Section 4 : Observers | Synchronization. ⓪

[Pecora-Carroll 91].

Lorenz-63: $\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = -xz + rx - y \\ \dot{z} = xy - bz. \end{cases}$ $u = (x, y, z)$ Soln's have abs. ball property. $|u| \leq R$.

Suppose $(\hat{x}, \hat{y}, \hat{z})$ is given by ...

$$\begin{cases} \dot{\hat{x}} = \sigma(y - \hat{x}) \\ \hat{y} = y \\ \dot{\hat{z}} = \hat{x}y - b\hat{z}. \end{cases} \qquad \hat{u} = (\hat{x}, \hat{y}, \hat{z}).$$

Then Let $w = u - \hat{u} = (w_1, w_2, w_3)$ satisfies,

$$\dot{w}_1 = -\sigma w_1, \quad w_2 = 0, \quad \dot{w}_3 = w_1 y - b w_3.$$

$\Rightarrow$ Clearly $|w| \to 0$ as $t \to \infty$.

~~This can be generalized to other classes interpolants~~

Synchronization for Lorenz-96 Model for "atmospheric observables".

[Law, Sanz-Alonso, ~~Stuart~~ Shukla, Stuart '15]

$$\frac{du^{(j)}}{dt} = u^{(j-1)}\left(u^{(j+1)} - u^{(j-2)}\right) - u^{(j)} + F. \quad j = 1, \ldots, n$$

Periodic BCs $\quad u^{(0)} = u^{(n)}$ etc...

We write this as

$$\frac{du}{dt} + \cancel{A}u + B(u,u) = f.$$

$$A = I, \quad f = \begin{pmatrix} F \\ \vdots \\ F \end{pmatrix}, \quad B(u,v) = -\frac{1}{2}\begin{pmatrix} \vdots \\ u^{(j-1)}v^{(j+1)} + u^{(j+1)}v^{(j-1)} - \left(u^{(j-2)}v^{(j-1)} + u^{(j-1)}v^{(j-2)}\right) \\ \vdots \end{pmatrix}$$

$: \mathbb{R}^n \to \mathbb{R}^n$ (Assume min $3|n$).

Let P be the projection operator

$$P\xi = (e_1\, e_2\, 0\, e_3\, e_4\, 0 \ldots)$$

$$Q = I - P = (0\, 0\, e_3\, 0\, 0\, e_6 \ldots)$$

Properties:

- $B(Qu, Qu) = 0 \quad \forall u.$
- $B(u,v) = B(v,u)$
- $2\langle B(u,v), u\rangle = -\langle B(u,u), v\rangle.$

③

We will build an observer $\hat{u}$ s.t. $|\hat{u} - u| \to 0$ exponentially as $t \to \infty$.

Let $\hat{u} = Pu + q$, where

$$\frac{dq}{dt} + q + QB(Pu+q, Pu+q) = Qf.$$

$$q(0) = Qq(0). \quad \therefore \quad Qq = q.$$

$\boxed{\text{Thm: } |\hat{u}-u| \to 0 \text{ (expo quickly)}}$

Let $W = \hat{u} - u$. ~~then~~

$$= (Pu+q) - (Pu+Qu) = q - Qu.$$

$\cancel{\frac{dw}{dt} + w + Qw = w}$ $\quad Qw = w$

$\cancel{\frac{dQu}{dt} + Qu + QB(u,u) = Qf}.$

$\therefore \quad \cancel{\frac{dw}{dt} + w} \quad \frac{dQu}{dt} + Qu + QB(u,u) = Qf.$

$\therefore \cancel{\frac{dw}{dt} + w}$

$$\frac{dW}{dt} + W + QB(Pu+q, Pu+q) - QB(u,u) = \cancel{Qf}\ 0.$$

$$Q\frac{dW}{dt} + QW + \underbrace{QB(Pu+q, Pu+q) - QB(u,u)}_{} = 0.$$

$$\to \cancel{QB(Pu, Pu)} + 2QB(Pu, q) + QB(q,q)$$
$$- \cancel{QB(Pu, Pu)} - 2QB(Pu, Qu) - QB(Qu, Qu)$$

④

$$= 2QB(Pu, q - Qu) = 2QB(Pu, Qw).$$

$$\therefore \frac{1}{2}\frac{d}{dt}|w|^2 = \frac{1}{2}\frac{d}{dt}|Qw|^2$$

$$= -|w|^2 - \langle 2QB(Pu, Qw), Qw\rangle$$

But $-2\langle QB(Pu, Qw), Qw\rangle$

$$= -2\langle B(Pu, Qw), Qw\rangle$$

$$= -2\langle B(Qw, Pu), Qw\rangle$$

$$= \langle B(Qw, Qw), Pu\rangle = 0.$$

$$\therefore \frac{1}{2}\frac{d}{dt}|w|^2 + |w|^2 = 0.$$

$\square$

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for NSE.

Synchronization using "interpolants"

[Azouani, Olson, Titi 13]. on $\Omega \subseteq \mathbb{R}^2$

We can write 2d div-free Navier-Stokes in the form:

$$\frac{du}{dt} + \nu A u + B(u,u) = f.$$

$A = -P_\sigma \Delta$ $\qquad$ P_σ Leray projection.

$B(u,v) = P_\sigma(u \cdot \nabla v)$.

Dirichlet BCs
- $\mathcal{V} = \mathcal{C}^\infty$ comp. supp. in Ω. VFs $: \Omega \to \mathbb{R}^2$ div-free.
- H = closure of $\mathcal{V}$ in $L^2(\Omega)$.
- V = closure of $\mathcal{V}$ in $H^1(\Omega)$.

H, V Hilbert spaces w/ inner-products

$$\langle u,v \rangle = \int_\Omega u\, v \quad , \quad \langle\langle u,v \rangle\rangle = \int_\Omega \partial_{x_j} u_i \, \partial_{x_j} v_i$$

Norms $|\cdot|$, $\|\cdot\|$.

We observe the solution u through a function

$$I_h : H^1(\Omega) \to L^2(\Omega).$$

which is linear and satisfies

$$\|\varphi - I_h(\varphi)\|_{L^2}^2 \leq c_0^2 h^2 \|\varphi\|_{H^1}^2 \quad \forall \varphi \in H^1$$

h is "spatial resolution" parameter.

eg. I_h projection onto Fourier modes $|k| \leq \frac{1}{h}$.

eg. Vol elements: $I_h(\varphi)(x) = \sum_{j=1}^{N} \bar{\varphi}_j \mathbb{1}_{Q_j}(x)$, $\bar{\varphi}_j = \frac{1}{|Q_j|}\int_{Q_j} \varphi(x)\,dx.$

Let $\hat{u}$ satisfy

$$\frac{d\hat{u}}{dt} + \nu A\hat{u} + B(\hat{u},\hat{u}) = f + \mu P_\sigma\left(I_h(u) - I_h(\hat{u})\right).$$

$\mu > 0$ some parameter.

Thm: ~~$|u-\hat{u}| \to 0$ exponentially~~

If $\mu \geq 5c^2 G^2 \nu \lambda_1$ and $\mu c_0^2 h^2 \leq \nu$ then

$|u-\hat{u}| \to 0$ exponentially as $t \to \infty$.

Properties :

- $\langle B(u,w),w\rangle = 0$
- $|\langle B(w,u),w\rangle| \leq c\,\|u\|\,.\,|w|\,.\,\|w\|$.

Let $G = \dfrac{1}{\nu^2\lambda_1} \limsup_{t\to\infty} \|f(t)\|_{L^2}$ (Grashof number).

λ_1 smallest eig. of A.

Then $\displaystyle\int_t^{t+T} \|u(s)\|^2 \, ds \leq 2(1+T\nu\lambda_1)\nu G^2$.

Proof : Let $w = u - \hat{u}$ then

$$\frac{dw}{dt} + \nu A w + B(u,w) + B(w,u) - B(w,w) = -\mu P_\sigma I_h(w).$$

Thus we have

$$\frac{1}{2}\frac{d}{dt}|w|^2 = \langle -\nu A w - B(u,w) - B(w,u) + B(w,w) - \mu P_\sigma I_h(w), w\rangle$$
$$= -\nu\|w\|^2 - \langle B(w,u), w\rangle - \mu\langle P_\sigma I_h(w), w\rangle.$$

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + \langle B(w,u), w\rangle$$
$$= \mu\langle P_\sigma(w - I_h(w)), w\rangle - \mu|w|^2.$$
$$\leq \frac{\mu}{2}|P_\sigma(w - I_h(w))|^2 - \frac{\mu}{2}|w|^2. \leq \frac{\mu c_0 h^2}{2}\|w\|^2 - \frac{\mu}{2}|w|^2$$
$$\leq \frac{\nu}{2}\|w\|^2 - \frac{\mu}{2}|w|^2.$$

Moreover, $|\langle B(w,u), w\rangle| \leq c\|u\|.|w|.\|w\|$
$$\leq \frac{c^2}{2\nu}\|u\|^2|w|^2 + \frac{\nu}{2}\|w\|^2 \quad \left(\begin{array}{l}\text{Young} \\ \text{w/ } \delta = c/\nu\end{array}\right)$$

$$\therefore \frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 \leq \nu\|w\|^2 - \left(\frac{\mu}{2} - \frac{c^2}{2\nu}\|u\|^2\right)|w|^2.$$

Let $\alpha(t) = \mu - \frac{c^2}{\nu}\|u\|^2$ $\therefore \frac{d}{dt}|w|^2 \leq -\alpha(t)|w|^2$.

Gronwall: If for some $T>0$ & $\liminf_{t\to\infty} \int_t^{t+T} \alpha(s)\,ds \geq \gamma > 0$

then $|w| \to 0$ exponentially as $t \to \infty$.

Take $T = (\nu\lambda_1)^{-1}$ then we know

$$\limsup_{t\to\infty} \int_t^{t+T} \|u\|^2\,ds \leq 2(1 + T\nu\lambda_1)\nu G^2 = 4\nu G^2 \cdot \text{~~(t large enough)~~}$$

$$\therefore \liminf_{t\to\infty} \int_t^{t+T} \alpha(s)\,ds \geq \mu T - \frac{c^2}{\nu}\int_t^{t+T}\|u\|^2\,ds$$

$$\geq \frac{\mu}{\nu\lambda_1} - 4c^2G^2$$

$$\therefore \liminf_{t\to\infty} \int_t^{t+T} \alpha(s)\,ds \geq \frac{\mu}{\nu\lambda_1} - 4c^2G^2 \geq 5c^2G^2 - 4c^2G^2 > 0.$$

$\square$