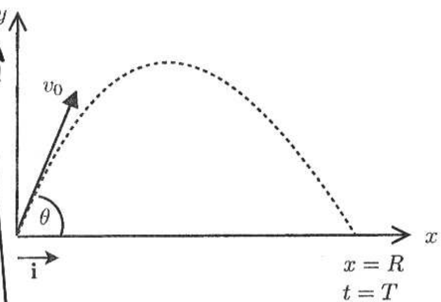
(a) Solve the initial value problem, and thus derive expressions for the horizontal and vertical velocities, as functions of t, of the projectile.

(b) Calculate the time T it takes for the projectile to be at its maximum height H.

deel deur met m

$$\frac{d\underline{v}}{dt} = -g\underline{j} - c\underline{v} \quad \text{met } c = \frac{k}{m}$$

$\underline{j}\uparrow$



Vektore in komponente

$$\boxed{\begin{array}{ll} \underline{a}(t) = x''(t)\,\underline{i} + y''(t)\,\underline{j} & \text{(versnelling)} \\ \underline{v}(t) = x'(t)\,\underline{i} + y'(t)\,\underline{j} & \text{(snelheid)} \\ \underline{s}(t) = x(t)\,\underline{i} + y(t)\,\underline{j} & \text{(verplasing)} \end{array}}$$

Stel in DV :

$$x''(t)\underline{i} + y''(t)\underline{j} = -g\underline{j} - c\left(x'(t)\,\underline{i} + y'(t)\underline{j}\right)$$

$$\left.\begin{array}{lll} \text{in die } \underline{i} \text{ rigting} & x'' = -cx' & \cdots \ (1) \\ \text{in die } \underline{j} \text{ rigting} & y'' = -g - cy' & \cdots \ (2) \end{array}\right\} \text{stelsel van DV's.}$$

Beskou eers ①

$$\frac{d(x')}{dt} = -c\,x' \qquad \text{(eksponensiële verval)}$$

$$x' = K e^{-ct}$$

as $t=0$ dan is $x' = v_0 \cos\theta$

$$\Rightarrow \boxed{x'(t) = v_0 \cos\theta\, e^{-ct}}$$

Beskou nou ②

$$\frac{d}{dt}(y') + cy' = -g$$

$$I(t) = e^{\int c\,dt} = e^{ct}$$

$$\Rightarrow e^{ct}\frac{d}{dt}(y') + ce^{ct}y' = -ge^{ct}$$

$$\frac{d}{dt}\left(e^{ct}y'\right) = -ge^{ct}$$

$$e^{ct}y' = -g\int e^{ct}\,dt = -\frac{g e^{ct}}{c} + K$$

$t=0 \quad y' = v_0 \sin\theta$

$$\Rightarrow \qquad v_0 \sin\theta = -\frac{g}{c} + K$$

$$K = v_0 \sin\theta + \frac{g}{c}$$

$$\Rightarrow \boxed{y'(t) = -\frac{g}{c} + \left(v_0 \sin\theta + \frac{g}{c}\right)e^{-ct}}$$

(b) Projektiel is by maksimum hoogte as

$$y'(t) = 0$$

$$\Rightarrow \quad \frac{g}{c} = \left(V_0 \sin\theta + \frac{g}{c}\right) e^{-ct}$$

$$e^{-ct} = \frac{g/c}{V_0 \sin\theta + g/c} = \frac{g}{cV_0 \sin\theta + g}$$

$$-ct = \ln\left(\frac{g}{cV_0 \sin\theta + g}\right)$$

$$t = \frac{1}{c} \ln\left(\frac{g + cV_0 \sin\theta}{g}\right)$$

$$\boxed{T = \frac{1}{c} \ln\left(1 + \frac{c}{g} V_0 \sin\theta\right)}$$

(c) Gegee dat die uitdrukkings

(c) *Suppose the expressions*

$$x(t) = \frac{v_0 \cos\theta}{c}\left(1 - e^{-ct}\right) \quad \ldots\ldots \quad (1)$$

$$y(t) = -\frac{g}{c}t + \frac{1}{c}\left(v_0 \sin\theta + \frac{g}{c}\right)\left(1 - e^{-ct}\right) \quad \ldots \quad (2)$$

onderskeidelik, die horisontale en vertikale verplasing van die projektiel beskryf (met $c = \frac{k}{m}$), herlei die baan vergelyking van die projektiel ('n uitdrukking waar y 'n funksie van x is).

describe the horizontal and vertical displacement, respectively, of the projectile (with $c = \frac{k}{m}$), derive the expression for the trajectory of the projectile (an expression where y is a function of x).

Gebruik (1) om t as 'n funksie van x te skryf

$$1 - e^{-ct} = \frac{xc}{V_0 \cos\Theta} \quad \ldots \quad (3)$$

$$e^{-ct} = +1 - \frac{cx}{V_0 \cos\Theta} = \frac{V_0\cos\Theta - cx}{V_0 \cos\Theta}$$

$$-ct = \ln\left(\frac{V_0\cos\Theta - cx}{V_0\cos\Theta}\right)$$

$$t = -\frac{1}{c}\ln\left(1 - \frac{cx}{V_0\cos\Theta}\right) \quad \ldots \quad (4)$$

Stel (3) en (4) in (2)

$$y(x) = -\frac{g}{c}\left(-\frac{1}{c}\ln\left(1 - \frac{cx}{V_0\cos\Theta}\right)\right) + \frac{1}{c}\left(V_0\sin\Theta + \frac{g}{c}\right)\left(\frac{xc}{V_0\cos\Theta}\right)$$

$$y(x) = \frac{g}{c^2}\ln\left(1 - \frac{cx}{V_0\cos\Theta}\right) + \left(V_0\sin\Theta + \frac{g}{c}\right)\left(\frac{x}{V_0\cos\Theta}\right)$$

Vraag 2 (4+2+4+2+2=12 punte)

(a) Beskou die aanvangswaardeprobleem

Question 2 *(4+2+4+2+2=12 marks)*

(a) *Consider the initial value problem*

$$\frac{dy}{dt} = 4 - t + 2y, \qquad y(0) = 1.$$

Bereken benaderings vir $y(\frac{1}{3})$, $y(\frac{2}{3})$ en $y(1)$ deur Euler se metode met staplengte $h = \frac{1}{3}$ te gebruik. WENK: $y_{j+1} = y_j + hf(t_j, y_j)$.

Calculate approximations for $y(\frac{1}{3})$, $y(\frac{2}{3})$ *and* $y(1)$ *by using Euler's method with interval length* $h = \frac{1}{3}$. HINT: $y_{j+1} = y_j + hf(t_j, y_j)$.

(b) Gegee dat die eksakte oplossing van die beginwaarde probleem in **(a)**

(b) *With the exact solution of the initial value problem in* **(a)** *given by*

$$y(t) = \frac{1}{4}\Big(-7 + 2t + 11e^{2t}\Big)$$

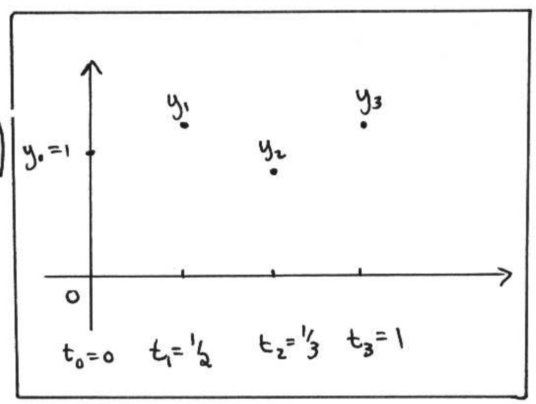
is, bereken die fout van Euler se metode by $y(1)$.

calculate the error made by Euler's method at $y(1)$.

(a) $y_1 = y_0 + \left(\frac{1}{3}\right)\left(4 - 0 + 2(y_0)\right)$

$= 1 + \frac{1}{3}\left(4 + 2(1)\right) = 1 + \frac{1}{3}(6) = \boxed{3}$

$y_2 = 3 + \frac{1}{3}\left(4 - \frac{1}{3} + 2(3)\right) = \frac{56}{9} = \boxed{6.22\dot{2}}$



$y_3 = \frac{56}{9} + \frac{1}{3}\left(4 - \frac{2}{3} + 2\left(\frac{56}{9}\right)\right)$

Benadering vir $y(1)$

$= \frac{310}{27}$

$= \boxed{11.48148}$

(b) $y(1) = \frac{1}{4}\left(-7 + 2 + 11e^2\right) \simeq 19.0699$

$y_3 = 11.48148$

4

fout $= |y(1) - y_3| \approx \boxed{7.5884}$.

(c) Bereken benaderings vir $y(\frac{1}{3})$, $y(\frac{2}{3})$ en $y(1)$ deur die **trapesiumreël** met staplengte $h = \frac{1}{3}$ te gebruik.
WENK: $y_{j+1} = y_j + \frac{h}{2}\Big(f(t_j, y_j) + f(t_{j+1}, y_{j+1})\Big)$

(d) Bereken die fout van die trapesiumreël by $y(1)$, en vergelyk dit met die fout van **(b)**.

(c) *Calculate approximations for* $y(\frac{1}{3})$, $y(\frac{2}{3})$ *and* $y(1)$ *by using the* **trapezoidal rule** *with interval length* $h = \frac{1}{3}$.
HINT: $y_{j+1} = y_j + \frac{h}{2}\Big(f(t_j, y_j) + f(t_{j+1}, y_{j+1})\Big)$

(d) *Calculate the error made by the trapezoidal rule at* $y(1)$ *and compare it to the error of* **(b)**.

$$y_{j+1} = y_j + \frac{h}{2}\Big[4 - t_j + 2y_j + 4 - (t_j + h) + 2(y_{j+1})\Big]$$

g

f is 'n lineêre funksie van y => kan ons dit herskryf om eksplisiet in terme van y_j, t_j te wees.

$$\Big[1 - \frac{h}{2}\cdot 2\Big] y_{j+1} = \Big[1 + \frac{h}{2}\cdot 2\Big] y_j + \frac{h}{2}\Big[4 - t_j + 4 - t_j - h\Big]$$

$$y_{j+1} = \frac{(1+h)y_j + \frac{h}{2}(8 - 2t_j - h)}{1-h}$$

met $h = 1/3$

$$y_{j+1} = \frac{\frac{4}{3}y_j + \frac{1}{2.3}(8 - 2t_j - 1/3)}{2/3}$$

$$= \frac{4y_j + 4 - t_j - 1/6}{2}$$

$$y_0 = 1$$

$$y_1 = \frac{4\times 1 + 4 - 0 - \frac{1}{6}}{2} = \frac{47}{12} = 3.916\dot{6}$$

$$y_2 = \frac{4 \times \frac{47}{12} + 4 - \frac{1}{3} - \frac{1}{6}}{2} = \frac{115}{12} = 9.583\dot{3}$$

$$y_3 = \frac{4 \times \frac{115}{12} + 4 - \frac{2}{3} - \frac{1}{6}}{2} = \frac{83}{4} = 20.75$$

(d) $\text{fout} = |y(1) - y_3| = 1.6801 < 7.5884$

die fout in (b).

(e) Toon aan hoe die volgende tweede orde Runge-Kutta metode uit die trapsiumreël van deel (c) herlei kan word

(e) *Show how the following second order Runge-Kutta method can be derived from the trapezoidal rule of part* (c)

$$y_{j+1} = y_j + \tfrac{1}{2}h\Big[f(t_j, y_j) + f(t_{j+1}, y_j + hf(t_j, y_j))\Big].$$

Euler se metode

$$y_{j+1} = y_j + hf(t_j, y_j)$$

Trapesiumreël

$$y_{j+1} = y_j + \frac{h}{2}\left(f(t_j, y_j) + f(t_{j+1}, y_{j+1})\right)$$

Die probleem met die trapesiumreël is dat ons y_{j+1} nodig het om vir y_{j+1} op te los, dus metode is implisiet.

Moontlike oplossing: gebruik Euler se metode om 'n skatting vir y_{j+1} te maak, dus

$$\tilde{y}_{j+1} = y_j + hf(t_j, y_j)$$

en gebruik hierdie in die RK van trap.reël

$$y_{j+1} = y_j + \frac{h}{2}\left[f(t_j, y_j) + f(t_{j+1}, \tilde{y}_{j+1})\right]$$

$$= y_j + \frac{h}{2}\left[f(t_j, y_j) + f(t_{j+1}, y_j + hf(t_j, y_j))\right]$$