Fast-slow systems with chaotic noise

David Kelly Ian Melbourne

Courant Institute
New York University
New York NY
www.dtbkelly.com

March 8, 2016

Probability seminar, CUNY.

Fast-slow systems

We consider fast-slow systems of the form

$$\frac{dX}{dt} = \varepsilon h(X, Y) + \varepsilon^2 f(X, Y)$$
$$\frac{dY}{dt} = g(Y) \, ,$$

where $\varepsilon \ll 1$.

$\frac{dY}{dt} = g(Y)$ be some **mildly chaotic** ODE with state space Λ and ergodic invariant measure μ. (**eg**. $3d$ Lorenz equations.)

$h, f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ and $\int h(x, y) \, \mu(dy) = 0$.

Our aim is to find a **reduced equation** for X.

Fast-slow systems

We consider fast-slow systems of the form

$$\frac{dX}{dt} = \varepsilon h(X, Y) + \varepsilon^2 f(X, Y)$$
$$\frac{dY}{dt} = g(Y) \, ,$$

where $\varepsilon \ll 1$.

$\frac{dY}{dt} = g(Y)$ be some **mildly chaotic** ODE with state space Λ and ergodic invariant measure μ. (**eg**. $3d$ Lorenz equations.)

$h, f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ and $\int h(x, y) \, \mu(dy) = 0$.

Our aim is to find a **reduced equation** for X.

Fast-slow systems

We consider fast-slow systems of the form

$$\frac{dX}{dt} = \varepsilon h(X, Y) + \varepsilon^2 f(X, Y)$$
$$\frac{dY}{dt} = g(Y) \,,$$

where $\varepsilon \ll 1$.

$\frac{dY}{dt} = g(Y)$ be some **mildly chaotic** ODE with state space Λ and ergodic invariant measure μ. (**eg**. $3d$ Lorenz equations.)

$h, f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ and $\int h(x, y)\, \mu(dy) = 0$.

Our aim is to find a **reduced equation** for X.

Fast-slow systems

We consider fast-slow systems of the form

$$\frac{dX}{dt} = \varepsilon h(X, Y) + \varepsilon^2 f(X, Y)$$
$$\frac{dY}{dt} = g(Y) \,,$$

where $\varepsilon \ll 1$.

$\frac{dY}{dt} = g(Y)$ be some **mildly chaotic** ODE with state space Λ and ergodic invariant measure μ. (**eg**. $3d$ Lorenz equations.)

$h, f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ and $\int h(x, y)\, \mu(dy) = 0$.

Our aim is to find a **reduced equation** for X.

Fast-slow systems

If we rescale to **large time scales** ($\sim \varepsilon^{-2}$) we have

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon, Y_\varepsilon) + f(X_\varepsilon, Y_\varepsilon)$$
$$\frac{dY_\varepsilon}{dt} = \varepsilon^{-2} g(Y_\varepsilon) \; ,$$

We turn X_ε into a random variable by taking $Y(0) \sim \mu$.

The aim is to characterise the **distribution** of the random path X_ε as $\varepsilon \to 0$.

Fast-slow systems

If we rescale to **large time scales** ($\sim \varepsilon^{-2}$) we have

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon, Y_\varepsilon) + f(X_\varepsilon, Y_\varepsilon)$$
$$\frac{dY_\varepsilon}{dt} = \varepsilon^{-2} g(Y_\varepsilon) ,$$

We turn X_ε into a random variable by taking $Y(0) \sim \mu$.

The aim is to characterise the **distribution** of the random path X_ε as $\varepsilon \to 0$.

Fast-slow systems

If we rescale to **large time scales** ($\sim \varepsilon^{-2}$) we have

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon, Y_\varepsilon) + f(X_\varepsilon, Y_\varepsilon)$$
$$\frac{dY_\varepsilon}{dt} = \varepsilon^{-2} g(Y_\varepsilon) \ ,$$

We turn X_ε into a random variable by taking $Y(0) \sim \mu$.

The aim is to characterise the **distribution** of the random path X_ε as $\varepsilon \to 0$.

Fast-slow systems as SDEs

Consider the simplified slow equation

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon) v(Y_\varepsilon) + f(X_\varepsilon)$$

where $h : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ and $v : \Lambda \to \mathbb{R}^d$ with $\int v(y)\mu(dy) = 0$.

If we write $W_\varepsilon(t) = \varepsilon^{-1} \int_0^t v(Y_\varepsilon(s)) ds$ then

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s)) dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s)) ds$$

where the integral is of Riemann-Stieltjes type ($dW_\varepsilon = \frac{dW_\varepsilon}{ds} ds$).

Fast-slow systems as SDEs

Consider the simplified slow equation

$$\frac{dX_\varepsilon}{dt} = \varepsilon^{-1} h(X_\varepsilon) v(Y_\varepsilon) + f(X_\varepsilon)$$

where $h : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ and $v : \Lambda \to \mathbb{R}^d$ with $\int v(y)\mu(dy) = 0$.

If we write $W_\varepsilon(t) = \varepsilon^{-1} \int_0^t v(Y_\varepsilon(s))ds$ then

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s)) dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

where the integral is of Riemann-Stieltjes type ($dW_\varepsilon = \frac{dW_\varepsilon}{ds} ds$).

Invariance principle for W_ε

We can write W_ε as

$$W_\varepsilon(t) = \varepsilon \int_0^{t/\varepsilon^2} v(Y(s))ds = \varepsilon \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor - 1} \int_j^{j+1} v(Y(s))ds$$

The assumptions on Y lead to **decay of correlations** for the sequence $\int_j^{j+1} v(Y(s))ds$.

For very general classes of chaotic Y, it is known that $W_\varepsilon \Rightarrow W$ in the sup-norm topology, where W is a multiple of Brownian motion.

We will call this class of Y **mildly chaotic**.

Invariance principle for W_ε

We can write W_ε as

$$W_\varepsilon(t) = \varepsilon \int_0^{t/\varepsilon^2} v(Y(s))ds = \varepsilon \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor - 1} \int_j^{j+1} v(Y(s))ds$$

The assumptions on Y lead to **decay of correlations** for the sequence $\int_j^{j+1} v(Y(s))ds$.

For very general classes of chaotic Y, it is known that $W_\varepsilon \Rightarrow W$ in the sup-norm topology, where W is a multiple of Brownian motion.

We will call this class of Y **mildly chaotic**.

Invariance principle for W_ε

We can write W_ε as

$$W_\varepsilon(t) = \varepsilon \int_0^{t/\varepsilon^2} v(Y(s))ds = \varepsilon \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor - 1} \int_j^{j+1} v(Y(s))ds$$

The assumptions on Y lead to **decay of correlations** for the sequence $\int_j^{j+1} v(Y(s))ds$.

For very general classes of chaotic Y, it is known that $W_\varepsilon \Rightarrow W$ in the sup-norm topology, where W is a multiple of Brownian motion.

We will call this class of Y **mildly chaotic**.

What about the SDE?

Since

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s))dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

This suggest a limiting SDE

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s)) \star dW(s) + \int_0^t f(\bar{X}(s))ds$$

But how should we interpret $\star dW$? Stratonovich? Itô? neither?

What about the SDE?

Since

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s))dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

This suggest a limiting SDE

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s)) \star dW(s) + \int_0^t f(\bar{X}(s))ds$$

But how should we interpret $\star dW$? Stratonovich? Itô? neither?

- For **additive noise** $h(x) = I$ the answer is simple.
- For **multiplicative noise** we require some new ideas.

- For **additive noise** $h(x) = I$ the answer is simple.
- For **multiplicative noise** we require some new ideas.

Continuity with respect to noise (Sussmann '78)

Consider

$$X(t) = X(0) + \int_0^t dU(s) + \int_0^t f(X(s))ds \;,$$

where U is a uniformly continuous path.

The above equation is well defined and moreover $\Phi : U \to X$ is continuous in the sup-norm topology.

If U is one dimensional, this also works in the multiplicative noise case: $h(X)dU$. The integral is of Stratonovich type.

Continuity with respect to noise (Sussmann '78)

Consider

$$X(t) = X(0) + \int_0^t dU(s) + \int_0^t f(X(s))ds \,,$$

where U is a uniformly continuous path.

The above equation is well defined and moreover $\Phi : U \to X$ is continuous in the sup-norm topology.

If U is one dimensional, this also works in the multiplicative noise case: $h(X)dU$. The integral is of Stratonovich type.

Continuity with respect to noise (Sussmann '78)

Consider

$$X(t) = X(0) + \int_0^t dU(s) + \int_0^t f(X(s))ds \ ,$$

where U is a uniformly continuous path.

The above equation is well defined and moreover $\Phi : U \to X$ is continuous in the sup-norm topology.

If U is one dimensional, this also works in the multiplicative noise case: $h(X)dU$. The integral is of Stratonovich type.

The simple case (Melbourne, Stuart '11 + Gottwald, Melbourne'13)

If the flow is mildly chaotic ($W_\varepsilon \Rightarrow W$) then $X_\varepsilon \Rightarrow \bar{X}$ in the sup-norm topology, where

$$d\bar{X} = dW + f(\bar{X})ds \; .$$

In the multiplicative $1d$ noise case, the limit is Stratonovich

$$d\bar{X} = h(\bar{X}) \circ dW + f(\bar{X})ds \; .$$

The simple case (Melbourne, Stuart '11 + Gottwald, Melbourne'13)

If the flow is mildly chaotic ($W_\varepsilon \Rightarrow W$) then $X_\varepsilon \Rightarrow \bar{X}$ in the sup-norm topology, where

$$d\bar{X} = dW + f(\bar{X})ds \ .$$

In the multiplicative $1d$ noise case, the limit is Stratonovich

$$d\bar{X} = h(\bar{X}) \circ dW + f(\bar{X})ds \ .$$

The strategy

The solution map takes irregular path space (C^α for $\alpha < 1/2$) to solution space

$$\Phi : W_\varepsilon \mapsto X_\varepsilon$$

If this map were **continuous** then we could lift the weak limit $W_\varepsilon \Rightarrow W$ to $X_\varepsilon \Rightarrow X$.

When the noise is both **multidimensional** and **multiplicative**, this strategy fails.

Stratonovich and sons

SDEs are very **sensitive** wrt approximations of BM.

Suppose

$$dX = h(X)dW + f(X)dt$$

and define an approximation

$$dX_n = h(X_n)dW_n + f(X_n)dt$$

where W_n is a piecewise smooth (C^1) approximation of W.

Taking $n \to \infty$, X_n can have many different limits, depending on the choice of approximation W_n.

Stratonovich and sons

SDEs are very **sensitive** wrt approximations of BM.

Suppose

$$dX = h(X)dW + f(X)dt$$

and define an approximation

$$dX_n = h(X_n)dW_n + f(X_n)dt$$

where W_n is a piecewise smooth (C^1) approximation of W.

Taking $n \to \infty$, X_n can have many different limits, depending on the choice of approximation W_n.

Eg. 1 (Wong-Zakai 65) If W_n is a linear interpolation of W, then

$$dX_n = h(X_n)\dot{W}_n dt + f(X_n)dt$$

converges to the **Stratonovich** SDE

$$dX = h(X)dW + \frac{1}{2}h'(X)h(X)dt + f(X)dt$$

Eg. 2 (McShane 72) Found a $W \in \mathbb{R}^2$ example with a non-linear interpolation of BM:

$$W_n(t) = W(t_j^n) + \psi\left(\frac{t - t_j^n}{t_{j+1}^n - t_j^n}\right)W(t_{j+1}^n)$$

the limit is **neither Ito nor Stratonovich**.

Eg. 1 (Wong-Zakai 65) If W_n is a linear interpolation of W, then

$$dX_n = h(X_n)\dot{W}_n dt + f(X_n)dt$$

converges to the **Stratonovich** SDE

$$dX = h(X)dW + \frac{1}{2}h'(X)h(X)dt + f(X)dt$$

Eg. 2 (McShane 72) Found a $W \in \mathbb{R}^2$ example with a non-linear interpolation of BM:

$$W_n(t) = W(t_j^n) + \psi\left(\frac{t - t_j^n}{t_{j+1}^n - t_j^n}\right)W(t_{j+1}^n)$$

the limit is **neither Ito nor Stratonovich**.

Eg 3 (Sussman 91) Anything is possible! For a large class of vector fields g, one can find an interpolation approximation W_n of W such that X_n converges to

$$dX = h(X)dW + g(X)dt + f(X)dt$$

It is not enough to know that $W_n \to BM$.

We need more information.

From smooth paths to rough paths

The usual framework for solving differential equations (with some smooth forcing signal) is: You provide the smooth signal U, the vector fields h, f and initial condition $X(0)$ and I'll give you a solution X satisfying

$$X(t) = X(0) + \int_0^t h(X(s)) dU(s) + \int_0^t h(X(s)) ds$$

When U is not smooth (like BM, for instance), this doesn't work anymore. We don't know how to define the integral.

From smooth paths to rough paths

The usual framework for solving differential equations (with some smooth forcing signal) is: You provide the smooth signal U, the vector fields h, f and initial condition $X(0)$ and I'll give you a solution X satisfying

$$X(t) = X(0) + \int_0^t h(X(s))dU(s) + \int_0^t h(X(s))ds$$

When U is not smooth (like BM, for instance), this doesn't work anymore. We don't know how to define the integral.

From smooth paths to rough paths

Rough path theory(Lyons '97) takes a slightly different approach:

In addition to $U, f, h, X(0)$, you provide me with an "*iterated integral*" of the rough path $\mathbb{U}$ and I'll give you a solution X.
The "iterated integral" is a matrix valued path

$$\mathbb{U}(t)^{ij} \stackrel{def}{=} \int_0^t U(0,r)^i dU(r)^j$$

where $U^i(0,r) = U^i(r) - U^i(0)$. The pair $\mathbf{U} = (U, \mathbb{U})$ is called a **rough path** and we write the solution as

$$X(t) = X(0) + \int_0^t h(X(s)) d\mathbf{U}(s) + \int_0^t h(X(s)) ds$$

this is called a **rough differential equation**.

From smooth paths to rough paths

Rough path theory(Lyons '97) takes a slightly different approach:

In addition to $U, f, h, X(0)$, you provide me with an "*iterated integral*" of the rough path $\mathbb{U}$ and I'll give you a solution X.

The "iterated integral" is a matrix valued path

$$\mathbb{U}(t)^{ij} \stackrel{def}{=} \int_0^t U(0,r)^i dU(r)^j$$

where $U^i(0,r) = U^i(r) - U^i(0)$. The pair $\mathbf{U} = (U, \mathbb{U})$ is called a **rough path** and we write the solution as

$$X(t) = X(0) + \int_0^t h(X(s)) d\mathbf{U}(s) + \int_0^t h(X(s)) ds$$

this is called a **rough differential equation**.

Rough path theory (Lyons '97)

The **rough path integral** is constructed using the second order approximation

$$\int_s^t h(X(r))dU(r)$$
$$\approx h(X(s))U(s,t) + h'(X(s))h(X(s))\int_s^t U(s,r)dU(r)$$

when $|s-t| \ll 1$.

Rough path theory (Lyons '97)

Eg. 1 If $U = W$ and $\mathbb{U} = \int W dW$ is the Ito iterated integral, then the constructed X is the solution to the Ito SDE.

Eg. 2 If $U = W$ and $\mathbb{U} = \int W \circ dW$ is the Stratonovich iterated integral, then the constructed X is the solution to the Stratonovich SDE.

Rough path theory **unifies** all the different notions of stochastic integration.

Rough path theory (Lyons '97)

Eg. 1 If $U = W$ and $\mathbb{U} = \int W dW$ is the Ito iterated integral, then the constructed X is the solution to the Ito SDE.

Eg. 2 If $U = W$ and $\mathbb{U} = \int W \circ dW$ is the Stratonovich iterated integral, then the constructed X is the solution to the Stratonovich SDE.

Rough path theory **unifies** all the different notions of stochastic integration.

Rough path theory (Lyons '97)

Most important property (for us) is that the map

$$\Phi : (U, \mathbb{U}) \mapsto X$$

is an **extension** of the classical solution map and is **continuous** with respect to the "rough path topology".

Convergence of fast-slow systems

Returning to the slow variables

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s))dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

If we let

$$\mathbb{W}_\varepsilon^{ij}(t) = \int_0^t W_\varepsilon^i(r)dW_\varepsilon^j(r)$$

then $X_\varepsilon = \Phi(W_\varepsilon, \mathbb{W}_\varepsilon)$.

Due to the continuity of Φ, if $(W_\varepsilon, \mathbb{W}_\varepsilon) \Rightarrow (W, \mathbb{W})$, then $X_\varepsilon \Rightarrow \bar{X}$, where

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s))d\mathbf{W}(s) + \int_0^t h(\bar{X}(s))ds$$

with $\mathbf{W} = (W, \mathbb{W})$.

Convergence of fast-slow systems

Returning to the slow variables

$$X_\varepsilon(t) = X_\varepsilon(0) + \int_0^t h(X_\varepsilon(s))dW_\varepsilon(s) + \int_0^t f(X_\varepsilon(s))ds$$

If we let

$$\mathbb{W}_\varepsilon^{ij}(t) = \int_0^t W_\varepsilon^i(r)dW_\varepsilon^j(r)$$

then $X_\varepsilon = \Phi(W_\varepsilon, \mathbb{W}_\varepsilon)$.

Due to the continuity of Φ, if $(W_\varepsilon, \mathbb{W}_\varepsilon) \Rightarrow (W, \mathbb{W})$, then $X_\varepsilon \Rightarrow \bar{X}$, where

$$\bar{X}(t) = \bar{X}(0) + \int_0^t h(\bar{X}(s))d\mathbf{W}(s) + \int_0^t h(\bar{X}(s))ds$$

with $\mathbf{W} = (W, \mathbb{W})$.

Theorem (K. & Melbourne '14)

If the fast dynamics are mildly chaotic, then $(W_\varepsilon, \mathbb{W}_\varepsilon) \Rightarrow (W, \mathbb{W})$ *where* W *is a Brownian motion and*

$$\mathbb{W}^{ij}(t) = \int_0^t W^i(s) dW^j(s) + \lambda^{ij} t$$

where the integral is Itô type and

$$\lambda^{ij} \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0))\, v^j(Y(s)) \}\, ds\ .$$

$$\mathrm{Cov}^{ij}(W) \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0)) v^j(Y(s)) + v^j(Y(0))\, v^i(Y(s)) \}\, ds$$

Homogenized equations

Corollary

Under the same assumptions as above, the slow dynamics $X_\varepsilon \Rightarrow \bar{X}$ where

$$d\bar{X} = h(\bar{X})dW + \left(f(\bar{X}) + \sum_{i,j,k} \lambda^{ij} \partial^k h^i(\bar{X}) h^{kj}(\bar{X}) \right) dt \ .$$

in Itô form, with $\lambda^{ij} \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0))\, v^j(Y(s)) \}\, ds$

Homogenized equations

Corollary

Under the same assumptions as above, the slow dynamics $X_\varepsilon \Rightarrow \bar{X}$ *where*

$$d\bar{X} = h(\bar{X}) \circ dW + \left(f(\bar{X}) + \sum_{i,j,k} \lambda^{ij} \partial^k h^i(\bar{X}) h^{kj}(\bar{X}) \right) dt$$

in Stratonovich form, with
$\lambda^{ij} \text{``}=\text{''} \int_0^\infty \mathbf{E}_\mu \{ v^i(Y(0))\, v^j(Y(s)) - v^j(Y(0))\, v^i(Y(s)) \}\, ds$.

Proof I : Find a martingale

The strategy is to decompose

$$W_\varepsilon(t) = M_\varepsilon(t) + A_\varepsilon(t)$$

where M_ε is a good martingale sequence (**Kurtz-Protter** 92):

$$\left(U_\varepsilon, M_\varepsilon, \int U_\varepsilon dM_\varepsilon\right) \Rightarrow \left(U, M, \int U dM\right)$$

where the integrals are of Itô type.

And $A_\varepsilon \to 0$ uniformly, but oscillates rapidly. Hence A_ε is like a corrector.

Proof I : Find a martingale

The strategy is to decompose

$$W_\varepsilon(t) = M_\varepsilon(t) + A_\varepsilon(t)$$

where M_ε is a good martingale sequence (**Kurtz-Protter** 92):

$$\left(U_\varepsilon, M_\varepsilon, \int U_\varepsilon dM_\varepsilon\right) \Rightarrow \left(U, M, \int U dM\right)$$

where the integrals are of Itô type.

And $A_\varepsilon \to 0$ uniformly, but oscillates rapidly. Hence A_ε is like a corrector.

Proof II : Martingale approximation (Gordin 69)

Introduce a Poincaré section Λ with Poincaré map T and return times τ_j.
Write

$$W_\varepsilon(t) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} \int_{\tau_j}^{\tau_{j+1}} v(Y(s))ds$$

$$= \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} \tilde{v}(T^j Y(0)) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} V_j \; .$$

We have a CLT sum for a stationary random sequence $\{V_j\}$ with natural filtration $\mathcal{F}_j = T^{-j}\mathcal{M}$ (where $\mathcal{M}$ is the σ-algebra for the $Y(0)$ probability space)

Proof II : Martingale approximation (Gordin 69)

Introduce a Poincaré section Λ with Poincaré map T and return times τ_j. Write

$$W_\varepsilon(t) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} \int_{\tau_j}^{\tau_{j+1}} v(Y(s))ds$$

$$= \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} \tilde{v}(T^j Y(0)) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} V_j \; .$$

We have a CLT sum for a stationary random sequence $\{V_j\}$ with natural filtration $\mathcal{F}_j = T^{-j}\mathcal{M}$ (where $\mathcal{M}$ is the σ-algebra for the $Y(0)$ probability space)

Proof II : Martingale approximation (Gordin 69)

Introduce a Poincaré section Λ with Poincaré map T and return times τ_j. Write

$$
\begin{aligned}
W_\varepsilon(t) &= \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} \int_{\tau_j}^{\tau_{j+1}} v(Y(s))ds \\
&= \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} \tilde{v}(T^j Y(0)) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} V_j \; .
\end{aligned}
$$

We have a CLT sum for a stationary random sequence $\{V_j\}$ with natural filtration $\mathcal{F}_j = T^{-j}\mathcal{M}$ (where $\mathcal{M}$ is the σ-algebra for the $Y(0)$ probability space)

Proof II : Martingale approximation (Gordin 69)

We use a martingale approximation to show that $\varepsilon \sum_{j=0}^{N_\varepsilon - 1} V_j \Rightarrow W$.

Suppose we can decompose $V_j = M_j + (Z_j - Z_{j+1})$ where $\mathbf{E}(M_j|\mathcal{F}_j) = 0$ and Z_j bdd. Then we have

$$\varepsilon \sum_{j=0}^{N_\varepsilon - 1} V_j = \varepsilon \sum_{j=0}^{N_\varepsilon - 1} M_j + \varepsilon(Z_0 - Z_{N_\varepsilon - 1}) .$$

A good choice (if it converges) is the series

$$Z_j = \sum_{k=0}^{\infty} \mathbf{E}(V_{j+k}|\mathcal{F}_j) .$$

Convergence of this series is guaranteed by decay of correlations for the Poincaré map.

Proof II : Martingale approximation (Gordin 69)

We use a martingale approximation to show that $\varepsilon \sum_{j=0}^{N_\varepsilon - 1} V_j \Rightarrow W$.

Suppose we can decompose $V_j = M_j + (Z_j - Z_{j+1})$ where $\mathbf{E}(M_j | \mathcal{F}_j) = 0$ and Z_j bdd. Then we have

$$\varepsilon \sum_{j=0}^{N_\varepsilon - 1} V_j = \varepsilon \sum_{j=0}^{N_\varepsilon - 1} M_j + \varepsilon(Z_0 - Z_{N_\varepsilon - 1}) \, .$$

A good choice (if it converges) is the series

$$Z_j = \sum_{k=0}^{\infty} \mathbf{E}(V_{j+k} | \mathcal{F}_j) \, .$$

Convergence of this series is guaranteed by decay of correlations for the Poincaré map.

Proof II : Martingale approximation (Gordin 69)

We use a martingale approximation to show that $\varepsilon \sum_{j=0}^{N_\varepsilon - 1} V_j \Rightarrow W$.

Suppose we can decompose $V_j = M_j + (Z_j - Z_{j+1})$ where $\mathbf{E}(M_j | \mathcal{F}_j) = 0$ and Z_j bdd. Then we have

$$\varepsilon \sum_{j=0}^{N_\varepsilon - 1} V_j = \varepsilon \sum_{j=0}^{N_\varepsilon - 1} M_j + \varepsilon (Z_0 - Z_{N_\varepsilon - 1}) \, .$$

A good choice (if it converges) is the series

$$Z_j = \sum_{k=0}^{\infty} \mathbf{E}(V_{j+k} | \mathcal{F}_j) \, .$$

Convergence of this series is guaranteed by decay of correlations for the Poincaré map.

Proof II : Martingale approximation (Gordin 69)

We use a martingale approximation to show that $\varepsilon \sum_{j=0}^{N_\varepsilon - 1} V_j \Rightarrow W$.

Suppose we can decompose $V_j = M_j + (Z_j - Z_{j+1})$ where $\mathbf{E}(M_j | \mathcal{F}_j) = 0$ and Z_j bdd. Then we have

$$\varepsilon \sum_{j=0}^{N_\varepsilon - 1} V_j = \varepsilon \sum_{j=0}^{N_\varepsilon - 1} M_j + \varepsilon (Z_0 - Z_{N_\varepsilon - 1}) \ .$$

A good choice (if it converges) is the series

$$Z_j = \sum_{k=0}^{\infty} \mathbf{E}(V_{j+k} | \mathcal{F}_j) \ .$$

Convergence of this series is guaranteed by decay of correlations for the Poincaré map.

Proof II : Martingale approximation (Gordin 69)

The good martingale is $M_\varepsilon(t) = \varepsilon \sum_{j=0}^{N_\varepsilon - 1} M_j$ and the corrector is $A_\varepsilon(t) = \varepsilon(Z_0 - Z_{N_\varepsilon - 1})$. We then get

$$W_\varepsilon(t) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} M_j + \varepsilon(Z_0 - Z_{N_\varepsilon - 1}) \Rightarrow W(t) + 0$$

by Martingale CLT and boundedness of Z.

We are sweeping a lot under the rug here since $\mathcal{F}_j \supseteq \mathcal{F}_{j+1}$. Need to reverse the martingales.

Proof II : Martingale approximation (Gordin 69)

The good martingale is $M_\varepsilon(t) = \varepsilon \sum_{j=0}^{N_\varepsilon - 1} M_j$ and the corrector is $A_\varepsilon(t) = \varepsilon(Z_0 - Z_{N_\varepsilon - 1})$. We then get

$$W_\varepsilon(t) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} M_j + \varepsilon(Z_0 - Z_{N_\varepsilon - 1}) \Rightarrow W(t) + 0$$

by Martingale CLT and boundedness of Z.

We are sweeping a lot under the rug here since $\mathcal{F}_j \supseteq \mathcal{F}_{j+1}$. Need to reverse the martingales.

Proof II : Martingale approximation (Gordin 69)

The good martingale is $M_\varepsilon(t) = \varepsilon \sum_{j=0}^{N_\varepsilon - 1} M_j$ and the corrector is $A_\varepsilon(t) = \varepsilon(Z_0 - Z_{N_\varepsilon - 1})$. We then get

$$W_\varepsilon(t) = \varepsilon \sum_{j=0}^{N(\varepsilon^{-2}t)-1} M_j + \varepsilon(Z_0 - Z_{N_\varepsilon - 1}) \Rightarrow W(t) + 0$$

by Martingale CLT and boundedness of Z.

We are sweeping a lot under the rug here since $\mathcal{F}_j \supseteq \mathcal{F}_{j+1}$. Need to reverse the martingales.

Proof III: Computing the iterated integral

To compute $\mathbb{W}_\varepsilon$ we decompose it

$$\int W_\varepsilon dW_\varepsilon = \int M_\varepsilon dM_\varepsilon + \int M_\varepsilon dA_\varepsilon + \int A_\varepsilon dM_\varepsilon + \int A_\varepsilon dA_\varepsilon$$

Since M_ε is a good martingale sequence

$$\int M_\varepsilon dM_\varepsilon \Rightarrow \int W dW \qquad \int A_\varepsilon dM_\varepsilon \Rightarrow 0 \, .$$

Even though $A_\varepsilon = O(\varepsilon)$, the iterated term $A_\varepsilon dA_\varepsilon$ does not vanish. The last two terms are computed as ergodic averages

$$\int M_\varepsilon dA_\varepsilon + \int A_\varepsilon dA_\varepsilon \to \lambda t \quad (a.s)$$

Proof III: Computing the iterated integral

To compute $\mathbb{W}_\varepsilon$ we decompose it

$$\int W_\varepsilon d W_\varepsilon = \int M_\varepsilon d M_\varepsilon + \int M_\varepsilon d A_\varepsilon + \int A_\varepsilon d M_\varepsilon + \int A_\varepsilon d A_\varepsilon$$

Since M_ε is a good martingale sequence

$$\int M_\varepsilon d M_\varepsilon \Rightarrow \int W d W \quad \int A_\varepsilon d M_\varepsilon \Rightarrow 0 \,.$$

Even though $A_\varepsilon = O(\varepsilon)$, the iterated term $A_\varepsilon d A_\varepsilon$ does not vanish. The last two terms are computed as ergodic averages

$$\int M_\varepsilon d A_\varepsilon + \int A_\varepsilon d A_\varepsilon \to \lambda t \quad (a.s)$$

Proof III: Computing the iterated integral

To compute $\mathbb{W}_\varepsilon$ we decompose it

$$\int W_\varepsilon dW_\varepsilon = \int M_\varepsilon dM_\varepsilon + \int M_\varepsilon dA_\varepsilon + \int A_\varepsilon dM_\varepsilon + \int A_\varepsilon dA_\varepsilon$$

Since M_ε is a good martingale sequence

$$\int M_\varepsilon dM_\varepsilon \Rightarrow \int W dW \quad \int A_\varepsilon dM_\varepsilon \Rightarrow 0 \; .$$

Even though $A_\varepsilon = O(\varepsilon)$, the iterated term $A_\varepsilon dA_\varepsilon$ does not vanish. The last two terms are computed as ergodic averages

$$\int M_\varepsilon dA_\varepsilon + \int A_\varepsilon dA_\varepsilon \to \lambda t \quad (a.s)$$

Proof III: Computing the iterated integral

To compute $\mathbb{W}_\varepsilon$ we decompose it

$$\int W_\varepsilon dW_\varepsilon = \int M_\varepsilon dM_\varepsilon + \int M_\varepsilon dA_\varepsilon + \int A_\varepsilon dM_\varepsilon + \int A_\varepsilon dA_\varepsilon$$

Since M_ε is a good martingale sequence

$$\int M_\varepsilon dM_\varepsilon \Rightarrow \int W dW \qquad \int A_\varepsilon dM_\varepsilon \Rightarrow 0 \; .$$

Even though $A_\varepsilon = O(\varepsilon)$, the iterated term $A_\varepsilon dA_\varepsilon$ does not vanish. The last two terms are computed as ergodic averages

$$\int M_\varepsilon dA_\varepsilon + \int A_\varepsilon dA_\varepsilon \to \lambda t \quad (a.s)$$

Extensions + Future directions

- The general fast-slow system (with $h(x, y)$) can be treated with infinite dimensional rough paths (or alternatively, rough flows - Bailleul+Catellier)
- Rough path tools can be adapted to address discrete-time fast-slow maps.
- Fast-slow systems with feedback. Ergodic properties of Y^X are poorly understood.
- Stochastic PDE limits; regularity structures.

Extensions + Future directions

- The general fast-slow system (with $h(x, y)$) can be treated with infinite dimensional rough paths (or alternatively, rough flows - Bailleul+Catellier)
- Rough path tools can be adapted to address discrete-time fast-slow maps.
- Fast-slow systems with feedback. Ergodic properties of Y^X are poorly understood.
- Stochastic PDE limits; regularity structures.

Extensions + Future directions

- The general fast-slow system (with $h(x, y)$) can be treated with infinite dimensional rough paths (or alternatively, rough flows - Bailleul+Catellier)
- Rough path tools can be adapted to address discrete-time fast-slow maps.
- Fast-slow systems with feedback. Ergodic properties of Y^X are poorly understood.
- Stochastic PDE limits; regularity structures.

Extensions + Future directions

- The general fast-slow system (with $h(x, y)$) can be treated with infinite dimensional rough paths (or alternatively, rough flows - Bailleul+Catellier)
- Rough path tools can be adapted to address discrete-time fast-slow maps.
- Fast-slow systems with feedback. Ergodic properties of Y^X are poorly understood.
- Stochastic PDE limits; regularity structures.

Extensions + Future directions

- The general fast-slow system (with $h(x, y)$) can be treated with infinite dimensional rough paths (or alternatively, rough flows - Bailleul+Catellier)
- Rough path tools can be adapted to address discrete-time fast-slow maps.
- Fast-slow systems with feedback. Ergodic properties of Y^X are poorly understood.
- Stochastic PDE limits; regularity structures.

References

1 - D. Kelly & I. Melbourne. *Smooth approximations of SDEs*. **Ann. Probab.** (2016).

2 - D. Kelly & I. Melbourne. *Deterministic homogenization of fast slow systems with chaotic noise*. arXiv (2015).

3 - D. Kelly. *Rough path recursions and diffusion approximations*. **Ann. App. Probab.** (2015).

All my slides are on my website (www.dtbkelly.com) **Thank you**!