TW324 **Tutoriaaltoets #3** **2004**

NAAM: ________________________________ US NR: ____________

INSTRUKSIES: Vier probleme, 20 punte, 45 minute. Toon alle berekenings by Probleme 3 en 4.

INSTRUCTIONS: *Four problems, 20 marks, 45 minutes. Show all work on Problems 3 and 4.*

WENKE (Gebruik enige plek sonder bewys):

HINTS *(Use anywhere without proof)*:

$$f(x) - p_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi_x)$$

$$\max_{x_0 \le x \le x_1} |f(x) - p_1(x)| \le \tfrac{1}{8} h^2 M_1, \qquad M_1 = \max_{x_0 \le x \le x_1} |f^{(2)}(x)|.$$

$$\max_{x_0 \le x \le x_2} |f(x) - p_2(x)| \le \tfrac{1}{9\sqrt{3}} h^3 M_2, \qquad M_2 = \max_{x_0 \le x \le x_2} |f^{(3)}(x)|.$$

$$T_n(x) = \cos(n \arccos x), \qquad -1 \le x \le 1$$

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2x\,T_n(x) - T_{n-1}(x) \end{aligned}$$

P1	P2	P3	P4	T
3	3	5	9	20

Probleem 1 (3 punte)

Beskou die deelverskiltabel

Consider the divided difference table

(a) $\frac{\alpha - 1}{4-1} = -\frac{1}{6} \Rightarrow \alpha = 1 - \frac{1}{2} = \frac{1}{2}$

(b) $\frac{\beta - 1}{2} = \alpha \Rightarrow \beta = 1 + 2\cdot\frac{1}{2} = 2$

x	$f(x)$		
1	0		
		1	
2	1		$-\frac{1}{6}$
		α	
4	β		

(c) $p_2(x) = 0 + 1.(x-1) - \frac{1}{6}(x-1)(x-2)$
$= (x-1)\left(1 - \frac{1}{6}(x-2)\right)$
$p_2(3) = 1.5\left(1 - \frac{1}{2}\right) = \frac{3}{4}$

(a) Bereken die vermiste inskrywing α. — *Compute the missing entry α.*

(A) $\alpha = 0$ (B) $\alpha = \frac{1}{2}$ (C) $\alpha = 1$ (D) $\alpha = \frac{3}{2}$ (E) $\alpha = 2$

(b) Bereken die vermiste inskrywing β. — *Compute the missing entry β.*

(A) $\beta = 0$ (B) $\beta = \frac{1}{2}$ (C) $\beta = 1$ (D) $\beta = \frac{3}{2}$ (E) $\beta = 2$

(c) Laat $p_2(x)$ die parabool wees wat die data interpoleer. Bereken $p_2(3)$. — *Let $p_2(x)$ be the parabola that interpolates the data. Compute $p_2(3)$.*

(A) $p_2(3) = \frac{1}{3}$ (B) $p_2(3) = \frac{2}{3}$ (C) $p_2(3) = \frac{4}{3}$ (D) $p_2(3) = \frac{5}{3}$ (E) $p_2(3) = \frac{7}{3}$

Probleem 2 (3 punte)

Laat $T_{10}(x)$ die Chebyhsev-polinoom van graad 10 voorstel.

Let $T_{10}(x)$ denote the Chebyhsev polynomial of degree 10.

(a) Bereken $T_{10}(\frac{1}{2})$ $= \cos(10 \arccos \frac{1}{2}) = \cos(10.\pi/3) = -\frac{1}{2}$ — *Compute $T_{10}(\frac{1}{2})$.*

(A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

(b) Bereken die grootste wortel van $T_{10}(x)$ (korrek tot 4 desimale). — *Compute the largest root of $T_{10}(x)$ (correct to 4 decimals).*

(A) 0.6738 (B) 0.9125 (C) 0.9877 (D) 0.9997 (E) 1.3174

(c) Bereken $\widetilde{T}_{10}(0)$, waar $\widetilde{T}_{10}(x)$ die moniese Chebyhsev-polinoom van graad 10 is. — *Compute $\widetilde{T}_{10}(0)$, where $\widetilde{T}_{10}(x)$ is the monic Chebyhsev polynomial of degree 10.*

(A) -2^{-8} (B) 2^{-8} (C) -2^{-9} (D) 2^{-9} (E) -2^{-10}

(b) $T_{10}(x) = \cos(10 \arccos x) = 0 \Rightarrow 10 \arccos x = \frac{\pi}{2} \Rightarrow$
$x = \cos\frac{\pi}{20} = 0.9877$

(c) $\widetilde{T}_{10}(x) = \frac{1}{2^9} T_{10}(x) \Rightarrow \widetilde{T}_{10}(0) = \frac{1}{2^9}\cos(10 \arccos 0)$
$= \frac{1}{2^9}\cos(10.\frac{\pi}{2}) = -\frac{1}{2^9}$

Probleem 3 (4 + 1 = 5 *punte*)

Laat a, b en c konstantes wees en beskou

Let a, b and c be constants, and consider

$$S(x) = \begin{cases} x^3 = S_0(x) & 0 \le x \le 1 \\ \underbrace{\tfrac{1}{2}(x-1)^3 + a(x-1)^2 + b(x-1) + c}_{S_1(x)} & 1 \le x \le 3 \end{cases}.$$

(a) Bepaal a, b en c sodat die funksie 'n *kubiese latfunksie* voorstel. Skryf asb jou finale antwoorde in die ruimte aangedui.

Determine a, b and c such that the function represents a cubic spline. Please write your final answer in the space indicated.

kontinuiteit by $x=1$: $S_0(1) = S_1(1) \Rightarrow 1 = c$

Kontinuiteit v. 1e afgeleide by $x=1$: $S_0'(x) = 3x^2$, $S_1'(x) = \frac{3}{2}(x-1)^2 + 2a(x-1) + b$

$S_0'(1) = S_1'(1) \Rightarrow 3 = b$

ditto vir 2e afgeleide: $S_0''(x) = 6x$, $S_1''(x) = 3(x-1) + 2a$

$S_0''(1) = S_1''(1) \Rightarrow 6 = 2a \Rightarrow a = 3$

Antwoord (a) $\longrightarrow$ $a =$ 3, $b =$ 3, $c =$ 1 $\longleftarrow$ Answer (a)

$S_0''(0) = 0 = S_1''(3)$?

(b) Is die funksie 'n **natuurlike** latfunksie? Motiveer jou antwoord.

Is the function a **natural** *spline? Justify your answer.*

$S_0''(x) = 6x \Rightarrow S_0''(0) = 0$

$S_1''(x) = 3(x-1) + 2a \Rightarrow S_1''(3) = 3\cdot 2 + 2\cdot 3 = 12 \neq 0$

Dus nie 'n natuurlike latfunksie nie.

Probleem 4 (3 + 4 + 2 = 9 punte)

Beskou die funksie $L(x)$, gedefinieer deur die integraal links, met funksiewaardes gegee in die tabel regs.

Consider the function $L(x)$, defined by the integral on the left, with function values given in the table on the right.

$$L(x) = \int_2^x \frac{dt}{\ln t},$$

x	2	3	4
$L(x)$	0	1.11842	1.92242

(a) Laat $p_2(x)$ die parabool wees wat aldrie die datapunte in die tabel interpoleer. Bereken $p_2(2.5)$ met **Neville** se algoritme. (Enige ander algoritme verdien hoogstens 1/3.)

Let $p_2(x)$ be the parabola that interpolates all three of the data values in the table. Compute $p_2(2.5)$ with **Neville**'s *algorithm. (Any other algorithm earns at most 1/3.)*

2	0		
		0.55921	
3	1.11842		0.59851 →
		0.71642	
4	1.92242		

Antwoord (a) ⟶ $p_2(2.5) =$ 0.59851 ⟵ Answer (a)

Probleem 4 (vervolg/*continued*)

(b) Begrens die fout in die benadering van deel (a). *Wenk:* Dit is nie nodig om enige afgeleides van $L(x)$ met die hand te bereken nie: afgeleides plus grafieke word op die agterblad getoon.

Bound the error in the approximation of part (a). Hint: It is not necessary to compute any derivatives of $L(x)$ by hand: derivatives plus graphs are shown on the back page.

$$f(x) - p_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(\xi_x) \qquad f = L$$

$x = 2.5$
$x_0 = 2$
$x_1 = 3$
$x_2 = 4$

$$f(2.5) - p_2(2.5) = \frac{(2.5-2)(2.5-3)(2.5-4)}{6} f^{(3)}(\xi)$$

$$(*) \qquad = 0.0625\, f^{(3)}(\xi) \qquad \xi \text{ in } [2,4]$$

Uit wenke op laaste bladsy volg

$$L'''(4) \leq f^{(3)}(\xi) \leq L'''(2)$$

Stel in (*) $\qquad = 0.07944 \qquad\qquad = \underbrace{2.02173\ldots}_{M_2}$

Antwoord (b) $\longrightarrow$ $\underline{0.00496\ldots} \leq L(2.5) - p_2(2.5) \leq \underline{0.126\ldots}$ $\longleftarrow$ Answer (b)

(c) Is dit veilig om $p_2(x)$ as benadering tot $L(x)$ te gebruik as die absolute fout nêrens in $[2,4]$ die waarde 0.2 mag oorskry nie? Motiveer

Is it safe to use $p_2(x)$ as approximation to $L(x)$ if the absolute error is nowhere in $[2,4]$ to exceed the value 0.2? Justify.

$$\max_{2 \leq x \leq 4} |f(x) - p_2(x)| \leq \frac{1}{9\sqrt{3}} \cdot \underbrace{1}_{h}^{3} \cdot \underbrace{2{,}02173\ldots}_{\text{hierdie bereken}}$$

$$= 0.12969\ldots \leq 0.2$$

Gebruik 3e wenk op voorblad.

Dus veilig.

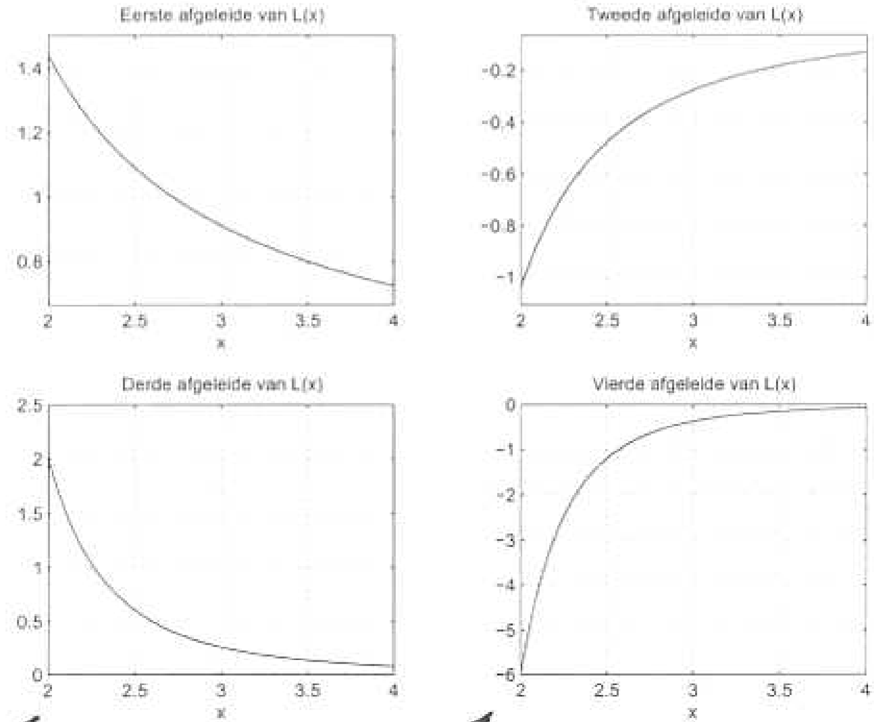
Probleem 4 (Wenke/*Hints*)

$$L(x) = \int_2^x \frac{dt}{\ln t}$$

Hieronder volg die afgeleides van $L(x)$, asook hulle grafieke. Hierdie inligting moet gebruik word om Probleem 4(b)–(c) te beantwoord. (Daar is meer inligting hier as wat in werklikheid nodig is. Ignoreer dit wat oorbodig is.)

Below are the derivatives of $L(x)$, as well as their graphs. This information is to be used in Problems 4(b)–(c). (There is more information here than actually necessary. Ignore that which is redundant.)

$$L'(x) = \frac{1}{\ln x} \qquad L''(x) = -\frac{1}{(\ln(x))^2 x}$$



$$L'''(x) = \frac{2}{(\ln(x))^3 x^2} + \frac{1}{(\ln(x))^2 x^2} \qquad L''''(x) = -\frac{6}{(\ln(x))^4 x^3} - \frac{6}{(\ln(x))^3 x^3} - \frac{2}{(\ln(x))^2 x^3}$$