Probleem 1:

(a) Vir $f(t) = \sin at$ is die Laplace transform dan

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} \sin at \, dt \\
&= e^{-st} \left(\frac{-\cos at}{a} \right) \bigg|_0^\infty - \int_0^\infty -se^{-st} \left(\frac{-\cos at}{a} \right) dt \quad \text{[deelwyse integrasie]} \\
&= \frac{1}{a} - \frac{s}{a} \int_0^\infty e^{-st} \cos at \, dt \quad (s > 0) \\
&= \frac{1}{a} - \frac{s}{a} \left[e^{-st} \left(\frac{\sin at}{a} \right) \bigg|_0^\infty - \int_0^\infty -se^{-st} \left(\frac{\sin at}{a} \right) dt \right] \\
&= \frac{1}{a} - \frac{s}{a} \left[0 + \frac{s}{a} \int_0^\infty e^{-st} \sin at \, dt \right] \\
&= \frac{1}{a} - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at \, dt
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad F(s) &= \frac{1}{a} - \frac{s^2}{a^2} F(s) \\
\Rightarrow \quad F(s) &= \frac{a}{s^2 + a^2}.
\end{aligned}
$$

(b) Vir $n = 0$ is $f(t) = 1$ en die Laplace transform is dan

$$
F(s) = \int_0^\infty e^{-st} dt = \frac{-e^{-st}}{s} \bigg|_0^\infty = \frac{1}{s}
$$

Ons het nou Nommer 1 bewys vir $n = 0$. Neem aan dat dit geld vir $n = k$, dws vir $f(t) = t^k$ is die Laplace transform

$$
F(s) = \frac{k!}{s^{k+1}}
$$

en ondersoek nou die Laplace transform van $f(t) = t^{k+1}$:

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} t^{k+1} \, dt \\
&= t^{k+1} \left(\frac{e^{-st}}{-s} \right) \bigg|_0^\infty - \int_0^\infty (k+1) t^k \frac{e^{-st}}{-s} \, dt \quad \text{[deelwyse integrasie]} \\
&= 0 + \frac{k+1}{s} \int_0^\infty t^k e^{-st} \, dt \quad (s > 0) \\
&= \frac{k+1}{s} \left(\frac{k!}{s^{k+1}} \right) = \frac{(k+1)!}{s^{k+2}}.
\end{aligned}
$$

$\Rightarrow$ Nommer 1 geld vir alle n.

Probleem 2:

$$L[\alpha f + \beta g](s)$$

$$= \int_0^\infty \left(\alpha f(t) + \beta g(t)\right) e^{-st}\, dt$$

$$= \int_0^\infty \alpha f(t) e^{-st}\, dt + \int_0^\infty \beta g(t) e^{-st}\, dt$$

$$= \alpha \int_0^\infty f(t) e^{-st}\, dt + \beta \int_0 g(t) e^{-st}\, dt$$

$$= \alpha L[f](s) + \beta L[g](s)$$

Probleem 3:

$$f(t) = 5e^{-2t} + 3\sin 4t$$

$$L[e^{-2t}](s) = \frac{1}{s+2}$$

$$L[\sin 4t](s) = \frac{4}{s^2+4^2} = \frac{4}{s^2+16}$$

$$\Rightarrow \quad L[f](s) = 5L[e^{-2t}](s) + 3L[\sin 4t](s)$$

$$= \frac{5}{s+2} + \frac{12}{s^2+16} \quad , \quad s > 0$$

Probleem 4:

$$F(s) = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}$$

Parsiële breuke

$$= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

$$= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}$$

$$\Rightarrow \quad f(t) = L^{-1}\left[\frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{s}{s^2+1}\right] - 3L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= t + \cos t - 3\sin t$$