Subsidy Schemes in Double Auctions*

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Abstract

We study uniform-price double auctions augmented with a class of subsidy schemes. Using these subsidies, any profile of linear demand schedules can be implemented as an equilibrium outcome. By revenue equivalence, all other mechanisms which implement linear auction equilibria are essentially equivalent to some subsidy scheme in our class. We show that, under a linear dependency condition on primitives, fully efficient, budget balanced, and individually rational trade is possible. Thus, the welfare loss from equilibria in uniform-price auctions without subsidies is a non-fundamental distortion, which can be fixed with better mechanism design. However, we show that monopolist trading platforms have incentives to reduce allocative efficiency to increase revenue, even as the number of traders becomes large.

Keywords: double auction, price impact, allocative efficiency

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1 Introduction

The uniform-price auction is a natural generalization of the second-price auction to multi-unit settings. It is a popular model of limit-order books, and is commonly used in treasury auctions, exchange opening and closing auctions, electricity auctions, and other settings. It is known to be inefficient, since agents have incentives to shade their bids. In practice, the uniform-price auction is often used in combination with various side payments, such as make- and take- fees or subsidies. Motivated by these mechanisms in practice, this paper studies the following questions. What are the equilibrium outcomes, which can be sustained by uniform-price auctions with subsidy schemes? How much can subsidy schemes improve efficiency in double auctions? Do market platforms have incentives to choose side payment schemes which are efficiency-improving?

In this paper, we analyze uniform-price double-auctions, augmented with three kinds of subsidies. Using these subsidies, any profile of linear bidding strategies can be implemented as an equilibrium. We find that fully efficient and budget balanced trade is achievable if and only if a certain parameter restriction relating the means of traders' endowments and their holding costs is satisfied. Special cases which satisfy this restriction are when traders are ex-ante symmetric or when the means of their endowments are zero which is assumed by many existing models in the literature. We find that if market operators are revenue-maximizing and have market power, they may even have incentives to introduce subsidies that even reduce efficiency relative to a double auction without subsidies in order to increase revenue. That is, subsidy schemes, and related mechanisms have the potential to improve efficiency in multi-unit trading settings, but revenue-maximizing market platforms may not have incentives to implement efficiency-improving outcomes.

We analyze these questions in a Gaussian-quadratic multi-unit trading game. Agents are endowed with a Gaussian inventory position in a single asset, and incur holding costs which are quadratic in their final asset positions. Agents' inventory positions are private information, but agents' holding capacities, and the distributions of agents' inventory positions, are common knowledge. We allow the magnitude of agents' holding capacities, and the means and variances of agents' inventory positions, to vary arbitrarily across agents.

We study uniform-price double auctions, with three types of subsidies (or taxes). *Quadratic subsidies* are payments proportional to the squared quantity of the asset bought or sold. *Slope subsidies* depend on the slope of agents' bids with respect to prices. Finally, *linear subsidies* pay agents proportional to the net quantity of the asset that they buy or sell. These subsidies encourage agents to either buy more or sell more, for a given inventory position.

These subsidy schemes are tractable, and can implement a large range of outcomes. First, we show that these subsidy schemes sustain Bayes-Nash equilibria in linear bidding strategies. The existence of linear equilibria under quadratic and linear subsidies is known in the literature: it follows because agents' ex-post best responses to uncertainty in other agents' types can be implemented using linear bidding strategy. We show that linear equilibria are still optimal, in expectation, under slope subsidies. Technically, we solve for equilibria by changing variables so that agents choose quantities as a function of the residual supply intercept, and then formulating the optimal bidding problem as a calculus-of-variations problem in this space.

In linear equilibria of double auctions, agents' bids are characterized by three quantities: an inventory sensitivity, a price sensitivity, and an intercept. All three components are important. Inventory sensitivities determine how much agents pass through their inventory shocks to the market. Price sensitivities determine how much agents absorb inventory in the market. We show that *any* pattern of linear bid strategies – intercepts, and price and inventory sensitivities for each agent – can be implemented using some subsidy scheme, which we can solve for in closed form. Thus, using subsidy schemes, the market operator can effectively choose any agent to be a liquidity provider or a liquidity taker, or to induce any agent to buy or sell more of the asset on average.

The intuition for this result is that the three kinds of subsidies have different effects on each of the three components of bids. Slope subsidies, combined with quadratic taxes, encourage agents to provide liquidity, but not to take liquidity: price sensitivities are high, but inventory sensitivities are low. Conversely, quadratic subsidies and slope taxes tend to induce agents to become liquidity takers: to bid with high inventory sensitivities, and low price sensitivities. Linear subsidies allow us to vary how much agents are buyers or sellers on average.

Next, we show that *revenue equivalence* holds in our setting. Thus, any other trading mechanism which implements identical allocation rules to our subsidy schemes is equivalent, in terms of expected revenues and utilities, to a uniform-price auction with some subsidy scheme. As a result, our results about subsidy schemes generalize to a broad class of trading games. If fully efficient trade is possible under any trading game, it can be achieved using some subsidy scheme in our class. Similarly, the subsidy scheme which maximizes the auction platform's revenue cannot be improved on, in expectation, by any other trading game which implements linear bidding equilibria.

We proceed to analyze the classic question of Myerson and Satterthwaite (1983) in our setting: can any trading mechanism achieve fully efficient, individually rational, and budget balanced trade? We show that efficient trade is possible if and only if a linear dependency condition holds: traders' risk capacities must be exactly proportional to the means of their inventory shocks.

In the efficient mechanism, the market operator charges fixed entry fees to market participants, and uses the revenue to subsidize more aggressive trade. When full efficiency is feasible, it can be achieved using a continuum of different subsidy schemes. In particular, we show that there is always an efficient subsidy schemes which is budget balanced ex-post, so the market operator faces no revenue risk; and there is always a scheme which is ex-post incentive compatible for bidders, so it gives agents robust bidding incentives. Thus, when full efficiency is achievable, the market operator can use different choices of implementing subsidies to trade off the goals of revenue risk and robust incentive provision. However, we also show that, in any fully efficient subsidy scheme, the market operator pays out exactly as much in subsidies as she charges in entry fees, so the market operator is left with no revenue surplus.

Assuming agents are symmetric, we analytically characterize the revenue-maximizing mechanism for the market operator. In the revenue-optimal mechanism, the market operator imposes taxes which limit agents' trading aggressiveness. This improves revenue substantially, relative to the uniform-price auction, but in fact decreases agents' expected welfare. The welfare loss from the optimal mechanism is large. While the uniform-price auction converges to the efficient allocation as the number of traders increases, the revenue-maximizing mechanism destroys a constant fraction of first-best trade surplus, independent of the number of traders.

Next, we analyze an extension of our model to a dynamic setting, and ask under what conditions efficient dynamic budget balanced trade is possible. By dynamic budget balance, we mean the conditional expected future cost of running the mechanism, following any history is equal to zero. We can not offer a sharp characterization as we could in the static model – we only provide sufficient conditions on parameters for budget balanced, incentive compatible, and individually rational trade.

We show that fully efficient trade is actually easier to achieve in the dynamic case than the static case: the set of parameters for which full efficiency is possible is strictly larger in the dynamic case. The intuition is that dynamic trade relaxes traders' participation constraints, allowing higher participation fees to be charged. This is because if traders do not participate in the first round of trade, they will never be allowed to participate thereafter, and thus must bear the exposure to risk of their inventory shocks forever after. Participation is more valuable for traders of all types because of the presence of future inventory shocks.

We explore two extensions to the model. First, we characterize the set of allocation rules which can be implemented by our subsidy schemes. Second we numerically solve for the revenue maximizing subsidy scheme and second-best welfare maximizing mechanism when traders are heterogeneous. Under a variety of parameter settings we find that subsidy schemes can significantly improve welfare and increase auction revenue.

The main implications of our results are as follows. First, better market design can improve trading efficiency in multi-unit auction settings. When agents' inventory shocks are mean-zero or ex-ante symmetric, subsidy schemes can implement fully efficiency and budget balanced trade. It is well known in the literature that the uniform-price auction is an inefficient mechanism, since agent have incentives to shade their bids, regardless of the distributions of agents' initial inventory positions. Our results imply that a large part of this inefficiency is non-fundamental: it is a weakness of the uniform-price auction mechanism, and can be alleviated with better trading game design. In the setting of our model, various side payment schemes in double auctions are able to substantially improve trading efficiency. By revenue equivalence, other mechanisms which encourage more aggressive bidding may have similar results.

However, we also show that market platforms' incentives are not aligned with those of market participants. When fully efficient mechanisms are possible, market operators pay out in subsidies as much as they collect in entry fees, so they are left with no revenue surplus. Subsidy schemes which maximizing market operators' revenue are inefficient: they are always worse than uniform-price auctions without subsidies, and the efficiency loss is not alleviated by bidder competition. Regulators should closely monitor the market design decisions of monopolist market platforms, as the revenue-optimal trading mechanisms for monopolist platforms can be detrimental for trading efficiency.

1.1 Literature review

This paper is related to a number of strands of literature. First, we contribute to the literature on multi-unit double auctions with quadratic utility functions.[1] In particular, our paper contributes to the literature on the effects of heterogeneous holding capacities in financial markets. For instance, Sannikov and Skrzypacz (2016) show that heterogeneous risk capacities give rise to phenomena such as momentum and front-running in dynamic markets. Malamud and Rostek (2017) shows that fragmented markets may be welfare superior to a centralized market if risk capacities are sufficiently heterogeneous. In contrast with these papers we focus on implementing equilibria by altering the trading mechanism. We show heterogeneous risk capacities limit the potential to improve allocative efficiency when the

[1] See, for example, Kyle (1989), Vayanos (1999), Vives (2011), Rostek and Weretka (2012a), Vives (2014), Rostek and Weretka (2015a), Manzano and Vives (2016), Sannikov and Skrzypacz (2016), Antill and Duffie (2017), Du and Zhu (2017), Duffie and Zhu (2017), Malamud and Rostek (2017), Lee and Kyle (2018), Chen and Duffie (2020), and Zhang (2020).

auction operator is constrained by a budget.

Our contribution, relative to this literature, is that our subsidy schemes allow us to study the revenue and efficiency properties of all possible linear auction equilibria. To our knowledge, our full implementation result is new to the literature. Other papers have analyzed the effects of certain subsidy schemes in double auctions; for example, quadratic subsidies are also discussed by Manzano and Vives (2016), who study double auctions with two kinds of bidders, studying how bidders' information precision and holding costs affect market power. Üslü (2019) also analyzes quadratic subsidies in a search-and-bargaining model of OTC markets. To our knowledge, we are the first to study slope subsidies, and to show that they admit linear auction equilibria.

A closely related paper is Antill and Duffie (2017), who study a particular "workup" mechanism which improves efficiency in dynamic double-auction settings. Their mechanism, in a static context, is ex-post individually rational, incentive compatible, budget balanced, and efficient; however, it requires that the platform operator observes the sum of all agents' inventory shocks. In contrast, we show that budget-balanced efficient trade is possible without assuming the aggregate inventory is observable to the platform operator.

Pycia and Woodward (2019) compare uniform-price and pay-as-bid auctions. However, most of the results apply to the case where agents have no private information. Woodward (2019) studies a hybrid mechanism which is a convex combination of uniform-price and pay-as-bid auctions. Andreyanov and Sadzik (2017) study a more general class of settings: agents' utility functions must have a single dimension of heterogeneity and satisfy single-crossing, but do not need to be quadratic. Agents' values may also be interdependent. In this setting, Andreyanov and Sadzik characterize "σ-Walrasian equilibrium" mechanisms, which are prior-robust mechanisms that are individually rational, incentive compatible, budget balanced, and have bounded efficiency losses, which converge to 0 as the number of agents increase. We study a less general setting than Andreyanov and Sadzik, as our setting requires the platform operator to know the distributions of agents' types; however, the benefit is that we can more sharply characterize exactly optimal and revenue-maximizing mechanisms in our setting.

By demonstrating our full-space and revenue equivalence results, we are able to analyze our mechanisms as if they were direct revelation mechanisms. In doing so, we bring together the double auctions literature with the classic mechanism design literature, which studies the possibility of budget-balanced, individually rational and incentive compatible efficient trade.[2] While the Gaussian-quadratic double auctions framework falls within the framework

[2]See, for example, d'Aspremont and Gérard-Varet (1979), Myerson (1981), Myerson and Satterthwaite (1983), Cramton, Gibbons, and Klemperer (1987), Krishna and Perry (1998), McAfee (1991), Segal and

of single-crossing mechanism design, to our knowledge, the literature has not attempted to apply the revelation principle and mechanism design in this setting. We take a somewhat roundabout approach: we study a specific class of subsidy and tax mechanisms, and show that this class can implement any affine rule. By revenue equivalence, the characterization of the subsidy/tax rules thus equivalently characterizes the entire space of incentive-compatible mechanisms, up to agent-specific constants.

This paper is also related to a number of papers on multi-unit auctions without quadratic utility, which argue that subsidies can encourage aggressive bidding and combat collusion.[3] Our dynamic model is also related to the literature on mechanism design in dynamic settings. See, for example, Bergemann and Välimäki (2010), Athey and Segal (2013), Pavan, Segal, and Toikka (2014), and Skrzypacz and Toikka (2015). In contrast to these papers, we provide an indirect implementation of the efficient allocation by a double auction with a dynamic subsidy scheme.

1.2 Outline

The rest of this paper proceeds as follows. In section 2 we describe the setup of our baseline static model. In section 3 we characterize linear equilibria of the model. In section 4, we prove our full-space and revenue equivalence results. In section 5, we provide conditions for when budget balanced efficient trade is attainable, and in section 6 we characterize mechanisms which maximize the platform operator's revenue. In section 7 we partially extend the results to a dynamic setting. In section 8 we analyze various extensions of the model. In section 9 we discuss our results and conclude.

2 Baseline model

In our baseline model, trade of a single perfectly divisible asset takes place in a single period. There are $N > 2$ traders, indexed by $i \in \{1, 2, ..., N\}$, who participate in the market. Prior to trade, a given trader i is endowed with an initial quantity $X_i \sim N(\mu_i, \sigma^2_{X,i})$ which is her private information. We assume that traders' endowments, $(X_i)_i = (X_1, ..., X_N)$, are jointly independent and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The joint distribution over endowments is common knowledge—all traders know the means $(\mu_i)_i$ and variances $(\sigma^2_{X,i})_i$ of endowments.

Whinston (2011), and Segal and Whinston (2016).

[3] See McAdams (2002), Kremer and Nyborg (2004), LiCalzi and Pavan (2005), McAdams (2007) and Levmore (2018).

Agents have private values. We assume that trader i's utility function is quasilinear in cash transfers with a quadratic holding cost incurred on a nonzero post-trade position:

$$-\frac{1}{2\kappa_i}(X_i + q_i)^2 + t_i(q_1, ..., q_N).$$

Preferences of this form are prevalent in the market microstructure literature. See Vives (2011), Rostek and Weretka (2012a), Du and Zhu (2012), and Sannikov and Skrzypacz (2016). The holding cost may for example represent costs associated with regulatory capital requirements, margin requirements, or aversion to risk. When κ_i is higher, trader i can better tolerate large asset positions and so we refer to κ_i as trader i's *holding capacity*. We assume that traders' holding capacities are common knowledge and allow for heterogeneity.

The trading mechanisms we analyze are uniform price double auctions, augmented with three kinds of subsidies. Each trader i submits a demand schedule to the double auction. The demand schedule is a measurable function $q_i : \mathbb{R}^2 \to \mathbb{R}$ specifying the quantity, $q_i(X_i, p)$, that trader i will purchase for each realization of the market clearing price, p, given her endowment, X_i. We assume that admissible demand schedules must be continuously differentiable, strictly monotone decreasing in price, and satisfy a technical decay condition[4]. We denote the set of admissible demand schedules by $\mathcal{M}$. Given the submitted demand schedules, the double auction computes the market clearing price, p^*, which satisfies

$$\sum_{i \in N} q_i(X_i, p^*) = 0.$$

Finally, traders receive the quantities specified by their demand schedules at the per-unit price, p^*. They also receive payments from three different types of subsidies: linear, quadratic, and slope subsidies.

The linear subsidy pays trader i

$$\tau_i q_i(X_i, p^*)$$

units which if τ_i is positive, encourages trader i to buy more of the asset. The quadratic subsidy pays trader i

$$c_i q_i(X_i, p^*)^2$$

units. If c_i is positive it encourages trader i to increase the magnitude of her trade. The slope subsidy pays trader i

$$\frac{\frac{\partial}{\partial p} q_i(X_i, p^*)}{\sum_{j \in N} \frac{\partial}{\partial p} q_j(X_j, p^*)} R_i$$

[4]See Appendix A for a formal statement of the condition. A more easily stated sufficient condition is that $\lim_{|p| \to \infty} e^{-\lambda p} q_i(x, p) = 0$ for each $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$.

units. If R_i is positive, the slope subsidy encourages trader i to submit a steeper demand schedule which is more sensitive to the market clearing price. Taking into account the subsidies, the total net transfer to trader i following trade is

$$t_i(q_1, ..., q_N) = -p^* q_i(X_i, p^*) + \frac{\frac{\partial}{\partial p} q_i(X_i, p^*)}{\sum_{j \in N} \frac{\partial}{\partial p} q_j(X_j, p^*)} R_i + \frac{c_i}{2} q_i(X_i, p^*)^2 + \tau_i q_i.$$

Note that we allow the coefficients $(R_i, c_i, \tau_i)_i$ to differ across traders. A double auction with subsidies is characterized by the profile of subsidy coefficients $(R_i, c_i, \tau_i)_i$ which we will refer to as a *subsidy scheme*[5].

3 Linear equilibria

In this section we characterize linear Bayes-Nash equilibria of the model. A *linear equilibrium* is a profile of coefficients $(a_i, y_i, w_i)_i$ which in turn defines a profile of linear demand schedules $(q_1, ..., q_N)$ where

$$q_i(X_i, p) = a_i - w_i X_i - y_i p \tag{1}$$

such that each trader i maximizes her expected utility by selecting (1), assuming that all other traders submit their equilibrium demand schedules. Formally, for each trader i, (1) must solve her demand selection problem:

$$\sup_{f \in \mathcal{M}} \mathbb{E}\left[-\frac{1}{2\kappa_i}(X_i + f(X_i, p_f^*))^2 + t_i(q_1, ..., f, ..., q_N)\right] \tag{2}$$

where p_f^* is the random variable defined by the market clearing condition

$$f(p_f^*) + \sum_{\{j \in N | j \neq i\}} a_j - w_j X_j - y_j p_f^* = 0. \tag{3}$$

In a linear equilibrium, each trader's bidding strategy is characterized by three numbers: a_i, y_i, w_i. Here, a_i is the quantity demanded absent consideration for endowment or price. The coefficient, w_i determines the proportion of trader i's endowment that will be unloaded in the exchange irrespective of the price. That is, w_i can be thought of as capturing trader i's demand for liquidity from the market. y_i determines how responsive i's demands are to market prices and thus how much of other traders' endowments trader i will absorb through the equilibrium price. Thus y_i can be thought of as a measure of how much liquidity trader

[5] We allow each subsidy coefficient to be either positive or negative. In the latter case the corresponding "subsidy" is really a tax but we do not make this distinction in our terminology.

i supplies to the market.

Given a linear equilibrium, using equations (1) and (3), we can characterize the equilibrium allocation, price, and residual supply curve facing each trader.

Lemma 1. *In a linear equilibrium,* $(a_i, y_i, w_i)_i$*,*

1. *the market clearing price is*

$$p^* = \frac{\left(\sum_{i \in N} a_i - w_i X_i\right)}{\sum_{i \in N} y_i}. \tag{4}$$

2. *trader i's post-trade inventory is:*

$$X_i + q_i = (1 - w_i) X_i + a_i - y_i \frac{\left(\sum_{j \in N} a_j - w_j X_j\right)}{\sum_{j \in N} y_j}. \tag{5}$$

In order to characterize linear equilibria, it is useful to work with equilibrium residual supply curves. From the perspective of agent i, any equilibrium induces a random residual supply curve, $q_{RSi}(p)$, which specifies the number of units of the underlying asset that i is able to trade at price p. The residual supply curve facing i is the negative of the sum of all other agents' demand schedules:

$$q_{RSi}(p) = -\sum_{\{j \in N | j \neq i\}} q_j(X_i, p) \tag{6}$$

If all agents' bids are linear, residual supply curves will be affine, taking the form:

$$q_{RSi}(p) = d_i p + \eta_i \tag{7}$$

The slope of residual supply, d_i, is the inverse of price impact: if agent i purchases 1 additional unit of the asset, she changes market clearing prices by $\frac{1}{d_i}$. The residual supply intercept, η_i, is a function of other agents' inventory positions X_j. The following lemma characterizes the equilibrium value of d_i, and the mean and variance of η_i, in terms of primitives $(\kappa_i, \mu_{Xi}, \sigma^2_{Xi})_i$ and the linear equilibrium parameters $(a_i, y_i, w_i)_i$.

Lemma 2. *Given* $(\kappa_i, \mu_{Xi}, \sigma^2_{Xi})_i$ *and* $(a_i, y_i, w_i)_i$*, the residual supply curve facing i satisfies:*

$$d_i = \sum_{\{j \in N | j \neq i\}} y_j \tag{8}$$

Hence, trader i's price impact is:

$$\frac{1}{d_i} = \frac{1}{\sum_{\{j\in N|j\neq i\}} y_j} \tag{9}$$

The residual supply intercept η_i is:

$$\eta_i = \sum_{\{j\in N|j\neq i\}} -a_j + w_j X_j \tag{10}$$

η_i thus has mean and variance:

$$\mu_{\eta_i} = \sum_{\{j\in N|j\neq i\}} -a_j + w_j \mu_{Xj} \tag{11}$$

$$\sigma^2_{\eta_i} = \sum_{\{j\in N|j\neq i\}} w_j^2 \sigma^2_{X,j}. \tag{12}$$

The following proposition provides necessary and sufficient conditions for a candidate set of demand coefficients $(a_i, y_i, w_i)_i$ to be a linear equilibrium for a given subsidy scheme $(R_i, c_i, \tau_i)_i$.

Proposition 1. *Given a subsidy scheme, $(R_i, c_i, \tau_i)_i$, a necessary and sufficient condition for $(a_i, y_i, w_i)_i$ to be a linear equilibrium is that for each i:*

$$\frac{1}{2\kappa_i} - \frac{c_i}{2} + \frac{1}{d_i} > 0 \tag{13}$$

$$a_i = \frac{\tau_i - \frac{\mu_{\eta_i} R_i}{\sigma^2_{\eta_i}}}{\left(\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}}\right)} \tag{14}$$

$$w_i = \frac{d_i}{\kappa_i + d_i - \kappa_i d_i \left(c_i + \frac{R_i}{\sigma^2_{\eta_i}}\right)} \tag{15}$$

$$y_i = \frac{\kappa_i d_i \left(1 + \frac{R_i d_i}{\sigma^2_{\eta_i}}\right)}{\kappa_i + d_i - \kappa_i d_i \left(c_i + \frac{R_i}{\sigma^2_{\eta_i}}\right)}. \tag{16}$$

A formal proof of Proposition 1 is contained in Appendix A.3. While it is known in the literature that linear equilibria exist under quadratic and linear subsidies (Manzano and Vives (2016)), to our knowledge, however, the analysis of slope subsidies is new to this paper. Nonzero slope subsidies imply that agents' payoffs depend on both the slope and the level of their own bids, so we cannot use the standard approach in the double-auctions literature, of finding a bid curve which is point-wise optimal given price impact.

Instead, we recognize that the problem of choosing quantities as a function of the market price, $q_i(p)$, is equivalent to selecting a quantity, $\tilde{q}_i(\eta_i)$, as a function of the residual supply intercept η_i. The key reason why slope subsidies are tractable is that under this change-of-variables the slope subsidy revenue for agent i is simply linear in $\tilde{q}_i'(\eta_i)$. As a result, when solving trader i's optimization problem using the calculus-of-variations, we find that agents' optimal bid curves turn out to be linear whenever η_i is Gaussian as in our model.

When slope subsidies R_i are nonzero, proposition 1 implies that agents' optimal bid parameters $(a_i, y_i, w_i)_i$ depend on both residual supply slopes, d_i, and the mean and variance of residual supply, μ_{η_i} and $\sigma^2_{\eta_i}$ respectively. From Lemma 2, $\left(d_i, \mu_{\eta_i}, \sigma^2_{\eta_i}\right)_i$ also depend on $(a_i, y_i, w_i)_i$. Thus, equilibrium conditions involve a fixed point simultaneously in $\left(d_i, \mu_{\eta_i}, \sigma^2_{\eta_i}\right)_i$ and $(a_i, y_i, w_i)_i$.

Intuitively, with slope subsidies, agents attempt to guess what the equilibrium price will be, and bid elastic bid curves going through the conjectured equilibrium price, in order to increase slope subsidy payoffs. This implies that agents' optimal bids depend on the mean and variance of residual supply: when residual supply variance is low, it is less costly to tilt away from ex-post optimality to harvest slope subsidies. Thus, slope subsidies affect bidding aggressiveness more when the variance of residual supply is low: from (15) and (16), the effect of a small change in R_i is larger when $\sigma^2_{\eta_i}$ is small.

Qualitatively, positive slope and quadratic subsidies both increase agents' bidding aggressiveness, and thus can improve trading efficiency relative to equilibrium without subsidies. Consider first the special case when all subsidy coefficients are zero. A higher price impact or equivalently lower d_i leads trader i to decrease w_i and thus the quantity of endowment he unloads in the auction. It also leads trader i to decrease y_i and thus provide less liquidity to the rest of the market. Price impact is therefore a key source of inefficiency because it inhibits the mutually beneficial redistribution of endowments across traders. This can be most easily seen in the case of symmetric holding capacities. Inspecting equations (5) and (15), the post-trade inventory of trader i is

$$\frac{\kappa}{\kappa + d} X_i + \frac{d}{\kappa + d} \frac{1}{N} \sum_{j \in N} X_j.$$

Clearly, in the symmetric case, the efficient allocation is one in which each trader's post-trade inventory is $\frac{1}{N} \sum_{j \in N} X_j$. Thus the higher is price impact, the lower is d and the less efficient is the equilibrium allocation.

As seen from equations (15) and (16), an increase in either c_i or R_i leads to increases in w_i

and y_i and thus reduces price impact faced by other traders and increases the aggressiveness with which all traders unload their endowments. Though both improve efficiency, c_i and R_i do so through different primary channels. R_i compensates traders proportional to y_i and thus encourages trader i to provide more liquidity to other traders—ie. reducing the price impact faced by other traders. Though an increase in c_i also increases y_i (an indirect effect), it has the direct effect of lowering the effective price impact faced by trader i himself. This is because price impact costs are quadratic and equal to $\frac{1}{d_i}q_i^2$, so a quadratic subsidy reduces trader i's effective price impact by c_i units.

4 Implementable allocations and revenue equivalence

Our next result shows that our subsidies allow the market operator to implement a large class of outcomes: any profile of bid strategies, $(a_i, y_i, w_i)_i$, can be implemented using some subsidy scheme, and we can solve for the implementing subsidies in closed form.

Proposition 2. *Given a profile of demand coefficients, $(a_i, y_i, w_i)_i$ such that $w_i > 0$, $y_i > 0$ for each $i \in N$, the unique subsidy scheme $(R_i, c_i, \tau_i)_i$ such that $(a_i, y_i, w_i)_i$ is a linear equilibrium satisfies*

$$c_i = \frac{1}{\kappa_i}(1 - \frac{1}{w_i}) - \frac{y_i}{\sum_{\{j \in N | j \neq i\}} y_j}\left(\frac{1}{\kappa_i w_i}\right) + \frac{2}{\sum_{\{j \in N | j \neq i\}} y_j} \tag{17}$$

$$R_i = \left[\frac{y_i}{\sum_{\{j \in N | j \neq i\}} y_j}\left(\frac{1}{\kappa_i w_i}\right) - \frac{1}{\sum_{\{j \in N | j \neq i\}} y_j}\right]\sigma_{\eta_i}^2 \tag{18}$$

$$\tau_i = \frac{a_i}{w_i \kappa_i} + \frac{\mu_{\eta_i} R_i}{\sigma_{\eta_i}^2} \tag{19}$$

for each $i \in N$, where d_i, μ_{η_i}, and $\sigma_{\eta_i}^2$ are defined in equations (8), (11), and (12).

A simple intuition for the proof of proposition 2 is the following. For any given set of subsidies (R_i, c_i, τ_i), solving the linear equilibrium conditions in proposition 1 is nontrivial, since it involves finding a fixed point in $(a_i, y_i, w_i)_i$ and the implied residual supply parameters $\left(d_i, \mu_{\eta_i}, \sigma_{\eta_i}^2\right)_i$. However, given a target value of $(a_i, y_i, w_i)_i$, we can calculate the residual supply parameters $\left(d_i, \mu_{\eta_i}, \sigma_{\eta_i}^2\right)_i$, and simply substititute them in to 1, and solve for the subsidies (R_i, c_i, τ_i) which implement this as an equilibrium. Hence, Proposition 2 implies that it is much easier to solve the inverse problem, of calculating the subsidy schemes which implement a given allocation, than to solve for equilibrium under a given set of subsidies.

Proposition 2 implies that our subsidies and taxes are flexible enough to implement a large family of bidding schemes, and thus a large family of allocation rules.[6] A natural question raised by Proposition 2 is whether there is any loss of generality in focusing on implementation using these particular subsidies and taxes. That is, if a given allocation rule can be implemented using these subsidy schemes, is it possible that a wholly different trading game could implement the same allocation rule, but with higher expected revenue for the market operator? The following proposition shows that this is not possible – *revenue equivalence* holds in our setting, so any two trading games which implement the same allocation have essentially the same implications for the market operator's revenue, and expected utilities for all market participants.

Before stating the proposition, we first define a mechanism in our model setting. A *mechanism* consists of

1. an *message space* $\mathcal{A}_i$,
2. an *allocation function*, $f_i : \mathcal{A}_1 \times ... \times \mathcal{A}_N \to \mathbb{R}$, which maps a profile of traders' actions to a post trade inventory for trader i,
3. a *payment function*, $g_i : \mathcal{A}_1 \times ... \times \mathcal{A}_N \to \mathbb{R}$, which maps a profile of all traders' actions to a net transfer to trader i

for each trader i. A *direct mechanism* is a mechanism where $\mathcal{A}_i = \mathbb{R}$ for each i. A mechanism *implements* an equilibrium with allocation $(q_1, ..., q_N)$ if there exists a BNE, $a = (a_1, ..., a_N) \in \mathcal{A}_1 \times ... \times \mathcal{A}_N$, where

$$(f_1(a), ..., f_N(a)) = (q_1, ..., q_N)$$

almost surely. By the revelation principle, if an allocation is implementable by a mechanism with a given set of equilibrium transfers, it is implementable in a truthful reporting equilibrium of a *direct mechanism* in which traders report their private endowments with the same transfers.

Proposition 3. *Suppose two direct mechanisms M and M' implement the same allocation in a truthful reporting equilibrium. Let U_i^M and $U_i^{M'}$ denote the equilibrium payoff of trader i in the truthful reporting equilibrium of M and M' respectively. Let $X = (X_1, ..., X_N)$. If for each i, f_i^M, $f_i^{M'}$, g_i^M, $g_i^{M'}$ are differentiable, then there exist constants $\delta_i \in \mathbb{R}$ such that*

$$\mathbb{E}[g_i^M(X)|X_i] = \mathbb{E}[g_i^{M'}(X)|X_i] + \delta_i$$

[6] In subsection 8.1 below, we characterize precisely the set of allocation rules which can be implemented in linear auction equilibria, and give examples of allocation rules which cannot be implemented.

and

$$\mathbb{E}[U_i^M|X_i] = \mathbb{E}[U_i^{M'}|X_i] - \delta_i$$

for $\omega \in \Omega$.

Appendix A.6 proves some additional results about revenue equivalence; we characterize linear equilibria which induce equivalent allocation rules, and show that a stronger version of revenue equivalence holds: any two subsidy schemes which implement the same allocation rule have the same expected revenues, utilities, and payments, even without agent-specific fixed fees.

Revenue equivalence implies that any other mechanism which implements linear bidding equilibria is expected-revenue- and expected-utility-equivalent to some subsidy scheme. Hence, only mechanisms which are nonlinear, or achieve other allocation rules, can improve upon our subsidy schemes.

5 Implementing the efficient allocation

In this section, we ask: under what conditions can a subsidy scheme implement the fully efficient allocation?

Since utilities are quasilinear in transfers, the relevant measure of allocative efficiency is the sum of traders' holding costs. It turns out that at the efficient allocation each trader's post trade inventory is proportional to her holding capacity.

Lemma 3. *At the efficient allocation, trader i's post-trade inventory is*

$$X_i + q_i = \frac{\kappa_i}{\sum_{j\in N}\kappa_j}(\sum_{j\in N} X_j).$$

The allocation of linear equilibrium $(a_i, y_i, w_i)_i$ *is efficient if and only if* $w_i = 1$, $y_i = \alpha\kappa_i$, *and* $a_i = \beta\kappa_i$ *for some* $\alpha, \beta \in \mathbb{R}$ *and* $\alpha > 0$ *for each* $i \in N$.

By Lemma 3 the efficient allocation is the outcome of several linear equilibria. Hence, using Proposition 2 we can derive the subsidy schemes that implement the efficient allocation.

Lemma 4. *A subsidy scheme implements the efficient allocation if and only if:*

$$R_i = \left(\frac{\alpha - 1}{\alpha}\right)\frac{\sum_{\{j\in N|j\neq i\}}\sigma^2_{X,j}}{\sum_{\{j\in N|j\neq i\}}\kappa_j} \tag{20}$$

$$c_i = \left(\frac{2-\alpha}{\alpha}\right)\frac{1}{\sum_{\{j\in N|j\neq i\}}\kappa_j} \tag{21}$$

$$\tau_i = K + \left(\frac{\alpha - 1}{\alpha}\right) \frac{\sum_{\{j \in N | j \neq i\}} \mu_{Xj}}{\sum_{\{j \in N | j \neq i\}} \kappa_j} \tag{22}$$

for each each $i \in N$ for some $\alpha \in \mathbb{R}_+$ and $K \in \mathbb{R}$.

As seen from the lemma, there are a continuum of subsidy schemes that are efficient.[7] Fixing an arbitrary choice we compute the expected cost of the subsidy scheme as well as the maximal participation fees the auction operator can charge subject to interim individual rationality (each trader's expected utility, conditional on her endowment, from participating in the auction is non-negative). In general, we find that the expected cost exceeds the revenue from participation fees whenever the traders' holding capacities are not proportional to the means of their endowments. Thus, by revenue equivalence, there generally does not exist a mechanism which implements the efficient allocation subject to ex-ante budget balance and individual rationality.

Proposition 4. *There exists a mechanism (with differentiable transfer rules) that implements the efficient allocation such that it is individually rational for traders to participate in the mechanism and is ex-ante budget balanced if and only if the vectors $[\kappa_1, ..., \kappa_N]$ and $[\mu_1, ..., \mu_N]$ are linearly dependent.*

Notice that, if budget-balanced full efficiency is not achievable by some subsidy scheme, then revenue equivalence in proposition 3 implies that budget-balanced and IR full efficiency cannot be achieved using any other mechanism. Hence, proposition 4 in fact characterizes when efficient trade is possible with *any* mechanism. To our knowledge, this result is new to the literature.

A special case when the linear dependency condition holds is the often-studied case when all traders are ex-ante symmetric. In that setting, full efficiency is achievable at zero expected cost—welfare losses due to price impact avoidance are a symptom of the double auction mechanism and not an inherent inefficiency of the model setting and can be eliminated with subsidies. This may at first glance seem surprising, in that it appears to violate the classic impossibility result of Myerson and Satterthwaite (1983). The intuition behind Proposition 4 is similar to another classic result due to Cramton, Gibbons, and Klemperer (1987) who show under certain conditions that when agents own shares of a partnership, the efficient outcome is achievable with budget balance. This is because, in that setting, each type of each agent earns positive expected surplus and provided this surplus is large enough, the mechanism designer can charge participation fees to cover the cost of any subsidies. The

[7] In fact, efficiency can be achieved by using only quadratic subsidies by setting α to 1 or by using only slope and linear subsidies by setting α to 2. For other inefficient allocations, we generally require all three subsidy types for implementation.

same logic applies to our model where, unlike in Myerson and Satterthwaite (1983), each type of each trader can be either a buyer or seller with positive probability and thus earns a surplus from participating in the auction. Indeed, under the linear dependency condition it can be shown that in expectation each trader is neither a net buyer or net seller. This ensures that the surplus is sufficiently high for the worst-off type of each trader so participation fees can cover the cost of the efficient subsidy scheme.

Though a continuum of subsidies can implement the efficient allocation as shown in Lemma 2 certain subsidy schemes have desirable properties. One desirable property is ex-post budget balance where for each $\omega \in \Omega$ the cost of operating the auction is non-positive. A second desirable property is implementation in dominant strategies. Both of these are achievable with subsidy schemes (but not simultaneously).

Proposition 5. *Suppose that $[\kappa_1, ..., \kappa_N]$ and $[\mu_1, ..., \mu_N]$ are linearly dependent. Then subsidy schemes with*

$$R_i = \frac{\sum_{\{j \in N | j \neq i\}} \sigma^2_{X,j}}{2 \sum_{\{j \in N | j \neq i\}} \kappa_j}, \ c_i = 0, \ \tau_i = K + \frac{\sum_{\{j \in N | j \neq i\}} \mu_{Xj}}{2 \sum_{\{j \in N | j \neq i\}} \kappa_j} \tag{23}$$

for some $K \in \mathbb{R}$ combined with a fixed entry fee for each trader i of

$$\frac{\kappa_i \left(\sum_{\{j \in N | j \neq i\}} \sigma^2_{X,j} \right)}{2 \left(\sum_{\{j \in N | j \neq i\}} \kappa_j \right) \left(\sum_{j=1}^{n} \kappa_j \right)}$$

implement fully efficient trade with individual rationality, and ex-post budget balance. Subsidy schemes with

$$R_i = 0, \ c_i = \frac{1}{\sum_{\{j \in N | j \neq i\}} \kappa_j}, \ \tau_i = K \tag{24}$$

for some $K \in \mathbb{R}$ combined with a fixed entry fee for each trader i of:

$$\frac{\kappa_i \left(\sum_{\{j \in N | j \neq i\}} \sigma^2_{X,j} \right)}{2 \left(\sum_{\{j \in N | j \neq i\}} \kappa_j \right) \left(\sum_{j=1}^{n} \kappa_j \right)}$$

implement fully efficient trade in dominant strategies with individual rationality and ex-ante budget balance. If fully efficient trade is possible, any mechanism which achieves full efficiency leaves the market operators with no expected revenue surplus.

In words, proposition 5 states that, when the linear dependency condition holds and budget-balanced full efficiency is achievable, it can be achieved with only slope subsidies, in (23), or only quadratic subsidies, in (24). Slope subsidies have the benefit that, since bid slopes are nonrandom in our setting, the auctioneer breaks even ex post, and thus faces no revenue risk. Quadratic subsidies have the advantage that agents' bid curves are ex-post best responses, so these mechanisms are somewhat more incentive-robust for bidders. The fact that there is a continuum of subsidies which implement the efficient allocation allows the market operator to trade off the objectives of revenue risk and incentive robustness, while maintaining full allocative efficiency.

By revenue equivalence, any two subsidy schemes which implement the efficient allocation are revenue-equivalent. Moreover, Proposition 4 implies that fully efficient subsidy schemes leave the market operator with exactly zero expected revenue: market operators charge entry fees to market participants, and in expectation, pay out all of the fee revenue in slope or quadratic subsidies. Naturally, revenue-maximizing market operators may not have incentives to choose efficient subsidy schemes. In the following subsection, we will characterize subsidy schemes which maximize the market operator's expected revenue, and show that they substantially decrease trading efficiency.

In summary, this section we have provided a sharp condition for when fully efficient trade is implementable with ex-ante budget balance and individual rationality. A natural question is: what is the second-best mechanism which maximizes efficiency subject to ex-ante budget balance and individual rationality when this condition is violated? Even when considering only our simple class of subsidy schemes answering this question seems intractable analytically. However, in an extension in Appendix B.6, we are able to solve for an upper bound on the welfare loss of the second best mechanism. The bound is constructive and is obtained by solving for the highest constant $\gamma < 1$ such that each trader buys a fraction γ of the quantity purchased in an efficient equilibrium.

Beyond this analytical result, we also solve for the second best subsidy scheme numerically for a range of parameters in Subsection 8.2. Subsidy schemes are amenable to numerical analysis because of the relatively low dimensionality of linear equilibria, which are characterized by three demand coefficients for each trader. We find that in several cases (when the linear dependency condition is violated) the second best subsidy scheme achieves a high fraction of first best gains.

6 Revenue-maximizing subsidy schemes

In this section we ask whether a (monopolistic) auction operator with revenue maximizing incentives would implement subsidy schemes that improve the welfare of traders. We find that the auction operator may actually have incentives to implement subsidy schemes that *reduce* welfare. For our analytical results we restrict attention to symmetric subsidies with ex-ante symmetric traders but our numerical analysis confirms the result for several parameters even when allowing for asymmetry[8].

The following proposition characterizes revenue-maximizing subsidy schemes and associated equilibrium welfare losses.

Proposition 6. *When traders are ex-ante symmetric, revenue-maximizing symmetric subsidy schemes implement linear equilibria with*

$$w_i = \frac{1}{2}.$$

Revenue-maximizing subsidy schemes satisfy

$$c + \frac{2\,(N+2)}{\sigma_X^2\,(N-1)} R = \frac{3-N}{(N-1)\,\kappa},\ \ \tau = K \tag{25}$$

for some $K \in \mathbb{R}$. The auction's expected revenue per trader is

$$\frac{(N-1)\,\sigma_X^2}{8\kappa N}. \tag{26}$$

Expected gains from trade per agent is

$$\frac{3\,(n-1)\,\sigma_X^2}{8\kappa n} \tag{27}$$

Expected welfare loss per trader relative to first best welfare is

$$\frac{(N-1)\,\sigma_X^2}{8\kappa N}. \tag{28}$$

Expected welfare loss per trader relative to the equilibrium welfare with no subsidies is

$$\frac{\sigma_X^2}{N\kappa}[\frac{N-1}{8} - \frac{1}{2(N-1)}] \geq 0. \tag{29}$$

Under any revenue-maximizing subsidy scheme, the sensitivity of traders' demand schedules to their endowments is equal to $1/2$ whereas in the first best equilibrium it is equal to 1 and in the equilibrium with no subsidies it is equal $N-1/N-2$. Thus, when $N>3$ traders trade even less aggressively than when there are no subsidies resulting in further welfare losses as seen in (29). Surprisingly, in the special case of $N=3$ the revenue maximizing subsidy scheme entails no subsidies. Note also that welfare losses do not converge to zero

[8] Our numerical analyses suggest that symmetric subsidies are optimal when traders are ex-ante symmetric though we have not yet attempted to prove this.

as the number of traders tends to infinity when price impact tends to zero. This contrasts with the equilibrium without subsidies which is asymptotically efficient. In fact, equations (27) and (28) imply that welfare losses relative to first best are a fraction 1/4 of the total possible gains from trade for all N. Thus auction operators may have incentives to reduce allocative efficiency in order to increase revenue.

Proposition 6 only applies to the case of ex-ante symmetric traders. In general, it is analytically intractable to solve for the revenue maximizing subsidy scheme when traders are asymmetric. However we are able to numerically solve this problem for a variety of parameter settings. We report these results in Subsection 8.2. For some asymmetric parameter values we do find that the equilibrium allocation of the revenue maximizing subsidy scheme is welfare superior to that of the no subsidy case, however this is not generally the case.

7 A dynamic model

In this section we extend our baseline model to study when efficient trade is achievable in a dynamic setting.

As before, we assume that there are $N > 2$ traders and a single asset. Time, t, is discrete with an infinite horizon taking values in $\{0, 1, 2, ...\}$. At each date, traders participate in a double auction with subsidies. Trader i begins with an initial inventory of the asset, $X_{i0} \sim N(\mu_i, \sigma_{iX}^2)$. Thereafter, her inventory evolves according to

$$X_{i,t+1} = X_{it} + q_{it} + \epsilon_{i,t+1}$$

where q_{it} is the amount purchased in period t, and $\epsilon_{i,t+1} \sim N(0, \sigma_{i\epsilon}^2)$ is an inventory shock. For simplicity we assume that $\sigma_{iX}^2 = \sigma_{i\epsilon}^2$ for each $i \in N$. We assume that all primitive random variables are jointly independent and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Analogous to the static setting, trader i's utility is equal to

$$\sum_{t=0}^{\infty} e^{-rt} \left(-\frac{1}{2\kappa_i} (X_{it} + q_{it})^2 + T_{it} \right)$$

where T_{it} is the time-t net cash transfer to trader i. At each t, given her information set consisting of her past endowments, demand schedules, and prices, trader i selects a continuously differentiable demand schedule which is strictly monotone decreasing in the time-t price and satisfies a technical decay condition[9]. In addition, we assume that admissible

[9]The technical decay condition is that $\lim_{|p|\to\infty} e^{-\lambda p} q_{it}(\omega, p) = 0$ for each $\omega \in \Omega$ and $\lambda \in \mathbb{R}_+$.

demand submission strategies satisfy a no-Ponzi scheme condition

$$lim_{t\to\infty} e^{-rt}\mathbb{E}[X_{it}^2] = 0.$$

In a dynamic setting, a subsidy scheme specifies (potentially stochastic) *processes* of linear, quadratic, and slope subsidies. We restrict attention to a comparatively small class of *stationary subsidy schemes* defined below.

Definition 1. A stationary subsidy scheme, $(\zeta_i, \omega_i, c_i, R_i)_i$ consists of

1. a linear subsidy process which pays trader i

$$\tau_{it} q_{it} = \left(\zeta_i \sum_{j\neq i} X_{j,t-1} + \omega_i q_{i,t-1}\right) q_{it}$$

   at each $t \in \{0, 1, 2, ...\}$ where $X_{j,-1} = \mu_i$.

2. a quadratic subsidy process which pays trader i

$$c_i q_{it}^2$$

   at each $t \in \{0, 1, 2, ...\}$.

3. a slope subsidy process which pays trader i

$$R_i \frac{q'_{it}}{\sum_{j\in N} q'_{jt}}$$

   at each $t \in \{0, 1, 2, ...\}$.

Above, the quadratic and slope subsidy coefficients, c_i and R_i, remain constant over time however the linear subsidy coefficient, τ_{it} evolves dynamically as a function of the aggregate inventory of the other traders and the quantity purchased by trader i at time $t-1$. More precisely, τ_{it} depends on the auction operator's inference of $\sum_{j\neq i} X_{j,t-1}$ from the demand schedules submitted at time $t-1$. However, in any equilibrium the operator's inference will be correct so for notational simplicity we do not distinguish between inferred and actual inventories. Later we will give intuition for why we assume τ_{it} evolves in this particular way.

In what follows we show that a stationary subsidy scheme can implement any stationary allocation of the aggregate inventory. That is given arbitrary positive constants γ_1,...,γ_N which sum to 1, there exists a stationary subsidy scheme such that, in a linear PBE, trader

i's post-trade inventory is

$$X_{it} + q_{it} = \gamma_i \sum_{j \in N} X_{jt}$$

for each t. Thus, trader i absorbs a constant fraction γ_i of the total inventory shock, $\sum_{j \in N} \epsilon_{jt}$ in each period. We focus on implementing allocations of this form primarily for tractability—with these allocations, the aggregate inventory is perfectly revealed to each trader at the end of each trading date. If this were not the case the model would run into the issue of infinite regress of beliefs as traders must infer other traders' inventories, other traders' beliefs about traders' inventories, beliefs about beliefs and so on.

For a given $(\gamma_i)_i$, we solve for a stationary subsidy scheme that implements a linear PBE in which traders' demand coefficients remain constant over time and are consistent with $(\gamma_i)_i$. To provide intuition behind the law of motion of τ_{it} note that in any such equilibrium, the mean of residual supply $\mu_{\eta,i,t}$ facing trader i necessarily evolves over time as the aggregate inventory is shocked. As seen from equation (14) of the static model the intercept a_i of a trader's demand schedule depends on the mean of residual supply—to ensure that this intercept does not evolve over time τ_{it} must evolve to offset changes in $\mu_{\eta,i,t}$. This is why we allow τ_{it} to evolve in our definition of a stationary subsidy scheme. τ_{it} evolves as a function of q^i_{s-1} and $\sum_{j \neq i} X^j_{s-1}$ because these are sufficient statistics from trader i's perspective for inferring the mean of residual supply. The other demand coefficients, as in the static model (see (15) and (16)), do not depend on the mean of residual supply and thus neither quadratic nor slope subsidy coefficients need to evolve over time for there to be stationary demand coefficients.

The following proposition characterizes the stationary subsidy schemes that implement a given stationary allocation.

Proposition 7. *Given a stationary allocation $\gamma_1, ..., \gamma_N$, the stationary subsidy schemes which implement it in a PBE are of the form*

$$\zeta_i = \frac{R_i}{\sum_{j \neq i} \sigma^2_{j\epsilon}} = -\omega_i = -\frac{1}{\Theta} \frac{1}{1-\gamma_i} \frac{-\frac{1}{\kappa_i}\Theta\gamma_i + 1 - e^{-r}}{1 - e^{-r}}$$

and

$$2c_i = \frac{1}{(1-\gamma_i)\Theta}[-2 + \frac{\frac{1}{\kappa_i}\Theta\gamma_i}{1 - e^{-r}}]$$

for each $i \in N$ for some $\Theta \in \mathbb{R}_+$.

The fully efficient allocation corresponds to the case when $\gamma_i = \frac{\kappa_i}{\sum_{j \in N} \kappa_j}$ for each i. Using Proposition 7, we can compute the expected cost of implementing the efficient allocation (allowing for participation fees subject to individual rational participation in each period[10]). We can then derive a condition on model primitives such that the implementation is *dynamic budget balanced* and participation is *individually rational.*

Definition 2. A mechanism is dynamic budget balanced if at any history, the conditional expectation of the future cost of running the mechanism conditional on reaching that history is zero. In the case of a stationary subsidy scheme with participation fees $(P_{it})_{it}$ the condition is that

$$\mathbb{E}\left[\sum_{s=t}^{\infty}\sum_{i\in N}\left(c_i q_{is}^2+\tau_i q_{is}+R_i\frac{y_i}{\sum_{j\in N} y_j}\right)-P_{is}\middle|\mathcal{F}_t\right]\leq 0$$

for all t where $\mathcal{F}_t$ is the auction operator's time t information set[11].

Definition 3. A mechanism is individually rational if for each trader i, at any history, trader i's continuation utility from participation exceeds her continuation utility from exiting the mechanism (and thus absorbing her own endowment shocks forever after).

Proposition 8 presents a sufficient condition for dynamically budget balanced and individually rational efficient trade to be implementable in a PBE.

Proposition 8. *If*

$$\frac{e^{-r}}{2(1-e^{-r})^2}\sum_{i\in N}\left(-\frac{1}{\sum_{j\in N}\kappa_j}+\frac{1}{\kappa_i}\right)\sigma_{i\epsilon}^2$$
$$> -\frac{\sum_{i\in N}\mu_i\sum_{j\neq i}\mu_j}{2(1-e^{-r})\sum_{j\in N}\kappa_j}+\sum_{i\in N}\frac{1}{2(1-e^{-r})}\frac{\kappa_i}{\sum_{j\in N}\kappa_j\sum_{j\neq i}\kappa_j}\left(\sum_{j\neq i}\mu_j\right)^2$$

then fully efficient, dynamically budget balanced, and individually rational trade is implementable in a PBE. When $[\kappa_1,...,\kappa_N]$ *is proportional to* $[\mu_i,...,\mu_j]$*, the above inequality is satisfied.*

The proposition implies that, after a first round of trade, it is *costless* for the operator to keep all traders at the efficient allocation thereafter since the efficient allocation is proportional to risk capacities. Moreover in a dynamic setting, there is a larger range of parameters for which fully efficient and budget balanced trade is achievable. The intuition behind this result is similar to that of Skrzypacz and Toikka (2015). With multiple rounds of trade, since traders expect future inventory shocks, participation is more attractive than in the

[10] We assume that if a trader does not participate in a given period she is excluded from participation forever after.

[11] $\mathcal{F}_t$ is the σ-algebra generated by the demand schedules of all traders and all prices prior to time t.

static model. Thus the auction operator can charge higher participation fees. Indeed the inequality in Proposition 8 is slackened when the variances of inventory shocks are higher.

Note however, that Proposition 8 only gives a sufficient condition. Unlike in the static model our characterization is not sharp. This is because revenue equivalence may not hold in our dynamic setting so there may possibly exist a more complicated mechanism that can implement the efficient allocation at a cheaper cost[12].

8 Extensions and additional results

8.1 What are the restrictions that linear auction equilibria impose on allocation rules?

Thus far, we have studied allocation rules implemented by linear equilibria of auctions, described by tuples of coefficients $\{(a_i, y_i, w_i)\}$. A natural question is how this restricts the set of allocation rules we consider. In this subsection, we characterize the set of allocation rules which can be implemented by linear auction equilibria, and present two examples of allocation rules are not in this set.

Building on the results of section 4, we consider truthful-reporting equilibria of direct mechanisms, which can be thought of as defining mappings from profiles of agents' types to amounts traded by each agent. Consider a general allocation function:

$$[q_1 (X_1 \dots X_n), \dots q_n (X_1 \dots X_n)]$$

where $q_i (X_1 \dots X_n)$ is the net amount of the asset traded by agent i, assuming that all other agents' types are $X_1 \dots X_n$. Since the net amount traded by agents must be 0, we require allocation functions to satisfy:

$$\sum_{i \in N} q_i (X_1 \dots X_n) = 0 \; \forall (X_1 \dots X_n) \tag{30}$$

The following lemma is a direct consequence of (5) in Lemma 1, and the full implementation result of Proposition 2.

Lemma 5. *An allocation rule can be implemented by a double auction with subsidies if and*

[12]We plan to investigate this soon.

only if it can be represented as:

$$\begin{pmatrix} q_1(x_1 \dots x_n) \\ \vdots \\ q_n(x_1 \dots x_n) \end{pmatrix} = \boldsymbol{k} + \boldsymbol{A} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{31}$$

Where the elements k_i of $\boldsymbol{k}$ satisfy:

$$\sum_{i \in N} k_i = 0 \tag{32}$$

And $\boldsymbol{A}$ has the form:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$a_{ij} = s_i w_j \ \forall i, j \neq i \tag{33}$$

$$a_{ii} = -(1 - s_i) w_i \ \forall i \tag{34}$$

Where $w_i \geq 0$, $s_i \geq 0$, and $\sum_{i \in N} s_i = 1$.

A more intuitive condition on allocation rules, which is implied by proposition 5, is the following.

Lemma 6. *Any allocation rule which can be implemented by a double auction with subsidies satisfies, for distinct indices i, j, k, l:*

$$\frac{\frac{\partial q_i}{\partial x_k}}{\frac{\partial q_j}{\partial x_k}} = \frac{\frac{\partial q_i}{\partial x_l}}{\frac{\partial q_j}{\partial x_l}} \tag{35}$$

Intuitively, lemma 6 implies that allocation rules implemented by auction equilibria must satisfy a kind of partial symmetry, or anonymity, condition, given in (35): if i's quantity traded is more responsive than j's quantity is to k's endowment, than i's quantity traded is proportionally more responsive than j's quantity traded to l's endowment as well. Another intuition for (35) is that, in auction equilibria, any given agent can choose to trade more or less aggressively, but she must trade more aggressively with all agents; agents cannot choose to trade preferentially more aggressively with some agents but not others.

The following two examples illustrate allocation rules which cannot be implemented by linear double-auction equilibria.

Example 1. Suppose $n = 4$, and the allocation rule is:

$$q_1 (X_1 \dots X_4) = \frac{X_1 + X_2}{2} - X_1$$

$$q_2 (X_1 \dots X_4) = \frac{X_1 + X_2}{2} - X_2$$

$$q_3 (X_1 \dots X_4) = \frac{X_3 + X_4}{2} - X_3$$

$$q_4 (X_1 \dots X_4) = \frac{X_3 + X_4}{2} - X_4$$

This allocation rule does not satisfy (35), since:

$$\frac{\frac{\partial q_1}{\partial X_2}}{\frac{\partial q_1}{\partial X_4}} = \frac{0.5}{0}, \ \frac{\frac{\partial q_3}{\partial X_2}}{\frac{\partial q_3}{\partial X_4}} = \frac{0}{0.5}$$

Intuitively, example 1 illustrates an allocation rule in which agents 1 and 2 trade with each other, and agents 3 and 4 trade with each other, but agents 1 and 2 cannot trade with agents 3 and 4. In financial settings, we might think of 1 and 2 as trading on a different exchange from agents 3 and 4. This example violates condition (35), because 1's trade quantity responds to 2's endowment, but not 4's, whereas 3's trade quantity responds to 4's but not 2's, so it cannot be implemented by a double-auction equilibrium.

Example 2. Suppose $n = 3$, and the allocation rule is:

$$\begin{aligned} q_1 (X_1 \dots X_3) &= -\frac{X_1}{2} + \frac{X_2}{2} + \frac{X_3}{2} \\ q_2 (X_1 \dots X_3) &= \frac{X_1}{4} - \frac{X_2}{2} \\ q_3 (X_1 \dots X_3) &= \frac{X_1}{4} - \frac{X_3}{2} \end{aligned} \tag{36}$$

This allocation rule cannot be implemented by an auction. To see this, note that:

$$\frac{\partial q_1}{\partial X_3} = s_1 w_3 = \frac{1}{2}$$

hence, $w_3 > 0$; moreover,

$$\frac{\partial q_2}{\partial X_1} = s_2 w_1 = \frac{1}{4}$$

hence, $s_2 > 0$; thus, we must have

$$\frac{\partial q_2}{\partial X_3} > 0$$

But from (36), we have $\frac{\partial q_2}{\partial X_3} = 0$.

The allocation rule in example 2 can be thought of as a dealer network: agents 2 and 3 trade their endowments with 1, but never trade with each other. Condition (35) trivially holds, since (35) requires at least 4 agents; however, this allocation rule cannot be represented in a way that satisfies (33) and (34).

Mathematically, restricting attention to double-auction equilibria simplifies the space of allocation rules substantially, since it reduces the dimensionality of the space of allocation rules from order N^2 to order N. Moreover, in our setting, the fully efficient allocation can always be attained by a non-discriminatory allocation rule (though the resultant mechanism may not satisfy budget balance). However, when budget-balanced full efficiency is not attainable, in principle, second-best efficiency could potentially be increased by considering a broader class of allocation rules than the ones we consider in this paper.

8.2 Numerical results

In a second extension, we numerically study how subsidy schemes perform for either revenue maximization, or welfare maximization subject to budget balance, under a variety of parameter settings. Proposition 2 shows that our subsidy schemes can be used to implement any linear equilibrium, not just the efficient allocation, and derives closed-form expressions for the implementing subsidies. Thus, given a conjectured linear equilibrium, we can derive analytic, though complex, expressions for welfare and the platform's total revenue; we show these expressions in appendix D.4. This allows us to formulate the welfare and revenue maximization problems as analytical optimization problems. To find the second-best mechanism, for a set of primitives, we seek the linear equilibrium which maximizes welfare, conditional on total revenue being nonnegative:

$$\max_{\{(a_i,y_i,w_i)\}} Welfare\left(\{(a_i,y_i,w_i)\};\left\{\left(\kappa_i,\mu_{Xi},\sigma^2_{Xi}\right)\right\}\right)$$
$$s.t.\ Revenue\left(\{(a_i,y_i,w_i)\};\left\{\left(\kappa_i,\mu_{Xi},\sigma^2_{Xi}\right)\right\}\right) \geq 0 \quad (37)$$

To find the revenue-maximizing problem, we solve:

$$\max_{\{(a_i,y_i,w_i)\}} Revenue\left(\{(a_i,y_i,w_i)\};\left\{\left(\kappa_i,\mu_{Xi},\sigma^2_{Xi}\right)\right\}\right) \quad (38)$$

For a variety of parameter settings, we solve (37) and (38) numerically using a convex optimization routine. Appendix D.3 describes additional details of our procedure.

Figure 1 shows our results for numerical welfare maximization, subject to budget balance. Essentially, this corresponds to solving numerically for "second-best" subsidy schemes, subject to budget balance. The y-axis of each plot in Figure 1 shows the ratio:

$$\frac{Welfare_{Second\ best} - Welfare_{No\ subsidies}}{Welfare_{First\ best} - Welfare_{No\ subsidies}}$$

in words, the ratio of the difference between expected welfare from the second-best and expected welfare from the uniform-price auction without subsidies, and the difference between first-best welfare and uniform-price auction welfare. That is, this is the welfare gain from the second-best mechanism relative to the uniform-price auction without subsidies, as a fraction of the total possible welfare gain.

To construct the left-most figure, for different values of the number of traders, N, we set

$$(\mu_{X_1}, ..., \mu_{X_N}) = (0, 0, \ldots k_1, -k_1), (\kappa_{X_1}, ..., \kappa_{X_N}) = (1, \ldots 1), (\sigma^2_{X_1}, ..., \sigma^2_{X_N}) = (1, \ldots 1)$$

and plot the welfare gain as we vary k_1 from 0 to 3. That is, we focus on how the welfare gain depends on means of traders endowments. To construct the middle figure we set

$$(\mu_{X_1}, ..., \mu_{X_N}) = (0, 0 \ldots 1, -1), (\kappa_{X_1}, ..., \kappa_{X_N}) = (1, \ldots k_2, k_2), (\sigma^2_{X_1}, ..., \sigma^2_{X_N}) = (1, \ldots 1)$$

and plot the welfare gain as we vary k_2 from 0 to 5. This allows us to see how the welfare gain depends on agents' risk capacities. Finally, we set

$$(\mu_{X_1}, ..., \mu_{X_N}) = (0, 0 \ldots 1, -1), (\kappa_{X_1}, ..., \kappa_{X_N}) = (1, \ldots 1), (\sigma^2_{X_1}, ..., \sigma^2_{X_N}) = (1, \ldots k_3, k_3)$$

where k_3 ranges from 0.1 to 5 to see how the welfare gain changes with the variances of agents endowments. For all graphs, different lines represent different values of the number of traders. Note that under any parameter specification, the linear dependency condition for fully efficient and budget balanced trade is violated, so fully efficient trade is impossible, and the y-axis must be lower than 100%. However, figure 1 shows that, across all parameter settings we tried, the second-best mechanism can achieve over 90% of the total welfare gap between the first-best mechanism and equilibrium in a double auction with no subsidies. This shows that, even when full efficiency is not achievable, optimal subsidies can improve efficiency substantially, relative to the uniform-price auction with no subsidies.

In Figure 2, we show results for the revenue-maximizing mechanism. The y-axis shows the ratio:

$$\frac{Revenue_{Rev\ max}}{Revenue_{No\ subsidies}}$$

That is, this is the ratio between the market operator's expected revenue from the revenue-

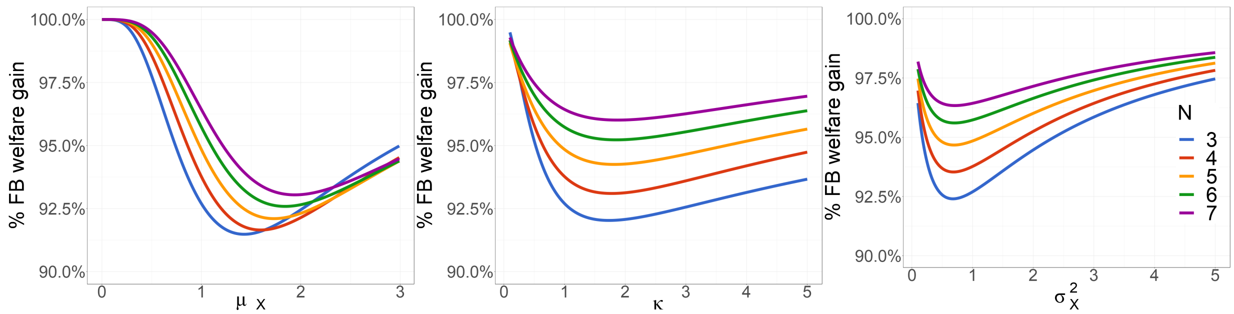

Figure 1: Welfare gain from the second best mechanism, as a percentage of the gap between first-best welfare and equilibrium welfare.

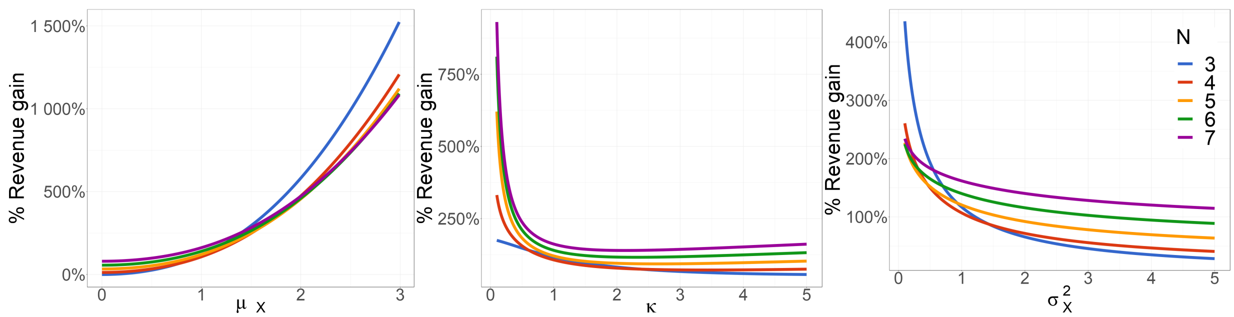

Figure 2: Revenue gain from the revenue-maximizing mechanism, as a percentage of equilibrium revenue in a double auction without subsidies but with optimal entry fees.

optimal subsidy schemes, compared to expected revenue from the uniform-price auction without subsidies, combined with optimal entry fees. We see that the revenue-maximizing mechanism is often able to substantially improve upon the uniform-price auction without subsidies: in some cases, optimal revenue is many times the revenue of a uniform-price auction without subsidies. The revenue gain is higher when the means of endowments are more disparate. It is also higher when the variance of endowments and risk capacities are low for the agents with extremal endowment means (in this case agents with μ_X equal to 1 or -1).

9 Discussion

9.1 Implications for market design

Besides the uniform-price auction, a broad variety of other mechanisms are used in multi-unit trading settings: discriminatory-price auctions, dark pools, workup, size-discovery mechanisms,[13] fragmented trade across multiple venues,[14] various OTC- or networked mechanisms,[15] payments for order flow and make-take fees,[16] and many other mechanisms. The basic idea behind many of these trading mechanisms which aim to improve trading allocative efficiency, such as dark pools and make-take fees, is to encourage agents to trade more aggressively, either by reducing agents' price impact, or by directly paying agents to trade more.

In this paper, we show that agents' behavior in multi-unit trading games can also be modified by simply augmenting uniform-price double auctions with a simple class of subsidy schemes. Our subsidies are simple to describe and implement, and in the context of our model, can implement a large space of outcomes. They have other desirable properties for the market operator: slope subsidies have low revenue uncertainty for the market operator, and quadratic subsidies give market participants robust incentives. An interesting direction for future research would be to test the performance of these subsidies, and to compare their performance to existing mechanisms, in lab experiments or in real-world settings.

Our results also suggest that monopolist market platforms may not have good incentives to develop mechanisms which improve trading efficiency. Thus, while better market design can improve market outcomes, free and unregulated markets – especially markets in which some platforms have substantial market power – may not naturally converge to efficiency-improving mechanisms without regulatory oversight.

9.2 Implications for mechanism design

In the literature on multi-unit trade, the uniform-price auction is known to be an inefficient mechanism. Bidders in double auctions have incentives to shade their bids in order to avoid price impact; in equilibrium, agents under-trade relative to the social optimum. At first glance, the efficiency losses from the uniform-price auction seem analogous to the intuition behind the Myerson-Satterthwaite impossibility theorem. Building on this intuition, a natural conjecture would be that the bid-shading effect of the double auction generalizes to a larger set of mechanisms, and that fully efficient and budget-balanced trade is impossible, under any mechanism, in multiple-unit settings with private information.

The results of this paper show that the story is somewhat more subtle. When a linear dependency condition between agents' inventory means and risk capacities is satisfied, fully

[13] See Duffie and Zhu (2017), Antill and Duffie (2017)

[14] See Budish, Lee, and Shim (2019), Degryse, De Jong, and van Kervel (2015), Gresse (2017), Pagnotta and Philippon (2018), Chen and Duffie (2020)

[15] See Malamud and Rostek (2017)

[16] See Colliard and Foucault (2012), Foucault, Kadan, and Kandel (2013), Malinova and Park (2015), Cardella, Hao, and Kalcheva (2015), Battalio, Corwin, and Jennings (2016), and Chao, Yao, and Ye (2019).

efficient and budget balanced trade is possible. This condition is satisfied in a number of settings studied in the literature, in which agents' endowments are ex-ante symmetric. When the linear dependency condition is not satisfied, no mechanism can achieve budget-balanced and fully efficient trade. However, our computational results show that the standard uniform-price auction is inefficient even in this setting: under a variety of parameter settings, the welfare loss of the standard uniform-price auction is much larger than that of the second-best mechanism.

Our results thus show that there are two sources of inefficiency in the uniform-price double auction mechanism. The first is a fundamental distortion caused by agents' private information, analogous to Myerson-Satterthwaite impossibility, which is independent of the trading mechanism used. The second is a distortion which results from the uniform-price double auction mechanism, and it can be fixed with better game design.

Our approach differs somewhat from the standard mechanism design approach. In a standard mechanism design paper, the revelation principle is applied to show that Bayes-Nash equilibria of arbitrary games can be represented as direct revelation mechanisms, which map profiles of agents' types to implementable allocation rules and implementing transfers. The analyst can then study all possible Bayes-Nash equilibria of trading games, by analyzing the set of incentive-compatible revelation mechanisms. A direct revelation mechanism is a mapping from profiles of agents' types into implementable allocation rules, with associated transfers calculated using the envelope theorem.

This paper takes a slightly different approach, but achieves a similar goal. Proposition 2 shows that any linear equilibrium can be implemented by some subsidy scheme. From proposition 3, this implies that subsidy schemes are expected revenue- and utility-equivalent to direct revelation mechanisms which implement linear equilibrium allocation rules. Our subsidies can thus be thought of as a particular representation of direct revelation mechanisms, associated with this subset of allocation rules. Relative to using direct revelation mechanisms, the subsidy approach also implies that any linear equilibria can be implemented using a fairly simple class of side payments. Our subsidy schemes can still be used in settings where the market operator is not completely certain about the underlying parameters, whereas incentive-compatible direct revelation mechanisms are very sensitive to the distributions of agents' types.

Our search for efficient and revenue-maximizing mechanism is limited in a number of ways. We restrict attention to allocation rules which can be implemented by linear double-auction equilibria. As subsection 8.1 shows, this rules out allocation rules implemented by some mechanisms, such as dealer networks or fragmented exchanges, which may be important in practice.[17] An interesting direction for future work would be relax this assumption, exploring optimal mechanism design for a broader class of allocation rules.

Our approach also rules out non-linear equilibria, and we restrict preferences to be linear-quadratic with Gaussian uncertainty. The literature has shown that the general nonlinear case is analytically complex. If agents' utility functions can be arbitrary nonlinear functions, even describing agents' type spaces becomes quite difficult. Moreover, there is a large applied theory literature analyzing the linear-quadratic case. We thus view the linear-quadratic case as an important special case to study.

We assume private values and common knowledge of risk capacities. An interesting direction for future work would be to explore how our subsidies perform in environments with interdependent values with generalized information structures.[18]

[17]See, for example, Glode and Opp (2016), Peivandi and Vohra (2014), Yoon (2017), Babus and Parlatore (2017), Wang (2016), Malamud and Rostek (2017), Chen and Duffie (2020)

[18]Some papers which analyze double auctions with interdependent values include Rostek and Weretka (2012b), Rostek and Weretka (2015b), and Bergemann, Heumann, and Morris (2015)

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Appendix

A Proofs and Supplementary Material for Sections 3 and 4

A.1 Proof of Lemma 1

The market clearing condition is

$$\sum_{i \in N} q_i(p^*) = \sum_{i \in N} a_i - w_i X_i - y_i p^* = 0.$$

Solving for p^* gives equation (4) and substituting p^* into traders' equilibrium demand schedules gives (5).

A.2 Proof of Lemma 2

Combining agents' demand schedules from (1) with the definition of residual supply in (6), we have:

$$q_{RSi}(p) = -\left[\sum_{\{j \in N | j \neq i\}} a_j - w_j X_j - y_j p\right] = \eta_i + d_i p$$

This gives (8) and (10). Taking the mean and variance of η_i, we get (11) and (12).

A.3 Proof of Proposition 1

Proof. The proof proceeds in two steps. First, we show that the condition that $\{(y_i, w_i)\}_{i \in N}$ satisfy the system of equations defined by (13), (15), and (16) is a necessary condition for a linear equilibrium. In the second step we show that it is a sufficient condition.

Necessity: Suppose all agents other than i are bidding:

$$q_i(p) = a_i - w_i X_i - y_i p_i. \tag{39}$$

To solve an agent's optimal demand submission problem, it is convenient to solve for an agent's optimal demand schedule as if he or she could condition the quantity on the residual supply intercept,

$$\eta_i \equiv q_i - d_i p \tag{40}$$

defined in (10) of lemma 2. Fixing the slope of residual supply d, any function $q_i\left(p\right)$ which is a continuous, differentiable, and strictly decreasing linear function of prices, can be uniquely represented as a function $\tilde{q}_i\left(\eta_i\right)$, which is a continuous, differentiable, and strictly increasing function of η_i. The function $\tilde{q}_i\left(\eta_i\right)$ is defined as:

$$\tilde{q}_i\left(\eta_i\right)=\left\{q_i\left(p_i\right):q_i\left(p_i\right)-d_ip=\eta_i\right\} \tag{41}$$

By assumption, $q_i\left(p\right)$ is continuously differentiable and strictly decreasing, so the function

$$q_i\left(p_i\right)-d_ip$$

is a cont. differentiable and strictly decreasing function of p, so $\tilde{q}_i\left(\eta_i\right)$ defined in (41) is also cont. differentiable and strictly increasing in η_i. Also, if two $q_i\left(p\right)$ functions differ for some p, the functions $\tilde{q}_i\left(\eta_i\right)$ constructed through (41) must also differ for some η_i.

Writing the agent's bidding problem in terms of $\tilde{q}_i\left(\eta_i\right)$ is useful because it simplifies the expression for slope subsidies. From (41), for any p, $\tilde{q}_i\left(\eta_i\right)$ satisfies:

$$\tilde{q}_i\left(q_i\left(p_i\right)-d_ip\right)=q_i\left(p\right)$$

Differentiating both sides with respect to p, we have:

$$\tilde{q}_i'\left(q_i\left(p_i\right)-d_ip\right)\left(q_i'\left(p\right)-d_i\right)=q_i'\left(p\right)$$

$$\tilde{q}_i'\left(\eta_i\right)=\frac{q_i'\left(p\right)}{q_i'\left(p\right)-d_i}.$$

Hence, we can write agent i's slope subsidy as:

$$\frac{-\frac{\partial}{\partial p}q_i(X_i,p^*)}{-\frac{\partial}{\partial p}q_i(X_i,p^*)+y_i}R_i=\frac{y_i}{d_i+y_i}R_i=R_i\tilde{q}_i'(\eta_i)$$

We can therefore write an agent's optimization problem as:

$$max_{\tilde{q}\in\mathcal{M}}\int_{-\infty}^{\infty}[\tau_i\tilde{q}_i-\frac{1}{2\kappa_i}(X_i+\tilde{q}_i)^2-p\tilde{q}_i+\tilde{q}_i'(\eta_i)R_i-\frac{c_i}{2}\tilde{q}_i^2]\phi(\eta_i)dr_i \tag{42}$$

where ϕ is the pdf of η_i; the mean and variance of η_i are characterized in lemma 2. The agent maximizes over $\mathcal{M}$, the set of strictly increasing and continuously differentiable functions, f, of sufficiently rapid decay: $lim_{r\to\infty}\phi(r_i)f(r_i)=0$ and $lim_{r_i\to-\infty}\phi(r_i)f(r_i)=0$.

First, from (40), we can write

$$p = \frac{\tilde{q}_i + \eta_i}{d_i}$$

So (42) becomes:

$$max_{\tilde{q}\in\mathcal{M}} \int_{-\infty}^{\infty} [-\frac{1}{2\kappa_i}(X_i + \tilde{q}_i)^2 - \left(\frac{\tilde{q}_i + \eta_i}{d_i}\right)\tilde{q}_i + \tilde{q}_i'(\eta_i)R_i + \frac{c_i}{2}\tilde{q}_i^2]\phi(\eta_i)d\eta_i \tag{43}$$

To solve (43), we derive the Euler-Lagrange conition. We take the variation of $\tilde{q}_i$ with an arbitrary function h in $\mathcal{M}$ and substitute into the objective function:

$$\int_{-\infty}^{\infty} [\tau_i\tilde{q}_i - \frac{1}{2\kappa_i}(X_i + \tilde{q}_i + \alpha h)^2 - (\frac{\eta_i + \tilde{q}_i + \alpha h}{d_i})(\tilde{q}_i + \alpha h) + (\tilde{q}_i' + \alpha h')R_i + \frac{c_i}{2}(\tilde{q}_i + \alpha h)^2]\phi(\eta_i)d\eta_i. \tag{44}$$

Where $\alpha \in \mathbb{R}$. A necessary condition for $\tilde{q}_i$ to solve (43) is that (44), viewed as a function of α, is maximized at $\alpha = 0$. The first order condition with respect to α gives the necessary condition,

$$\int_{-\infty}^{\infty} [\tau_i - \frac{1}{\kappa_i}(X_i + \tilde{q}_i)h - (\frac{\eta_i + \tilde{q}_i}{d_i})h - \frac{\tilde{q}_i}{d_i}h + h'R_i + c_i\tilde{q}_ih]\phi(\eta_i)d\eta_i = 0.$$

We integrate by parts to get:

$$\int_{-\infty}^{\infty} [\tau_i - \frac{1}{\kappa_i}(X_i + \tilde{q}_i) - (\frac{\eta_i + \tilde{q}_i}{d_i}) - (\frac{1}{d_i} - c_i)\tilde{q}_i]\phi(\eta_i) + \phi'(\eta_i)R_i d\eta_i = 0$$

where we have used that $h \in \mathcal{M}$ is of sufficiently rapid decay that $h\phi(r_i)]_{-\infty}^{\infty} = 0$. Since this must hold for all $h \in \mathcal{M}$, we derive that a necessary condition for optimality is the Euler-Lagrange condition:

$$[\tau_i - \frac{1}{\kappa_i}(X_i + \tilde{q}_i) - (\frac{\eta_i + \tilde{q}_i}{d_i}) - (\frac{1}{d_i} - c_i)\tilde{q}_i]\phi(r_i) = -\phi'(\eta_i)R_i.$$

Since $\phi(\eta_i)$ is the normal pdf, it satisfies:

$$\phi'(\eta_i) = \frac{\eta_i - \mu_\eta}{\sigma_\eta^2}\phi(\eta_i)$$

Therefore we have:

$$[\tau_i - \frac{1}{\kappa_i}(X_i + \tilde{q}_i) - p - (\frac{1}{d_i} - c_i)\tilde{q}_i] = -\frac{\eta_i - \mu_{\eta_i}}{\sigma_\eta}R_i$$

Now plug in:

$$\eta_i = q_i - pd_i$$

$$[\tau_i - \frac{1}{\kappa_i}(X_i + \tilde{q}_i) - p - (\frac{1}{d_i} - c_i)\tilde{q}_i] = \frac{\mu_{\eta_i} + pd_i - q_i}{\sigma^2_{\eta_i}} R_i$$

Solve for $\tilde{q}$:

$$q = \left(\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_\eta}\right)^{-1} \left(\tau_i - \frac{R_i \mu_{\eta_i}}{\sigma^2_{\eta_i}} - \frac{X_i}{\kappa_i} - p\left(1 + \frac{R_i d_i}{\sigma^2_{\eta_i}}\right)\right) \tag{45}$$

Expression (45) relates the optimal quantity q to X_i, p_i, and constant terms. Extracting the coefficients on X_i and p and the constant, and simplifying somewhat, we get (14), (15) and (16) of proposition 1.

Sufficiency: Now, we show that the Euler-Lagrange condition is a sufficient condition. We compute the second derivative of (44) with respect to α to derive

$$\int_{-\infty}^{\infty} \left(-\frac{1}{2\kappa_i} - \frac{1}{d_i} + \frac{c_i}{2}\right) h^2(\eta_i)\phi(\eta_i)d\eta_i \leq 0$$

This implies that $\frac{1}{2\kappa_i} + \frac{1}{d_i} - \frac{c_i}{2} > 0$ is a necessary condition that any linear equilibrium must satisfy. If it is equal to zero, then the objective function is linear in α and optimality can not be achieved at $\alpha = 0$. Similarly, if $\frac{1}{2\kappa_i} + \frac{c_i}{2} + \frac{1}{d_i} < 0$ then the objective function is globally convex in α and can not be maximized at $\alpha = 0$ for any $\tilde{q}$, a necessary condition for a demand schedule to be optimal. Put differently, any demand schedule which satisfies the Euler-Lagrange condition can not be an optimal demand schedule, but the Euler-Lagrange condition is a necessary condition as we argued earlier.

We now prove sufficiency of the Euler-Lagrange condition under the assumption $\frac{1}{2\kappa_i} + \frac{1}{d_i} - \frac{c_i}{2} > 0$. Recall that h is an arbitrary function in $\mathcal{M}$. The inequality is strict as long as h is not equal to zero on a set of positive $\mathbb{P}$-measure. This implies that (44) is a strictly concave function of α for each $h \in \mathcal{M}$. Suppose $\tilde{q}_i$ solves the Euler-Lagrange condition. Suppose for contradiction that there exists a $k \in \mathcal{M}$ which achieves a higher value of the objective function than $\tilde{q}_i$ but does not satisfy the Euler-Lagrange conditions. Take h to be $k - \tilde{q}_i$ in (44). By assumption the objective function is higher at $\alpha = 1$ than at $\alpha = 0$. However since (44) is strictly concave in α, a first order condition is both necessary and sufficient for optimality. Since $\tilde{q}_i$ satisfies the Euler-Lagrange condition by construction the first order condition is satisfied at $\alpha = 0$ which contradicts the assumption that the objective achieves a higher value at $\alpha = 1$ than at $\alpha = 0$. This completes step 2.

□

A.4 Example of equilibrium nonexistence

To show that equilibrium nonexistence is possible, suppose agents have equal risk capacities, $\kappa_i = \kappa$, and have arbitrary endowment means and variances. We consider symmetric quadratic subsidies, so $c_i = c > 0$ and $R_i = 0$ for all i. In equilibrium, agents' demand slopes will be symmetric, $y_i = y$, and by proposition 1, they satisfy:

$$y = \frac{\kappa d}{\kappa + d - \kappa d c}$$

where $d = (n-1)\,y$. Substituting for d and solving, we have:

$$y = \kappa \frac{(n-2)}{(1-\kappa c)\,(n-1)}$$

We see that $y > 0$, so there does not exist an equilibrium with linear downward-sloping demand curves if $c > \frac{1}{\kappa}$.

A.5 Sufficient conditions for equilibrium existence

Proposition 9. *A sufficient condition for existence of a linear equilibrium is that the following parameter conditions hold for each $i \in N$:*

1.
$$R_i \geq 0,$$

2.
$$\frac{1}{\kappa_i} - c_i > 0,$$

3.
$$\frac{1}{\kappa_i} - c_i - \frac{R_i}{\sigma^2_{\eta_{max,i}}} > 0$$

4.
$$\frac{\left(\frac{1}{\kappa_i} - c_i - \frac{R_i}{\sigma^2_{\eta_{max,i}}}\right)}{\frac{1}{\kappa_i} - c_i} < \frac{N-1}{N}.$$

where

$$\sigma^2_{\eta_{max,i}} = \sum_{\{j \in N | j \neq i\}} \sigma^2_{X,j} \left(\frac{\frac{1}{\kappa_i}}{\frac{1}{\kappa_i} - c_i - \frac{R_i}{\sigma^2_\epsilon}}\right)^2 + \sigma^2_\epsilon. \tag{46}$$

Proof. Suppose the parameter conditions 1 through 4 in the statement of the theorem are satisfied. To prove existence, it suffices to prove that there exists a solution to the system of equations defined by (15) and (16) such that $\frac{1}{2\kappa_i} + \frac{1}{d_i} - \frac{c_i}{2} > 0$ holds for each $i \in N$. To start we divide the numerator and denominator of equations (15), and (16) by $\kappa_i d_i$ to derive the expressions

$$w_i = \frac{\frac{1}{\kappa_i}}{\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}}} \tag{47}$$

and

$$y_i = \frac{1 + \frac{R_i d_i}{\sigma^2_{\eta_i}}}{\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}}}. \tag{48}$$

Recall that $\sigma^2_{\eta_i} := \sum_{\{j \in N | j \neq i\}} \sigma^2_{X,j} w_j^2 + \sigma^2_\epsilon$. Next, we can, using the definition of d_i, express

$$y_i = \frac{1}{N-1} \sum_{j \in N} d_j - d_i$$

Using this equation together with (48) we have

$$\frac{1}{N-1} \sum_{j \in N} d_j - d_i = \frac{1 + \frac{R_i d_i}{\sigma^2_{\eta_i}}}{\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}}}. \tag{49}$$

Rearranging, we derive

$$d_i = \frac{\frac{1}{N-1} \sum_{j \in N} d_j - \frac{1}{\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}}}}{1 + \frac{\frac{R_i}{\sigma^2_{\eta_i}}}{\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}}}}. \tag{50}$$

Rearranging further, we have

$$d_i = \frac{\frac{1}{N-1} \sum_{j \in N} d_j [\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}}] - 1}{\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i}$$

$$\Leftrightarrow$$

$$d_i[\frac{1}{\kappa_i} - c_i] + 2 = \frac{1}{N-1} \sum_{j \in N} d_j [\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}}]$$

$$\Leftrightarrow$$

$$d_i^2[\frac{1}{\kappa_i} - c_i] + 2d_i = d_i \frac{1}{N-1} \sum_{j \in N} d_j [\frac{1}{\kappa_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}}] + \frac{1}{N-1} \sum_{j \in N} d_j$$

$$\Leftrightarrow$$

$$d_i^2[\frac{1}{\kappa_i} - c_i] + d_i[2 - \frac{1}{N-1} \sum_{j \in N} d_j (\frac{1}{\kappa_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}})] - \frac{1}{N-1} \sum_{j \in N} d_j = 0. \tag{51}$$

Define $b = 2 - \frac{1}{N-1}\sum_{j\in N} d_j(\frac{1}{\kappa_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}})$. Then by the quadratic equation we have

$$d_i = \frac{-b + \sqrt{b^2 + 4(\frac{1}{\kappa_i} - c_i)\frac{1}{N-1}\sum_{j\in N} d_j}}{2[\frac{1}{\kappa_i} - c_i]}. \tag{52}$$

As long as $\sum_{j\in N} d_j$ is positive, by our assumption that $\frac{1}{\kappa_i} - c_i > 0$ the above expression is well defined and a positive real number.

Consider the following map, Φ, which takes as input a candidate $(w_1, ..., w_N, d_1, ..., d_N)$. Then, using equations (47) and (52), Φ computes a new candidate $(w_1', ..., w_N', d_1', ..., d_N')$ as output. That is,

$$w_i' = \frac{\frac{1}{\kappa_i}}{\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}}} \tag{53}$$

and

$$d_i' = \frac{-b + \sqrt{b^2 + 4(\frac{1}{\kappa_i} - c_i)\frac{1}{N-1}\sum_{j\in N} d_j}}{2[\frac{1}{\kappa_i} - c_i]} \tag{54}$$

for each $i \in N$. Let $\mathcal{M}_C$ denote the set

$$[0, \frac{\frac{1}{\kappa_1}}{\frac{1}{\kappa_1} - c_1 - \frac{R_2}{\sigma^2_\epsilon}}] \times [0, \frac{\frac{1}{\kappa_2}}{\frac{1}{\kappa_2} - c_2 - \frac{R_2}{\sigma^2_\epsilon}}] \times ... \times [0, \frac{\frac{1}{\kappa_N}}{\frac{1}{\kappa_N} - c_N - \frac{R_N}{\sigma^2_\epsilon}}] \times [0, C]^N$$

where C is a strictly positive constant in $\mathbb{R}$. We argue that there exists C such that Φ maps from $\mathcal{M}_C$ into $\mathcal{M}_C$.

By the assumption that $\frac{1}{\kappa_i} - c_i - \frac{R_i}{\sigma^2_\epsilon} > 0$ and $R_i > 0$ we have

$$0 < w_i' = \frac{\frac{1}{\kappa_i}}{\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}}} < \frac{\frac{1}{\kappa_i}}{\frac{1}{\kappa_i} - c_i - \frac{R_i}{\sigma^2_\epsilon}}$$

for each $i \in N$.

Consider (54). By inspection, the right-hand side is strictly increasing in $\sum_{j\in N} d_j$. For large values of $\sum_{j\in N} d_j$, b is approximately equal to $-\frac{1}{N-1}\sum_{j\in N} d_j(\frac{1}{\kappa_i} + c_i - \frac{R_i}{\sigma^2_{\eta_i}})$. Thus, the right-hand side of (54) is approximately equal to the left-hand side of

$$\frac{\frac{1}{N-1}\sum_{j\in N} d_j(\frac{1}{\kappa_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}})}{\frac{1}{\kappa_i} - c_i} < \frac{\frac{1}{N-1}\sum_{j\in N} d_j(\frac{1}{\kappa_i} - c_i - \frac{R_i}{\sigma^2_{\eta_{max,i}}})}{\frac{1}{\kappa_i} - c_i}$$

with the approximation becoming arbitrarily "good" as $\sum_{j\in N} d_j$ diverges. The inequality uses the assumption that $R_i \geq 0$. Suppose each $d_j \in [0, C]$ for some C finite arbitrarily large. Then the right hand side of the above inequality is less than

$$\frac{\frac{N}{N-1}C(\frac{1}{\kappa_i}-c_i-\frac{R_i}{\sigma^2_{\eta_{max,i}}})}{\frac{1}{\kappa_i}-c_i} \tag{55}$$

Note that the above expression is strictly less than C since it was assumed that

$$\frac{\frac{N}{N-1}(\frac{1}{\kappa_i}-c_i-\frac{R_i}{\sigma^2_{\eta_{max,i}}})}{\frac{1}{\kappa_i}-c_i}<1.$$

Thus, if we take C to be sufficiently large but finite, then the output $d_i' \in [0, C]$ for all $i \in N$ if the input $d_i \in [0, C]$ for each $i \in N$. For such a C, Φ maps from $\mathcal{M}_C$ into $\mathcal{M}_C$. By the Brower's fixed point theorem, there exists a fixed point of Φ which in turn implies the existence of a solution to the system of equations defined by (15) and (16) (take the fixed point of Φ and define each y_i by the equation (49)). This solution constitutes a linear equilibrium, since $\frac{1}{2\kappa_i}+\frac{1}{d_i}-\frac{c_i}{2}>0$ is satisfied.

□

A.6 Linear equilibria which induce equivalent allocation rules

The following lemma characterizes linear equilibria which implement identical allocation rules.

Lemma 7. *Two linear equilibria, $\{(a_i, y_i, w_i)\}$ and $\{(\tilde{a}_i, \tilde{y}_i, \tilde{w}_i)\}$ implement the same allocation if and only if $w_i = \tilde{w}_i$, and $y_i = \alpha\tilde{y}_i$ for each $i \in N$ for some $\alpha \in \mathbb{R}$, and $\tilde{a}_i = a_i - \frac{y_i}{\sum_{j\in N} y_j}\beta$ for some $\beta \in \mathbb{R}$.*

Lemma 7 implies that there are two dimensions of redundancy in linear equilibria: agents' bids can be scaled up or down, or shifted in parallel, without changing the allocation rule that is implemented. Thus, the space of allocation rules is $(3N-2)$-dimensional. This implies that, for general N, all three of our subsidies are needed to span the full space of allocation rules. An immediate corollary of proposition 3 is that, if two subsidy schemes implement the same allocation, they are revenue- and utility-equivalent.

In fact, under our subsidy schemes, a stronger version of revenue equivalence holds. The following lemma shows that any two subsidy schemes which implement the same allocation rule have the same expected revenues, utilities, and payments, even without agent-specific fixed fees.

Lemma 8. *Let $\mathcal{M}$ and $\mathcal{M}'$ be any two double auctions with subsidy schemes that implement the same allocation. Then for each $i \in N$,*

$$\mathbb{E}[t_i^{\mathcal{M}} | X_i = x_i] = \mathbb{E}[t_i^{\mathcal{M}'} | X_i = x_i]$$

and

$$\mathbb{E}[U_i^{\mathcal{M}} | X_i = x_i] = \mathbb{E}[U_i^{\mathcal{M}'} | X_i = x_i]$$

for each $x_i \in \mathbb{R}$ where $t_i^{\mathcal{M}}$, $t_i^{\mathcal{M}'}$, $U_i^{\mathcal{M}}$, $U_i^{\mathcal{M}'}$ are defined analogously to those of proposition 3.

A.6.1 Proof of lemma 7

Proof. Suppose that $\{(a_i, y_i, w_i)\}$ and $\{(\tilde{a}_i, \tilde{y}_i, \tilde{w}_i)\}$ both induce the same equilibrium allocation. The final inventory of agent $i \in N$ is

$$(1 - w_i + \frac{y_i w_i}{\sum_{j \in N} y_j}) X_i + \sum_{j \neq i} \frac{y_i w_j}{\sum_{j \in N} y_j} X_j + \frac{y_i}{\sum_{j \in N} y_j} \epsilon - \frac{y_i}{\sum_{j \in N} y_j} \sum_{j \in N} a_j + a_i$$

which by assumption is equal to

$$(1 - \tilde{w}_i + \frac{\tilde{y}_i \tilde{w}_i}{\sum_{j \in N} \tilde{y}_j}) X_i + \sum_{j \neq i} \frac{\tilde{y}_i \tilde{w}_j}{\sum_{j \in N} \tilde{y}_j} X_j + \frac{\tilde{y}_i}{\sum_{j \in N} \tilde{y}_j} \epsilon - \frac{y_i}{\sum_{j \in N} y_j} \sum_{j \in N} \tilde{a}_j + \tilde{a}_i$$

for each $\omega \in \Omega$. Fix an arbitrary $i \in N$. For an arbitrary $j \in N$ distinct from i, consider $\omega \in \Omega$ such that $X_j \neq 0$ but $X_l = 0$ for all $l \in N$ such that $l \neq j$ and $\epsilon = 0$. Then it must be that

$$\frac{w_j y_i}{\sum_{j \in N} y_j} X_j - \frac{y_i}{\sum_{j \in N} y_j} \sum_{j \in N} a_j + a_i = \frac{\tilde{w}_j \tilde{y}_i}{\sum_{j \in N} \tilde{y}_j} X_j - \frac{\tilde{y}_i}{\sum_{j \in N} \tilde{y}_j} \sum_{j \in N} \tilde{a}_j + \tilde{a}_i \tag{56}$$

for all $\omega \in \Omega$. This is only possible if both the slope of X_j and the intercept are the same on both sides of the equality.

That is,

$$\frac{w_j y_i}{\sum_{j \in N} y_j} = \frac{\tilde{w}_j \tilde{y}_i}{\sum_{j \in N} \tilde{y}_j}$$
$$\Leftrightarrow$$
$$\frac{w_j}{\tilde{w}_j} = \frac{\frac{\tilde{y}_i}{\sum_{j \in N} \tilde{y}_j}}{\frac{y_i}{\sum_{j \in N} y_j}}. \tag{57}$$

Moreover, since j was arbitrary, this must hold for all $j \in N$ such that $j \neq i$. However $i \in N$

was arbitrary. Fix an arbitrary $j \in N$. Then it must be by (57)

$$\frac{\tilde{y}_i}{\sum_{j \in N} \tilde{y}_j} = \frac{\beta y_i}{\sum_{j \in N} y_j}$$

for all $i \in N$ such that $i \neq j$ for some constant $\beta := \frac{w_j}{\tilde{w}_j}$. Next, we show that

$$\frac{\tilde{y}_j}{\sum_{j \in N} \tilde{y}_j} = \frac{\beta y_j}{\sum_{j \in N} y_j}$$

also holds. Fix an arbitrary $k \in N$ such that $k \neq j$. Then we have that

$$\frac{\tilde{y}_i}{\sum_{j \in N} \tilde{y}_j} = \frac{w_k}{\tilde{w}_k} \frac{y_i}{\sum_{j \in N} y_j} \tag{58}$$

for all $i \in N$ such that $i \neq k$. Now take i distinct from both j and k. Then we have

$$\frac{\tilde{y}_i}{\sum_{j \in N} \tilde{y}_j} = \frac{w_k}{\tilde{w}_k} \frac{y_i}{\sum_{j \in N} y_j} = \beta \frac{y_i}{\sum_{j \in N} y_j}$$

which implies that $\beta = \frac{w_k}{\tilde{w}_k}$. Therefore by (58)

$$\frac{\tilde{y}_j}{\sum_{j \in N} \tilde{y}_j} = \frac{\beta y_j}{\sum_{j \in N} y_j}$$

as claimed. Summing over $j \in N$ on both sides of the above equality implies that $\beta = 1$. By (57) this implies $w_j = \tilde{w}_j$ for all $j \in N$. It also implies that

$$y_j = \frac{\sum_{j \in N} y_j}{\sum_{j \in N} \tilde{y}_j} \tilde{y}_j$$

which is of the form $y_j = \alpha \tilde{y}_j$ for some $\alpha \in \mathbb{R}$.

In order for the intercepts in equation (56) to be equal for each $i \in N$ it must be that

$$-\frac{y_i}{\sum_{j \in N} y_j} \sum_{j \in N} a_j + a_i = -\frac{y_i}{\sum_{j \in N} y_j} \sum_{j \in N} \tilde{a}_j + \tilde{a}_i$$

for all $i \in N$. Equivalently,

$$\frac{y_i}{\sum_{j \in N} y_j} \sum_{j \in N} (a_j - \tilde{a}_j) = a_i - \tilde{a}_i$$

for each $i \in N$. Then the above equality is equivalent to $\tilde{a}_i = a_i - \frac{y_i}{\sum_{j \in N} y_j} \beta$ holding for some arbitrary constant β. $\square$

A.6.2 Proof of lemma 8

Proof. From lemma 7, the equivalence class of subsidy schemes which implement the same allocation is: $\{(a_i + \frac{y_i}{\sum_{j\in N} y_j}\beta, \alpha y_i, w_i)\}$ for arbitrary α and β in $\mathbb{R}$. We will show that the total expected cost of any subsidy scheme that implements an equilibrium in the class $\{(a_i + \frac{y_i}{\sum_{j\in N} y_j}\beta, \alpha y_i, w_i)\}$, for some α and β, does not depend on α and β. First, note that we can write total subsidy revenue paid out to bidders by a subsidy scheme as:

$$\sum_{i\in N}(\frac{c_i}{2}\mathbb{E}[q_i^2] + \frac{R_i y_i}{d + y_i} + \tau_i \mathbb{E}[q_i])$$

We can decompose this into two pieces:

$$= \left[\sum_{i\in N}\left(\frac{c_i}{2}\text{Var}[q_i] + \frac{R_i y_i}{d_i + y_i}\right)\right] + \left[\sum_{i\in N}\left(\frac{c_i}{2}\mathbb{E}[q_i]^2 + \tau_i \mathbb{E}[q_i]\right)\right] \tag{59}$$

We will show that each piece separately does not depend on α or β.

Calculating $\frac{c_i}{2}$Var$[q_i] + \frac{R_i y_i}{d_i + y_i}$

Calculating $\frac{c_i}{2}$Var$[q_i]$: From proposition 2, the quadratic subsidy which implements the linear equilibrium $\{(a_i + \frac{y_i}{\sum_{j\in N} y_j}\beta, \alpha y_i, w_i)\}$ is:

$$c_i = \frac{1}{\kappa_i}(1 - \frac{1}{w_i}) - \frac{y_i}{\sum_{\{j\in N|j\neq i\}} y_j}\left(\frac{1}{\kappa_i w_i}\right) + \frac{2}{\sum_{\{j\in N|j\neq i\}} \alpha y_j} \tag{60}$$

Also, from (5) of lemma 1, the quantity traded by an agent on the exchange in equilibrium is

$$q_i = -w_i X_i + \left(a_i - \frac{y_i}{\sum_{j\in N} y_j}\beta\right) + \frac{y_i}{\sum_{j\in N} y_j}\left(\sum_{j\in N} - \left(a_j - \frac{y_j}{\sum_{j\in N} y_j}\beta\right) + w_j X_j\right) \tag{61}$$

Thus, $Var\,(q_i)$ is:

$$Var[q_i] = \left(\frac{\sum_{\{j\in N|j\neq i\}} y_i}{\sum_{j\in N} y_j} w_i\right)^2 \sigma_{X,i}^2 + \left(\frac{y_i}{\sum_{j\in N} y_j}\right)^2 \sigma_{\eta_i}^2 \tag{62}$$

where we have used the definition of $\sigma^2_{\eta_i}$ from lemma 2. Hence, we have:

$$\frac{c}{2}Var\left(q_i\right) = \sum_{i\in N}\frac{1}{2}\left(\frac{1}{\kappa_i}\left(1-\frac{1}{w_i}\right)-\frac{\alpha y_i}{\sum_{\{j\in N|j\neq i\}}\alpha y_j}\left(\frac{1}{\kappa_i w_i}\right)+\frac{2}{\sum_{\{j\in N|j\neq i\}}\alpha y_j}\right) \left(\left(\frac{\sum_{\{j\in N|j\neq i\}}\alpha y_i}{\sum_{j\in N}\alpha y_j}w_i\right)^2\sigma^2_{X,i}+\left(\frac{\alpha y_i}{\sum_{j\in N}\alpha y_j}\right)^2\sigma^2_{\eta_i}\right) \tag{63}$$

Expression (63) does not depend on β, and the the only piece which depends on α is:

$$\sum_{i\in N}\frac{1}{\sum_{\{j\in N|j\neq i\}}\alpha y_j}\left(\left(-w_i+\frac{y_i}{\sum_{j\in N}y_j}w_i\right)^2\sigma^2_{X,i}+\left(\frac{y_i}{\sum_{j\in N}y_j}\right)^2\sigma^2_{\eta_i}\right)$$

$$=\frac{1}{\alpha}\sum_{i\in N}\left(\frac{\sum_{\{j\in N|j\neq i\}}y_j}{\left(\sum_{j\in N}y_j\right)^2}w_i^2\sigma^2_{X,i}+\frac{1}{\sum_{\{j\in N|j\neq i\}}y_j}(\frac{y_i}{\sum_{j\in N}y_j})^2\sigma^2_{\eta_i}\right) \tag{64}$$

Now, we can write the left piece of (64) as:

$$\frac{1}{\left(\sum_{j\in N}y_j\right)^2}\sum_i\left(\sum_{\{j\in N|j\neq i\}}y_j\right)w_i^2\sigma^2_{X,i} = \frac{1}{\left(\sum_{j\in N}y_j\right)^2}\sum_i\left(\sum_{\{j\in N|j\neq i\}}w_j^2\sigma^2_{Xj}\right)y_i = \frac{1}{\left(\sum_{j\in N}y_j\right)^2}\sum_i y_i\sigma^2_{\eta_i} \tag{65}$$

To see this, note that:

$$\sum_{i\in N}\sum_{j=1}^n y_j w_i^2\sigma^2_{Xi} = \sum_{j=1}^n\sum_{i\in N}w_i^2\sigma^2_{Xi}y_j$$

And:

$$\sum_{i\in N}\sum_{j=1}^n y_j w_i^2\sigma^2_{Xi} - \sum_{i\in N}\left(\sum_{\{j\in N|j\neq i\}}y_j\right)w_i^2\sigma^2_{Xi} = \sum_i y_i w_i^2\sigma^2_{Xi}$$

$$\sum_{j=1}^n\sum_{i\in N}w_i^2\sigma^2_{Xi}y_j - \sum_{j=1}^n\sum_{i\neq j}\left(w_i^2\sigma^2_{Xi}\right)y_j = \sum_i y_i w_i^2\sigma^2_{Xi}$$

Thus, using (65), (64) can be written as:

$$= \frac{1}{\alpha} \sum_i \left[\frac{y_i}{\left(\sum_{j \in N} y_j\right)^2} \sigma_{\eta_i}^2 + \frac{1}{\sum_{\{j \in N | j \neq i\}} y_j} \left(\frac{y_i}{\sum_{j \in N} y_j} \right)^2 \sigma_{\eta_i}^2 \right]$$

This further simplifies to:

$$= \frac{1}{\alpha} \sum_i \left[\frac{y_i}{\left(\sum_{j \in N} y_j\right) \sum_{\{j \in N | j \neq i\}} y_j} \right] \sigma_{\eta_i}^2 \tag{66}$$

Hence, $\frac{c_i}{2}\mathrm{Var}[q_i]$ is equal to a constant plus (66).

Calculating $\frac{R_i y_i}{d_i + y_i}$: Now, the expected slope subsidy payment is:

$$\sum_i R_i \frac{y_i}{d_i + y_i}$$

From (18) of proposition 2, we have:

$$R_i = \left[\frac{y_i}{\sum_{\{j \in N | j \neq i\}} y_j} \left(\frac{1}{\kappa_i w_i} \right) - \frac{1}{\sum_{\{j \in N | j \neq i\}} \alpha y_j} \right] \sum_{\{j \in N | j \neq i\}} w_j \sigma_{Xj}^2$$

Thus,

$$\sum_i R_i \frac{y_i}{d_i + y_i} = \sum_i \left[\frac{\alpha y_i}{\sum_{\{j \in N | j \neq i\}} \alpha y_j} \left(\frac{1}{\kappa_i w_i} \right) - \frac{1}{\sum_{\{j \in N | j \neq i\}} \alpha y_j} \right] \left[\sum_{\{j \in N | j \neq i\}} w_j \sigma_{Xj}^2 \right] \left[\frac{\alpha y_i}{\sum_{j=1}^n \alpha y_j} \right] \tag{67}$$

Expression (67) does not depend on β, and the only part which depends on α is:

$$\frac{1}{\alpha} \sum_i \frac{y_i}{\sum y_i} \left[-\frac{1}{\sum_{\{j \in N | j \neq i\}} y_j} \right] \sigma_{\eta_i}^2 \tag{68}$$

(68) is exactly the negative of (66). Thus, we have shown that the sum

$$\sum_{i \in N} \left(\frac{c_i}{2} \mathrm{Var}[q_i] + \frac{R_i y_i}{d_i + y_i} \right)$$

does not depend on α and β.

Calculating $\sum_{i \in N} \left(\frac{c_i}{2} \mathbb{E}[q_i]^2 + \tau_i \mathbb{E}[q_i] \right)$

Calculating $\frac{c_i}{2}\mathbb{E}[q_i]^2$: From (60) above, we have

$$-c_i = \frac{1}{\kappa_i}(\frac{1}{w_i} - 1) + \frac{y_i}{\sum_{\{j\in N|j\neq i\}} y_j}[\frac{1}{\kappa_i w_i}] - \frac{2}{\alpha \sum_{\{j\in N|j\neq i\}} y_j}$$

which does not depend on β. Thus, the piece of $\frac{c_i}{2}\mathbb{E}[q_i]^2$ which depends on α can be written as:

$$\frac{1}{\alpha}\sum_{i\in N}\frac{1}{\sum_{\{j\in N|j\neq i\}} y_j}\mathbb{E}[q_i]^2. \tag{69}$$

Calculating $\tau_i\mathbb{E}[q_i]$: From (19) of proposition 2, we have:

$$\tau_i = \frac{a_i + \beta\frac{y_i}{\sum_j y_j}}{w_i\kappa_i} + \frac{\mu_{\eta_i} R_i}{\sigma^2_{\eta_i}}$$

Substituting for R_i, $\sigma^2_{\eta_i}$, and μ_{η_i}, we have:

$$\begin{aligned}\tau_i = &\frac{a_i + \beta\frac{y_i}{\sum_j y_j}}{w_i\kappa_i} + \\ &\left(\sum_{\{j\in N|j\neq i\}} -\left(a_j + \beta\frac{y_i}{\sum_j y_j}\right) + w_j\mu_{Xj}\right)\left[\frac{y_i}{\sum_{\{j\in N|j\neq i\}} y_j}\left(\frac{1}{\kappa_i w_i}\right) - \frac{1}{\sum_{\{j\in N|j\neq i\}} \alpha y_j}\right]\end{aligned} \tag{70}$$

Now, we can ignore all components of this that do not depend on β or α. This leaves us with:

$$\begin{aligned}&\frac{\beta\frac{y_i}{\sum_j y_j}}{w_i\kappa_i} + \left(\sum_{\{j\in N|j\neq i\}} -a_j + w_j\mu_{Xj}\right)\left(-\frac{1}{\sum_{\{j\in N|j\neq i\}} \alpha y_j}\right) - \\ &\qquad \beta\frac{\sum_{\{j\in N|j\neq i\}} y_i}{\sum_j y_j}\left[\frac{y_i}{\sum_{\{j\in N|j\neq i\}} y_j}\left(\frac{1}{\kappa_i w_i}\right) - \frac{1}{\sum_{\{j\in N|j\neq i\}} \alpha y_j}\right]\end{aligned} \tag{71}$$

This simplifies to:

$$\frac{\beta}{w_i\kappa_i}\frac{1}{\alpha\sum_j y_j} + \left(\sum_{\{j\in N|j\neq i\}} -a_j + w_j\mu_{Xj}\right)\left(-\frac{1}{\sum_{\{j\in N|j\neq i\}} \alpha y_j}\right)$$

Now the term

$$\frac{\beta}{w_i\kappa_i}\frac{1}{\alpha\sum_j y_j}$$

does not vary across agents; thus,

$$\sum_i \left(\frac{\beta}{w_i\kappa_i}\frac{1}{\alpha\sum_j y_j}\right)\mathbb{E}[q_i] = \left(\frac{\beta}{w_i\kappa_i}\frac{1}{\alpha\sum_j y_j}\right)\sum_i \mathbb{E}[q_i] = 0$$

since total trade quantities q_i always sum to 0 across agents. Thus, we are left with the term:

$$\frac{1}{\alpha}\frac{\sum_{\{j\in N|j\neq i\}} a_j - w_j\mu_{Xj}}{\sum_{\{j\in N|j\neq i\}} \alpha y_j}$$

The component of expected cost summed across agents which depends on α and β is thus:

$$\frac{1}{\alpha}\sum_i \frac{\sum_{\{j\in N|j\neq i\}} a_j - w_j\mu_{Xj}}{\sum_{\{j\in N|j\neq i\}} \alpha y_j} E\left[q_i\right] \tag{72}$$

Now, note that from the definition of residual supply in (7), we have:

$$p = \frac{q_i - \eta_i}{d_i} = \frac{q_i - \left[\sum_{\{j\in N|j\neq i\}} -a_j + w_j X_j\right]}{\sum_{\{j\in N|j\neq i\}} y_j}$$

Hence,

$$\frac{\sum_{\{j\in N|j\neq i\}} a_j + w_j X_j}{\sum_{\{j\in N|j\neq i\}} y_j} = p - \frac{q_i}{\sum_{\{j\in N|j\neq i\}} y_j}$$

Taking expectations,

$$\frac{\sum_{\{j\in N|j\neq i\}} a_j + w_j\mu_{Xj}}{\sum_{\{j\in N|j\neq i\}} y_j} = E\left[p\right] - \frac{E\left[q_i\right]}{\sum_{\{j\in N|j\neq i\}} y_j}$$

Thus, (72) is equal to

$$\frac{1}{\alpha}\sum_{i\in N}(\mathbb{E}[p] - \frac{\mathbb{E}[q_i]}{\sum_{\{j\in N|j\neq i\}} y_j})\mathbb{E}[q_i] \tag{73}$$

Now,

$$\sum_{i\in N}\mathbb{E}[p]\mathbb{E}[q_i] = \mathbb{E}[p]\sum_{i\in N}\mathbb{E}[q_i] = 0$$

so (73) simplifies further to:

$$= \frac{1}{\alpha}\sum_{i\in N} - \frac{\mathbb{E}[q_i]^2}{\sum_{\{j\in N|j\neq i\}} y_j}.$$

This is exactly the negative of (69); thus, the sum

$$\sum_{i\in N}(\frac{c_i}{2}\mathbb{E}[q_i]^2+\tau_i\mathbb{E}[q_i])$$

does not depend on α or β. This proves that all components of expected revenue, (59), are independent of α and β, proving lemma 8.

□

A.7 Proof of Proposition 2

Proof. Fix $\{(a_i, y_i, w_i)\}$ such that $w_i > 0$ and $y_i > 0$ for each $i \in N$. We will demonstrate that the subsidy-tax scheme, $\{(R_i, c_i, \tau_i)\}$, given in the statement of the proposition implements $\{(a_i, y_i, w_i)\}$ by demonstrating that conditions in Proposition 1 are satisfied. Using equation 15, we compute

$$w_i = \frac{\frac{1}{\kappa_i}}{\frac{1}{\kappa_i}+\frac{1}{d_i}-c_i-\frac{R_i}{\sigma^2_{\eta_i}}}.$$

Rearranging, we have

$$\frac{1}{\kappa_i}+\frac{1}{d_i}-c_i-\frac{R_i}{\sigma^2_{\eta_i}}=\frac{1}{\kappa_i w_i}$$

and rearranging again we obtain,

$$-c_i-\frac{R_i}{\sigma^2_{\eta_i}}=\frac{1}{\kappa_i}(\frac{1}{w_i}-1)-\frac{1}{\sum_{\{j\in N|j\neq i\}}y_j}. \tag{74}$$

Next, using equation 16, we have

$$y_i^* = \frac{1+\frac{R_i d_i}{\sigma^2_{\eta_i^*}}}{\frac{1}{\kappa_i}+\frac{1}{d_i}-c_i-\frac{R_i}{\sigma^2_{\eta_i^*}}}.$$

The following series of rearrangements give

$$y_i[\frac{1}{\kappa_i}+\frac{1}{d_i}-c_i-\frac{R_i}{\sigma^2_{\eta_i}}]=1+\frac{R_i d_i}{\sigma^2_{\eta_i}}$$

$$\implies y_i[\frac{1}{\kappa_i}+\frac{1}{\sum_{\{j\in N|j\neq i\}}y_j}+\frac{1}{\kappa_i}(\frac{1}{w_i}-1)-\frac{1}{\sum_{\{j\in N|j\neq i\}}y_j}]=1+\frac{R_i d_i}{\sigma^2_{\eta_i}}$$

$$\implies R_i = \left[\frac{y_i}{\sum_{\{j\in N|j\neq i\}} y_j} [\frac{1}{\kappa_i} + \frac{1}{\kappa_i}(\frac{1}{w_i} - 1)] - \frac{1}{\sum_{\{j\in N|j\neq i\}} y_j} \right] \sigma^2_{\eta_i}. \tag{75}$$

Substituting into (74), we obtain

$$-c_i = \frac{1}{\kappa_i}(\frac{1}{w_i} - 1) + \frac{y_i}{\sum_{\{j\in N|j\neq i\}} y_j} [\frac{1}{\kappa_i} + \frac{1}{\kappa_i}(\frac{1}{w_i} - 1)] - \frac{2}{\sum_{\{j\in N|j\neq i\}} y_j} \tag{76}$$

Simplifying (75) and (76), we get (17) and (18).

Solving (14) for τ_i, we get:

$$\tau_i = a_i \left(\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}} \right) + \frac{\mu_{\eta_i} R_i}{\sigma^2_{\eta_i}}$$

Now, plugging in for R_i and c_i using (17) and (18), we have:

$$c_i + \frac{R_i}{\sigma^2_{\eta_i}}$$

$$= \frac{1}{\kappa_i}(1 - \frac{1}{w_i}) - \frac{y_i}{\sum_{\{j\in N|j\neq i\}} y_j} \left(\frac{1}{\kappa_i w_i} \right) + \frac{2}{\sum_{\{j\in N|j\neq i\}} y_j} + \left[\frac{y_i}{\sum_{\{j\in N|j\neq i\}} y_j} \left(\frac{1}{\kappa_i w_i} \right) - \frac{1}{\sum_{\{j\in N|j\neq i\}} y_j} \right]$$

$$= \frac{1}{\kappa_i} + \frac{1}{d_i} - \frac{1}{w_i \kappa_i}$$

Hence,

$$\frac{1}{\kappa_i} + \frac{1}{d_i} - c_i - \frac{R_i}{\sigma^2_{\eta_i}} = \frac{1}{w_i \kappa_i}$$

thus, we can write τ_i as:

$$\tau_i = \frac{a_i}{w_i \kappa_i} + \frac{\mu_{\eta_i} R_i}{\sigma^2_{\eta_i}}$$

proving (19).

Finally, we check that

$$\frac{1}{2\kappa_i} - \frac{c_i}{2} + \frac{1}{d_i} > 0$$

holds for all $i \in N$. Substituting in the expression for c_i given in (76) yields

$$\frac{1}{2\kappa_i} + \frac{1}{2\kappa_i}(\frac{1}{w_i} - 1) + \frac{y_i}{\sum_{\{j\in N|j\neq i\}} y_j} [\frac{1}{2\kappa_i} + \frac{1}{2\kappa_i}(\frac{1}{w_i} - 1)] - \frac{1}{\sum_{\{j\in N|j\neq i\}} \alpha y_j} + \frac{1}{\alpha \sum_{\{j\in N|j\neq i\}} y_j} > 0$$

which is equivalent to the condition that.

$$\frac{1}{2\kappa_i}\frac{1}{w_i}(1+\frac{y_i}{\sum_{\{j\in N|j\neq i\}} y_j})>0$$

The above is satisfied if

$$\frac{1}{2\kappa_i w_i}\frac{\sum_{j\in N} y_j}{\sum_{\{j\in N|j\neq i\}} y_j}>0.$$

which is the case since for each $i \in N$ it was assumed that $w_i > 0$ and $y_i > 0$ for each $i \in N$.

□

A.8 Proof of Proposition 3

Proof. Fix an arbitrary direct mechanism with allocation rule $\{q_i\}$ and transfer rule $\{t_i\}$ such that each q_i and t_i is differentiable. In equilibrium, truthful reporting must be incentive compatible which implies that

$$X_i = argmax_{\tilde{X}_i} - \mathbb{E}[\frac{1}{2\kappa_i}(X_i + q_i(\tilde{X}_i, X_{-i}))^2 - t_i(\tilde{X}_i, X_{-i})|X_i].$$

Taking a first order condition with respect to $\tilde{X}_i$ and evaluating at X_i we have

$$-\mathbb{E}[\frac{1}{2\kappa_i}(X_i + q_i(X_i, X_{-i}))\frac{\partial}{\partial \tilde{X}_i}q_i(X_i, X_{-i})|X_i] = \mathbb{E}[\frac{\partial}{\partial \tilde{X}_i}t_i(X_i, X_{-i})]$$

which is a necessary condition for truthful reporting to be optimal.[19] By the fundamental theorem of calculus, we have

$$\mathbb{E}[t_i(X_i, X_{-i}) - t_i(0, X_{-i})|X_i] = \int_0^{X_i} -\mathbb{E}[\frac{1}{2\kappa_i}(x + \alpha_i(x, X_{-i}))\alpha_i'(x, X_{-i})|X_i]dx$$

which holds for each $X_i \in \mathbb{R}$. Proposition 3 is an obvious implication of the above equation.

□

[19] Need to assume some minor technical conditions in order to bring the derivative inside the expectation

B Proofs and Supplementary Material for Sections 5 and 6

B.1 Proof of Lemma 3

Proof. The sum of strategic traders' holding costs is:

$$\sum_{i=1}^{N} \frac{1}{2\kappa_i}(X_i + q_i)^2.$$

We form the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{2\kappa_i}\left(X_i + q_i\right)^2 - \lambda\left[\sum_{i=1}^{N} q_i - \epsilon\right] \tag{77}$$

Differentiating (77) with respect to q_i, we obtain:

$$\frac{1}{\kappa_i}\left(X_i + q_i\right) - \lambda = 0$$

which gives

$$q_i = \frac{\frac{1}{\kappa_i}X_i - \lambda}{-\frac{1}{\kappa_i}} = -X_i + \lambda\kappa_i.$$

Now we substitute into the market clearing condition to compute λ.

$$\sum_{i=1}^{N} -X_i + \lambda\kappa_i = \epsilon$$

which gives

$$\lambda = \frac{\sum_{i=1}^{N} X_i + \epsilon}{\sum_{i=1}^{N} \kappa_i}.$$

Thus,

$$X_i + q_i = \frac{\kappa_i\left(\sum_{j=1}^{N} X_j + \epsilon\right)}{\sum_{j=1}^{N} \kappa_j}$$

as desired.

Then, using (5) of proposition 1, a linear equilibrium induces efficient allocations if and only if:

$$\frac{\kappa_i\left(\sum_{j=1}^{N} X_j + \epsilon\right)}{\sum_{j=1}^{N} \kappa_j} = (1 - w_i)\, X_i + a_i - y_i \frac{\left(\sum_{j\in N} a_j - w_j X_j\right)}{\sum_{j\in N} y_i}$$

One bid profile which implements the efficient allocation is for all agents to bid honestly, setting:

$$a_i = 0, w_i = 1, y_i = \kappa_i \tag{78}$$

This collection of demand schedules clearly satisfies (B.1). Applying lemma 7, the set of all bid profiles which implements the efficient allocation is thus described by:

$$a_i = \beta\kappa_i, w_i = 1, y_i = \alpha\kappa_i$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha > 0$.

□

B.2 Proof of Lemma 4

Proof. Lemma 3 characterizes the set of bid profiles that implements the fully efficient outcome. Plugging in

$$a_i = \beta\kappa_i, w_i = 1, y_i = \alpha\kappa_i$$

to the implementing subsidies formulas (17) and (18) of proposition 2, we get:

$$c_i = \frac{1}{\sum_{\{j\in N|j\neq i\}} \kappa_j}\left(\frac{2-\alpha}{\alpha}\right)$$

$$R_i = \frac{\sum_{\{j\in N|j\neq i\}} \sigma^2_{X,j}}{\sum_{\{j\in N|j\neq i\}} \kappa_j}\left(\frac{\alpha-1}{\alpha}\right)$$

This proves (20) and (21). For τ_i, plugging in for d_i, μ_{η_i} using lemma 2, we have:

$$\tau_i = \beta\kappa_i\left(\frac{1}{\kappa_i}\right) + \frac{1}{\sum_{\{j\in N|j\neq i\}} \kappa_j}\left(\frac{\alpha-1}{\alpha}\right)\left(\sum_{\{j\in N|j\neq i\}} -\beta\kappa_j + \mu_{Xj}\right)$$

This simplifies to:

$$\tau_i = \frac{\beta}{\alpha} + \left(\frac{\alpha-1}{\alpha}\right)\frac{\sum_{\{j\in N|j\neq i\}} \mu_{Xj}}{\sum_{\{j\in N|j\neq i\}} \kappa_j}$$

Replacing $\frac{\beta}{\alpha}$ with K, we get (eq:efftausubs).

□

B.3 Proof of Proposition 4

Proof. Full efficiency is achieved by any subsidy scheme satisfying the conditions of proposition 4. We will choose a particular set of conditions, setting $\beta = 0$ and $\alpha = 2$, implying that the implementing subsidies are:

$$c_i = 0$$
$$R_i = \frac{1}{2\sum_{j\neq i}\kappa_j(\sum_{j\neq i}\sigma^2_{X,j} + \sigma^2_\epsilon)}$$
$$\tau_i = \frac{\sum_{j\neq i}\mu_j}{2\sum_{j\neq i}\kappa_j}$$

This choice is convenient because it sets $c_i = 0$. Notice that when the linear dependency condition is satisfied τ is a constant. We will show that there do not exist participation fees coupled with the above subsidy scheme which satisfy individual rationality. By revenue equivalence, no other subsidy scheme which implements the efficient allocation can be combined with participation fees which satisfy individual rationality, proving Proposition 4.

Notice that R_i is positive for each i and that we can charge participation fees equal to $R_i\frac{\kappa_i}{\sum_{j\in N}\kappa_j}$ for each i to cover the cost of the slope subsidy. We now show that it is possible to charge additional participation fees while satisfying IR to cover the cost of the linear subsidy if and only if the linear dependency condition in the statement of the proposition holds. This is equivalent to showing that the inequality

$$\sum_{i\in N} min_{X_i\in\mathbb{R}}\ \mathbb{E}[-\frac{1}{2\kappa_i}(X_i+q_i)^2 - pq_i + \tau_i q_i|X_i] - \frac{1}{2\kappa_i}X_i^2 \leq \mathbb{E}[\sum_{i\in N}\tau_i q_i] \tag{79}$$

holds and that equality is achieved if and only if the linear dependency condition is satisfied. This implies that, if the linear dependency condition is satisfied, efficient trade is possible and leaves the market operator with 0 expected revenue; if it is not satisfied, budget-balanced and fully efficient trade is impossible.

The left hand side is the maximum additional agent-specific participation fees that can be charged while satisfying IR summed across agents. The right hand side is the cost of the linear subsidy. Towards this end, we first show that the right hand side is strictly positive when the linear dependency condition does not hold and is zero otherwise. We then show that the left hand side is zero. We have

$$\mathbb{E}[\sum_{i\in N}\tau_i q_i] = \sum_{i\in N}\frac{\sum_{j\neq i}\mu_j}{2\sum_{j\neq i}\kappa_j}[-\mu_i + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}(\sum_{j\in N}\mu_j)]$$

$$= \sum_{i\in N} \frac{\sum_{j\neq i}\mu_j}{2}[-\frac{\mu_i}{\sum_{j\neq i}\kappa_j} + (\frac{1}{\sum_{j\neq i}\kappa_j} - \frac{1}{\sum_{j\in N}\kappa_j})\sum_{j\in N}\mu_j]$$

$$= \sum_{i\in N} \frac{\sum_{j\neq i}\mu_j}{2}[\frac{\sum_{j\neq i}\mu_j}{\sum_{j\neq i}\kappa_j} - \frac{1}{\sum_{j\in N}\kappa_j}\sum_{j\in N}\mu_j]$$

$$= \frac{1}{2}\sum_{i\in N}\frac{(\sum_{j\neq i}\mu_j)^2}{\sum_{j\neq i}\kappa_j} - \frac{1}{2\sum_{j\in N}\kappa_j}(N-1)^2(\sum_{j\in N}\mu_j)^2$$

$$= \frac{1}{2}[\sum_{i\in N}\frac{g_i^2}{z_i^2} - \frac{(\sum_{i\in N} g_i)^2}{\sum_{i\in N} z_i^2}]$$

where $g_i = \sum_{j\neq i}\mu_j$ and $z_i = \sqrt{\sum_{j\neq i}\kappa_j}$. Thus, by the Cauchy-Schwarz inequality,

$$\mathbb{E}[\sum_{i\in N}\tau_i q_i] \geq 0$$

with equality holding if and only if the vector of z_i^2's and the vector of g_i's are linearly dependent or (equivalently $[\mu_1, ..., \mu_N]$ and $[\kappa_1, ..., \kappa_N]$ being linearly dependent).

We now prove that the left hand side of (79) is zero by showing that

$$min_{X_i\in\mathbb{R}}\ \mathbb{E}[-\frac{1}{2\kappa_i}(X_i+q_i)^2 - pq_i + \tau_i q_i|X_i] - \frac{1}{2\kappa_i}X_i^2 = 0 \tag{80}$$

for each $i \in N$. We have

$$\mathbb{E}[-\frac{1}{2\kappa_i}(X_i+q_i)^2 - pq_i + \tau_i q_i|X_i] - \frac{1}{2\kappa_i}X_i^2$$

$$\Leftrightarrow$$

$$\mathbb{E}[-\frac{1}{2\kappa_i}(X_i^2 + 2X_iq_i + q_i^2) + \frac{X_i+q_i}{2\kappa_i}q_i) + \tau_i q_i|X_i] - \frac{1}{2\kappa_i}X_i^2$$

$$\Leftrightarrow$$

$$(-\frac{X_i}{2\kappa_i} + \tau_i)\mathbb{E}[q_i|X_i]$$

$$\Leftrightarrow$$

$$(-\frac{X_i}{2\kappa_i} + \tau_i)(-\frac{\sum_{j\neq i}\kappa_j}{\sum_{j\in N}\kappa_j}X_i + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j)$$

$$\Leftrightarrow$$

$$2\kappa_i \frac{\sum_{j \neq i} \kappa_j}{\sum_{j \in N} \kappa_j} \left(-\frac{X_i}{2\kappa_i} + \tau_i\right)^2$$

Thus, $X_i = 2\kappa_i \tau_i$ makes the objective in (80) equal to 0 and is the arg-min proving (80).

$\square$

B.4 Proof of Lemma 5

We prove (23) and (24) separately.

Ex-post budget balance: The subsidy scheme in (23) corresponds to the efficient subsidies of proposition (4), with $\alpha = 2$. Thus, they implement the fully efficient allocation, and by proposition 4, they are budget-balanced ex ante. Now, if $[\kappa_1, ..., \kappa_N]$ and $[\mu_1, ..., \mu_N]$ are linearly dependent, then

$$\tau_i = K + \frac{\sum_{\{j \in N | j \neq i\}} \mu_{Xj}}{2 \sum_{\{j \in N | j \neq i\}} \kappa_j}$$

is some constant τ. The total amount paid out by the market platform in linear subsidies is thus 0, since

$$\sum_i \tau_i q_i = \tau \sum_i q_i = 0$$

Then, by the result of lemma 3, in any efficient equilibrium, we have

$$y_i = \alpha \kappa_i$$

this implies that the amount paid out to agent i in slope subsidies is:

$$\frac{y_i}{\sum_j y_j} R_i = \frac{\kappa_i}{\sum_j \kappa_j} R_i = \frac{\kappa_i \left(\sum_{\{j \in N | j \neq i\}} \sigma_{X,j}^2\right)}{2 \left(\sum_{\{j \in N | j \neq i\}} \kappa_j\right) \left(\sum_{j=1}^n \kappa_j\right)}$$

If we charge agents this much in fixed entry fees, the market platform exactly breaks even, for any realization of $X_1 \dots X_N$.

Ex-post incentive compatibility: To prove (24), we prove a more general statement: any subsidy scheme for which $R_i = 0$ induces agents to bid ex-post optimally.

Suppose $R_i = 0$. From (42) in appendix A.3, agents' optimization problem can be written:

$$max_{\tilde{q} \in \mathcal{M}} \int_{-\infty}^{\infty} [\tau_i \tilde{q}_i - \frac{1}{2\kappa_i}(X_i + \tilde{q}_i)^2 - \left(\frac{\tilde{q}_i + \eta_i}{d_i}\right) \tilde{q}_i - \frac{c_i}{2} \tilde{q}_i^2] \phi(\eta_i) dr_i$$

Since there is no $\tilde{q}_i'$ term, this problem can be solved pointwise in η_i. Differentiate the integrand with respect to $\tilde{q}_i$, to get:

$$\tau_i - \frac{1}{\kappa_i}\left(X_i + \tilde{q}_i\right) - \left(\frac{\tilde{q}_i + \eta_i}{d_i}\right) - \frac{\tilde{q}_i}{d_i} - c_i \tilde{q}_i = 0$$

Now, since

$$\frac{\tilde{q}_i + \eta_i}{d_i} = p$$

this is:

$$\tau_i - \frac{1}{\kappa_i}\left(X_i + \tilde{q}_i\right) - p - \frac{\tilde{q}_i}{d_i} - c_i \tilde{q}_i = 0$$

Solve for q, to get:

$$q_i\left(c_i + \frac{1}{d_i} + \frac{1}{\kappa_i}\right) = \tau_i - \frac{X_i}{\kappa_i} - p$$

Hence, the demand schedule:

$$q_i\left(X_i, p\right) = \frac{\tau_i - \frac{X_i}{\kappa_i} - p}{c_i + \frac{1}{d_i} + \frac{1}{\kappa_i}}$$

is ex-post optimal, in the sense that it is optimal regardless of the realization of η_i.

B.5 Proof of proposition 6

Total expected revenue of the market platform is equal to the maximal entry fees charged to the agent, less the total revenue paid out in subsidies.

Under symmetry, equilibria are characterized simply by w and y; we set $a = 0$ by setting $\tau = 0$. Now, by strong revenue equivalence, any way of implementing a given w produces the same revenue. We will implement different choices of w by varying c, as this is analytically simpler.

We will use expressions from appendix D.4 below. From lemma 11, price and quantity are

$$p\left(X, \eta\right) = \frac{-wX - \eta}{d + y}, \; q\left(X, \eta\right) = \frac{-dwX + y\eta}{d + y} \tag{81}$$

Equilibrium under c subsidies in the symmetric case: From proposition 1, equilibrium conditions are:

$$w = \frac{d}{\kappa + d - \kappa d\left(c + \frac{R}{\sigma_\eta^2}\right)}, \; y = \frac{\kappa d\left(1 + \frac{Rd}{\sigma_\eta^2}\right)}{\kappa + d - \kappa d\left(c + \frac{R}{\sigma_\eta^2}\right)}$$

Setting $R = 0$, and using $d = (n-1)\,y$, we have:

$$y = \frac{\kappa\,(n-1)\,y}{\kappa + (n-1)\,y - \kappa\,(n-1)\,yc}$$

Solving this for y and w, we have:

$$y = \frac{\kappa\,(n-2)}{(1-c\kappa)\,(n-1)} \tag{82}$$

$$w = \frac{1}{1-c\kappa}\left(\frac{n-2}{n-1}\right) \tag{83}$$

In order for the second-order condition (13) to hold, we need:

$$\frac{1}{2\kappa} - \frac{c}{2} + \frac{1}{(n-1)\,\kappa} > 0$$

$$\frac{2n-1}{2n-2} > c\kappa \tag{84}$$

Entry fees: The entry fees that the platform operator can charge to each agent are pinned down by IR constraints: the worst-off type X of each agent must have nonnegative expected utility gains from the mechanism. The total expected utility gain of type X of an agent is:

$$E\left[U \mid X\right] = E\left[-\frac{1}{2\kappa}\left(2Xq + q^2\right) - pq + \tau q + \frac{c}{2}q^2 + R\frac{y}{d+y} \mid X\right] \tag{85}$$

In the symmetric case, with $\tau = 0$, the agent with lowest utility will always be the agent with $X = 0$. The expected utility of this agent is:

$$E\left[U \mid X = 0\right] = E\left[-\frac{1}{2\kappa}\left(q^2\right) - pq + \frac{c}{2}q^2 + R\frac{y}{d+y} \mid X\right]$$

Substituting for q using (81), we have:

$$E\left[-\frac{1}{2\kappa}\left(\frac{y}{d+y}\eta\right)^2 - \left(\frac{-\eta}{d+y}\right)\left(\frac{y\eta}{d+y}\right) + \frac{c}{2}\left(\frac{y}{d+y}\eta\right)^2 \mid X\right]$$

$$= \sigma_\eta^2\left[-\frac{1}{2\kappa}\left(\frac{y}{d+y}\right)^2 + \left(\frac{y}{(d+y)^2}\right) + \frac{c}{2}\left(\frac{y}{d+y}\right)^2\right] \tag{86}$$

Now, in any symmetric equilibrium, we have $d = (n-1)\,y$, so (86) simplifies substantially, to:

$$= \frac{\sigma_\eta^2}{n^2}\left[-\frac{1}{2\kappa}+\frac{1}{y}+\frac{c}{2}\right]$$

Substituting for σ_η^2, we have entry fees:

$$= \frac{n-1}{n^2}w^2\sigma_X^2\left[-\frac{1}{2\kappa}+\frac{1}{y}+\frac{c}{2}\right] \tag{87}$$

Total expenditures: The total expected expenditures on subsidies, for any one agent, is:

$$E\left[\frac{cq^2}{2}\right] = E\left[\frac{c}{2}\left(\frac{-d}{d+y}wX+\frac{y}{d+y}\eta\right)^2\right]$$

$$= \frac{c}{2}\left(\left(\frac{d}{d+y}\right)^2 w^2\sigma_X^2+\left(\frac{y}{d+y}\right)^2\sigma_\eta^2\right)$$

$$= \frac{c}{2}\left(\left(\frac{n-1}{n}\right)^2 w^2\sigma_X^2+\left(\frac{1}{n}\right)^2(n-1)\,w^2\sigma_X^2\right) \tag{88}$$

Total revenue: The total revenue of the platform operator is equal to the sum difference between entry fees, (87), and the total amount paid out in subsidies, (88). This difference is:

$$Totrev = \frac{n-1}{n^2}w^2\sigma_X^2\left[-\frac{1}{2\kappa}+\frac{1}{y}-\frac{c}{2}(n-1)\right]$$

Now, we substitute for w^2 and y in terms of c, using (82) and (83), to get:

$$Totrev = \frac{(n-2)^2}{n^2(n-1)}\sigma_X^2\left(\frac{1}{1-c\kappa}\right)^2\left[-\frac{1}{2\kappa}+\frac{1-c\kappa}{\kappa}\frac{n-1}{n-2}-\frac{c}{2}(n-1)\right] \tag{89}$$

Differentiating with respect to c, we have:

$$\frac{dTotrev}{dc} = \frac{(n-2)}{2n(n-1)}\sigma_X^2\left[\frac{3-n-ck(n-1)}{(1-ck)^3}\right] \tag{90}$$

Setting (90) to 0 and solving for c, we get:

$$c^* = \frac{3-n}{\kappa(n-1)} \tag{91}$$

This always satisfies (84). Moreover, (90) is negative for $c > c^*$, and positive for $c < c^*$.

Thus, (91) describes the revenue-maximizing choice of c. Plugging this into (83), we get:

$$w = \frac{1}{2}$$

To calculate optimal expected revenue per agent for the platform operator, we plug the optimized value c^* into the revenue expression (89), to get:

$$\frac{(n-1)\,\sigma_X^2}{8\kappa n}$$

This proves (26). To find the set of all subsidy schemes which can implement the revenue-maximizing allocation, we solve:

$$w = \frac{(n-1)\,y}{n\kappa - (n-1)\,y - \kappa\,(n-1)\,y\left(\frac{R}{(n-1)w^2\sigma_X^2}\right)}$$

$$y = \frac{\kappa\,(n-1)\,y\left(1 + \frac{R(n-1)y}{(n-1)w^2\sigma_X^2}\right)}{\kappa + (n-1)\,y - \kappa\,(n-1)\,y\left(c + \frac{R}{(n-1)w^2\sigma_X^2}\right)}$$

with $w = 0.5$; this proves (25). Note that τ is arbitrary, because linear subsidies only affect the constant term a_i, shifting it uniformly upwards or downwards, which does not affect allocations, and thus does not affect the platform's revenue.

Welfare: To calculate welfare, note that agents' trade quantity is:

$$q\,(X,\eta) = \frac{-dwX + y\eta}{d+y} = -\frac{n-1}{n}wX + \frac{1}{n}\eta$$

and we have:

$$\sigma_\eta^2 = w^2\sigma_X^2 = \frac{(n-1)\,\sigma_X^2}{4}$$

Expected total holding costs for each agent are:

$$E\left[-\frac{1}{2\kappa}\left(2Xq + q^2\right)\right]$$

Expanding, we have:

$$E\left[-\frac{1}{2\kappa}\left(2X\left(-\frac{n-1}{n}wX + \frac{1}{n}\eta\right) + \left(-\frac{n-1}{n}wX + \frac{1}{n}\eta\right)^2\right)\right]$$

Taking the expectation and setting $w = 1/2$, this becomes:

$$\frac{1}{2\kappa}\left[\frac{3}{4}\frac{n-1}{n}\sigma_X^2\right]$$

This proves (27). The maximal welfare gain can be found by setting $w = 1$ and repeating the calculations above; we get:

$$\frac{1}{2\kappa}\frac{n-1}{n}\sigma_X^2$$

Hence, the expected welfare loss per agent in the revenue-maximizing mechanism, compared to the social optimum, is:

$$\frac{1}{2\kappa}\left[\frac{1}{4}\frac{n-1}{n}\sigma_X^2\right]$$

proving (28).

Welfare loss compared to equilibrium without subsidies: (29) follows from (27), and the following lemma.

Lemma 9. *In equilibrium without subsidies, expected gains from trade per agent are:*

$$\frac{(n-2)}{2\kappa(n-1)}\sigma_X^2 \tag{92}$$

The expected welfare loss per agent, relative to the efficient allocation, is:

$$\frac{\sigma_X^2}{2\kappa(n-1)n} \tag{93}$$

Proof. In the absence of subsidies, the equilibrium conditions in proposition 1 reduce to:

$$a = 0,\ w = \frac{d}{\kappa + d},\ y = \frac{\kappa d}{\kappa + d}$$

Given that $d_i = (n-1)\kappa_i$, equilibrium bids are thus:

$$a = 0,\ w = \frac{n-2}{n-1},\ y = \frac{n-2}{n-1}\kappa$$

expected holding costs for each agent are:

$$E\left[-\frac{1}{2\kappa}\left(2X\left(-\frac{n-1}{n}wX + \frac{1}{n}\eta\right) + \left(-\frac{n-1}{n}wX + \frac{1}{n}\eta\right)^2\right)\right]$$

Taking expectations, with $w = \frac{n-2}{n-1}$, we have:

$$\frac{(n-2)}{2\kappa(n-1)}\sigma_X^2$$

The welfare loss per agent, relative to the efficient allocation, is thus:

$$\frac{\sigma_X^2}{2\kappa(n-1)n}$$

$\square$

B.6 An upper-bound for second best efficiency loss

For general parameter values, when efficiency is not obtainable, finding the optimal subsidy scheme subject to expected budget balance is analytically intractable. Nonetheless, we are able to provide a lower bound on the efficiency loss which becomes tight as risk capacities become proportional to the means of traders' endowments. The lower bound is obtained constructively—for each $\gamma \in \mathbb{R}$, we solve for the subsidy scheme such that each trader i purchases γq_i^* where q_i^* is the efficient trade quantity:

$$q_i^* = -X_i + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\in N} X_j.$$

We next solve for the γ such that allocative efficiency is highest (sum of traders' holding costs is lowest) subject to expected budget balance.

To start, we solve for a set of demand schedule coefficients such that trader i purchases γq_i^* units. By revenue equivalence, it suffices to consider implementing any choice of demand schedule coefficients in this set. It is straightforward to see that coefficients of the form

$$w_i = \gamma,$$

$$y_i = \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\Theta,$$

and

$$a_i = 0$$

where Θ is an arbitrary positive constant work. For simplicity, we set $\Theta = 1$. By an earlier

Proposition (fill in), the implementing subsidy scheme sets

$$c_i = -\frac{1}{\kappa_i\gamma_i} + \frac{1}{\kappa_i} - \frac{1}{\sum_{j\neq i}\kappa_j}\frac{1}{\gamma_i} + \frac{2\sum_{j\in N}\kappa_j}{\sum_{j\neq i}\kappa_j} = -\frac{\sum_{j\in N}\kappa_j}{\kappa_i\sum_{j\neq i}\kappa_j}\frac{1}{\gamma} + \frac{1}{\kappa_i} + \frac{2\sum_{j\in N}\kappa_j}{\sum_{j\neq i}\kappa_j} \tag{94}$$

and

$$\frac{R_i}{\sigma^2_{\eta_i}} = \frac{1}{\sum_{j\neq i}\kappa_j\gamma} - \frac{\sum_{j\in N}\kappa_j}{\sum_{j\neq i}\kappa_j}. \tag{95}$$

and

$$\tau_i = \frac{\sum_{j\neq i}\mu_j}{\sum_{j\neq i}\kappa_j}[1-\gamma\sum_{j\in N}\kappa_j]. \tag{96}$$

Given this subsidy scheme, we seek to compute the expected cost of operation. Since we can always charge each trader i the cost of the slope subsidies up front as participation fees, this entails computing the maximum chargeable participation fees net of slope-subsidy costs, the cost of the quadratic subsidies, and the cost of the linear subsidies.

To compute maximum participation fees net of the cost of the linear subsidies we solve

$$min_{X_i}\mathbb{E}[-\frac{1}{2\kappa_i}(X_i+q_i)^2 - pq_i + \frac{c}{2}q_i^2 + \tau_i q_i|X_i] + \frac{1}{2\kappa_i}X_i^2$$

$$\Leftrightarrow$$

$$min_{X_i}\mathbb{E}[-\frac{1}{2\kappa_i}(X_i^2+2X_iq_i+q_i^2) - pq_i + \frac{c}{2}q_i^2 + \tau_i q_i|X_i] + \frac{1}{2\kappa_i}X_i^2$$

Recall that

$$q_i = -\gamma X_i - \frac{\kappa_i}{\sum_{j\in N}\kappa_j}p$$

$$\Leftrightarrow$$

$$p = -\frac{q_i+\gamma X_i}{\frac{\kappa_i}{\sum_{j\in N}\kappa_j}}.$$

Then we have

$$min_{X_i}\mathbb{E}[-\frac{1}{2\kappa_i}(X_i^2+2X_iq_i+q_i^2) + \frac{\sum_{j\in N}\kappa_j}{\kappa_i}(\gamma X_i+q_i)q_i + \frac{c_i}{2}q_i^2 + \tau_i q_i|X_i] + \frac{1}{2\kappa_i}X_i^2$$

$$\Leftrightarrow$$

$$min_{X_i}\mathbb{E}[-\frac{1}{2\kappa_i}(2X_iq_i+q_i^2) + \frac{\sum_{j\in N}\kappa_j}{\kappa_i}(\gamma X_i+q_i)q_i + \frac{c_i}{2}q_i^2 + \tau_i q_i|X_i]$$

To compute the relevant moments in the objective, we recall that

$$\mathbb{E}[q_i|X_i] = \gamma[-\frac{\sum_{j\neq i}\kappa_j}{\sum_{j\in N}\kappa_j}X_i + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j].$$

and

$$\mathbb{E}[q_i^2|X_i] = \gamma^2(-\frac{\sum_{j\neq i}\kappa_j}{\sum_{j\in N}\kappa_j}X_i + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j)^2 + \gamma^2(\frac{\kappa_i}{\sum_{j\in N}\kappa_j})^2\sum_{j\neq i}\sigma_{j\epsilon}^2.$$

Substituting into the objective and taking a first order condition, we find that the minimizing X_i is

$$X_{min} = \frac{\kappa_i\sum_{j\neq i}\mu_j}{\sum_{j\neq i}\kappa_j}.$$

For this value of X_i

$$\mathbb{E}[q_i|X_i = X_{min}] = 0$$

and

$$\mathbb{E}[q_i^2|X_i = X_{min}] = \gamma^2(\frac{\kappa_i}{\sum_{j\in N}\kappa_j})^2\sum_{j\neq i}\sigma_{j\epsilon}^2.$$

Substituting these moments into the objective, we find that the minimum participation fee which can be charged is

$$\left[\frac{1}{\kappa_i} - \frac{1}{\kappa_i\sum_{j\neq i}\kappa_j}\frac{1}{\gamma} + \frac{1}{\sum_{j\neq i}\kappa_j}\right]\gamma^2\frac{\kappa_i^2}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\sigma_{j\epsilon}^2$$

$$\Leftrightarrow$$

$$\left[\frac{\sum_{j\in N}\kappa_j}{\kappa_i\sum_{j\neq i}\kappa_j} - \frac{1}{\kappa_i\sum_{j\neq i}\kappa_j}\frac{1}{\gamma}\right]\gamma^2\frac{\kappa_i^2}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\sigma_{j\epsilon}^2$$

$$\Leftrightarrow$$

$$\gamma^2\frac{\kappa_i}{\sum_{j\neq i}\kappa_j}\sum_{j\neq i}\sigma_{j\epsilon}^2 - \gamma\frac{\kappa_i}{\sum_{j\neq i}\kappa_j\sum_{j\in N}\kappa_j}\sum_{j\neq i}\sigma_{j\epsilon}^2.$$

We now compute the cost of the quadratic subsidy and linear subsidy. We have

$$\begin{aligned}&\mathbb{E}[\frac{c_i}{2}q_i^2]\\ &= [-\frac{\sum_{j\in N}\kappa_j}{2\kappa_i\sum_{j\neq i}\kappa_j}\frac{1}{\gamma}+\frac{1}{2\kappa_i}+\frac{\sum_{j\in N}\kappa_j}{\sum_{j\neq i}\kappa_j}][\gamma^2(-\frac{\sum_{j\neq i}\kappa_j}{\sum_{j\in N}\kappa_j}\mu_i+\frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j)^2+\gamma^2(\frac{\kappa_i}{\sum_{j\in N}\kappa_j})^2\sum_{j\neq i}\sigma_{j\epsilon}^2]\end{aligned}$$

and

$$\mathbb{E}[\tau_i q_i] = \frac{\sum_{j\neq i}\mu_j}{\sum_{j\neq i}\kappa_j}[1-\gamma\sum_{j\in N}\kappa_j]\gamma[-\frac{\sum_{j\neq i}\kappa_j}{\sum_{j\in N}\kappa_j}\mu_i + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j].$$

Thus the total cost of unit and quadratic subsidies is

$$\mathbb{E}[\frac{c_i}{2}q_i^2] + \mathbb{E}[\tau_i q_i] = A\gamma + B\gamma^2$$

where

$$A = [-\frac{\sum_{j\in N}\kappa_j}{2\kappa_i\sum_{j\neq i}\kappa_j}][(-\frac{\sum_{j\neq i}\kappa_j}{\sum_{j\in N}\kappa_j}\mu_i + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j)^2 + (\frac{\kappa_i}{\sum_{j\in N}\kappa_j})^2\sum_{j\neq i}\sigma_{j\epsilon}^2]$$

$$+\frac{\sum_{j\neq i}\mu_j}{\sum_{j\neq i}\kappa_j}[-\frac{\sum_{j\neq i}\kappa_j}{\sum_{j\in N}\kappa_j}\mu_i + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j]$$

and

$$B = -\frac{\sum_{j\neq i}\mu_j}{\sum_{j\neq i}\kappa_j}[-\sum_{j\neq i}\kappa_j\mu_i + \kappa_i\sum_{j\neq i}\mu_j]$$

$$+[\frac{1}{2\kappa_i} + \frac{\sum_{j\in N}\kappa_j}{\sum_{j\neq i}\kappa_j}][(-\frac{\sum_{j\neq i}\kappa_j}{\sum_{j\in N}\kappa_j}\mu_i + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j)^2 + (\frac{\kappa_i}{\sum_{j\in N}\kappa_j})^2\sum_{j\neq i}\sigma_{j\epsilon}^2]$$

Next, we observe that the γ which is most efficient must exactly budget the balance. This is because the expected sum of holding costs has a single trough as a function of γ—in fact is quadratic in γ:

Lemma 10. *The expected sum of holding cost as a function of γ is single troughed since it is a quadratic function of γ.*

Proof. The expected sum of holding costs is

$$\mathbb{E}[\sum_{i\in N}\kappa_i[(1-\gamma)X_i + \gamma\frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\in N}X_j]^2].$$

□

Thus if the most efficient γ did not satisfy budget balance, it could either be increased or decreased to improve efficiency while still balancing the budget which is a contradiction. The budget balance condition is

$$\sum_{i\in N}A_i + \gamma\sum_{i\in N}B_i = \gamma\sum_{i\in N}\frac{\kappa_i}{\sum_{j\neq i}\kappa_j}\sum_{j\neq i}\sigma_{j\epsilon}^2 - \sum_{i\in N}\frac{\kappa_i}{\sum_{j\neq i}\kappa_j\sum_{j\in N}\kappa_j}\sum_{j\neq i}\sigma_{j\epsilon}^2$$

Note to arrive at the above condition we divided γ from both sides so $\gamma = 0$ was also always a valid solution which is intuitively obvious.

Solving for the γ for which the above holds gives

$$\gamma^* = \frac{\sum_{i\in N} A_i + \sum_{i\in N} \frac{\kappa_i}{\sum_{j\neq i}\kappa_j \sum_{j\in N}\kappa_j} \sum_{j\neq i}\sigma^2_{j\epsilon}}{\sum_{i\in N} \frac{\kappa_i}{\sum_{j\neq i}\kappa_j}\sum_{j\neq i}\sigma^2_{j\epsilon} - \sum_{i\in N} B_i}.$$

Thus the most efficient γ is either γ^* or 0. We therefore have a bound on inefficiency given by substituting γ^* into the expected sum of holding costs.

C Proofs and Supplementary Material for Section 7

C.1 Proof of Proposition 7

Proof. By inspection, a linear equilibrium with $a_i = 0$, $y_i = \gamma_i\Theta$ and $w_i = 1$ achieves the stationary allocation $(\gamma_i)_i$. Thus, it suffices to compute the stationary subsidy scheme that implements this equilibrium. To do this we consider trader i's optimal demand submission problem and derive conditions on the stationary subsidy scheme such that there is no profitable one shot deviation from the equilibrium strategy of submitting

$$q_{it}(p_t) = -X_i - \gamma_i\Theta p_t$$

in each period.

We observe that a one shot deviation in period t will have no effects on the expected flow utility in period $t+1$ and onward. This is because trader i reverts to the equilibrium strategy in period $t+1$ so her post trade inventory in period $t+1$ will be $\gamma_i \sum_{j\in N} X_{j,t+1}$. Thus in period $t+2$

$$q_{i,t+2} = \gamma_i(\sum_{j\in N} X^j_{t+2} - \sum_{j\in N} X^j_{t+1}) = \gamma_i \sum_{j\in N} \epsilon_{j,t+2}$$

regardless of the deviation. Thus her expected transfers from the time $t+2$ linear subsidy is zero since q^i_{t+2} is in expectation zero and conditionally independent of τ_t. The expected transfers from the slope and quadratic subsidy are also unaffected by the deviation since q^i_{t+2} is unaffected.

In light of this it suffices to show there is no one shot deviation in period t which achieves a higher expected value of the sum of flow utilities in periods t and $t+1$ than under the equilibrium strategy. That is, it suffices to consider the objective

$$\mathbb{E}_t[\sum_{s=0}^{1} e^{-rs}(-\frac{1}{2\kappa_i}(X_{i,t+s}+q_{i,t+s})^2 - p_{t+s}q_{i,t+s} + c_i q_{i,t+s}^2 + \tau_s^i q_{i,t+s} + R_i \frac{q'_{i,t+s}}{\sum_{j\in N} q'_{j,t+s}})]. \quad (97)$$

The subscript t means that the expectation is conditional on the past history of prices and quantities traded by trader i and on the residual supply in period t. In order to derive an optimality condition for q_t^i, we will transform the objective function to look like the objective function in the static case (but with different coefficients) in order to apply Proposition 1. This entails rewriting the flow utility in period $t+1$ in terms of q_t^i, p_t^i and X_t^i.

Note that

$$R_i \frac{(q_{i,t+1})'}{\sum_{j\in N}(q_{j,t+1})'} = R_i\gamma_i$$

and

$$(X_{t+1}^i + q_{t+1}^i) = \gamma_i \sum_{j\in N} X_{t+1}^j.$$

Neither of these terms are affected by q_t^i so we can drop them from consideration in the objective function. Thus it suffices to rewrite

$$-p_{t+1}q_{t+1}^i + c_i(q_{t+1}^i)^2 + \tau_{t+1}^i q_{t+1}^i$$

in terms of q_t^i, p_t^i and X_t^i. We split this task into two steps. In the first step, we rewrite $-p_{t+s}q_{t+s}^i + c_i(q_{t+s}^i)^2 + \tau_s^i q_{t+s}^i - \omega_i q_{s-1}^i q_s^i$ and in the second step we rewrite $\omega_i q_{s-1}^i q_s^i$.

Step 1: For $s > t$ we have

$$p_s q_s^i = \frac{1}{\Theta}(-\sum_{j\in N} X_j^s)(\gamma_i \sum_{j\in N} X_s^j - X_s^i) = \frac{1}{\Theta}[-\gamma_i(\sum_{j\in N} X_s^j)^2 + X_s^i \sum_{j\in N} X_s^j], \quad (98)$$

$$(q_s^i)^2 = (\gamma_i \sum_{j\in N} X_s^j - X_s^i)^2 = \gamma_i^2(\sum_{j\in N} X_s^j)^2 + (X_s^i)^2 - 2\gamma_i X_s^i \sum_{j\in N} X_s^j \quad (99)$$

and

$$\begin{aligned}
\tau_s^i q_s^i - \omega_i q_{s-1}^i q_s^i &= \zeta_i \sum_{j\neq i} X_{s-1}^j(\gamma_i \sum_{j\in N} X_s^j - X_s^i) \\
&= \zeta_i \sum_{j\neq i} X_{s-1}^j(\gamma_i \sum_{j\neq i} X_{s-1}^j - (1-\gamma_i)X_s^i - \gamma_i q_{s-1}^i + \gamma_i \sum_{j\neq i}\epsilon_{sj}) \\
&= \zeta_i\gamma_i(\sum_{j\neq i} X_{s-1}^j)^2 - \zeta_i(1-\gamma_i)\sum_{j\neq i} X_{s-1}^j X_s^i - \gamma_i\zeta_i \sum_{j\neq i} X_{s-1}^j q_{s-1}^i + \zeta_i\gamma_i \sum_{j\neq i} X_s^j \sum_{j\neq i}\epsilon_{sj}, \quad (100)
\end{aligned}$$

Using the above equations, we derive that for $s > t$,

$$\tau_s^i q_s^i - \omega_i q_{s-1}^i q_s^i + c_i(q_s^i)^2 - p_s q_s^i = \\ (\frac{\gamma_i}{\Theta} + c_i\gamma_i^2)(\sum_{j\in N} X_s^j)^2 - \zeta_i(1-\gamma_i)\sum_{j\neq i} X_{s-1}^j X_s^i + c_i(X_s^i)^2 \\ -\gamma_i\zeta_i \sum_{j\neq i} X_{s-1}^j q_{s-1}^i - (2c_i\gamma_i + \frac{1}{\Theta})X_s^i \sum_{j\in N} X_s^j + \zeta_i\gamma_i(\sum_{j\neq i} X_{s-1}^j)^2 + \zeta_i\gamma_i \sum_{j\neq i} X_s^j \sum_{j\neq i} \epsilon_{sj}. \quad (101)$$

Next for $s = t+1$ we express the right hand side of the above equation in terms of q_t^i, X_t^i, and p_t^i. This will allow us to transform the objective function into a form that is similar to the objective function in the static case so that we can apply Proposition 1. The terms in (101) that we will re-express are $(\sum_{j\in N} X_s^j)^2$, $\sum_{j\neq i} X_{s-1}^j X_s^j$, $(X_s^i)^2$, $\sum_{j\neq i} X_{s-1}^j q_{s-1}^i$, $X_s^i \sum_{j\in N} X_s^j$, and $(\sum_{j\neq i} X_{s-1}^j)^2$ for $s = t+1$.

At date $t+1$, of these terms, the only ones whose expectation will be affected by q_t^i are ($\approx$ indicates a term which does not depend on q_t^i in expectation given the time t information set is dropped)[20]:

$$\sum_{j\neq i} X_t^j X_{t+1}^i = \sum_{j\neq i} X_t^j (X_t^i + q_t^i + \epsilon_{i,t+1}) \approx \sum_{j\neq i} X_t^j q_t^i \approx (q_t^i)^2 - p_t q_t^i \Theta(1-\gamma_i)^{21} \quad (102)$$

$$(X_{t+1}^i)^2 \approx (X_t^i + q_t^i)^2 \quad (103)$$

$$\sum_{j\neq i} X_t^j q_t^i = (q_t^i)^2 - p_t q_t^i \Theta(1-\gamma_i) \quad (104)$$

$$X_{t+1}^i \sum_{j\in N} X_{t+1}^j = (X_t^i + q_t^i)\sum_{j\in N} X_t^j \approx X_t^i q_t^i + (q_t^i)^2 - p_t q_t^i \Theta(1-\gamma_i) \quad (105)$$

Using equations (102)—(105) with (101) we find that the part of $\tau_s^i q_s^i + c_i(q_s^i)^2 - p_s q_s^i$ for $s = t+1$ whose time t expectation depends on q_t^i can be expressed as

$$[(q_t^i)^2 - p_t q_t^i \Theta(1-\gamma_i)](-\zeta_i - 2c_i\gamma_i - \frac{1}{\Theta}) - (2c_i\gamma_i + \frac{1}{\Theta})X_t^i q_t^i + c_i(X_t^i + q_t^i)^2. \quad (106)$$

[20] Any terms on the right hand side which do not depend on q_t^i can be dropped from consideration since a period t deviation in the choice q_t^i will not affect them.

[21] We use the fact that $p_t\Theta(1-\gamma_i) - q_t^i = -\sum_{j\neq i} X_t^j$

Step 2: For $s > t$

$$\omega_i q^i_{s-1}(\gamma_i \sum_{j\in N} X^j_s - X^i_s) = \omega_i q^i_{s-1}[\gamma_i \sum_{j\neq i} X^j_{s-1} - (1-\gamma_i)X^i_s - \gamma_i q^i_{s-1} + \gamma_i \sum_{j\neq i}\epsilon_{sj}]$$
$$= \omega_i\gamma_i \sum_{j\neq i} X^j_{s-1}q^i_{s-1} - \omega_i(1-\gamma_i)q^i_{s-1}X^i_s - \gamma_i\omega_i(q^i_{s-1})^2 + \gamma_i\omega_i q^i_{s-1}\sum_{j\neq i}\epsilon_{sj}. \quad (107)$$

Inspecting the above equation, the terms we must work with are $\sum_{j\neq i} X^j_{s-1}q^i_{s-1}$, $X^i_s q^i_{s-1}$, and $(q^i_{s-1})^2$. At date $t+1$, of these terms the ones whose time t expectation are effected by q^i_t are

$$(q^i_t)^2$$

and

$$q^i_t X^i_{t+1} = q^i_t(X^i_t + q^i_t + \epsilon_{i,t+1}) = X^i_t q^i_t + (q^i_t)^2$$

and

$$\sum_{j\neq i} X^j_t q^i_t = (q^i_t)^2 - p_t q^i_t \Theta(1-\gamma_i)$$

Inserting these expressions into (107) gives

$$-\gamma_i\omega_i(q^i_t)^2 - \omega_i(1-\gamma_i)(X_t q^i_t + (q^i_t)^2) + \omega_i\gamma_i[(q^i_t)^2 - p_t q^i_t\Theta(1-\gamma_i)]$$

or equivalently,

$$-\omega_i(q^i_t)^2 - \omega_i(1-\gamma_i)X_t q^i_t + \omega_i\gamma_i[(q^i_t)^2 - p_t q^i_t\Theta(1-\gamma_i)] \quad (108)$$

Combining the results of steps 1 and 2:

By summing equations 106 and 108, the part of $\tau^i_{t+1}q^i_{t+1} + c_i(q^i_{t+1})^2 - p_{t+1}q^i_{t+1}$ whose time t expectation depend on q^i_t is

$$[(q^i_t)^2 - p_t q^i_t\Theta(1-\gamma_i)](-\zeta_i - 2c_i\gamma_i - \frac{1}{\Theta} + \omega_i\gamma_i)$$
$$- (2c_i\gamma_i + \frac{1}{\Theta} + \omega_i(1-\gamma_i))X^i_t q^i_t + c_i(X^i_t + q^i_t)^2 - \omega_i(q^i_t)^2. \quad (109)$$

Using this expression and inserting it into the objective (97), we find that it suffices, when deriving conditions for no profitable one shot deviation, to consider the objective function which is linear in $p_t q_{it}$, q_{it}, $(X_{it}+q_{it})^2$, q^2_{it}, and $\frac{(q^i_s)'}{\sum_{j\in N}(q^j_s)'}$ with the following coefficients.

The coefficient on $p_t q_t^i$ is

$$-1 - e^{-r}(-\zeta_i - 2c_i\gamma_i - \frac{1}{\Theta} + \omega_i\gamma_i)(1-\gamma_i)\Theta \tag{110}$$

The coefficient on q_t^i is

$$\tau_i - e^{-r}(2c_i\gamma_i + \frac{1}{\Theta} + \omega_i(1-\gamma_i))X_t^i \tag{111}$$

The coefficient on $(X_t^i + q_t^i)^2$ is

$$-\frac{1}{2\kappa_i} + e^{-r}c_i$$

The coefficient on $(q_t^i)^2$ is

$$c_i + e^{-r}[-\zeta_i - 2c_i\gamma_i - \frac{1}{\Theta} + \omega_i\gamma_i - \omega_i]. \tag{112}$$

and the coefficient on $\frac{(q_s^i)'}{\sum_{j\in N}(q_s^j)'}$ is

$$R_i.$$

Thus the objective is almost of the same form as in the static model. To get it into that form we find it easiest to ensure that the coefficient on q_t^i is τ_i. We observe that $(X_t^i + q_t^i)^2 = (X_t^i)^2 + 2X_t^i q_t^i + (q_t^i)^2$. Thus, ince $(X_t^i)^2$ is unaffected by the choice of q_t^i it suffices to adjust the coefficients on $(q_t^i)^2$ and $(X_t^i + q_t^i)^2$ accordingly to ensure that the coefficient on q_t^i is τ_i. The adjusted coefficients are as follows.

The coefficient on $p_t q_t^i$ is

$$-1 - e^{-r}(-\zeta_i - 2c_i\gamma_i - \frac{1}{\Theta} + \omega_i\gamma_i)(1-\gamma_i)\Theta \tag{113}$$

The coefficient on q_t^i is

$$\tau_i \tag{114}$$

The coefficient on $(X_t^i + q_t^i)^2$ is

$$-\frac{1}{2\kappa_i} + e^{-r}c_i - \frac{e^{-r}}{2}(2c_i\gamma_i + \frac{1}{\Theta} + \omega_i(1-\gamma_i))$$

The coefficient on $(q_t^i)^2$ is

$$c_i + e^{-r}[-\zeta_i - 2c_i\gamma_i - \frac{1}{\Theta} + \omega_i\gamma_i - \omega_i] + \frac{e^{-r}}{2}(2c_i\gamma_i + \frac{1}{\Theta} + \omega_i(1-\gamma_i)) \tag{115}$$

and the coefficient on $\frac{(q_s^i)'}{\sum_{j\in N}(q_s^j)'}$ is

$$R_i.$$

Continuing to transform the objective

Immediately by the results of the static case (after renormalizing the coefficients so that the coefficient on $p_t q_t^i$ is -1), we see that

$$\zeta_i = -\omega_i = \frac{R_i}{\sum_{j\neq i}\sigma_{j\epsilon}^2}{}^{22}. \tag{116}$$

Using this result with equations (110)-(112), we find:
The coefficient on $p_t q_t^i$ is

$$-1 - e^{-r}(-\zeta_i - 2c_i\gamma_i - \frac{1}{\Theta} - \zeta_i\gamma_i)(1-\gamma_i)\Theta \tag{117}$$

The coefficient on q_t^i is

$$\tau_i \tag{118}$$

The coefficient on $(X_t^i + q_t^i)^2$ is

$$-\frac{1}{2\kappa_i} + e^{-r}c_i - \frac{e^{-r}}{2}(2c_i\gamma_i + \frac{1}{\Theta} - \zeta_i(1-\gamma_i))$$

The coefficient on $(q_t^i)^2$ is

$$c_i + e^{-r}[-\zeta_i - 2c_i\gamma_i - \frac{1}{\Theta} - \zeta_i\gamma_i + \zeta_i]. \tag{119}$$

$$+\frac{e^{-r}}{2}(2c_i\gamma_i + \frac{1}{\Theta} - \zeta_i(1-\gamma_i))$$

and the coefficient on $\frac{(q_s^i)'}{\sum_{j\in N}(q_s^j)'}$ is

$$R_i.$$

For ease of analysis, we define $A_p^i, B_p^i, C_p^i, A_q^i, B_q^i, C_q^i, A_X^i, B_X^i, C_X^i, A_{sq}^i, B_{sq}^i, C_{sq}^i$ so that the coefficient on $p_t q_t^i$ is

$$-[A_p^i + B_p^i\zeta_i + C_p^i c^i],$$

the coefficient on $(X_t^i + q_t^i)^2$ is

$$A_X^i + B_X^i\zeta_i + C_X^i c^i,$$

[22] See the equation for a_i at the end of section D.2 of the Appendix in the main draft.

and the coefficient on $(q_t^i)^2$ is

$$\frac{1}{2}[A_{sq}^i + B_{sq}^i \zeta_i + C_{sq}^i c^i].$$

That is,

$$-A_p^i = -1 + e^{-r}(1-\gamma_i) \tag{120}$$

$$-B_p^i = e^{-r}(1-\gamma_i)\Theta(1+\gamma_i) \tag{121}$$

$$-C_p^i = 2e^{-r}\gamma_i(1-\gamma_i)\Theta \tag{122}$$

$$A_X^i = -\frac{1}{2\kappa_i} - \frac{e^{-r}}{2}\frac{1}{\Theta} \tag{123}$$

$$B_X^i = \frac{e^{-r}}{2}(1-\gamma_i) \tag{124}$$

$$C_X^i = e^{-r} - e^{-r}\gamma_i \tag{125}$$

$$A_{sq}^i = -e^{-r}\frac{1}{\Theta} \tag{126}$$

$$B_{sq}^i = 2[-e^{-r}\gamma_i - \frac{e^{-r}}{2}(1-\gamma_i)] \tag{127}$$

$$\Leftrightarrow$$

$$B_{sq}^i = -e^{-r}(1+\gamma_i)$$

$$C_{sq}^i = 2[1-\gamma_i e^{-r}] \tag{128}$$

Using the static case of Proposition 1 to derive optimality conditions:
Define

$$\frac{1}{2\tilde{\kappa}_i} = -\frac{[A_X^i + B_X^i \zeta_i + C_X^i c^i]}{A_p^i + B_p^i \zeta_i + C_p^i c^i}$$

and

$$\tilde{c}_i = \frac{A_{sq}^i + B_{sq}^i \zeta_i + C_{sq}^i c^i}{A_p^i + B_p^i \zeta_i + C_p^i c^i}$$

and

$$\tilde{R}_i = \frac{R_i}{A_p^i + B_p^i \zeta_i + C_p^i c^i}.$$

Here we have simply renormalized the objective function so that the coefficient on $p_t q_t^i$ is -1 and relabeled the coefficients to parallel those of the objective function in the static case. Then by the results of the static case we have the following opitmality conditions:

$$1 = \frac{\frac{1}{\tilde{\kappa_i}}}{\frac{1}{\tilde{\kappa_i}} + \frac{1}{(1-\gamma_i)\Theta} - \tilde{c}_i - \frac{\tilde{R}_i}{\sigma^2_{\eta_i}}}, \tag{129}$$

$$\gamma_i \Theta = \frac{1 + (1-\gamma_i)\Theta \frac{\tilde{R}_i}{\sigma^2_{\eta_i}}}{\frac{1}{\tilde{\kappa_i}} + \frac{1}{(1-\gamma_i)\Theta} - \tilde{c}_i - \frac{\tilde{R}_i}{\sigma^2_{\eta_i}}} \tag{130}$$

where

$$\sigma^2_{\eta_i} \equiv \sum_{j \neq i} \sigma^2_{\epsilon j},$$

Earlier, we had already used the condition

$$\tau_t^i = (\sum_{j \neq i} X^j_{t-1} - q^i_{t-1}) \frac{R_i}{\sum_{j \neq i} \sigma^2_{j\epsilon}}$$

to deduce that $\zeta_i = -\omega_i = \frac{R_i}{\sum_{j \neq i} \sigma^2_{j\epsilon}}$ to derive (116).

Using the optimality conditions to solve for c_i and ζ_i

Using (129) and (130) we derive that

$$\frac{1}{(1-\gamma_i)\Theta} - \tilde{c}_i - \frac{\tilde{R}_i}{\sigma^2_{\eta_i}} = 0 \tag{131}$$

and

$$\gamma_i \Theta = \tilde{\kappa}_i + \tilde{\kappa}_i (1-\gamma_i)\Theta \frac{\tilde{R}_i}{\sigma^2_{\eta_i}}. \tag{132}$$

Then using (131) we have

$$\frac{1}{(1-\gamma_i)\Theta} - \frac{A^i_{sq} + B^i_{sq}\zeta_i + C^i_{sq}c^i}{A^i_p + B^i_p \zeta_i + C^i_p c^i} - \frac{\zeta_i}{A^i_p + B^i_p \zeta_i + C^i_p c^i} = 0$$

$$\Leftrightarrow$$

$$(A^i_p + B^i_p \zeta_i + C^i_p c^i) \frac{1}{(1-\gamma_i)\Theta} = A^i_{sq} + B^i_{sq}\zeta_i + C^i_{sq}c^i + \zeta_i$$

$$\Leftrightarrow$$

$$c_i [C^i_p \frac{1}{(1-\gamma_i)\Theta} - C^i_{sq}] = -\frac{A^i_p}{(1-\gamma_i)\Theta} + A^i_{sq} + (B^i_{sq} + 1 - B^i_p \frac{1}{(1-\gamma_i)\Theta})\zeta_i \tag{133}$$

$$\Leftrightarrow$$

$$2c_i = \frac{-1}{(1-\gamma_i)\Theta} + \zeta_i$$

By (132) we also have the condition that

$$-2\gamma_i\Theta\frac{(A^i_X + B^i_X\zeta_i + C^i_X c^i)}{A^i_p + B^i_p\zeta_i + C^i_p c^i} = 1 + (1-\gamma_i)\Theta\frac{\zeta_i}{A^i_p + B^i_p\zeta_i + C^i_p c^i}$$

$$\Leftrightarrow$$

$$-2\gamma_i\Theta(A^i_X + B^i_X\zeta_i + C^i_X c^i) = A^i_p + B^i_p\zeta_i + C^i_p c^i + (1-\gamma_i)\Theta\zeta_i$$

$$\Leftrightarrow$$

$$c_i[-2\gamma_i\Theta C^i_X - C^i_p] = 2\gamma_i\Theta A^i_X + A^i_p + [2\gamma_i\Theta B^i_X + (1-\gamma_i)\Theta + B^i_p]\zeta_i$$

$$\Leftrightarrow$$

$$0 = 2\gamma_i\Theta A^i_X + A^i_p + [2\gamma_i\Theta B^i_X + (1-\gamma_i)\Theta + B^i_p]\zeta_i$$

$$\Leftrightarrow$$

$$2\gamma_i\Theta(-\frac{1}{2\kappa_i} - \frac{e^{-r}}{2}\frac{1}{\Theta}) + 1 - e^{-r}(1-\gamma_i) \\ + [2\gamma_i\Theta\frac{e^{-r}}{2}(1-\gamma_i) + (1-\gamma_i)\Theta - e^{-r}(1-\gamma_i)\Theta(1+\gamma_i)]\zeta_i = 0 \quad (134)$$

$$\Leftrightarrow$$

$$\zeta_i = -\frac{1}{\Theta}\frac{-\frac{1}{\kappa_i}\Theta\gamma_i - e^{-r}\gamma_i + 1 - e^{-r}(1-\gamma_i)}{\gamma_i e^{-r}(1-\gamma_i) + (1-\gamma_i) - e^{-r}(1-\gamma_i)(1+\gamma_i)}$$

$$\Leftrightarrow$$

$$\zeta_i = -\frac{1}{\Theta}\frac{1}{1-\gamma_i}\frac{-\frac{1}{\kappa_i}\Theta\gamma_i - e^{-r}\gamma_i + 1 - e^{-r}(1-\gamma_i)}{\gamma_i e^{-r} + 1 - e^{-r}(1+\gamma_i)}$$

$$\Leftrightarrow$$

$$\zeta_i = -\frac{1}{\Theta}\frac{-\frac{1}{\kappa_i}\Theta\gamma_i + 1 - e^{-r}}{1-\gamma_i}\frac{1}{1-e^{-r}}$$

Then using (133) we have an explicit expression for c_i:

$$2c_i = -\frac{1}{(1-\gamma_i)\Theta} - \frac{1}{\Theta}\frac{-\frac{1}{\kappa_i}\Theta\gamma_i + 1 - e^{-r}}{1-\gamma_i}\frac{1}{1-e^{-r}}$$

$$\Leftrightarrow$$

$$2c_i = \frac{1}{(1-\gamma_i)\Theta}[-2 + \frac{\frac{1}{\kappa_i}\Theta\gamma_i}{1-e^{-r}}].$$

Thus we have derived the implementing stationary subsidy scheme. □

C.2 Proof of Proposition 8

Proof. By setting $\Theta = 2(1-e^{-r})\sum_{j\in N}\kappa_j$ it is possible to implement the efficient allocation without quadratic subsidies($c_i = 0$). Since

$$\mathbb{E}[q_0^i] = \mathbb{E}[\frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\in N} X_0^j - X_0^i] = 0$$

the expected cost of the linear subsidy is 0 in period 0 if traders' means of initial inventories is proportional to risk capacities. (We set $\sum_{j\neq i} X_{s-1}^i$ to $\sum_{j\neq i}\mu_j$ for each j in period 0 and q_{s-1} to 0 in period 0). It is also zero in all future periods as well since the efficient allocation satisfies this condition. Since R-subsidy costs can always be recouped with a participation costs budget balance is achievable in the case when μ_i and κ_i are proportional. We give a more formal proof below that gives a sufficient condition on model primitives for budget balanced trade.

We set $\Theta = 2(1-e^{-r})\sum_{j\in N}\kappa_j$. Then

$$\begin{aligned}\mathbb{E}[p_0 q_{i0}|X_{i0}] &= -\frac{1}{\Theta}\mathbb{E}[\gamma_i(\sum_{j\in N} X_{j0})^2 - X_{i0}\sum_{j\in N} X_{j0}] \\ &= -\frac{\gamma_i}{\Theta}\left[(X_{i0} + \sum_{j\neq i}\mu_j)^2 + \sum_{j\neq i}\sigma_{j\epsilon}^2\right] + \frac{1}{\Theta}[X_{i0}^2 + X_{i0}\sum_{j\neq i}\mu_j]. \quad (135)\end{aligned}$$

We compute the net present value of all future payments. We have for $t > 0$

$$q_{it} = -(1-\gamma_i)\epsilon_{it} + \gamma_i\sum_{j\neq i}\epsilon_{jt}$$

since trader i unloads a fraction $1-\gamma_i$ of his own inventory shock and absorbs a fraction γ of the other traders' inventory shocks. The price is

$$p_t = -\frac{\sum_{j\in N} X_{jt}}{2\sum_{j\in N}\kappa_j(1-e^{-r})} = \frac{-\sum_{j\in N} X_{j,t-1} - \sum_{j\in N}\epsilon_{jt}}{2\sum_{j\in N}\kappa_j(1-e^{-r})}.$$

We therefore have, for $t > 0$

$$-\mathbb{E}[p_t q_{it}|X_{i0}] = \frac{-(1-\gamma_i)\sigma_{i\epsilon}^2 + \gamma_i \sum_{j\neq i}\sigma_{j\epsilon}^2}{2\sum_{j\in N}\kappa_j(1-e^{-r})} = \frac{-\frac{\sum_{j\neq i}\kappa_j}{\sum_{j\in N}\kappa_j}\sigma_{i\epsilon}^2 + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\sigma_{j\epsilon}^2}{2\sum_{j\in N}\kappa_j(1-e^{-r})}. \tag{136}$$

Then

$$\mathbb{E}[\sum_{t>0} -e^{-rt} p_t q_{it}|X_{i0}] = -\frac{e^{-r}\sum_{j\neq i}\kappa_j}{2(\sum_{j\in N}\kappa_j)^2(1-e^{-r})^2}\sigma_{i\epsilon}^2 + \frac{e^{-r}\kappa_i\sum_{j\neq i}\sigma_{j\epsilon}^2}{2(\sum_{j\in N}\kappa_j)^2(1-e^{-r})^2}. \tag{137}$$

Next, we compute the net present value of quadratic holding costs. Using,

$$\begin{aligned}&\mathbb{E}[(\sum_{j\in N} X_{jt})^2|X_{i0}] \\ &\quad= \mathbb{E}[(\sum_{j\in N} X_{j0} + \sum_{s=1}^{t}\sum_{j\in N}\epsilon_{jt})^2|X_{i0}] = (X_{i0} + \sum_{j\neq i}\mu_j)^2 + \sum_{j\neq i}\sigma_{j\epsilon}^2 + t\sum_{j\in N}\sigma_{j\epsilon}^2. \end{aligned}\tag{138}$$

we have

$$\begin{aligned}&-\frac{\gamma_i^2}{2\kappa_i}\mathbb{E}[\sum_{t=0}^{\infty} e^{-rt}(\sum_{j\in N} X_{jt})^2|X_{i0}] \\ &\quad= -\frac{1}{2\kappa_i}\gamma_i^2\frac{(X_{i0}+\sum_{j\neq i}\mu_j)^2 + \sum_{j\neq i}\sigma_{j\epsilon}^2}{1-e^{-r}} - \frac{1}{2\kappa_i}\gamma_i^2\sum_{j\in N}\sigma_{j\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2}\end{aligned}\tag{139}$$

Next we compute the expected cost of the τ-subsidies. The expected cost is zero for all $t > 0$ so we only need to compute the expected cost of the τ-subsidies at $t = 0$.

$$\mathbb{E}[\tau_0^i q_{i0}] = \frac{1}{2(1-e^{-r}\sum_{j\neq i}\kappa_j}\sum_{j\neq i}\mu_j[-\frac{\sum_{j\neq i}\kappa_j\mu_i}{\sum_{j\in N}\kappa_j} + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j]. \tag{140}$$

Therefore the total sum of costs due to τ-subsidies is

$$\sum_{i\in N}\tau_0^i q_{i0} = -\frac{\sum_{i\in N}\mu_i\sum_{j\neq i}\mu_j}{2(1-e^{-r})\sum_{j\in N}\kappa_j} + \sum_{i\in N}\frac{1}{2(1-e^{-r})}\frac{\kappa_i}{\sum_{j\in N}\kappa_j\sum_{j\neq i}\kappa_j}(\sum_{j\neq i}\mu_j)^2. \tag{141}$$

The expected cost of the τ-subsidies for a trader of type X_{i0} is

$$\mathbb{E}[\tau_0^i q_{i0}|X_{i0}] = \frac{1}{2(1-e^{-r})\sum_{j\neq i}\kappa_j}\sum_{j\neq i}\mu_j[-\frac{\sum_{j\neq i}\kappa_j X_{i0}}{\sum_{j\in N}\kappa_j} + \frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j]. \tag{142}$$

The expected utility of trader i if he does not partcipate in the mechanism is

$$-[\frac{1}{2\kappa_i}\frac{1}{1-e^{-r}}X_{i0}^2+\frac{1}{2\kappa_i}\sigma_{i\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2}]. \tag{143}$$

The maximum participation fee is the minimum over X_{i0} of

$$\begin{aligned}
&\frac{1}{2\kappa_i}\frac{1}{1-e^{-r}}X_{i0}^2+\frac{1}{2\kappa_i}\sigma_{i\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2}\\
&\qquad+\frac{1}{2(1-e^{-r})\sum_{j\neq i}\kappa_j}\sum_{j\neq i}\mu_j[-\frac{\sum_{j\neq i}\kappa_j X_{i0}}{\sum_{j\in N}\kappa_j}+\frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j]\\
&\qquad-\frac{e^{-r}\sum_{j\neq i}\kappa_j}{2(\sum_{j\in N}\kappa_j)^2(1-e^{-r})^2}\sigma_{i\epsilon}^2+\frac{e^{-r}\kappa_i\sum_{j\neq i}\sigma_{j\epsilon}^2}{2(\sum_{j\in N}\kappa_j)^2(1-e^{-r})^2}\\
&+\frac{\kappa_i}{2(1-e^{-r})(\sum_{j\in N}\kappa_j)^2}\left[(X_{i0}+\sum_{j\neq i}\mu_j)^2+\sum_{j\neq i}\sigma_{j\epsilon}^2\right]-\frac{1}{2(1-e^{-r})\sum_{j\in N}\kappa_j}[X_{i0}^2+X_{i0}\sum_{j\neq i}\mu_j]\\
&\qquad-\frac{\kappa_i}{2(\sum_{j\in N}\kappa_j)^2}\frac{(X_{i0}+\sum_{j\neq i}\mu_j)^2+\sum_{j\neq i}\sigma_{j\epsilon}^2}{1-e^{-r}}-\frac{\kappa_i}{2(\sum_{j\in N}\kappa_j)^2}\sum_{j\in N}\sigma_{j\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2}.
\end{aligned} \tag{144}$$

or equivalently, the minimum over X_{i0} of

$$\begin{aligned}
&\frac{1}{2\kappa_i}\frac{1}{1-e^{-r}}X_{i0}^2+\frac{1}{2\kappa_i}\sigma_{i\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2}\\
&\qquad+\frac{1}{2(1-e^{-r})\sum_{j\neq i}\kappa_j}\sum_{j\neq i}\mu_j[-\frac{\sum_{j\neq i}\kappa_j X_{i0}}{\sum_{j\in N}\kappa_j}+\frac{\kappa_i}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j]\\
&\qquad-\frac{e^{-r}\sum_{j\neq i}\kappa_j}{2(\sum_{j\in N}\kappa_j)^2(1-e^{-r})^2}\sigma_{i\epsilon}^2+\frac{e^{-r}\kappa_i\sum_{j\neq i}\sigma_{j\epsilon}^2}{2(\sum_{j\in N}\kappa_j)^2(1-e^{-r})^2}\\
&\qquad-\frac{1}{2(1-e^{-r})\sum_{j\in N}\kappa_j}[X_{i0}^2+X_{i0}\sum_{j\neq i}\mu_j]-\frac{\kappa_i}{2(\sum_{j\in N}\kappa_j)^2}\sum_{j\in N}\sigma_{j\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2}.
\end{aligned} \tag{145}$$

Taking a first order condition with respect to X_{i0} we have

$$\frac{1}{\kappa_i}X_{i0}-\frac{1}{\sum_{j\in N}\kappa_j}\sum_{j\neq i}\mu_j-\frac{1}{\sum_{j\in N}\kappa_j}X_{i0}=0$$

$$\Leftrightarrow$$

$$\frac{\sum_{j\neq i}\kappa_j}{\kappa_i}X_{i0}-\sum_{j\neq i}\mu_j=0$$

$$\Leftrightarrow$$

$$X_{i0} = \frac{\kappa_i \sum_{j \neq i} \mu_j}{\sum_{j \neq i} \kappa_j}.$$

We now substitute into the expression for the participation cost to obtain

$$\begin{aligned}
&\frac{1}{2\kappa_i}\frac{1}{1-e^{-r}}X_{i0}^2 + \frac{1}{2\kappa_i}\sigma_{i\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2} \\
&\qquad - \frac{e^{-r}\sum_{j \neq i}\kappa_j}{2(\sum_{j \in N}\kappa_j)^2(1-e^{-r})^2}\sigma_{i\epsilon}^2 + \frac{e^{-r}\kappa_i\sum_{j \neq i}\sigma_{j\epsilon}^2}{2(\sum_{j \in N}\kappa_j)^2(1-e^{-r})^2} \\
&\qquad - \frac{1}{2(1-e^{-r})\sum_{j \in N}\kappa_j}[X_{i0}^2 + X_{i0}\sum_{j \neq i}\mu_j] - \frac{\kappa_i}{2(\sum_{j \in N}\kappa_j)^2}\sum_{j \in N}\sigma_{j\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2}
\end{aligned} \tag{146}$$

$$\Leftrightarrow$$

$$\begin{aligned}
&\frac{\kappa_i}{2(\sum_{j \neq i}\kappa_j)^2}\frac{1}{1-e^{-r}}(\sum_{j \neq i}\mu_j)^2 + \frac{1}{2\kappa_i}\sigma_{i\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2} \\
&\qquad - \frac{e^{-r}\sum_{j \neq i}\kappa_j}{2(\sum_{j \in N}\kappa_j)^2(1-e^{-r})^2}\sigma_{i\epsilon}^2 + \frac{e^{-r}\kappa_i\sum_{j \neq i}\sigma_{j\epsilon}^2}{2(\sum_{j \in N}\kappa_j)^2(1-e^{-r})^2} \\
&\qquad - \frac{1}{2(1-e^{-r})}\frac{\kappa_i}{(\sum_{j \neq i}\kappa_j)^2}(\sum_{j \neq i}\mu_j)^2 - \frac{\kappa_i}{2(\sum_{j \in N}\kappa_j)^2}\sum_{j \in N}\sigma_{j\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2}
\end{aligned} \tag{147}$$

$$\Leftrightarrow$$

$$\begin{aligned}
&\frac{1}{2\kappa_i}\sigma_{i\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2} \\
&\qquad - \frac{e^{-r}\sum_{j \neq i}\kappa_j}{2(\sum_{j \in N}\kappa_j)^2(1-e^{-r})^2}\sigma_{i\epsilon}^2 + \frac{e^{-r}\kappa_i\sum_{j \neq i}\sigma_{j\epsilon}^2}{2(\sum_{j \in N}\kappa_j)^2(1-e^{-r})^2} \\
&\qquad - \frac{\kappa_i}{2(\sum_{j \in N}\kappa_j)^2}\sum_{j \in N}\sigma_{j\epsilon}^2\frac{e^{-r}}{(1-e^{-r})^2}
\end{aligned} \tag{148}$$

$$\Leftrightarrow$$

$$\frac{e^{-r}}{2(1-e^{-r})^2}(-\frac{1}{\sum_{j \in N}\kappa_j} + \frac{1}{\kappa_i})\sigma_{i\epsilon}^2 > 0.$$

Note that since $(\mu_i)_i$ were arbitrary, given our infinite horizon setting, the above implies that it is also individually rational for traders to pay the participation fee equal to the value of the R-subsidy at all future dates. Comparing this expression with the expected cost of the τ_0 subsidies gives the expression in the statement of the proposition.

□

D Proofs and Supplementary Material for Section 8

D.1 Proof of lemma 5

From lemma 1, in any linear equilibrium, the quantity agent i trades is:

$$q_i = a_i - w_i x_i - \frac{y_i}{\sum_{j=1}^n y_j}\left(\sum_{j=1}^n (a_j - w_j x_j)\right)$$

Rearranging, we can write this as:

$$q_i = \left(a_i - \frac{y_i}{\sum_{j=1}^n y_j}\sum_{j=1}^n a_j\right) - w_i\left(1 - \frac{y_i}{\sum_{j=1}^n y_j}\right)x_i + \frac{y_i}{\sum_{j=1}^n y_j}\left(\sum_{\{j\in N|j\neq i\}} w_j x_j\right) \tag{149}$$

This corresponds to a $\boldsymbol{k}$-$\boldsymbol{A}$ allocation function, with intercept vector:

$$\boldsymbol{k} = \begin{pmatrix} a_1 - \frac{y_1}{\sum_{j=1}^n y_j}\sum_{j=1}^n a_j \\ \dots \\ a_n - \frac{y_n}{\sum y_i}\sum_j a_j \end{pmatrix}$$

The sum of these elements is:

$$\sum_i \left(a_i - \frac{y_i}{\sum_{j=1}^n y_j}\sum_{j=1}^n a_j\right) = \left(\sum_{i\in N} a_i\right) - \sum_{j=1}^n a_j\left(\sum_{i\in N}\frac{y_i}{\sum_{j=1}^n y_j}\right) = 0$$

Hence, (32) always holds. Moreover, from (149), the elements of the $\boldsymbol{A}$ matrix are:

$$a_{ij} = \frac{dq_i}{dx_j} = \frac{w_j y_i}{\sum_j y_j}\ \forall j \neq i$$

$$a_{ii} = \frac{dq_i}{dx_i} = -w_i\left(1 - \frac{y_i}{\sum_{j=1}^n y_j}\right)\ \forall i$$

Hence, (33) and (34) hold with

$$s_i \equiv \frac{y_i}{\sum_{j=1}^n y_j}$$

Note that $y_i \geq 0$ implies that $s_i \geq 0$, and $\sum_{i\in N} s_i = 1$ always holds.

D.2 Proof of lemma 6

For any allocation rule of the form 31, we have:

$$\frac{\frac{\partial q_i}{\partial x_k}}{\frac{\partial q_j}{\partial x_k}} = \frac{a_{ik}}{a_{jk}} = \frac{s_i w_k}{s_j w_k} = \frac{s_i}{s_j}$$

$$\frac{\frac{\partial q_i}{\partial x_l}}{\frac{\partial q_j}{\partial x_l}} = \frac{a_{il}}{a_{jl}} = \frac{s_i w_l}{s_j w_l} = \frac{s_i}{s_j}$$

D.3 Details on numerical results

In Appendices D.4.2 and D.4.3, we derive analytical expressions for expected welfare and revenue, allowing us to express (37) and (38) as analytical, though very complex, optimization problems. We thus solve these problems numerically, using convex optimization routines.

One issue for implementing the optimization problem is problem (37) searches over the $3N$-dimensional space of linear equilibria; as lemma 7 shows, the space of allocation rules is only $3N-2$-dimensional. From the revenue equivalence results of propositions 2 and 3, any two linear equilibria induce identical allocations will also produce identical revenue for both the platform operator and all agents. Hence, we need to impose two constraints so that we do not search over equilibria which induce identical allocations; we normalize a_1 to 0, and we normalize the sum of all y_i to equal 1. As the start point for finding the second-best mechanism, we use the equilibrium values of $\{(a_i, y_i, w_i)\}$ with no subsidies.

To construct Figure 1, we solve problem (37) under three sets of parameter assumptions stated in the extensions section of the main text. Table 1 shows the second-best subsidies and equilibrium parameters for different choices of primitives. The main difference between the second-best mechanism and the no-subsidy equilibrium is that the second-best mechanism increases w_i, inventory sensitivities, for all agents, fixing the bid shading distortion; in fact, w_i is quite close to 1 for all choices of primitives we tried. a_i is nonzero at the second-best mechanism, while it is 0 for the no-subsidy mechanism. In terms of transfers, net revenue for the platform is positive for the agent with $\mu_{Xi} = 0$, and negative to agents with positive and negative values of μ_{Xi}: on average, the platform operator takes revenue from the agent who is neither a net buyer or seller, who expects with high probability to profit from trade, and uses it to subsidize the agents with extremal values of μ_{Xi}, who are net buyers and sellers, and thus tend to distort their bids more.

Similarly, we solve the revenue-maximization problem, (38), using a convex optimization algorithm. Analogous to the second-best mechanism, we normalize a_1 to 0, and we normalize the sum of all y_i to equal 1. To construct Figure 2 we then solve (38) under the same three sets of parameter settings that we use for the efficiency maximization problem, (37). As the start point of the optimization, we use the second-best values of $\{(a_i, y_i, w_i)\}$, which have total revenue equal to 0. Table 2 shows the second-best subsidies and equilibrium parameters for different choices of primitives.

D.4 Analytical expressions for welfare and revenue

D.4.1 Residual supply, implementing subsidies, trade quantities

Given any vector $\{(a_i, y_i, w_i)\}$, and primitives $\{(\kappa_i, \mu_{Xi}, \sigma^2_{Xi})\}$, we can calculate the residual supply parameters $\left\{\left(d_i, \mu_{\eta i}, \sigma^2_{\eta i}\right)\right\}$ facing agents, using lemma 2. Using proposition 2, we can then find the unique set of subsidies $\{(R_i, c_i, \tau_i)\}$ which implements $\{(a_i, y_i, w_i)\}$.

We can then express trade quantities and prices, from each agents' perspective, as functions of X_i and η_i. Setting bids equal to residual supply, we have:

$$q_i\left(X_i, p\right) = q_{RS}\left(i\right)$$

$$\implies a - wX - yp = \eta + dp$$

we have omitted i subscripts for simplicity; through all of the derivations in this section, expressions apply to individual agents, so we will omit subscripts i. Solving for prices and quantities, we have the following lemma:

Lemma 11. *Trade quantities and prices are:*

$$p\left(X, \eta\right) = \frac{a - wX - \eta}{d + y} \tag{150}$$

And:

$$q\left(X, \eta\right) = \frac{ad - dwX + y\eta}{d + y} \tag{151}$$

D.4.2 Welfare

The net welfare from a mechanism can be calculated as the sum of agents' holding costs, ignoring all transfers. The decrease in holding costs of agent i for buying q units of the asset is:

$$-\frac{1}{2\kappa}\left[\left(X+q\right)^2 - X^2\right]$$

$$-\frac{1}{2\kappa}\left(2Xq + q^2\right)$$

Table 1: Second-best mechanism

μ_X	κ	σ_X^2	NS a_i	NS w_i	NS y_i	SB a_i	SB w_i	SB y_i	SB c_i	SB R_i	SB τ_i	Entry fee	Payments	Net Rev
0.00	1.00	1.00	0.000	0.500	0.500	0.000	1.000	0.333	2.500	-2.000	-0.000	0.167	0.167	0.000
0.00	1.00	1.00	0.000	0.500	0.500	-0.000	1.000	0.333	2.500	-2.000	-0.000	0.167	0.167	0.000
-0.00	1.00	1.00	0.000	0.500	0.500	-0.000	1.000	0.333	2.500	-2.000	-0.000	0.167	0.167	0.000
0.00	1.00	1.00	0.000	0.500	0.500	0.000	0.922	0.333	2.374	-2.039	-0.000	0.192	0.050	0.142
1.00	1.00	1.00	0.000	0.500	0.500	0.216	1.032	0.333	2.546	-1.945	1.038	0.155	0.226	-0.071
-1.00	1.00	1.00	0.000	0.500	0.500	-0.216	1.032	0.333	2.546	-1.945	-1.038	0.155	0.226	-0.071
0.00	1.00	1.00	0.000	0.695	0.695	0.000	0.917	0.143	2.061	-2.201	-0.000	0.029	0.369	-0.340
1.00	3.00	1.00	0.000	0.379	1.137	0.273	1.057	0.429	3.281	-2.963	1.272	0.099	-0.071	0.170
-1.00	3.00	1.00	0.000	0.379	1.137	-0.273	1.057	0.429	3.281	-2.963	-1.272	0.099	-0.071	0.170
0.00	1.00	1.00	0.000	0.500	0.500	0.000	0.931	0.333	2.389	-5.419	0.000	0.504	-0.599	1.103
1.00	1.00	3.00	0.000	0.500	0.500	0.145	0.968	0.333	2.451	-3.621	0.960	0.317	0.868	-0.552
-1.00	1.00	3.00	0.000	0.500	0.500	-0.145	0.968	0.333	2.451	-3.621	-0.960	0.317	0.868	-0.552

Notes. Subsidies and payments for the second-best mechanism, for different parameter settings. "NS" columns describe equilibrium with no subsidies, and "SB" columns describe the second-best mechanism. "Entry fee" is the maximal fee charged to participants, "Payments" is total expected payments to agent i, and "Net Rev" is entry fees minus payments.

Table 2: Revenue-maximizing mechanism

μ_{Xi}	κ_i	σ^2_{Xi}	NS a_i	NS w_i	NS y_i	RM a_i	RM w_i	RM y_i	RM c_i	RM R_i	RM τ_i	Entry fee	Payments	Net Rev
0.00	1.00	1.00	0.000	0.500	0.500	0.000	0.500	0.333	1.000	-0.250	-0.000	0.083	-0.000	0.083
0.00	1.00	1.00	0.000	0.500	0.500	-0.000	0.500	0.333	1.000	-0.250	-0.000	0.083	-0.000	0.083
-0.00	1.00	1.00	0.000	0.500	0.500	-0.000	0.500	0.333	1.000	-0.250	-0.000	0.083	0.000	0.083
0.00	1.00	1.00	0.000	0.500	0.500	0.000	0.500	0.333	1.000	-0.528	-0.000	0.176	-0.062	0.238
1.00	1.00	1.00	0.000	0.500	0.500	0.400	0.727	0.333	1.936	-0.632	0.816	0.089	-0.063	0.152
-1.00	1.00	1.00	0.000	0.500	0.500	-0.400	0.727	0.333	1.936	-0.632	-0.816	0.089	-0.063	0.152
0.00	1.00	1.00	0.000	0.695	0.695	0.000	0.500	0.143	1.000	-0.927	-0.000	0.026	-0.029	0.056
1.00	3.00	1.00	0.000	0.379	1.137	0.447	0.746	0.429	3.051	-1.141	0.623	0.058	-0.036	0.094
-1.00	3.00	1.00	0.000	0.379	1.137	-0.447	0.746	0.429	3.051	-1.141	-0.623	0.058	-0.036	0.094
0.00	1.00	1.00	0.000	0.500	0.500	0.000	0.500	0.333	1.000	-1.031	0.000	0.344	-0.174	0.517
1.00	1.00	3.00	0.000	0.500	0.500	0.305	0.586	0.333	1.441	-0.829	0.702	0.182	0.016	0.166
-1.00	1.00	3.00	0.000	0.500	0.500	-0.305	0.586	0.333	1.441	-0.829	-0.702	0.182	0.016	0.166

Notes. Subsidies and payments for the revenue-maximizing mechanism, for different parameter settings. “NS” columns describe equilibrium with no subsidies, and “RM” columns describe the revenue maximizing mechanism. “Entry fee” is the maximal fee charged to participants, “Payments” is total expected payments to agent i, and “Net Rev” is entry fees minus payments.

Substituting for q using (151), we have:

$$\frac{ad(d(w-1)-y)}{\kappa(d+y)^2}X - \frac{y(d(w-1)-y)}{\kappa(d+y)^2}\eta X - \frac{a^2d^2}{2\kappa(d+y)^2} - \frac{ady}{\kappa(d+y)^2}\eta - \frac{(d^2(w-2)w-2dwy)}{2\kappa(d+y)^2}X^2 - \frac{y^2}{2\kappa(d+y)^2}\eta^2$$

Taking expectations over X and η, we have:

$$\frac{ad(d(w-1)-y)}{\kappa(d+y)^2}\mu_X + \frac{y(d(w-1)-y)}{\kappa(d+y)^2}\mu_X\mu_\eta - \frac{a^2d^2}{2\kappa(d+y)^2} - \frac{ady}{\kappa(d+y)^2}\mu_\eta - \frac{(d^2(w-2)w-2dwy)}{2\kappa(d+y)^2}\left(\mu_X^2+\sigma_X^2\right) - \frac{y^2}{2\kappa(d+y)^2}\left(\mu_\eta+\sigma_\eta^2\right) \quad (152)$$

The total welfare gain is the sum of (152) over agents i.

D.4.3 Revenue

Total revenue of the platform operator can be calculated as the total entry fees that can be charged to agents, minus the total net payments spent on subsidies.

$$\mathit{Revenue}\left(\{(a_i,y_i,w_i)\};\left\{\left(\kappa_i,\mu_{Xi},\sigma_{Xi}^2\right)\right\}\right) = \sum_{i\in N}\mathit{Entryfees}_i\left(\{(a_i,y_i,w_i)\};\left\{\left(\kappa_i,\mu_{Xi},\sigma_{Xi}^2\right)\right\}\right) - \mathit{Payments}_i\left(\{(a_i,y_i,w_i)\};\left\{\left(\kappa_i,\mu_{Xi},\sigma_{Xi}^2\right)\right\}\right)$$

D.4.4 Entry fees

The entry fees that the platform operator can charge to each agent are pinned down by IR constraints: the worst-off type X of each agent must have nonnegative expected utility gains from the mechanism. The total utility gain of type X of an agent is:

$$E\left[U \mid X\right] = E\left[-\frac{1}{2\kappa}\left(2Xq+q^2\right) - pq + \tau q + \frac{c}{2}q^2 + R\frac{y}{d+y} \mid X\right] \quad (153)$$

Substituting for q and p using lemma 11, and taking expectations over η, we can write the expectations of each piece in (153) as:

$$E\left[-\frac{1}{2\kappa}\left(2Xq+q^2\right) \mid X\right] = -\frac{d^2(a-(w-2)X)(a-wX) + 2dy(a(\mu_\eta+X)+X(\mu_\eta(-w)+\mu_\eta-wX)) + y^2\left(\mu_\eta^2+2\mu_\eta X+\sigma_\eta^2\right)}{2\kappa(d+y)^2}$$

$$E\left[-pq \mid X\right] = \frac{y(\mu_\eta(-a+\mu_\eta+wX)+\sigma_\eta^2)-d(a-wX)(a-\mu_\eta-wX)}{(d+y)^2}$$

$$E\left[\tau q \mid X\right] = \frac{\tau(ad-dwX+\mu_\eta y)}{d+y}$$

$$E\left[\frac{c}{2}q^2 \mid X\right] = \frac{c\left(d^2(a-wX)^2+2d\mu_\eta y(a-wX)+y^2\left(\mu_\eta^2+\sigma_\eta^2\right)\right)}{2(d+y)^2}$$

$$E\left[R\frac{y}{d+y} \mid X\right] = R\frac{y}{d+y}$$

To find the worst-off type, we differentiate the sum of the five pieces above with respect to X, set the derivative to 0, and solve for X, to get:

$$X_{min} = \frac{ad(dw(c\kappa-1)+d-2\kappa w+y)+d\mu_\eta(wy(c\kappa-1)+\kappa w+y)+d^2\kappa\tau w+d\kappa\tau wy+\mu_\eta y(y-\kappa w)}{dw(d(w(c\kappa-1)+2)+2(y-\kappa w))}$$

We plug X_{min} into (153) to calculate the total entry fees we can charge an agent. We also have to ensure that utility is a convex function of X, so in the optimization process, we require the second derivative of (153) with respect to X to be nonnegative, that is:

$$\frac{dw(\kappa w(cd-2)+d(-w)+2d+2y)}{\kappa(d+y)^2} \geq 0$$

D.4.5 Payments

The total amount paid out by the platform operator to agents is the sum over all three subsidies:

$$\tau q + \frac{c}{2}q^2 + R\frac{y}{d+y}$$

Again, we substitute for q using lemma 11, and take expectations over X and η. These subsidies can be written as:

$$\tau E\left[q\right] + \frac{c}{2}\left\{E\left[q\right]^2 + Var\left[q\right]\right\} + R\frac{y}{d+y}$$

Where, using lemma 11,

$$E\left(q\right) = \frac{ad-dw\mu_X+y\mu_\eta}{d+y}$$

$$Var\left(q\right) = \frac{d^2w^2\sigma_X^2+y^2\sigma_\eta^2}{\left(d+y\right)^2}$$