Probleem 1:

LAPLACE TRANSFORMS OF DERIVATIVES

Denote $\mathscr{L}\{y(x)\}$ by $Y(s)$. Then under very broad conditions, the Laplace transform of the nth-derivative ($n = 1, 2, 3, \ldots$) of $y(x)$ is

$$\mathscr{L}\left\{\frac{d^n y}{dx^n}\right\} = s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0) \tag{17.1}$$

If the initial conditions on $y(x)$ at $x = 0$ are given by

$$y(0) = c_0, \qquad y'(0) = c_1, \ldots, y^{(n-1)}(0) = c_{n-1} \tag{17.2}$$

then (*17.1*) can be rewritten as

$$\mathscr{L}\left\{\frac{d^n y}{dx^n}\right\} = s^n Y(s) - c_0 s^{n-1} - c_1 s^{n-2} - \cdots - c_{n-2}s - c_{n-1} \tag{17.3}$$

For the special cases of $n = 1$ and $n = 2$, Eq. (*17.3*) simplifies to

$$\mathscr{L}\{y'(x)\} = sY(s) - c_0 \tag{17.4}$$

$$\mathscr{L}\{y''(x)\} = s^2 Y(s) - c_0 s - c_1 \tag{17.5}$$

$$y'' + 4\,y = 0, \quad y(0) = 2, \quad y'(0) = 2.$$

(a) Taking Laplace transforms, we have $\mathscr{L}\{y''\} + 4\mathscr{L}\{y\} = \mathscr{L}\{0\}$. Then, using Eq. (*17.5*) with $c_0 = 2$ and $c_1 = 2$, we obtain

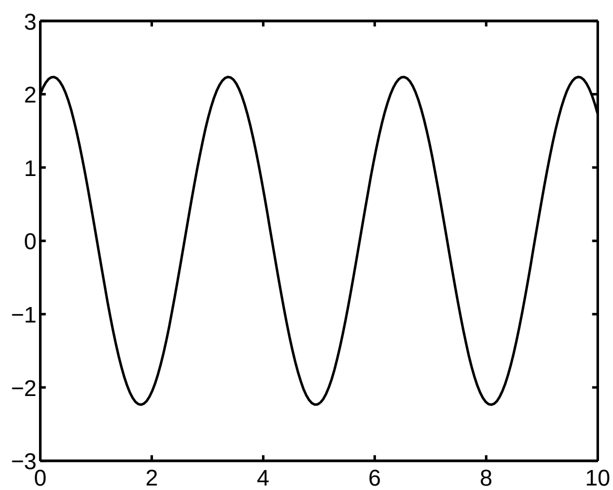
$$[s^2 Y(s) - 2s - 2] + 4Y(s) = 0$$

or

$$Y(s) = \frac{2s + 2}{s^2 + 4} = \frac{2s}{s^2 + 4} + \frac{2}{s^2 + 4}$$

Finally, taking the inverse Laplace transform, we obtain

$$y(x) = \mathscr{L}^{-1}\{Y(s)\} = 2\mathscr{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} + \mathscr{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\} = 2\cos 2x + \sin 2x$$

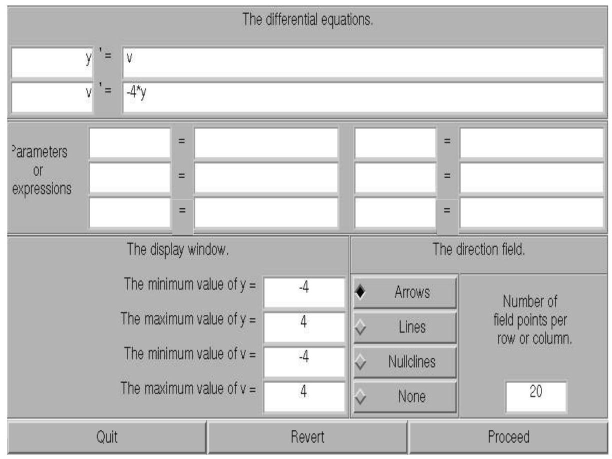


(b) Laat $v = \frac{dy}{dt}$, dan kan die gegewe tweede order DV herskryf word as die volgende stelsel van DV's:

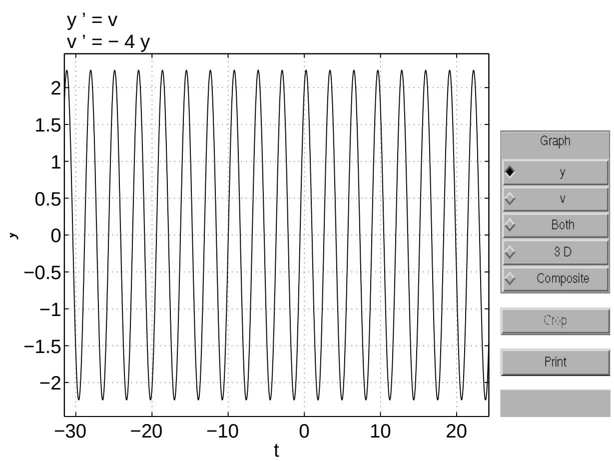
$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -4y$$

met beginwaardes $y(0) = 2$, $v(0) = 2$.



Gaan na `Solutions` $\mapsto$ `Keyboard input`, en voor dan die beginwaardes in. Gaan nou na `Graph` $\mapsto$ `y vs t` en klik dan op $y(0) = 2$, $v(0) = 2$.



Probleem 2:

(a) Taking Laplace transforms, we have $\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{0\}$. Then, using Eq. (17.5) with $c_0 = 2$ and $c_1 = 2$, we obtain

$$[s^2Y(s) - 2s - 2] + 4Y(s) = 0$$

or

$$Y(s) = \frac{2s+2}{s^2+4} = \frac{2s}{s^2+4} + \frac{2}{s^2+4}$$

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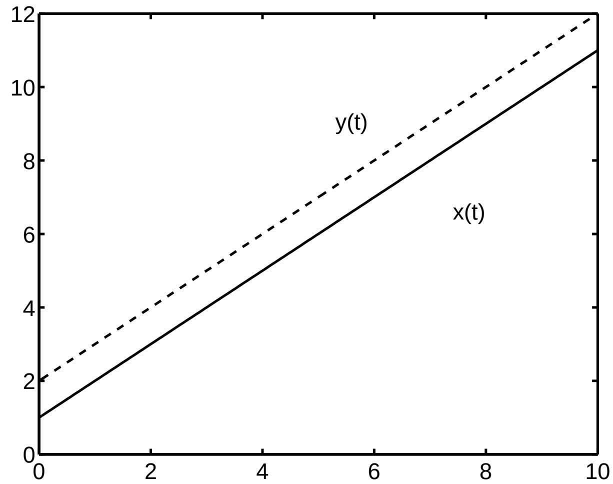
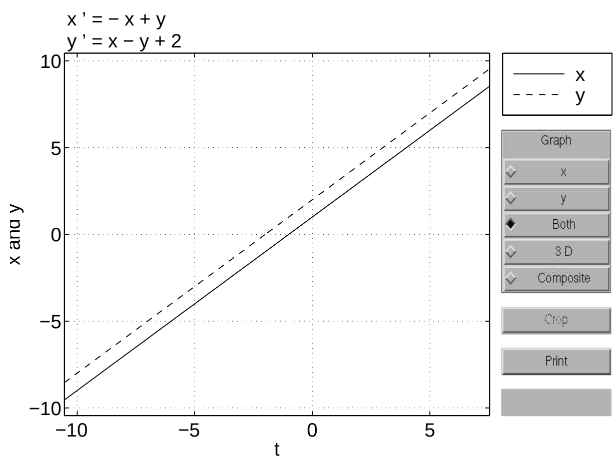
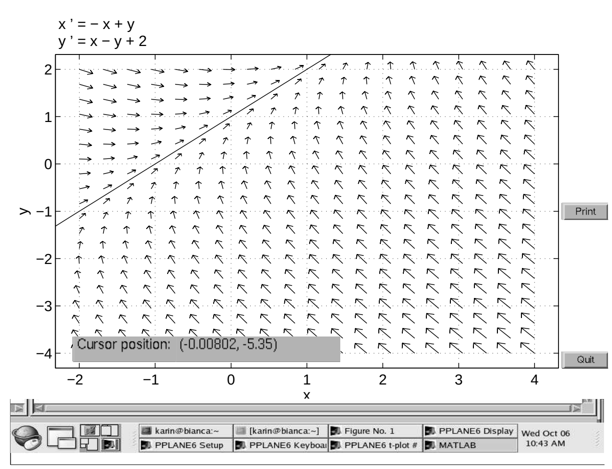


(c) Hierdie stelsel DV's is ekwivalent aan die vergelyling

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

wat nie homogeen is nie. Ons teorie het ons alleenlik vir homogene vergelykings afgelei. Laat asb hierdie voorbeeld uit, en sien HW #11 Probleem 1, vir 'n voorbeeld van die oplos van 'n stelsel DV's mbv eiewaardes en eievektore.





Probleem 3:

(a) Laat

$$\begin{array}{ll} x(t) & \text{liters kleurstof teenwoordig in tenk A op tydstip } t \\ y(t) & \text{liters kleurstof teenwoordig in tenk B op tydstip } t \end{array}$$

$$\begin{array}{ll} \dfrac{x(t)}{100} & \text{konsentrasie van kleurstof in tenk A op tydstip } t \\ \dfrac{y(t)}{100} & \text{konsentrasie van kleurstof in tenk B op tydstip } t \end{array}$$

Die DV is dan, volgens die behoud van kleurstof,

$$\begin{aligned} \frac{dx}{dt} &= 1 - \tfrac{x}{100} \times 4 + \tfrac{y}{100} \times 3 \\ \frac{dy}{dt} &= \tfrac{x}{100} \times 4 - \tfrac{y}{100} \times 3 + \tfrac{y}{100} \times 2 \end{aligned}$$

$$\begin{aligned} \frac{dx}{dt} &= 1 - \tfrac{4}{100}\, x + \tfrac{3}{100}\, y \\ \frac{dy}{dt} &= \tfrac{4}{100}\, x - \tfrac{5}{100}\, y \end{aligned}$$

met beginwaardes $x(0) = 0$ en $y(0) = 0$.

(b) Laat $X(s) = \mathcal{L}\{x(t)\}$ en $Y(s) = \mathcal{L}\{y(t)\}$ en neem dan die Laplace transform van beide DV's

$$\begin{aligned} sX(s) &= \tfrac{1}{s} + \tfrac{3}{100}Y(s) - \tfrac{4}{100}X(s) \\ sY(s) &= \tfrac{4}{100}X(s) - \tfrac{5}{100}Y(s) \end{aligned}$$

of, ekwivalent

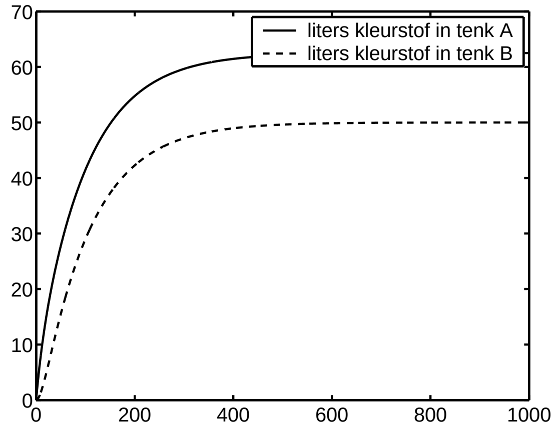
$$\begin{aligned} (s + \tfrac{4}{100})X(s) &= \tfrac{3}{100}Y(s) + \tfrac{1}{s} \\ (s + \tfrac{5}{100})Y(s) &= \tfrac{4}{100}Y(s) \end{aligned}$$

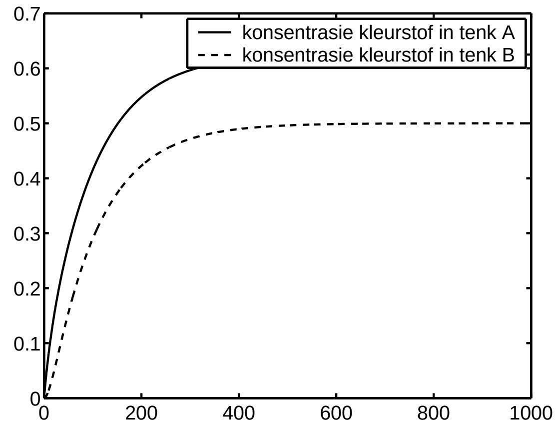
Die oplossing van hierdie gelyktydige vergelykings is

$$\begin{aligned} X(s) &= \frac{s + \frac{5}{100}}{s(s + \frac{1}{100})(s + \frac{8}{100})} = \frac{\frac{5}{8}(100)}{s} + \frac{-\frac{4}{7}(100)}{s + \frac{1}{100}} \frac{-\frac{3}{56}(100)}{s + \frac{8}{100}} \\ Y(s) &= \frac{\frac{4}{100}}{s(s + \frac{1}{100})(s + \frac{8}{100})} = \frac{\frac{1}{2}(100)}{s} + \frac{-\frac{4}{7}(100)}{s + \frac{1}{100}} \frac{\frac{1}{14}(100)}{s + \frac{8}{100}}. \end{aligned}$$

Neem nou die inverse transforms om die oplossing tot die gegewe DV te kry

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{X(s)\} = 100\Big(\tfrac{5}{8} - \tfrac{4}{7}e^{-\frac{1}{100}t} - \tfrac{3}{56}e^{-\frac{8}{100}t}\Big) \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} = 100\Big(\tfrac{1}{2} - \tfrac{4}{7}e^{-\frac{1}{100}t} + \tfrac{1}{14}e^{-\frac{8}{100}t}\Big). \end{aligned}$$





(c) $x(t) \to \frac{125}{2}$ en $y(t) \to 50$