Multi-Line Fitting and Straight Edge Detection using Polynomial Phase Signals*

K. Abed-Meraim and Y. Hua

Department of Electrical and Electronic Engineering
University of Melbourne, Parkville, Victoria 3052 Australia
E-mail: abed, yhua@ee.mu.oz.au

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Abstract

In this paper, a new signal processing method is developed for solving the multi-line fitting problem in a two dimensional image. We first reformulate the former problem in a special parameter estimation framework such that a first order or a second order polynomial phase signal structure is obtained. Then, the recently developed algorithms in that formalism can be exploited to produce accurate estimates for line parameters. The signal representation employed in this formulation can be generalized to handle both problems of line fitting (in which a set of binary valued discrete pixels is given) and straight edge detection (in which one starts with a grey scale image).

A main advantage of the proposed method is its ability to estimate the parameters of parallel lines with different offsets which can not be handled by the sensor array processing technique introduced by Aghajan et al. Simulation results are presented to demonstrate the usefulness of the proposed method.

1 Introduction

A recurring problem in computer picture processing is the detection of straight lines in digitized images. This problem arises in many application areas such as road tracking in robotic vision, mask-wafer alignment in semi-conductor manufacturing, text alignment in document analysis, particle tracking in bubble chambers, etc. In the simplest case, the digitized image may contain a number of discrete, black figure points lying on a white background (i.e., discrete '1' pixels lying on a '0' background). Our objective is to detect and estimate the parameters of straight lines that fit groups of colinear or almost colinear '1' pixels. Many detection/estimation techniques have been proposed in the literature which include the total least squares methods [7], the Hough-transform method [8, 9], and the maximum likelihood method [10]. These methods generally suffer from their high computational cost or their poor resolution (accuracy) estimation.

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More recently, a high resolution technique has been introduced in [6]. This technique is shown to be both computationally and statistically effective. However, a major drawback of the technique is that it deals only with non-parallel straight lines case and does not take into account the quantization noise. To overcome this drawback, we present in this paper a new approach based on the introduction of a perfect mathematical analogy between the multi-line fitting problem and the problem of estimating the phase parameters of multi-components polynomial phase signal. A number of standard methods exist for solving the latter problem, e.g., [1, 2, 3] for linear phase (sinusoidal) signal and [4, 5, 11] for quadratic phase (chirp) signal.

We present here a new technique to estimate the phase parameters of the signals. This technique is a two steps procedure that estimates first the line angles then the line offsets. It uses the well known Matrix Pencils (MP) method [2] applied to properly chosen 1-D signals processed from the recorded 2-D image. Better than to the technique in [6], the proposed method can handle the case of parallel straight lines.

2 Problem formulation and data model

Let $r(x, y)$ be the recorded image, defined on the Euclidean plane (X, Y). We model $r(x, y)$ as an image composed of d striated patterns corrupted by uniformly distributed additive noise.

We assume that the digitized image $r(x, y)$ contains only '1' and '0'-valued pixels. The '1' pixels represent pixels either almost colinear with each other in a finite number of groups, or outlier pixels, while the '0' pixels correspond to background. The 2D image is represented by an $N \times M$ matrix (N and M being the sizes of the image in the Y-direction and X-direction, respectively) with '1' or '0'-valued entries, where each entry corresponds to one pixel in the image.

The generalization to the problem of straight edge detection in grey-scale images can be done as in [6]. This is done by assigning an amplitude to the propagated signal from each pixel proportional to the gray-scale value of the pixel. Also, as mentioned in [6], a first step of edge enhancement may be used to attenuate background contributions.

The line parameterization used in the sequel is depicted in Figure 1, where a line is characterized by its X-axis offset and the angle it makes with the normal to the X-axis at the interception point (we use the conventional trigonometric orientation for the angles). The line equation (using continuous coordinates) is given by

$$x = y \tan\theta + x_0 \tag{1}$$

However, in the digitized image x and y take integer values and thus equation (1) becomes:

$$\begin{aligned} x &= \lceil y \tan\theta + x_0 \rceil \\ &= y \tan\theta + x_0 + \epsilon(y) \end{aligned} \tag{2}$$

where $\lceil x \rceil$ denotes the closest integer to x and $\epsilon(y)$ is the quantization noise that can be modeled as a random variable uniformly distributed in $[-0.5, 0.5]$.

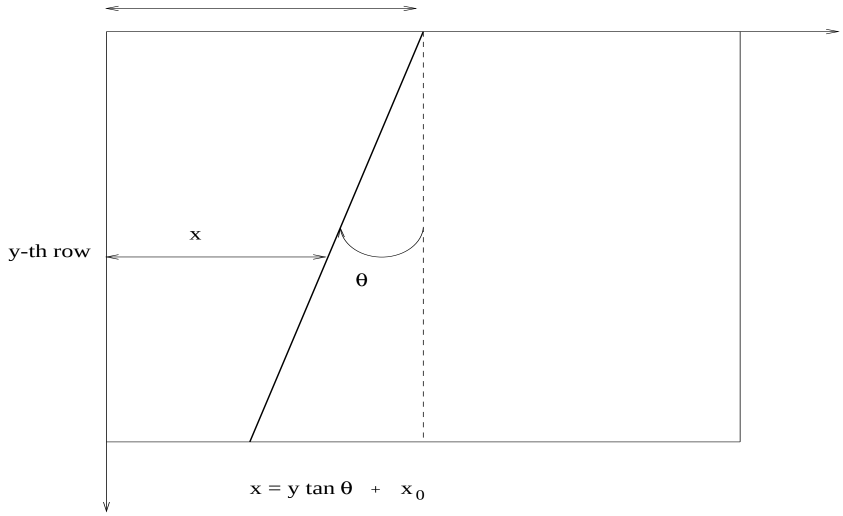

Figure 1.

We focus here on the problem of estimating[1] the line parameters $\theta_1, \cdots, \theta_d$ and $x_1, \cdots, x_d$ given the noisy image $r(x, y)$.

3 Reformulation of line fitting problem

In this section we introduce a perfect mathematical analogy between the multi-line fitting problem and the problem of estimating the phase parameters of multi-components polynomial phase signal. From the $N \times M$ matrix (2D problem) we construct a $N \times 1$ vector $\mathbf{z} = [z(0), \cdots, z(N-1)]^T$ (1D problem) according to the following transformation: If there are L nonzero pixels on the kth row of the image matrix located on columns $q_1(k), \cdots, q_L(k)$ respectively, then the kth entry of vector $\mathbf{z}$ is given by

$$z(k) = \sum_{i=1}^{L} e^{jP(q_i(k))} \tag{3}$$

where $P(x)$ is a properly chosen polynomial function of x. Now, consider the noiseless case of d lines with angles $\{\theta_i\}_{1 \le i \le d}$ and offsets $\{x_i\}_{1 \le i \le d}$ (see Figure 1). Provided that the line width is such that the line gives rise to only one nonzero pixel per row, the kth entry of $\mathbf{z}$ will be

$$z(k) = \sum_{i=1}^{d} e^{jP(k \tan\theta_i + x_i + \epsilon(k))} \tag{4}$$

which is a polynomial phase signal with random amplitude. In the following, we only use polynomials of degree one or two, i.e., $P_1(x) = \mu_1 x$ and $P_2(x) = \mu_2 x^2$ (μ_1 and μ_2 are properly chosen constant parameters). Let $\mathbf{z}_1$ and $\mathbf{z}_2$ be the $N \times 1$ vectors given by (4) for $P = P_1$ and $P = P_2$, respectively. We have:

$$z_1(k) = \sum_{i=1}^{d} A_{1i}(k) e^{j a_{1i} k} \tag{5}$$

$$z_2(k) = \sum_{i=1}^{d} A_{2i}(k) e^{j(a_{2i} k + b_{2i} k^2)} \tag{6}$$

[1]That includes implicitly the estimation of the number of straight lines d.

where the amplitude parameters A_{1i}, A_{2i} and the phase parameters a_{1i}, a_{2i}, b_{2i} are given by

$$
\begin{aligned}
A_{1i}(k) &= e^{j\mu_1(x_i+\epsilon_i(k))} \\
a_{1i} &= \mu_1 \tan\theta_i \\
A_{2i}(k) &= e^{j\mu_2[(x_i+\epsilon(k))^2+2k\tan\theta_i\epsilon(k)]} \\
a_{2i} &= 2\mu_2 x_i \tan\theta_i \\
b_{2i} &= \mu_2 \tan^2\theta_i
\end{aligned}
$$

In the presence of outlier pixels in the image, the signal will be

$$w(k) = z(k) + n(k) \tag{7}$$

where $n(k)$ represents the noise effect in the kth row[2].

Some comments: 1- The transform (3) can be applied in general to estimate the parameters of a large class of patterns other than straight lines, e.g., hyperbolic curves, parabolic curves, elliptic curves, etc.

2- From (5), we can see that two parallel lines correspond to sinusoidal signals with the same frequency and different amplitudes. In this case, using degree one polynomial (i.e., $\mathbf{z}_1$) is not sufficient to correctly estimate the number of straight lines and their parameters. On the other hand, we can see from (6) that two parallel lines correspond to chirp signals which have the same second order phase parameter but different first order phase parameters. Therefore, it is possible to correctly estimate from $\mathbf{z}_2$ the number of straight lines and their parameters using techniques such as the quadratic phase transform [5] (see also [11] for an alternative method).

3- The parameters μ_1 and μ_2 should be chosen large enough to increase the resolution (and thus the accuracy) of the estimation. On the other hand, the values of μ_1 and μ_2 should be small enough to avoid phase ambiguity problem, i.e., the phase parameters must satisfy, for all $l = 1, \cdots, d$:

$$|a_{1l}| < \pi, \quad |a_{2l}| < \pi, \quad \text{and} \quad |b_{2l}| < \pi$$

Also, the smaller μ_1 and μ_2 are, the larger are the mean values of the random amplitudes $A_{1l}(k)$ and $A_{2l}(k)$. For example, the mean and variance of $A_{1l}(k)$ are given by

$$
\begin{aligned}
m_{1l} &\stackrel{\text{def}}{=} E(A_{1l}(k)) = e^{jx_l}\frac{\sin(\mu_1/2)}{\mu_1/2}, \\
\sigma_{1l}^2 &\stackrel{\text{def}}{=} \operatorname{var}(A_{1l}(k)) = 1 - \left(\frac{\sin(\mu_1/2)}{\mu_1/2}\right)^2
\end{aligned}
$$

which means in particular, that for small values of μ_1, the sinusoidal signal $\mathbf{z}_1$ can be modeled as a sum of constant amplitude sinusoids plus additive noise that is due to the fluctuations of the amplitude functions around their mean values (e.g., for $\mu_1 = 0.5$ we have $|m_{1l}| = 0.99$ and $\sigma_{1l}^2 = -16.85$dB).

In our experiments, we observed that μ_1 is better to be chosen to have a value around unity while μ_2 is chosen with much smaller values around $1/M$, M being the image dimension in the X-direction.

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[2]It is shown in [6] that for $P = P_1$ and if the noise pixels are uniformly distributed on the image plane, $n(k)$ can be approximated by a Gaussian noise.

We have now transformed the original straight line detection problem into an estimation problem of the parameters of the multicomponent sinusoidal signal $\mathbf{z}_1$ or multicomponent chirp signal $\mathbf{z}_2$. Next, we present our estimation technique based on the matrix pencils method [2].

3.1 A two step estimation method

In this section, we briefly recall the principle of the matrix pencil (MP) method as shown in [2], then we show how we can apply the MP method to estimate the straight line parameters.

Matrix pencil method: Consider a noise free exponential data sequence which can be described by:

$$x(t) = \sum_{i=1}^{d} A_i e^{jf_it}, \qquad 0 \le t \le N-1$$

where A_i and f_i are the amplitude and frequency of the i-th sinusoids, respectively (with $f_i \neq f_j$ for $i \neq j$). The matrix pencil approach relies on the following model inherent in the exponential signals [2]: Define

$$\begin{aligned} \mathbf{x}(t) &= [x(t), \cdots, x(N-L+t-1)]^T \\ \mathbf{X}_0 &= [\mathbf{x}(L-1), \cdots, \mathbf{x}(0)] \\ \mathbf{X}_1 &= [\mathbf{x}(L), \cdots, \mathbf{x}(1)] \end{aligned}$$

where "T" denotes the transpose and L is a chosen parameter called the pencil parameter (it satisfies $d \le L \le N-d$). It is shown in [2] that

$$\begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_L\mathbf{A} \\ \mathbf{Z}_L\mathbf{A}\mathbf{Z} \end{bmatrix} \mathbf{Z}_R \tag{8}$$

where

$$\begin{aligned} \mathbf{Z}_L &= \begin{bmatrix} 1 & \cdots & 1 \\ e^{jf_1} & \cdots & e^{jf_d} \\ \vdots & & \vdots \\ e^{jf_1(N-L-1)} & \cdots & e^{jf_d(N-L-1)} \end{bmatrix} \\ \mathbf{A} &= \operatorname{diag}(A_1, \cdots, A_d) \\ \mathbf{Z} &= \operatorname{diag}(e^{jf_1}, \cdots, e^{jf_d}) \\ \mathbf{Z}_R &= \begin{bmatrix} e^{jf_1(L-1)} & e^{jf_1(L-2)} & \cdots & 1 \\ \vdots & & & \vdots \\ e^{jf_d(L-1)} & e^{jf_d(L-2)} & \cdots & 1 \end{bmatrix} \end{aligned}$$

From (8), we can see that e^{jf_i}, $i = 1, \cdots, d$ are the rank reducing number of the matrix pencil $\mathbf{X}_1 - z\mathbf{X}_0$ (see [2] for more details). To estimate the frequency parameter (i.e., $\mathbf{Z}$), we can proceed as follows:

- Estimate $\mathbf{E}$ the matrix given by the d [3] dominant singular (left) eigenvectors of $\mathbf{X} \stackrel{\text{def}}{=} [\mathbf{X}_0^T \ \mathbf{X}_1^T]^T$. We have

$$\operatorname{range}(\mathbf{E}) = \operatorname{range}(\mathbf{X}) = \operatorname{range}\left(\begin{bmatrix} \mathbf{Z}_L\mathbf{A} \\ \mathbf{Z}_L\mathbf{A}\mathbf{Z} \end{bmatrix}\right)$$

[3]The number of sinusoids d can be estimated by using the recently developed LS detection method [12].

and thus, there exists a non-singular matrix $\mathbf{T}$ satisfying

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_0 \\ \mathbf{E}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_L\mathbf{A} \\ \mathbf{Z}_L\mathbf{A}\mathbf{Z} \end{bmatrix} \mathbf{T}$$

- Estimate $e^{jf_1}, \cdots, e^{jf_d}$ as the eigenvalues of

$$\mathbf{U} = \mathbf{E}_0^{\#}\mathbf{E}_1 = \mathbf{T}^{-1}\mathbf{Z}\mathbf{T}$$

where $\mathbf{E}_0^{\#}$ denotes the pseudo-inverse of $\mathbf{E}_0$.

First step: Estimation of the line angles: To estimate the line angles, we apply the MP method to the noisy signal

$$\mathbf{w}_1(k) = z_1(k) + n_1(k)$$

where $n_1(k)$ is the noise due to outliers pixels and $z_1(k)$ is the signal given in (5). Let $\hat{\theta}_1, \cdots, \hat{\theta}_{d'}$ denote the estimated angles (d' being the number of sinusoids: $d' < d$ in the case of parallel lines).

Assuming a good angle estimation, it is possible to estimate the quantization errors $\epsilon_1(k), \cdots, \epsilon_d(k)$ from the angle estimates. In fact, we have for $i = 1, \cdots, d$

$$\begin{aligned} \epsilon_i(k) &= \lceil k \tan\theta_i + x_i \rceil - (k \tan\theta_i + x_i) \\ &= \lceil k \tan\theta_i \rceil - k \tan\theta_i \end{aligned}$$

and thus we can estimate the quantization errors as

$$\hat{\epsilon}_i(k) = \lceil k \tan\hat{\theta}_i \rceil - k \tan\hat{\theta}_i$$

These estimates are used next to improve the estimation of the line offsets.

Second step: Estimation of the line offsets: The estimation of the line offsets depends on whether the image contains parallel lines or not. In the latter case (i.e., no parallel lines), a simple estimation procedure consists in using a least-squares fitting approach, i.e.

$$\arg\min_{\mathbf{m}} \|\mathbf{z}_1 - \mathbf{Z}_L\mathbf{m}\|^2 \Longleftrightarrow \mathbf{m} = \mathbf{Z}_L^{\#}\mathbf{z}_1$$

where $\mathbf{Z}_L$ is the $N \times d$ Vandermonde matrix constructed from e^{jf_i}, $f_i = \mu_1 \tan\hat{\theta}_i$, $i = 1, \cdots, d$ and $\mathbf{m} = [m_{11}, \cdots, m_{1d}]^T$, m_{1i} being the mean value of the amplitude given by

$$m_{1i} = e^{j\mu_1 x_i} \frac{\sin(\mu_1/2)}{\mu_1/2} \Rightarrow x_i = \arg(m_{1i})/\mu_1$$

Note that we can use a smaller value of μ_1 to estimate the line offset than the one used for the line angles estimation to avoid the phase ambiguity problem and to decrease the quantization noise effect.

Unfortunately, the condition of non-parallel straight lines is very restrictive in practice (see for example [10]). Thus, we propose here alternative methods that proceed as follows:

Method 1: For $i = 1, \cdots, d'$:

- Demodulate the noisy chirp signal $w_2(k) = z_2(k) + n_2(k)$ using the previously estimated values of the i-th line angle $\hat{\theta}_i$ and the i-th quantization noise $\hat{\epsilon}_i(k)$:

$$
\begin{aligned}
y_i(k) &= w_2(k)e^{-j\mu_2(2k\tan\hat{\theta}_i\hat{\epsilon}_i(k)+k^2\tan^2\hat{\theta}_i)} \\
&\approx \sum_{\{l|\tan\theta_l=\tan\theta_i\}} b_l(k)e^{jf_lk} + \text{noise terms}
\end{aligned}
$$

where $b_l(k) = e^{j\mu_2(x_l+\epsilon_l(k))^2}$ and $f_l = 2\mu_2\tan\theta_i x_l$.

- Estimate the frequencies f_l by applying the MP method to $y_i(k)$. The line offsets are then computed as

$$
\hat{x}_l = \frac{\hat{f}_l}{2\mu_2\tan\hat{\theta}_i}
$$

Method 2: For $i = 1, \cdots, d'$:

- Estimate the $N \times 1$ signal:

$$
\begin{aligned}
y'_i(k) &= \sum_{i=1}^{L} e^{jk\mu_1(q_i(k)-\lceil k\tan\hat{\theta}_i\rceil)} \\
&\approx \sum_{\{l|\tan\theta_l=\tan\theta_i\}} b'_l(k)e^{jf'_lk} + \text{noise terms} \\
f'_l &= \mu_1 x_l, \qquad b'_l = 1
\end{aligned}
$$

- Estimate the frequencies f'_l by applying the MP method to $y'_i(k)$, then the line offsets as

$$
\hat{x}_l = \hat{f}'_l/\mu_1
$$

It is worth noting that other approaches for estimating the phase parameters of the multi-component time varying amplitude chirp signal $\mathbf{z}_2$ are possible (see [14, 13, 11]).

4 Simulation results

We present here some simulation results to illustrate the performance of the proposed method.

In figure 2, we show a simulation example in the case of a square image of dimension $N = 250$ containing two straight lines at $(\theta_1, x_1) = (15^o, 30)$ and $(\theta_2, x_2) = (50^o, -20)$ corrupted by uniformly distributed additive noise. We chose the parameters $\mu_1 = 1$ and $\mu'_1 = 0.1$. The estimated line parameters are $(\hat{\theta}_1, \hat{x}_1) = (14.8^o, 31)$ and $(\hat{\theta}_2, \hat{x}_2) = (50.5^o, -22)$.

Figure 3 shows another simulation example in the case of an image containing parallel lines. The image size is $N = 250$. The lines are located at $(\theta_1, x_1) = (15^o, 30)$, $(\theta_2, x_2) = (15^o, 50)$, and $(\theta_3, x_3) = (30^o, 10)$. We assume here that the quantization noise is negligible. We chose the parameters $\mu_1 = 1$ and $\mu_2 = 0.05$. In this case (i.e., negligible quantization noise), the results given by the proposed method (Method 1) are very accurate: the estimated line parameters are $(\hat{\theta}_1, \hat{x}_1) = (14.98^o, 30)$, $(\hat{\theta}_2, \hat{x}_2) = (14.98^o, 50)$, and $(\hat{\theta}_3, \hat{x}_3) = (30.02^o, 10)$.

However, we have observed throughout our simulation experiments that the estimation accuracy of the proposed method is drastically affected by the quantization noise. The good results shown in [6] in the absense of quantization noise are not any more valid here. Reducing the effect of quantization noise is a challenging and open problem.

5 Conclusion

In this paper, we have presented a new two-step procedure for multi-line fitting and straight edge detection. This approach can be seen as an extension of the array processing method in [6] to deal with the case of parallel lines. The new approach is based on an original problem reformulation that casts the 2-D multi-line fitting problem into a 1-D parameter estimation problem of multi-component chirp signals. This problem reformulation is shown to be a powerful technique that can be used to estimate the parameters of various geometric patterns such as parabolic, hyperbolic, or elliptic curves. Computer simulation results have been presented to illustrate the performance of the proposed method.

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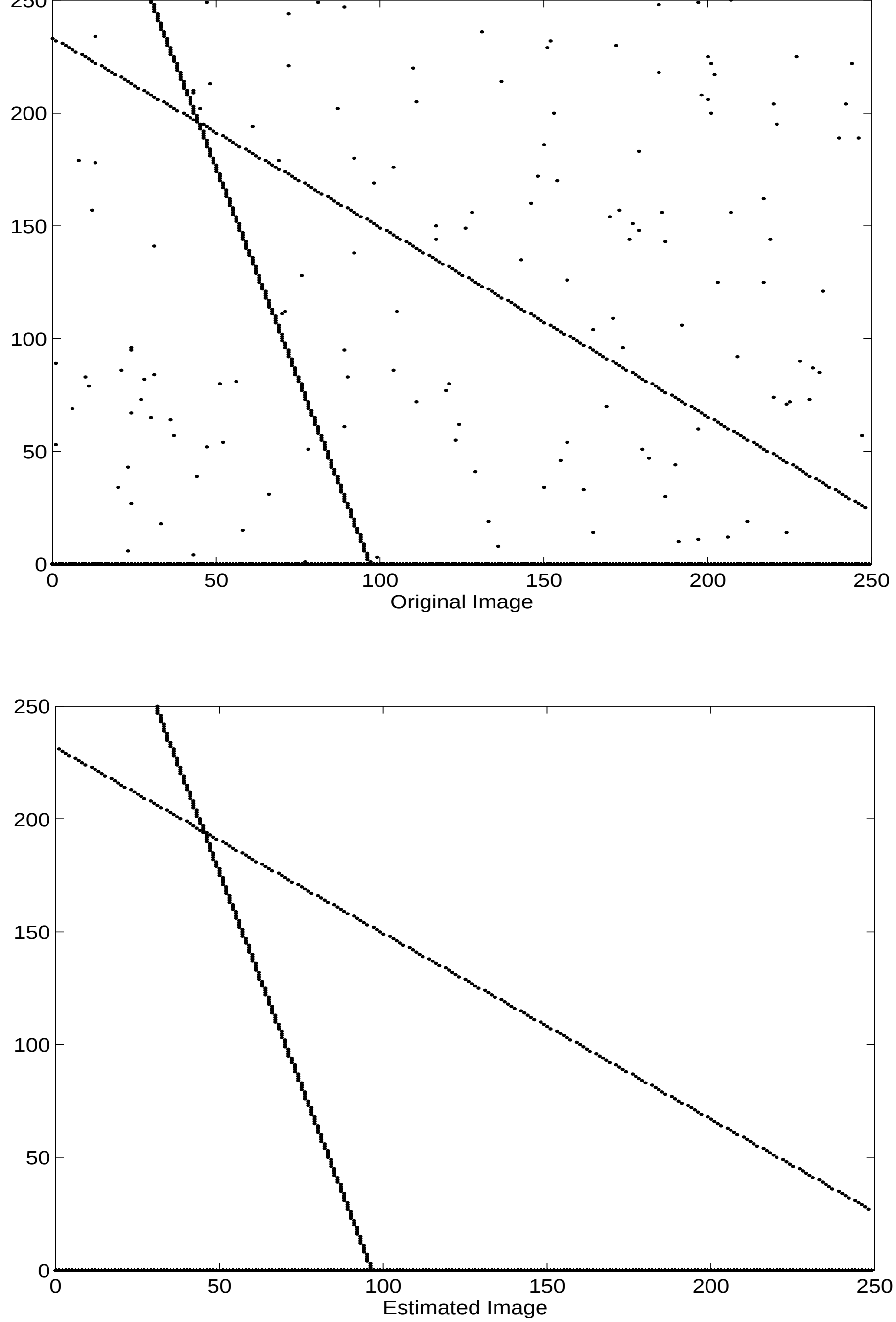

Figure 2: $d = 2$ lines at $(\theta_1, x_1) = (15^o, 30)$ and $(\theta_2, x_2) = (50^o, -20)$: $N = 250$
The estimated parameters are $(\hat{\theta}_1, \hat{x}_1) = (14.8^o, 31)$ and $(\hat{\theta}_2, \hat{x}_2) = (50.5^o, -22)$

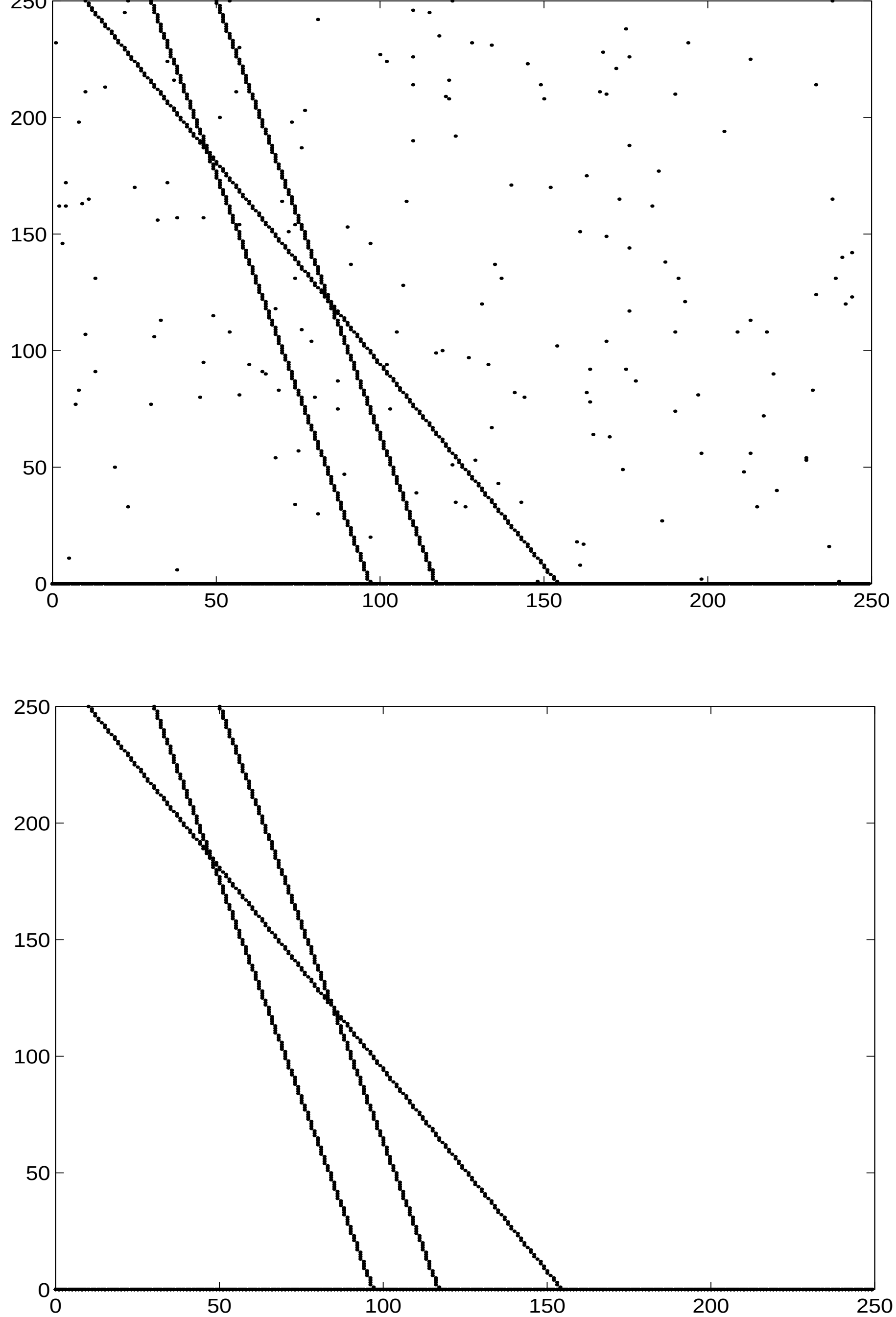

Figure 3: $d = 3$ lines at $(\theta_1, x_1) = (15^o, 30)$, $(\theta_2, x_2) = (15^o, 50)$, and $(\theta_3, x_3) = (30^o, 10)$: $N = 250$
The estimated parameters are $(\hat{\theta}_1, \hat{x}_1) = (14.98^o, 30)$, $(\hat{\theta}_2, \hat{x}_2) = (14.98^o, 50)$, and $(\hat{\theta}_3, \hat{x}_3) = (30.02^o, 10)$