Probleem 1:

Laat $f(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$ enige kubiese polinoom wees. Dan

$$\int_a^b f(x)\mathrm{d}x \;=\; \alpha \int_a^b x^3 \mathrm{d}x + \beta \int_a^b x^2 \mathrm{d}x + \gamma \int_a^b x \mathrm{d}x + \delta \int_a^b 1 \mathrm{d}x.$$

Die integrale $\int_a^b x^2 \mathrm{d}x$, $\int_a^b x \mathrm{d}x$ en $\int_a^b 1 \mathrm{d}x$ word almal eksak deur Simpson geïntegreer, want die metode is herlei uit paraboliese interpolasie. Dit is dus net nodig om aan te toon dat $\int_a^b x^3 \mathrm{d}x$ ook eksak geïntegreer word deur Simpson. Ons wil dus bewys dat

$$\int_a^b x^3 \mathrm{d}x \;=\; \frac{h}{3}\left(f(a) + 4f(\frac{a+b}{2}) + f(b)\right) \tag{1}$$

waar $h = (b-a)/2$ en $f(x) = x^3$.

Nou, die linkerkant van (1) is

$$\begin{aligned} L &= \int_a^b x^3 \mathrm{d}x \;=\; \frac{1}{4}x^4\Big|_a^b \;=\; \frac{1}{4}(b^4 - a^4) \\ &= \frac{1}{4}(b^2 - a^2)(b^2 + a^2) \;=\; \frac{1}{4}(b-a)(b+a)(b^2+a^2). \end{aligned}$$

Die regterkant van (1) is

$$\begin{aligned} R &= \frac{h}{3}\left(a^3 + 4(\frac{a+b}{2})^3 + b^3\right) \\ &= \frac{b-a}{6}\left(a^3 + \frac{1}{2}a^3 + \frac{3}{2}a^2b + \frac{3}{2}ab^2 + \frac{1}{2}b^3 + b^3\right) \\ &= \frac{b-a}{6}\frac{3}{2}(a^3 + a^2b + ab^2 + b^3) \\ &= \frac{b-a}{4}(a+b)(a^2+b^2) \;=\; L, \end{aligned}$$

en dus geld (1).

Probleem 2:

(a)

$$\int_{-1}^{1} f(x)\mathrm{d}x \;\approx\; w_0 f(-1) + w_1 f(1)$$

Ons vereis dat die formule die funksie $f = x^0$ eksak moet integreer,

$$\int_{-1}^{1} \mathrm{d}x \;=\; w_0 \cdot 1 + w_1 \cdot 1 \quad\Longrightarrow\quad 2 \;=\; w_0 + w_1, \tag{2}$$

en ook die funksie $f = x^1$,

$$\int_{-1}^{1} x\,\mathrm{d}x \;=\; w_0(-1) + w_1(1) \quad\Longrightarrow\quad 0 \;=\; -w_0 + w_1. \tag{3}$$

Los w_0 en w_1 op uit (2) en (3): $w_0 = 1$ en $w_1 = 1$. Die reël word dus gegee deur

$$\int_{-1}^{1} f(x)\mathrm{d}x \;=\; f(-1) + f(1).$$

Gestel die fout is van die vorm $Cf''(\xi)$, d.w.s.

$$\int_{-1}^{1} f(x)\mathrm{d}x - (f(-1) + f(1)) \;=\; Cf''(\xi).$$

Pas hierdie formule toe op $f = x^2$ (dus $f' = 2x$ en $f'' = 2$).

$$\begin{aligned} \int_{-1}^{1} x^2\mathrm{d}x - (1+1) \;&=\; C \cdot 2 \\ \frac{2}{3} - 2 \;&=\; 2C \quad\Longrightarrow\quad C \;=\; -\frac{2}{3}. \end{aligned}$$

Dus

$$\int_{-1}^{1} f(x)\mathrm{d}x - (f(-1) + f(1)) \;=\; -\frac{2}{3}f''(\xi),$$

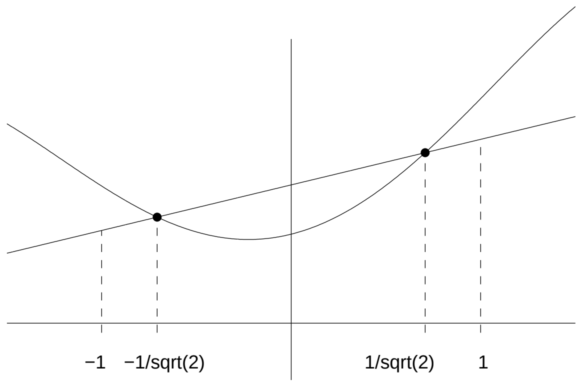
wat identies is aan die Trapesiumreël op $[a, b] = [-1, 1]$.

(b)

Net soos in deel (a) volg dat $w_0 = 1$ en $w_1 = 1$, sodat die reël gedefinieer word deur

$$\int_{-1}^{1} f(x)\mathrm{d}x \;\approx\; f(-\frac{1}{\sqrt{2}}) + f(\frac{1}{\sqrt{2}}).$$

Hierdie is 'n gewysigde trapesiumreël, soos in die volgende skets gesien kan word:



Ons bepaal die foutkonstante C, deur

$$\int_{-1}^{1} f(x)\mathrm{d}x - \left(f(-\frac{1}{\sqrt{2}}) + f(\frac{1}{\sqrt{2}}) \right) \; = \; Cf''(\xi)$$

toe te pas op $f = x^2$:

$$\begin{aligned} \int_{-1}^{1} x^2\mathrm{d}x - \left(\frac{1}{2} + \frac{1}{2}\right) \; &= \; C \cdot 2 \\ \frac{2}{3} - 1 \; &= \; 2C \quad \Longrightarrow \quad C \; = \; -\frac{1}{6}. \end{aligned}$$

Dus

$$\int_{-1}^{1} f(x)\mathrm{d}x - \left(f(-\frac{1}{\sqrt{2}}) + f(\frac{1}{\sqrt{2}}) \right) \; = \; -\frac{1}{6}f''(\xi),$$

wat 'n baie kleiner foutkonstante het as die trapesiumreël ($\frac{1}{6}$ vs. $\frac{2}{3}$). Dit is te verwagte, want die Chebyshev punte is optimaal vir interpolasie.

(c)

$$\int_{-1}^{1} f(x)\mathrm{d}x \; \approx \; w_0 f(-\frac{1}{\sqrt{3}}) + w_1 f(\frac{1}{\sqrt{3}})$$

Net soos in deel (a) volg dat $w_0 = 1$ en $w_1 = 1$. Probeer die foutkonstante bepaal uit

$$\int_{-1}^{1} f(x)\mathrm{d}x - \left(w_0 f(-\frac{1}{\sqrt{3}}) + w_1 f(\frac{1}{\sqrt{3}}) \right) \; = \; Cf''(\xi),$$

met $f = x^2$ (dus $f' = 2x$ en $f'' = 2$):

$$\frac{2}{3} - \left(\frac{1}{3} + \frac{1}{3}\right) \;=\; C \cdot 2 \quad\Longrightarrow\quad C \;=\; 0.$$

Wysig dus die regterkant van die foutformule na $Cf'''(\xi)$,

$$\int_{-1}^{1} f(x)\mathrm{d}x - \left(w_0 f(-\frac{1}{\sqrt{3}}) + w_1 f(\frac{1}{\sqrt{3}})\right) \;=\; Cf'''(\xi),$$

en pas toe op $f = x^3$ (dus $f' = 3x^2$, $f'' = 6x$ en $f''' = 6$):

$$\begin{aligned} \int_{-1}^{1} x^3\mathrm{d}x - \left(-\frac{1}{3\sqrt{3}} + \frac{1}{3\sqrt{3}}\right) \;&=\; C \cdot 6 \\ 0 - 0 \;&=\; 6C \quad\Longrightarrow\quad C \;=\; 0. \end{aligned}$$

Wysig dus die foutformule na $Cf^{(4)}(\xi)$,

$$\int_{-1}^{1} f(x)\mathrm{d}x - \left(w_0 f(-\frac{1}{\sqrt{3}}) + w_1 f(\frac{1}{\sqrt{3}})\right) \;=\; Cf^{(4)}(\xi),$$

en pas toe op $f = x^4$ (dus $f' = 4x^3$, $f'' = 12x^2$, $f''' = 24x$ en $f^{(4)} = 24$):

$$\begin{aligned} \int_{-1}^{1} x^4\mathrm{d}x - \left(-\frac{1}{9} + \frac{1}{9}\right) \;&=\; C \cdot 24 \\ \frac{2}{5} - \frac{2}{9} \;&=\; 24C \quad\Longrightarrow\quad C \;=\; \frac{1}{135}. \end{aligned}$$

Dus

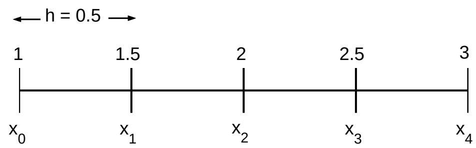
$$\int_{-1}^{1} f(x)\mathrm{d}x - \left(w_0 f(-\frac{1}{\sqrt{3}}) + w_1 f(\frac{1}{\sqrt{3}})\right) \;=\; \frac{1}{135} f^{(4)}(\xi), \quad \xi \in [-1, 1].$$

(d)

Die formule van deel (c) integreer alle polinome tot en met kubies eksak. Die formules van dele (a) en (b) integreer slegs polinome tot en met lineêr eksak. Verder is die foukonstante van die metode van (c) $\frac{1}{135}$, wat baie kleiner is as die foutkonstantes van die ander twee metodes ($\frac{2}{3}$ en $\frac{1}{6}$). Die metode van deel (c) is dus strate beter.

Probleem 3:

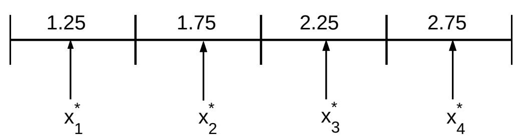
(a)



$$\begin{aligned}
T_4 &= h\left(\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4)\right) \\
&= 0.5\left(\frac{1}{2}\sqrt{1+1} + \sqrt{1+1.5^3} + \sqrt{1+2^3} + \sqrt{1+2.5^3} + \frac{1}{2}\sqrt{1+3^3}\right) \\
&= 6.2609\ldots
\end{aligned}$$

$$|I - T_4| \;=\; 3.1 \times 10^{-2}.$$

(b)



$$\begin{aligned}
M_4 &= h\left(f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*)\right) \\
&= 0.5\left(\sqrt{1+1.25^3} + \sqrt{1+1.75^3} + \sqrt{1+2.25^3} + \sqrt{1+2.75^3}\right) \\
&= 6.2145\ldots
\end{aligned}$$

$$|I - M_4| \;=\; 1.5 \times 10^{-2}.$$

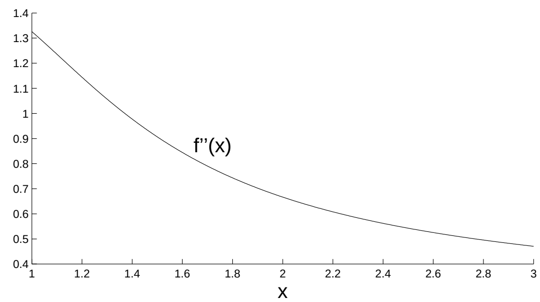
[LW: die fout met die middelpuntreël is ongeveer 2 maal kleiner as die fout met die trapesiumreël, soos voorspel deur die teoretiese foutskattings.]

(c)

$$
\begin{aligned}
S_4 &= \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)\\
&= \frac{0.5}{3}\left(\sqrt{1+1} + 4\sqrt{1+1.5^3} + 2\sqrt{1+2^3} + 4\sqrt{1+2.5^3} + \sqrt{1+3^3}\right)\\
&= 6.2303\ldots
\end{aligned}
$$

$$|I - S_4| \;=\; 3.4 \times 10^{-4}.$$

(d)



$$\Longrightarrow \quad \max_{1\le x\le 3} |f''(x)| \;=\; f''(1) \;=\; -\frac{9}{4}\cdot\frac{1}{2^{3/2}} + 3\cdot\frac{1}{\sqrt{2}} \;=\; \frac{15}{16}\sqrt{2}$$

$$
\begin{aligned}
I - T_n &= -\frac{b-a}{12}h^2 f''(\xi)\\
|I - T_n| &\le \frac{b-a}{12}h^2 \max_{a\le x\le b}|f''(x)|.
\end{aligned}
$$

Met $a = 1$, $b = 3$, $n = 4$ en $h = \frac{3-1}{4} = \frac{1}{2}$, volg

$$
\begin{aligned}
|I - T_4| &\le \frac{3-1}{12}\left(\frac{1}{2}\right)^2 \frac{15}{16}\sqrt{2}\\
&= \frac{5}{128}\sqrt{2}\\
&= 5.5 \times 10^{-2},
\end{aligned}
$$

wat groter is as die werklike fout, 3.1×10^{-2}, soos bereken in deel (a), soos dit hoort.

(e)

$$\begin{aligned}
\frac{b-a}{12}h^2 \max_{a\le x\le b}|f''(x)| &\le 10^{-4} \\
\frac{1}{6}\left(\frac{2}{n}\right)^2 \frac{15}{16}\sqrt{2} &\le 10^{-4} \\
n^2 &\ge \frac{5}{8}\sqrt{2}\times 10^4 \\
n &\ge 94.02\ldots
\end{aligned}$$

Gebruik dus ten minste 95 intervalle.

[In die praktyk word gevind dat 71 intervalle voldoende is vir hierdie spesifieke integraal.]