Geometric vs non-geometric rough paths

David Kelly Martin Hairer

Mathematics Institute
University of Warwick
Coventry UK CV4 7AL

dtbkelly@gmail.com

November 5, 2012

Data Assimilation 2012, Oxford.

The problem

We are interested in equations of the form

$$dY_t = \sum_i f_i(Y_t)dX^i_t \ ,$$

where $X : [0, T] \to V$ is path with *some* Hölder exponent $\gamma \in (0, 1)$, $Y : [0, T] \to U$ and $f_i : U \to U$ are smooth vector fields.

The theory of **rough paths** (Lyons) tells us that we should think of the equation as

$$dY_t = \sum_i f_i(Y_t)d\mathbb{X}_t \ , \qquad (\dagger)$$

where $\mathbb{X}$ is an object containing X as well as information about the iterated integrals of X. We call $\mathbb{X}$ a **rough path** above X.

The problem

We are interested in equations of the form

$$dY_t = \sum_i f_i(Y_t) dX^i_t \ ,$$

where $X : [0, T] \to V$ is path with *some* Hölder exponent $\gamma \in (0, 1)$, $Y : [0, T] \to U$ and $f_i : U \to U$ are smooth vector fields.

The theory of **rough paths** (Lyons) tells us that we should think of the equation as

$$dY_t = \sum_i f_i(Y_t) d\mathbb{X}_t \ , \qquad (\dagger)$$

where $\mathbb{X}$ is an object containing X as well as information about the iterated integrals of X. We call $\mathbb{X}$ a **rough path** above X.

What is a rough path

- $\mathbb{X}$ lives in the tensor product space $\mathbb{X} : [0, T] \to V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes N}$ where N is the largest integer such that $N\gamma \leq 1$.
- $\mathbb{X}$ lives above X in that $\langle \mathbb{X}_t, e_i \rangle = X^i_t$.
- The tensor components encode the **iterated integrals** of X

$$\langle \mathbb{X}_t, e_{ij} \rangle \text{ ``}=\text{''} \int_0^t \int_0^r dX^i_r dX^j_r$$

$$\text{and} \quad \langle \mathbb{X}_t, e_{ijk} \rangle \text{ ``}=\text{''} \int_0^t \int_0^r \int_0^u dX^i_v dX^j_u dX^k_r$$

- $\mathbb{X}$ is usually assumed to be **geometric**, which means that the integrals obey the "usual laws of calculus".

What is a rough path

- $\mathbb{X}$ lives in the tensor product space $\mathbb{X} : [0, T] \to V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes N}$ where N is the largest integer such that $N\gamma \leq 1$.
- $\mathbb{X}$ lives above X in that $\langle \mathbb{X}_t, e_i \rangle = X^i_t$.
- The tensor components encode the **iterated integrals** of X

$$\langle \mathbb{X}_t, e_{ij} \rangle \text{ ``}=\text{''} \int_0^t \int_0^r dX^i_r dX^j_r$$

$$\text{and} \quad \langle \mathbb{X}_t, e_{ijk} \rangle \text{ ``}=\text{''} \int_0^t \int_0^r \int_0^u dX^i_v dX^j_u dX^k_r$$

- $\mathbb{X}$ is usually assumed to be **geometric**, which means that the integrals obey the "usual laws of calculus".

What is a rough path

- $\mathbb{X}$ lives in the tensor product space $\mathbb{X} : [0, T] \to V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes N}$ where N is the largest integer such that $N\gamma \leq 1$.
- $\mathbb{X}$ lives above X in that $\langle \mathbb{X}_t, e_i \rangle = X^i_t$.
- The tensor components encode the **iterated integrals** of X

$$\langle \mathbb{X}_t, e_{ij} \rangle \text{ “ $=$ ” } \int_0^t \int_0^r dX^i_r dX^j_r$$

$$\text{and} \quad \langle \mathbb{X}_t, e_{ijk} \rangle \text{ “ $=$ ” } \int_0^t \int_0^r \int_0^u dX^i_v dX^j_u dX^k_r$$

- $\mathbb{X}$ is usually assumed to be **geometric**, which means that the integrals obey the “usual laws of calculus”.

What is a rough path

- $\mathbb{X}$ lives in the tensor product space $\mathbb{X} : [0, T] \to V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes N}$ where N is the largest integer such that $N\gamma \leq 1$.
- $\mathbb{X}$ lives above X in that $\langle \mathbb{X}_t, e_i \rangle = X^i_t$.
- The tensor components encode the **iterated integrals** of X

$$\langle \mathbb{X}_t, e_{ij} \rangle \text{``}=\text{''} \int_0^t \int_0^r dX^i_r dX^j_r$$

$$\text{and} \quad \langle \mathbb{X}_t, e_{ijk} \rangle \text{``}=\text{''} \int_0^t \int_0^r \int_0^u dX^i_v dX^j_u dX^k_r$$

- $\mathbb{X}$ is usually assumed to be **geometric**, which means that the integrals obey the "usual laws of calculus".

Non-geometric rough paths

What if the integrals in equations like ($\dagger$) **don't** obey the usual laws of calculus?

Eg. Riemann-sum integrals for non-semimartingales (Burdzy, Swanson), Regularised integrals (Russo, Vallois).

This still fits into the framework of rough paths, but we need to add a few more **components** to $\mathbb{X}$.

Non-geometric rough paths

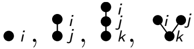
What if the integrals in equations like $(\dagger)$ **don't** obey the usual laws of calculus?

Eg. Riemann-sum integrals for non-semimartingales (Burdzy, Swanson), Regularised integrals (Russo, Vallois).

This still fits into the framework of rough paths, but we need to add a few more **components** to $\mathbb{X}$.

Non-geometric rough paths

Instead of tensors, the components of $\mathbb{X}$ are indexed by labelled **trees**



$, \ldots$

with the same labels used to index the basis of V.

Non-geometric rough paths

Instead of tensors, the components of $\mathbb{X}$ are indexed by labelled **trees**

$$\bullet_i\,,\ \substack{\bullet_i\\|\\\bullet_j}\,,\ \substack{\bullet_i\\|\\\bullet_j\\|\\\bullet_k}\,,\ \substack{\bullet_i\ \ \bullet_j\\ \diagdown\diagup\\ \bullet_k}\,,\ \ldots$$

with the same labels used to index the basis of V. And we have

$$\langle \mathbb{X}_t, \bullet_i \rangle = X^i_t \quad , \qquad \langle \mathbb{X}_t, \substack{\bullet_i\\|\\\bullet_j} \rangle = \int_0^t \int_0^r dX^i_u dX^j_r$$

$$\langle \mathbb{X}_t, \substack{\bullet_i\\|\\\bullet_j\\|\\\bullet_k} \rangle = \int_0^t \int_0^r \int_0^u dX^i_v dX^j_u dX^k_r \quad , \quad \langle \mathbb{X}_t, \substack{\bullet_i\ \ \bullet_j\\ \diagdown\diagup\\ \bullet_k} \rangle = \int_0^t X^i_r X^j_r dX^k_r$$

The object $\mathbb{X}$ is known as a **branched rough path** (Gubinelli).

Removing the branches

Theorem (MH,DK)

Every branched rough path can be encoded in a geometric rough path.

ie.

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\psi"] & \bar{\mathbb{X}} \\ X \arrow[u] \arrow[r, mapsto] & \bar{X} \arrow[u] \end{tikzcd}$$

For some X with a **branched** rough path $\mathbb{X}$ above it. There exists a path

$$\bar{X} = (X, \dots)$$

with a **geometric** rough path $\bar{\mathbb{X}}$ above it, satisfying

$$\langle \mathbb{X}_t, \tau \rangle = \langle \bar{\mathbb{X}}_t, \psi(\tau) \rangle \ .$$

for every tree τ.

The components of $\bar{\mathbb{X}}$ above X can be **any** geometric rough path.

Removing the branches

Theorem (MH,DK)

Every branched rough path can be encoded in a geometric rough path.

ie.

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, "\psi"] & \bar{\mathbb{X}} \\
X \arrow[u] \arrow[r, mapsto] & \bar{X} \arrow[u]
\end{tikzcd}
$$

For some X with a **branched** rough path $\mathbb{X}$ above it. There exists a path

$$\bar{X} = (X, \ldots)$$

with a **geometric** rough path $\bar{\mathbb{X}}$ above it, satisfying

$$\langle \mathbb{X}_t, \tau \rangle = \langle \bar{\mathbb{X}}_t, \psi(\tau) \rangle \ .$$

for every tree τ.

The components of $\bar{\mathbb{X}}$ above X can be **any** geometric rough path.

Removing the branches

Theorem (MH,DK)

Every branched rough path can be encoded in a geometric rough path.

ie.

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\psi"] & \bar{\mathbb{X}} \\ X \arrow[u] \arrow[r, mapsto] & \bar{X} \arrow[u] \end{tikzcd}$$

For some X with a **branched** rough path $\mathbb{X}$ above it. There exists a path

$$\bar{X} = (X, \dots)$$

with a **geometric** rough path $\bar{\mathbb{X}}$ above it, satisfying

$$\langle \mathbb{X}_t, \tau \rangle = \langle \bar{\mathbb{X}}_t, \psi(\tau) \rangle \ .$$

for every tree τ.

The components of $\bar{\mathbb{X}}$ above X can be **any** geometric rough path.

Removing the branches

Theorem (MH,DK)

Every branched rough path can be encoded in a geometric rough path.

ie.

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\psi"] & \bar{\mathbb{X}} \\ X \arrow[u] \arrow[r, mapsto] & \bar{X} \arrow[u] \end{tikzcd}$$

For some X with a **branched** rough path $\mathbb{X}$ above it. There exists a path

$$\bar{X} = (X, \dots)$$

with a **geometric** rough path $\bar{\mathbb{X}}$ above it, satisfying

$$\langle \mathbb{X}_t, \tau \rangle = \langle \bar{\mathbb{X}}_t, \psi(\tau) \rangle \ .$$

for every tree τ.

The components of $\bar{\mathbb{X}}$ above X can be **any** geometric rough path.

Removing the branches

Theorem (MH,DK)

Every branched rough path can be encoded in a geometric rough path.

ie.

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\psi"] & \bar{\mathbb{X}} \\ X \arrow[u] \arrow[r, mapsto] & \bar{X} \arrow[u] \end{tikzcd}$$

For some X with a **branched** rough path $\mathbb{X}$ above it. There exists a path

$$\bar{X} = (X, \dots)$$

with a **geometric** rough path $\bar{\mathbb{X}}$ above it, satisfying

$$\langle \mathbb{X}_t, \tau \rangle = \langle \bar{\mathbb{X}}_t, \psi(\tau) \rangle \ .$$

for every tree τ.

The components of $\bar{\mathbb{X}}$ above X can be **any** geometric rough path.

Consequences for rough DEs

Corollary (generalised Itô-Stratonovich correction formula)

Y is a solution to

$$dY_t = \sum_i f_i(Y_t) dX^i_t \quad \textit{(driven by } \mathbb{X}\textit{)}$$

if and only if Y is a solution to the rough DE

$$dY = \sum_i f_i(Y) \circ dX^i_t + \sum_\tau \bar{f}_\tau(Y_t) \circ d\bar{X}^\tau_t \quad \textit{(driven by } \bar{\mathbb{X}}\textit{)}\,,$$

where $\bar{\mathbb{X}}$ is the geometric rough path derived above.

nb. Should really be called *any non-geometric integral - any geometric integral* correction formula.

Consequences for rough DEs

Corollary (generalised Itô-Stratonovich correction formula)

Y is a solution to

$$dY_t = \sum_i f_i(Y_t) dX^i_t \quad \textit{(driven by } \mathbb{X}\textit{)}$$

if and only if Y is a solution to the rough DE

$$dY = \sum_i f_i(Y) \circ dX^i_t + \sum_\tau \bar{f}_\tau(Y_t) \circ d\bar{X}^\tau_t \quad \textit{(driven by } \bar{\mathbb{X}}\textit{)}\,,$$

where $\bar{\mathbb{X}}$ is the geometric rough path derived above.

nb. Should really be called *any non-geometric integral - any geometric integral* correction formula.