~~Lecture I:~~ Section I:

Two problems that illustrate important aspects of data assimilation:

Problem 1: We have a state $u_j \in X$ that is evolving in time $(j = 0, 1, 2, \dots)$

The state evolves according some model

$$u_{j+1} = \Psi(u_j; \omega) \quad \text{(possibly stochastic)}.$$

w/ an unknown initial condition $u_0 \sim N(m_0, C_0)$.

We also have data from observations

$$y_{j+1} = H u_{j+1} + \xi_{j+1},$$

$\xi_j \sim N(0, \Gamma)$ iid, $H: X \to Y$ linear, Y subspace of X.

The aim is to characterize the distribution

$$u_j \text{ given } y_{1:j} = \{y_1, \dots, y_j\}.$$

for each $j = \cancel{0}, 1, \dots, \cancel{N}T$.

This is called the filtering problem.

eg$_1$ Suppose $u_j = u(jh)$, $h>0$, where

$$du = -\nabla V(u)\,dt + \sigma dW.$$

$$u=(x,y), \quad V(x,y) = (1-x^2-y^2)^2.$$

Obs: $y_{j+1} = x((j+1)h) + \xi_{j+1}$. (only observe the x variable).

$\boxed{\text{Animation}}$

eg$_2$. Suppose
$$u_{j+1} = Mu_j + \eta_j, \quad \eta_j \sim N(0,\Sigma) \text{ iid}$$
$$y_{j+1} = Hu_{j+1} + \xi_{j+1}$$

Then the distribution $u_j | y_{1:j}$ can be computed explicitly. It is a Gaussian measure w/ explicitly computable mean and cov.

The solution is known as the Kalman filter.

(We will derive it later on.)

Observer / Synchronization problems

Suppose (for example) that u satisfies an evolution eqn of the form

$$\frac{du}{dt} + Au + B(u,u) = f \quad , \quad u : [0,T] \to \mathcal{X}.$$

(eg. 2d - Navier Stokes, Lorenz - 63 / 96).
big in geophysical models

Suppose we observe the state continuously

$$y = Hu \quad , \quad H : \cancel{\mathcal{X}} \to \mathcal{Y}.$$

Can we write down an evolution equation of the form

$$\frac{d\hat{u}}{dt} + A\hat{u} + B(\hat{u},\hat{u}) = f + C(\hat{u}, y).$$

In such a way that $|u - \hat{u}| \to 0$ as $t \to \infty$.

The observer problem / Synchronization.

Section I: Bayesian approach to filtering in discrete time.

We assume a model of the form

$$u_{j+1} = \Psi(u_j) + \eta_j \quad , \quad \eta_j \sim N(0,\Sigma) \text{ iid} \quad \cancel{, \Psi \text{ deterministic}}$$
$$y_{j+1} = H u_{j+1} + \xi_{j+1} \quad , \quad \xi_{j+1} \sim N(0,\Gamma) \text{ iid}.$$

$\Psi : \mathbb{R}^n \to \mathbb{R}^n$ deterministic, $H \in \mathbb{R}^{k \times n}$.

The smoothing problem:

Determine the distribution of the trajectory $u_{0:j} = \{u_0, u_1, \dots, u_j\}$ given $\cancel{\{y_1}$ $y_{1:j} = \{y_1, \dots, y_j\}$.

We can write down the density $P(u_{0:j} | y_{1:j})$ using Bayes formula.

$$P(u_{0:j} | y_{1:j}) = \frac{P(y_{1:j} | u_{0:j}) P(u_{0:j})}{P(y_{1:j})}$$

Prior: $P(u_{0:j}) = P(u_j | u_{j-1}) P(u_{0:j-1})$

$$= \left(\prod_{i=1}^{j} P(u_i | u_{i-1}) \right) P(u_0).$$

~~But~~ $$= \left(\prod_{i=1}^{j} Z_{\Sigma}^{-1} \exp\left(-\frac{1}{2}\left|u_i - \Psi(u_{i-1})\right|_{\Sigma}^{2}\right)\right) \cdot Z_{C_0} \exp\left(-\frac{1}{2}\left|u_0 - m_0\right|_{C_0}^{2}\right)$$

(Notation: $|\cdot|_{\Sigma} = |\Sigma^{-1/2}\cdot|$, $Z_{\Sigma} = (2\pi)^{d/2}(\det\Sigma)^{1/2}$.

$$\cancel{=} \propto \exp\left(\cancel{-\frac{1}{2}}\sum_{i=1}^{j} -\frac{1}{2}\left|u_i - \Psi(u_{i-1})\right|_{\Sigma}^{2} - \frac{1}{2}\left|u_0 - m_0\right|_{C_0}^{2}\right).$$

$\boxed{\text{Likelihood}}$: $$P(y_{1:j} \mid u_{1:j}) = \prod_{i=1}^{j} P(y_i \mid u_i)$$

$$= \prod_{i=1}^{j} Z_{\Gamma}^{-1} \exp\left(-\frac{1}{2}\left|y_i - Hu_i\right|_{\Gamma}^{2}\right).$$

$$\propto \exp\left(-\frac{1}{2}\sum_{i=1}^{j}\left|y_i - Hu_i\right|_{\Gamma}^{2}\right).$$

Back to Bayes:

$$P(u_{0:j} \mid y_{1:j}) \propto \exp\left(-\frac{1}{2}\sum_{i=1}^{j}\left|u_i - \Psi(u_{i-1})\right|_{\Sigma}^{2} + \left|y_i - Hu_i\right|_{\Gamma}^{2} - \cancel{+}\frac{1}{2}\left|u_0 - m_0\right|_{C_0}^{2}\right).$$

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Filtering problem: ~~Calculate~~ Estimate the current state u_j given all information up to present $y_{1:j}$

The "filtering update" is to compute $\mathbb{P}(u_{j+1} | y_{0:j+1})$ from $\mathbb{P}(u_j | y_{1:j})$. It consists of two steps:

(1) Forecast: Compute $P(u_{j+1} | y_{0:j})$.

eg. If $u_{j+1} \sim P(u_{j+1} | u_j)$. then we have.

$$P(u_{j+1} | y_{0:j}) = \int P(u_{j+1} | u_j) P(u_j | y_{0:j}) du_j$$

(2) Analysis: Compute $P(u_{j+1} | y_{0:j+1})$ by incorporating the data.

Bayes
$$\mathbb{P}(u_{j+1} | y_{1:j+1}) = \frac{\mathbb{P}(y_{j+1} | u_{j+1}, y_{0:j}) \mathbb{P}(u_{j+1} | y_{1:j})}{\mathbb{P}(y_{j+1} | y_{1:j})}.$$

$$\propto \mathbb{P}(y_{j+1} | u_{j+1}) \mathbb{P}(u_{j+1} | y_{1:j}).$$

Rmk: The filtering distribution is a marginal of the smoothing distribution

$$\mathbb{P}(u_j | y_{1:j}) = \int \mathbb{P}(u_{0:j} | y_{1:j}) du_{0:j-1}.$$

The Kalman filter

Suppose we have

$$u_{j+1} = M u_j + \eta_j \qquad \eta_j \sim N(0, \Sigma)$$
$$y_{j+1} = H u_{j+1} + \xi_{j+1} \qquad \xi_{j+1} \sim N(0, \Gamma).$$

and $M \in \mathbb{R}^{n \times n}$. The filtering problem has an explicit solution which is called the Kalman filter

Thm: $\mathbb{P}(u_j \mid y_{1:j})$ is a Gaussian with mean/cov (m_j, C_j) satisfing:

$$\cancel{C_j^{-1} = H^T\Gamma^{-1}H + \Sigma^{-1}}$$

$$\cancel{m_{j+1} = C_{j+1}(\hat{C}_{j+1}}$$

$$\cancel{C_{j+1}^{-1} = (M C_j M^T + \Sigma)^{-1} + H^T\Gamma^{-1}H}$$

$$\cancel{m_{j+1} = (M C_j M^T + \Sigma)^{-1}}(M m_j + H^T\Gamma^{-1} y_{j+1}).$$

$$\cancel{\text{where } \hat{C}_{j+1} = M C_j M^T + \Sigma}.$$

$$\boxed{\begin{aligned} C_{j+1}^{-1} &= \hat{C}_{j+1}^{-1} + H^T\Gamma^{-1}H. \\ C_{j+1}^{-1} m_{j+1} &= \hat{C}_{j+1}^{-1} M m_j + H^T\Gamma^{-1} y_{j+1} \\ \text{where } \hat{C}_{j+1} &= M C_j M^T + \Sigma \end{aligned}}$$

Proof: Assume inductively that $u_j | y_{1:j}$ is Gaussian ~~and~~ ; with mean and covariance (m_j, C_j) and that $C_j > 0$ (pos-def, sym.).

It follows that $u_{j+1} | y_{1:j}$ is also Gaussian.

$$\left[u_{j+1} | y_{1:j} = M u_j | y_{1:j} + \eta_j \right].$$

w/ mean/cov $(\hat{m}_{j+1}, \hat{C}_{j+1})$ given by

$$\hat{m}_{j+1} = E(u_{j+1} | y_{1:j}) = E(M u_j + \eta_j | y_{1:j})$$

$$= M E(u_j | y_{1:j}) = M m_j$$

$$\hat{C}_{j+1} = E\left((u_{j+1} - \hat{m}_{j+1})(u_{j+1} - \hat{m}_{j+1})^T | y_{1:j}\right)$$

$$= E\left((M u_j - M m_j + \eta_j)(M u_j - M m_j + \eta_j)^T | y_{1:j}\right).$$

$$= M E\left((u_j - m_j)(u_j - m_j)^T M^T | y_{1:j}\right) + \Sigma$$

$$= M C_j M^T + \Sigma$$

It follows that $\hat{C}_{j+1} > 0$.

Now compute $\mathbb{P}(u_{j+1} | y_{1:j+1})$ using Bayes:

$$\mathbb{P}(u_{j+1} | y_{1:j+1}) \propto \exp\left(-\tfrac{1}{2} |y_{j+1} - H u_{j+1}|_{\Gamma}^{2} - \tfrac{1}{2} |u_{j+1} - \hat{m}_{j+1}|_{\hat{C}_{j+1}}^{2}\right)$$

Complete the square

$$\propto \exp\left(-\tfrac{1}{2} |u_{j+1} - m_{j+1}|_{C_{j+1}}^{2}\right).$$

where $C_{j+1}^{-1} = H^{T}\Gamma^{-1}H + \hat{C}_{j+1}^{-1}$ and

$$C_{j+1}^{-1} m_{j+1} = \hat{C}_{j+1}^{-1} \hat{m}_{j+1} + H^{T}\Gamma^{-1} y_{j+1} \quad \square.$$

<u>Corollary</u>: We can also write

$$m_{j+1} = (I - K_{j+1} H)\hat{m}_{j+1} + K_{j+1} y_{j+1}$$

$$C_{j+1} = (I - K_{j+1} H)\hat{C}_{j+1}$$

where $K_{j+1} = \hat{C}_{j+1} H^{T} (H \hat{C}_{j+1} H^{T} + \Gamma)^{-1}$ (Kalman gain).

<u>Proof</u>: [See LSZ].