NAAM/*Name*: Oplossings US NR. ____________

INSTRUKSIES:

(a) Een uur, toeboek, vier probleme, 25 punte.

(b) Vandag is alle probleme van die uitskryf tipe. Alle berekenings moet getoon word en alle stappe gemotiveer word. 'n Korrekte antwoord verdien nie volpunte sonder die nodige verduideliking nie.

(c) Beantwoord alle probleme in die toetsboek wat voorsien word.

(d) Na inhandiging van die toetsboek moet u voortgaan met Huiswerk #5.

(e) Moenie omblaai voordat u aangesê word om dit te doen nie.

INSTRUCTIONS:

(a) One hour, closed book, four problems, 25 marks.

(b) Today all problems are of the write-out type. All calculations have to be shown and all steps must be justified. A correct answer does not earn full marks without the necessary justification.

(c) Solve all problems in the test book that will be provided.

(d) After handing in the test you are expected to continue with Home Work #5.

(e) Do not turn the page until you are told to do so.

Vraag 1 (6 punte) / **Question 1** *(6 marks)*

Beskou die data in die tabel

Consider the data in the table

x	0	1	2	3
y	1	1	2	6

Bepaal die parameters α en λ in die eksponensiële model

Determine the parameters α en λ in the exponential model

$$y = \alpha e^{\lambda x}$$

deur van die metode van kleinste kwadrate gebruik te maak.

by using the method of least squares.

$$\ln y = \ln(\alpha e^{\lambda x}) = \ln\alpha + \lambda x \qquad (1)$$

Stel data in vgl (1):

$$\ln\alpha + \lambda(0) = \ln 1$$

$$\ln\alpha + \lambda(1) = \ln 1$$

$$\ln\alpha + \lambda(2) = \ln 2$$

$$\ln\alpha + \lambda(3) = \ln 6$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} \ln\alpha \\ \lambda \end{bmatrix} = \underbrace{\begin{bmatrix} \ln 1 \\ \ln 1 \\ \ln 2 \\ \ln 6 \end{bmatrix}}_{b} \qquad (2)$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ln 2 \\ \ln 6 \end{bmatrix} = \begin{bmatrix} 2.4849 \\ 6.7616 \end{bmatrix}$$

$\Rightarrow$ kleinste-kwadrate oplossing tot lineêre stelsel (2) word gegee deur die oplossing tot

$$\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \ln \alpha \\ \lambda \end{bmatrix} = \begin{bmatrix} 2.4849 \\ 6.7616 \end{bmatrix}$$

$$\begin{bmatrix} \ln \alpha \\ \lambda \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 2.4849 \\ 6.7616 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2890 \\ 0.6069 \end{bmatrix}$$

$\Rightarrow \boxed{\lambda = 0.6069}$

en $\ln \alpha = -0.2890$

$\Rightarrow \boxed{\alpha = 0.7490}$

Vraag 2 ($3+4=7$ punte)

(a) Vind 'n integrasiefaktor vir die volgende DV

Question 2 *($3+4=7$ marks)*

Find an integration factor for the following DE

$$(\cos x)\frac{dy}{dx} = -\sin x\left(y + \cos x\right)$$

Vereenvoudig u antwoord sover as moontlik. Dit is onnodig om die algemene oplossing te bepaal. *Wenk:* $I(x) = e^{\int p(x)\,dx}$.

Simplify your answer as far as possible. It is not necessary to determine the general solution. Hint: $I(x) = e^{\int p(x)\,dx}$.

(b) Aanvaar sonder bewys dat $I(x) = (x^2+1)^{3/2}$ 'n integrasiefaktor is vir die DV

(b) Assume without proof that $I(x) = (x^2+1)^{3/2}$ *is an integrafion factor for the DE*

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1},$$

en bepaal dan die algemene oplossing.

and then determine the general solution.

(a)

$$(\cos x)\frac{dy}{dx} = -(\sin x)y - \cos x \sin x$$

$$\frac{dy}{dx} + (\tan x)y = -\sin x \qquad \text{(in normaalvorm)}$$

$$\Rightarrow \text{Integrasie faktor } I(x) = e^{\int \tan x\, dx} = e^{\ln \sec x} = \sec x$$

(b)

$$(x^2+1)^{3/2}\frac{dy}{dx} + (x^2+1)^{1/2}\,3xy = (x^2+1)^{1/2}\,6x$$

$$\frac{d}{dx}\left((x^2+1)^{3/2}y\right) = (x^2+1)^{1/2}\,6x$$

$$(x^2+1)^{3/2}y = \int (x^2+1)^{1/2}\,6x\,dx$$

$$= 3\int (x^2+1)^{1/2}\,2x\,dx \qquad (*)$$

3

Stel $u = x^2+1$

$\Rightarrow du = 2x\,dx$

$$\Rightarrow \quad \int (x^2+1)^{1/2}\, 2x\, dx = \int u^{1/2}\, du$$

$$= \frac{u^{3/2}}{3/2} + k$$

$$= \underline{\frac{2}{3}(x^2+1)^{3/2} + k.}$$

Stel terug in (*)

$$(x^2+1)^{3/2}\, y = 3 \cdot \frac{2}{3}(x^2+1)^{3/2} + k$$

$$\boxed{y(x) = 2 + \frac{k}{(x^2+1)^{3/2}}}$$

Vraag 3 (2 + 2 = 4 punte)

'n Swamkultuur bestaan aanvanklik uit α selle. Die groeitempo is direk eweredig aan die aantal selle teenwoordig.

(a) Skryf die DV neer wat hierdie situasie modelleer, en los dit op. (Met enige metode, insluitend memorisering.)

(b) Gestel na 1 uur het die aantal selle met 50% vermeerder. Hoe lank sal dit duur vir die aantal selle om te ver**drie**dubbel?

Question 3 *(2 + 2 = 4 marks)*

A culture of fungi consists initially of α cells. The growth rate is directly proportional to the number of cells present.

(a) Write down the DE that models this situation, and solve it. (By any method, including memorization.)

(b) Suppose after 1 hour the number of cells has increased by 50%. How long will it take for the number of cells to ***triple****?*

(a) Gestel $N = N(t)$ = aantal selle teenwoordig.

$$\frac{dN}{dt} = kN, \qquad N(0) = \alpha.$$

$$N(t) = \alpha e^{kt}.$$

(b) $N(1) = \frac{3}{2}\alpha \Rightarrow \frac{3}{2}\alpha = \alpha e^{k(1)}$

$$k = \ln\frac{3}{2}$$

$$N(t) = \alpha e^{(\ln\frac{3}{2})t}.$$

Bepaal t sodat $N(t) = 3\alpha$

$$3\alpha = \alpha e^{(\ln\frac{3}{2})t}$$

$$\ln 3 = (\ln\tfrac{3}{2})t$$

$$t = \frac{\ln 3}{\ln 3/2} \approx 2.7095 \text{ ure}$$

4

Hierdie antwoord maak sin wat >1 uur

Vraag 4 (4 + 4 = 8 punte)

'n Tenk bevat 400 liter bier met alkohol inhoud 3% alkohol per liter. Bier met 6% alkohol per liter word in die tenk gepomp teen 3 liter per minuut, terwyl die goed gemengde bier teen 3 liter per minuut uitgepomp word.

(a) Herlei 'n aanvangswaardeprobleem wat die hoeveelheid liter alkohol, $A(t)$, op enige tyd t in die tenk beskryf.

(b) Los die DV van (a) op en bepaal die persentasie alkohol in die tenk na 60 minute.

Question 4 *(4 + 4 = 8 marks)*

A tank contains 400 litres beer with alcohol content 3% alcohol per litre. Beer with 6% alcohol per litre is pumped into the tank at a rate of 3 litres per minute, while the well-mixed beer is pumped out at a rate of 3 litres per minute.

(a) Derive an initial value problem that describes the amount of litres of alcohol, $A(t)$, in the tank at time t.

(b) Solve the DE in (a) to determine the percentage alcohol in the tank after 60 minutes.

(a) Laat $A(t)$ die hoeveelheid liter alkohol, op tyd t, in die tenk wees.

Dan is

$$\begin{pmatrix}\text{die verandering}\\ \text{in } \ell \text{ alkohol}\end{pmatrix} = \begin{pmatrix}\ell \text{ alkohol}\\ \text{in}\end{pmatrix} - \begin{pmatrix}\ell \text{ alkohol}\\ \text{uit}\end{pmatrix}$$

$$\frac{dA}{dt} = \left(\frac{6}{100} \times 3\right) - \left(\frac{A}{400} \times 3\right)$$

$$= \frac{18}{100} - \frac{3}{400}A$$

$$A(0) = \frac{3}{100} \times 400 = 12 \ \ell$$

(b) $$\frac{dN}{dt} + \frac{3}{400} N = \frac{18}{100}$$

$$I(x) = e^{\int 3/400 \, dt} = e^{3t/400}$$

$$e^{3t/400} \frac{dN}{dt} + e^{3t/400} \frac{3}{400} N = \frac{18}{100} e^{3t/400}$$

$\boxed{\text{Laat } k = \frac{3}{400}}$

$$\frac{d}{dt}\left(e^{3t/400} N \right) = \frac{18}{100} e^{3t/400}$$

$$e^{kt} . N = \frac{18}{100} \int e^{kt} \, dt$$

$$e^{kt} N = \frac{18}{100} \frac{1}{k} e^{kt} + C$$

$$= \frac{18}{100} \frac{400}{3} e^{\frac{3}{400}t} + C$$

$$e^{\frac{3}{400}t} N = 24 e^{\frac{3}{400}t} + C$$

$N(0) = 12$ $$12 = 24 + C$$

$$\Rightarrow C = -12 .$$

$$\boxed{N(t) = 24 - 12 e^{-\frac{3}{400}t}}$$

As $t \to \infty$, $N(t) \to 24$

wat sin maak want mengsel wat in kom is 6% alkohol $\Rightarrow \frac{6}{100} \times 400 = 24 \ell$.

$$N(60) = 24 - 12e^{-\frac{3}{400}(60)}$$

$$= 16.348 \ \ell$$

persentasie alkohol na 60 minute $= 4.087$