Probleem 1: (a) $e^{-x}\,y' - y^2 = 0$
Oplossing:

$$\frac{1}{y} = -e^x + C \quad \Rightarrow \quad y = \frac{1}{-e^x + C}$$

(b) $y\,y' - xy^2\cos(x^2) = 0.$
Oplossing:

$$\ln y = \frac{1}{2}\sin x^2 + C \quad \Rightarrow \quad y = e^{\frac{1}{2}\sin x^2}\,K$$

Probleem 2:

(a) Die differensiaal vergelyking

$$z^4 u'(z) + 2z^3 u(z) = 1$$

oftewel

$$u'(z) + 2\frac{1}{z}\,u(z) = \frac{1}{z^4}$$

het integrasiefaktor

$$\mu(z) = e^{\int \frac{2}{z}dz} = e^{2\ln|z|} = z^2.$$

Dus

$$\begin{aligned} z^2\,u'(z) + 2z\,u(z) &= \frac{1}{z^2} \\ \frac{d}{dz}\left(z^2 u(z)\right) &= \frac{1}{z^2} \\ \Rightarrow \quad z^2 u(z) &= \int \frac{1}{z^2}\,dz + c = -\frac{1}{z} + c \\ u(z) &= -\frac{1}{z^3} + c\frac{1}{z^2} \\ &= g(z) + cf(z) \end{aligned}$$

Vir $u(z) = f(z) = \frac{1}{z^2}$

$$z^4 u'(z) + 2z^3 u(z) = z^4\left(-2\frac{1}{z^3}\right) + 2z^3\left(\frac{1}{z^2}\right) = -2z + 2z = 0$$

'n oplossing van die homogene DV.
Vir $u(z) = g(z) = -\frac{1}{z^3}$

$$z^4 u'(z) + 2z^3 u(z) = z^4\left(\frac{3}{z^4}\right) + 2z^3\left(-\frac{1}{z^3}\right) = 3 - 2 = 1$$

'n partikuliere integraal.

(b) Die differensiaal vergelyking

$$(\cos\theta)r'(\theta) - (\sin\theta)r(\theta) = 3\sin\theta\,\cos^2\theta$$

oftewel

$$r'(\theta) - (\tan\theta)r(\theta) = 3\sin\theta\,\cos\theta.$$

het integrasiefaktor

$$\mu(z) = e^{\int \tan\theta d\theta} = e^{|\cos\theta|} = \cos\theta.$$

Dus

$$\begin{aligned}
(\cos\theta)r'(\theta) - (\sin\theta)r(\theta) &= 3\sin\theta\,\cos^2\theta \\
\frac{d}{d\theta}\left(\cos\theta\, r(\theta)\right) &= 3\sin\theta\,\cos^2\theta \\
\Rightarrow \quad \cos\theta\, r(\theta) &= \int 3\sin\theta\,\cos^2\theta\, dz + c = -\cos^3\theta + c \\
r(\theta) &= -\cos^2\theta + c\sec\theta \\
&= g(\theta) + cf(\theta)
\end{aligned}$$

Vir $r(\theta) = f(\theta) = \sec\theta$

$$\begin{aligned}
(\cos\theta)r'(\theta) - (\sin\theta)r(\theta) &= (\cos\theta)(\sec\theta\tan\theta) - \sin\theta\,(\sec\theta) \\
&= \tan\theta - \tan\theta = 0
\end{aligned}$$

'n oplossing van die homogene DV.

Vir $r(\theta) = g(\theta) = -\cos^2\theta$

$$\begin{aligned}
(\cos\theta)r'(\theta) - (\sin\theta)r(\theta) &= (\cos\theta)(-2\cos\theta\sin\theta) - \sin\theta\left(-\cos^2\theta\right) \\
&= 2\cos^2\theta\sin\theta + \cos^2\theta\sin\theta \\
&= 3\cos^2\theta\sin\theta
\end{aligned}$$

'n partikuliere integraal.

Vir die randvoorwaarde $r(\pi) = 3$ is $c = -4$.

Probleem 3: Polonium (^{210}Po) is 'n radioaktiewe isotoop met halveertyd 140 dae.

Die tempo van verval is direk eweredig aan die hoeveelheid teenwoordig, dws as N die hoeveelheid Polonium is op tydstip t (t in dae) dan is

$$\begin{aligned}
\frac{dN}{dt} &= -k\,N \\
\Rightarrow \quad \int \frac{1}{N} dN &= -k\int dt \\
\ln|N| &= -k\,t + c \\
N(t) &= e^{-kt}C
\end{aligned}$$

en die helfte van enige gegewe hoeveelheid sal disintegreer in 140 dae, dws

$$\begin{aligned}
N(t) &= 2\,N(t+140) \\
e^{-kt}C &= 2\,e^{-k(t+140)}C \\
\frac{1}{2} &= e^{-140k} \quad \Rightarrow \quad k = \frac{\ln 2}{140}.
\end{aligned}$$

Stel k in vergelyking vir N

$$N(t) = e^{-\frac{\ln 2}{140}t}C \tag{1}$$

(a) Ons word gegee dat $N(0) = 20$. Gebruik hierdie in (1) om te kry dat

$$C = 20,$$

dus die formule vir die massa wat na t dae oorbly in

$$N(t) = 20\, e^{-\frac{\ln 2}{140} t}$$

(b) na twee weke is $t = 14$,

$$N(14) = 20\, e^{-\frac{\ln 2}{140} 14} = 20\, e^{-\frac{\ln 2}{10}} = 19$$

(afgerond tot die naaste mg).

(c) Ons soek t sodat $N(t) = 7$:

$$7 = 20\, e^{-\frac{\ln 2}{140} t} \quad \Rightarrow \quad t = 212 \text{ dae.}$$

Probleem 4: Gestel $y(t)$ is die hoeveelheid sout (in kg) op tydstip t. Ons begin deurom 'n formule vir dy/dt, die tempo van verandering van die hoeveelheid sout in die tenk op tydstip t, te bepaal. Dit is duidelik dat

$$\frac{dy}{dt} = \text{tempo in} - \text{tempo uit}, \tag{2}$$

en

$$\text{tempo in} = (2\, kg/\ell)(5\, \ell/min) = 10\, kg/min.$$

Op tydstip t, bevat die mengsel $y(t)\, kg$ sout un 100ℓ, dus is die konsentrasie van sout op tydstip t $y(t)/100\, kg/\ell$ en

$$\text{tempo uit} = \left(\frac{y(t)}{100}\, kg/\ell\right)(5\, \ell/min) = \frac{y(t)}{20}\, kg/min.$$

Nou kan ons (2) skryf as

$$\frac{dy}{dt} = 10 - \frac{y}{20}$$

of

$$\frac{dy}{dt} + \frac{y}{20} = 10,$$

wat 'n eerste orde lineêre DV is met beginwaarde

$$y(0) = 4.$$

Integrasie faktor vir hierdie DV $\mu = e^{\int \frac{1}{20} dt} = e^{\frac{t}{20}}$ wat dan gee

$$\begin{aligned}
\frac{d}{dt}(e^{\frac{t}{20}} y) &= 10\, e^{\frac{t}{20}} \\
e^{\frac{t}{20}} y &= \int 10\, e^{\frac{t}{20}}\, dt = 200\, e^{\frac{t}{20}} + C \\
y &= 200 + C\, e^{-\frac{t}{20}}.
\end{aligned}$$

Die beginwaarde gee

$$C = -196,$$

dus

$$y(t) = 200 - 196\, e^{-\frac{t}{20}}.$$

Dan vir $t = 10\, min$ is die hoeveelheid sout in die tenk

$$y(10) = 200 - 196\, e^{-\frac{10}{20}} \approx 81.1\, kg.$$