MOMENT CLOSURE ANALYSIS OF SIRS DISEASE MODEL ON HETEROGENEOUS NETWORKS

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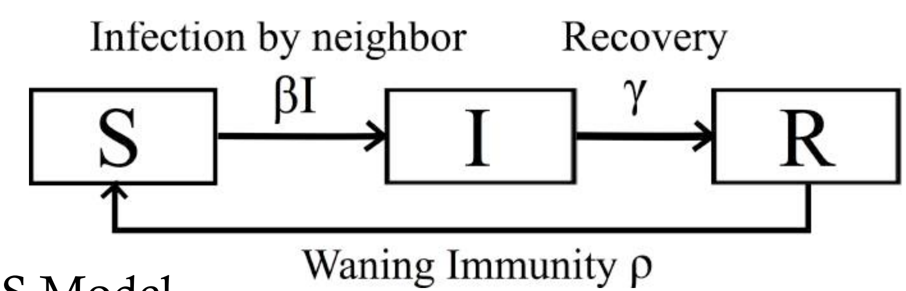
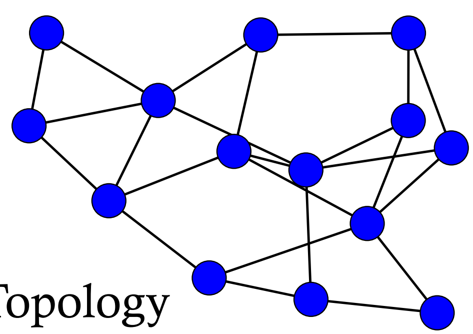
SIR-Type Models of Disease Spread

- How does disease spread through a population?
- Each individual takes on disease state
  - (S)usceptible – can become infected through contact
  - (I)nfected – infect susceptible nodes through contact
  - (R)ecovered – no longer infected, cannot become infected again
- Transition between **extinct** and **active** phases
  - Extinct: outbreak affects finite population, dies out in finite time
  - Active: outbreak affects macroscopic fraction of population, persists



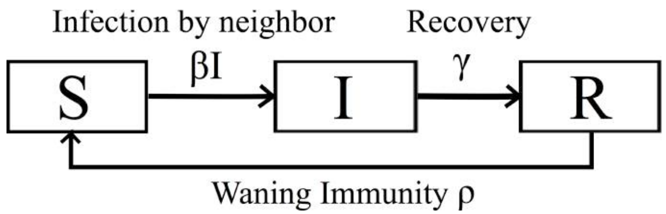
SIRS Model

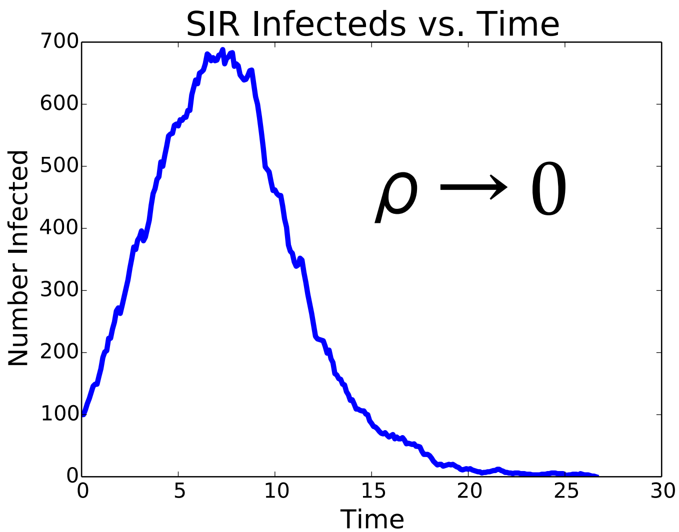


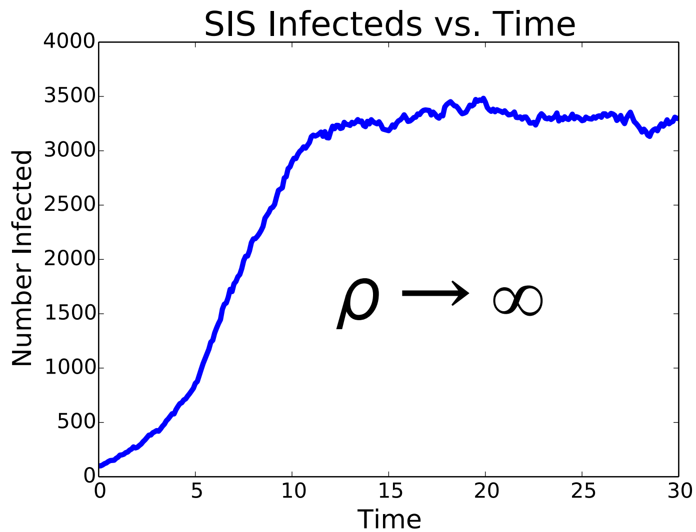
Network Topology

SIRS Model – Disease Persistence

- SIRS → SIR as $\rho \to 0$, single outbreak
- SIRS → SIS as $\rho \to \infty$, endemic disease



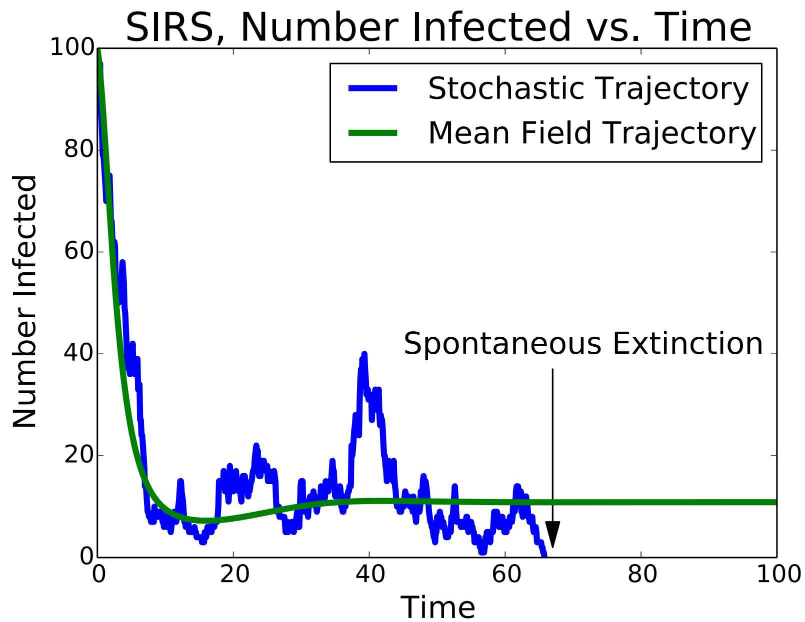




- **What happens for intermediate ρ ?**
- **When does the SIRS model sustain endemic disease?**

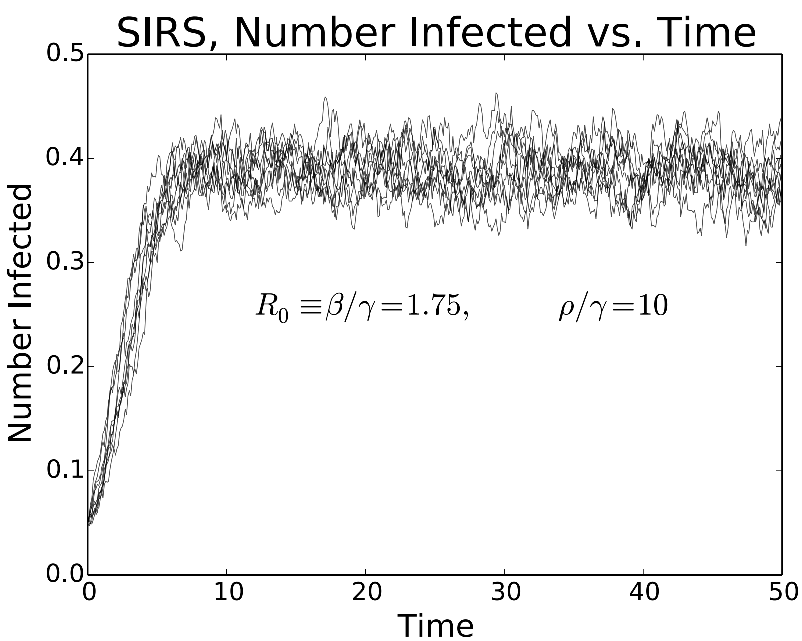
Spontaneous Extinction from Fluctuations

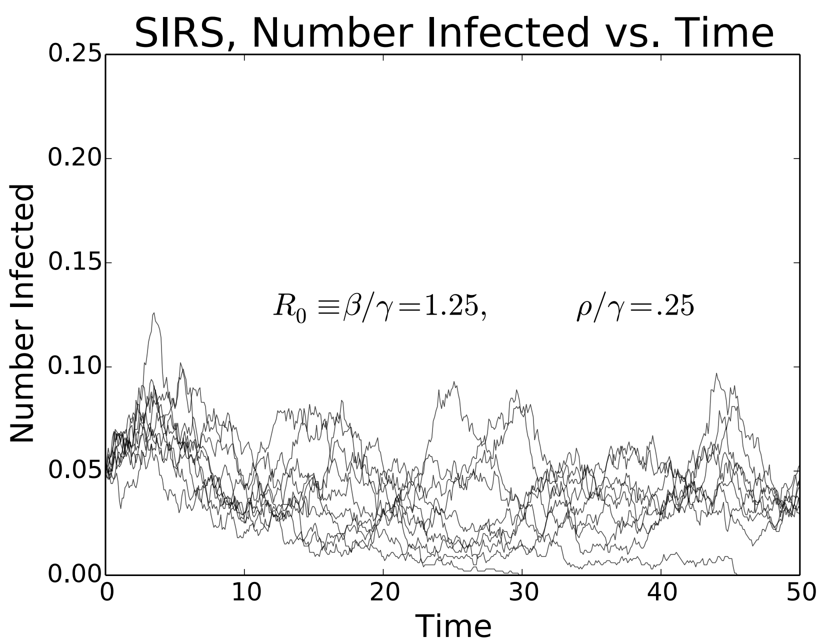
- Stochastic SIRS model – trajectories fluctuate
- Fluctuations not present in deterministic model
- Fluctuations lead to extinction events, end persistence



- Persists with probability given by model parameters
- **To understand disease persistence, we must understand fluctuation size compared to the mean**

Ensembles of Trajectories

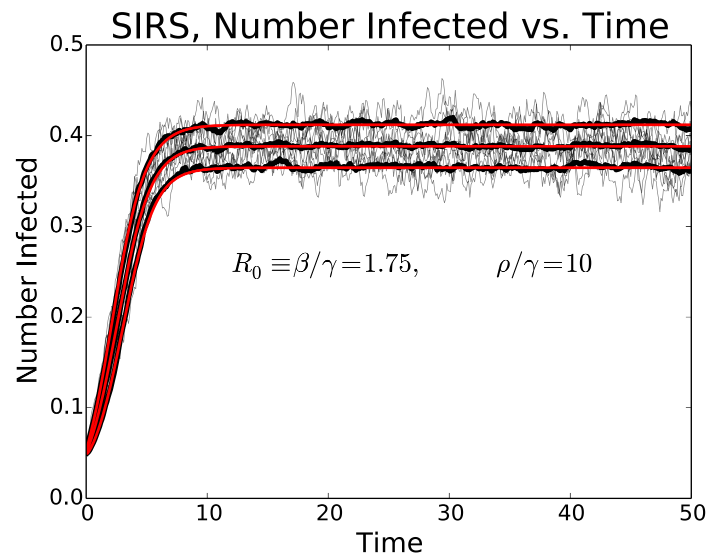


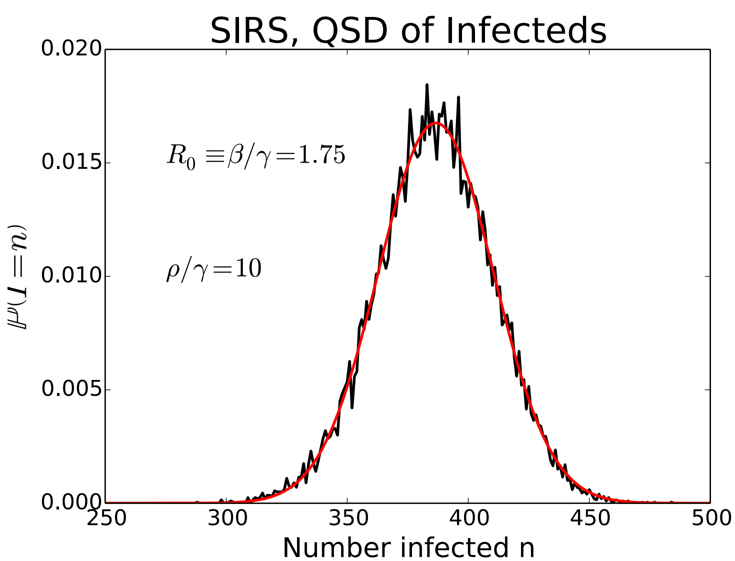


- Stochastic model trajectories drawn from an ensemble
- Want to describe the ensemble's distribution
- How does the ensemble depend on model parameters?

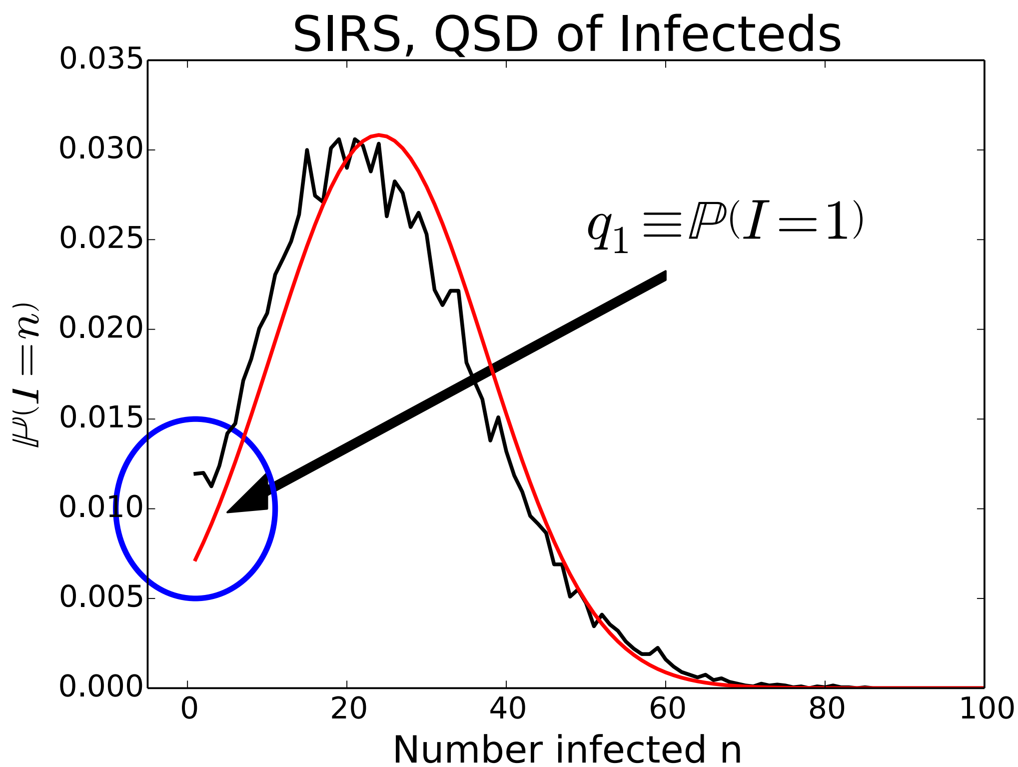
Moment Closure

- "Quasistatic Distribution" (QSD)
- Approximate QSD
- Assume Gaussian distribution
- Perform Moment Closure
- Obtain ODEs describing behavior of QSD mean (μ_I) and variance (σ_I)
- Produce analytical approximation to QSD as function of parameters



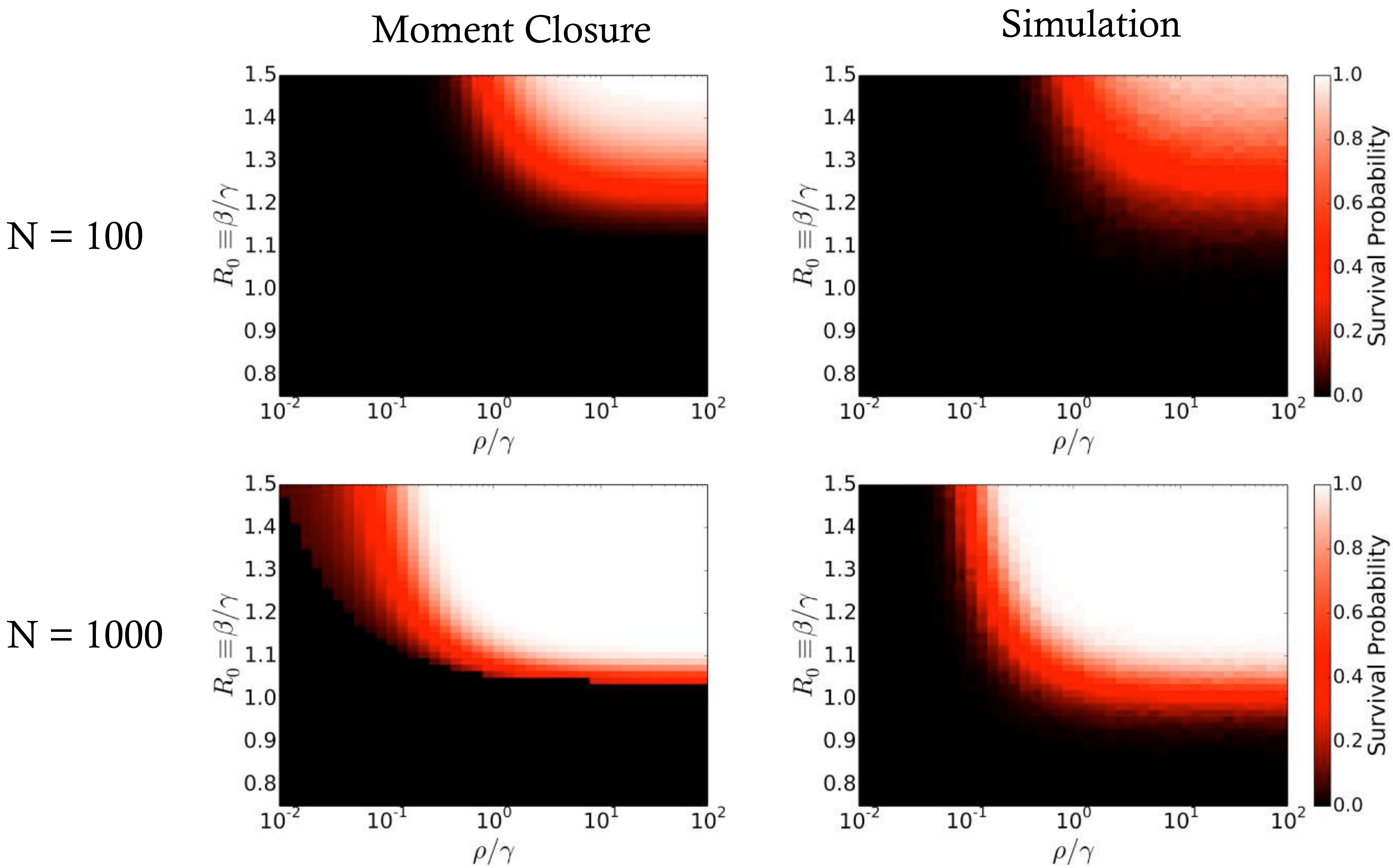


Extinction Rates



- Moment closure gives us mean and variance
- Can analytically approximate q_1
- $q_1 \propto$ Rate at which trajectories become extinct
- Allows us to predict probability of persistence at time *t*

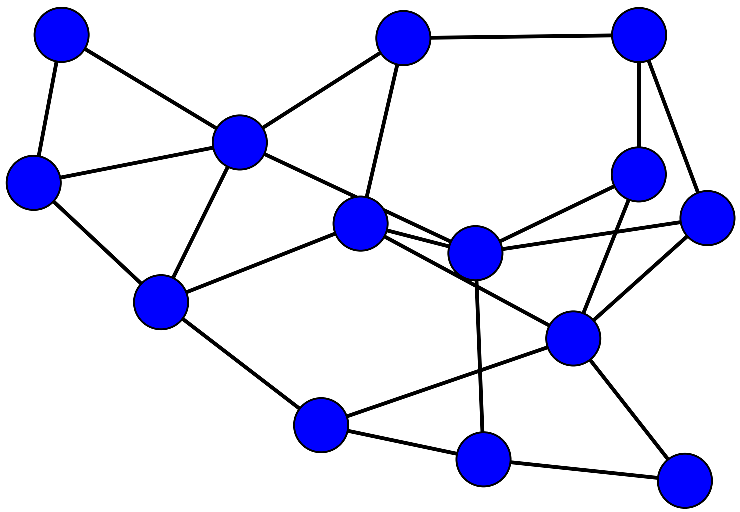
Persistence of Endemic Disease



- Bright regions show where disease persists at t = 50

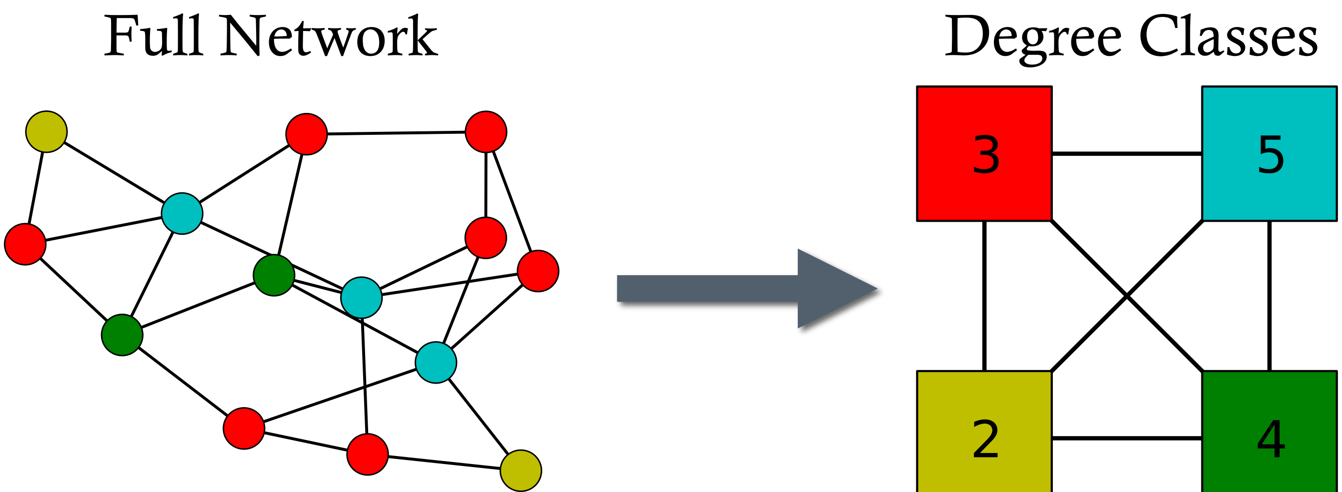
Disease Spread on Networks

- Can we use Moment Closure to predict persistence on networks?
- Need to account for contact heterogeneity



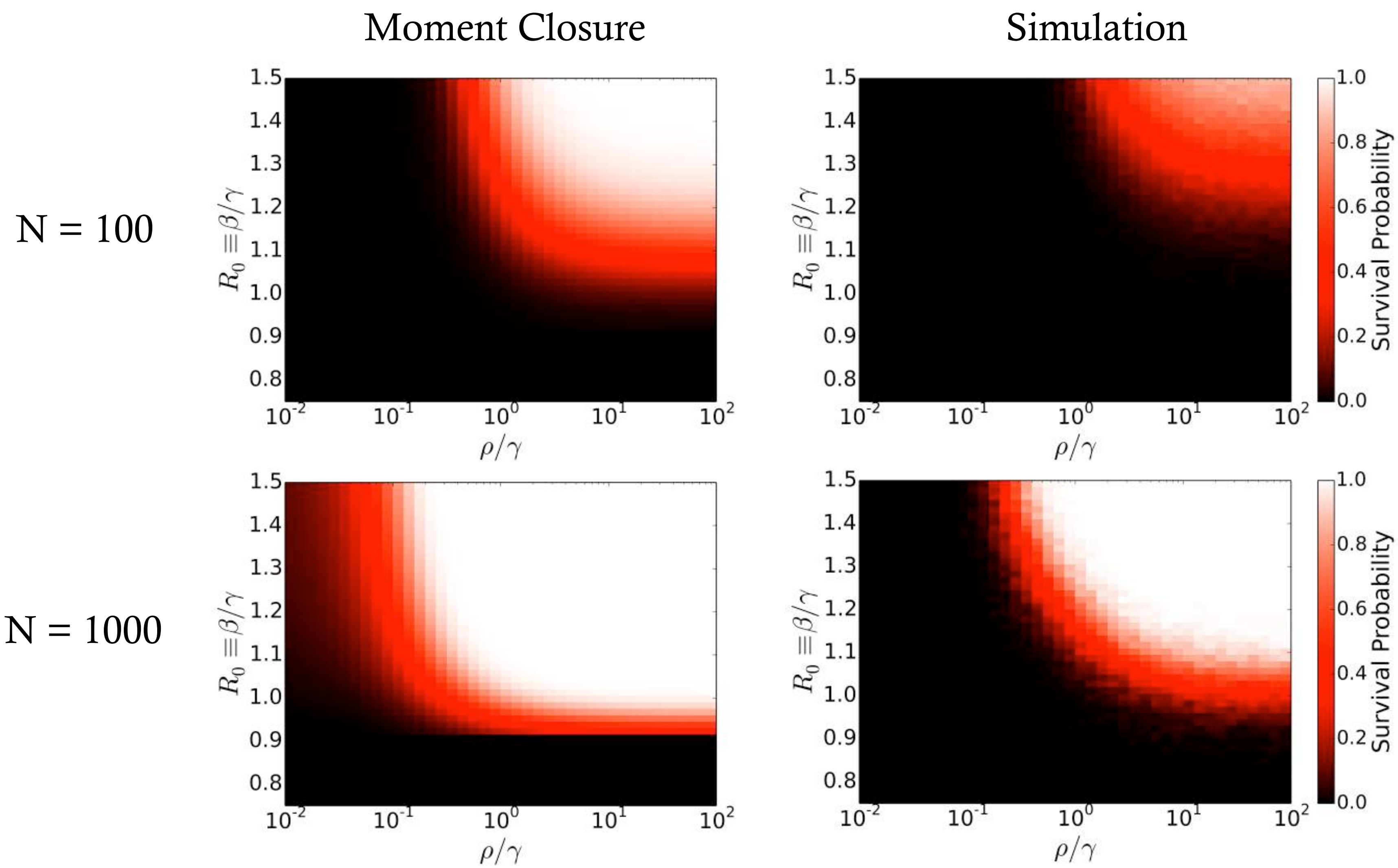
Disease Spread on Networks

- Heterogeneous MFT
- Separate network into K coupled degree classes



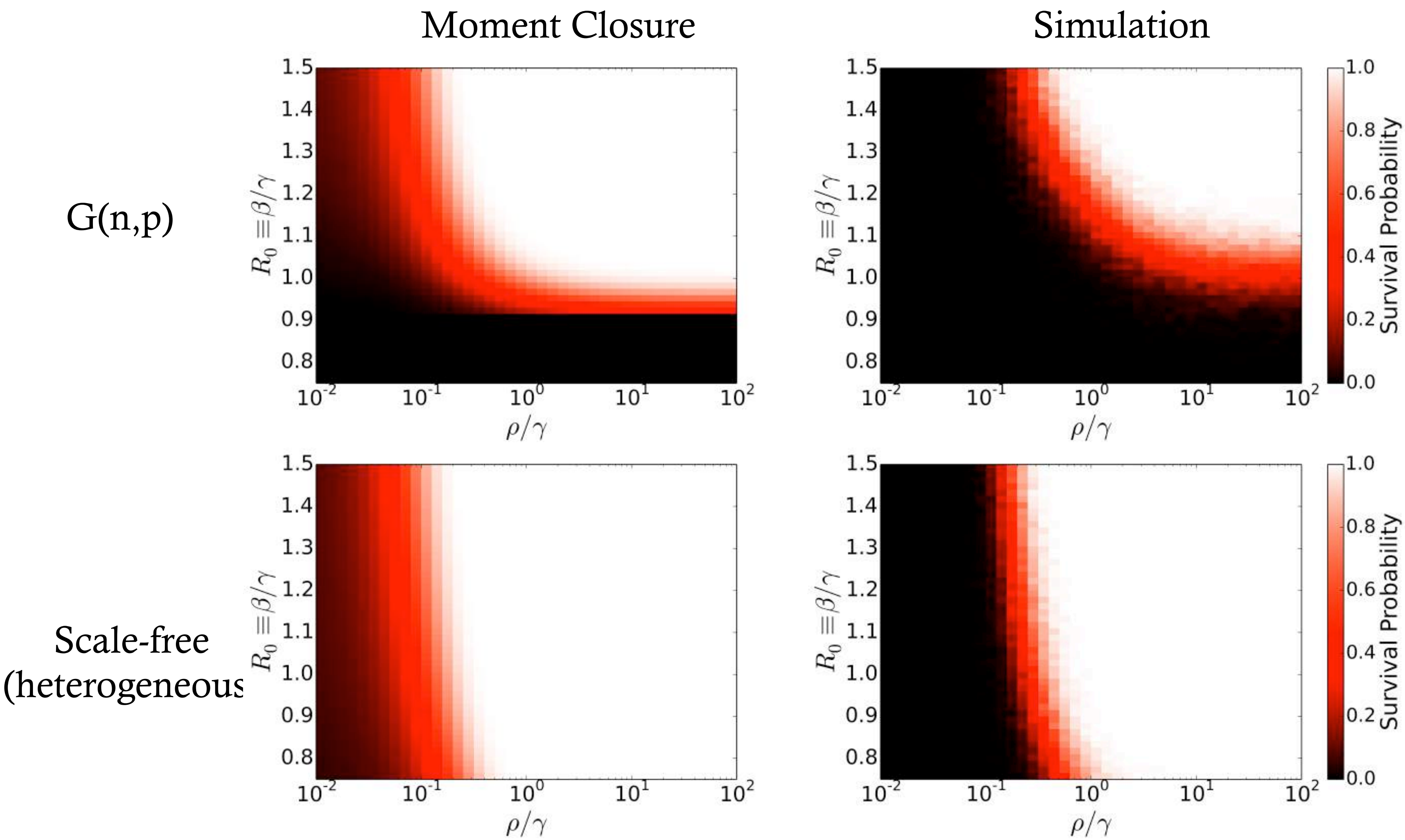
- QSD now a K-dimensional multivariate Gaussian
- Repeat Moment Closure calculations
- Obtain means and variances for all degree classes

Persistence of Endemic Disease, G(n,p) graphs



- Bright regions indicate persistence at t = 50

Network Topology Affects Persistence



- Bright regions indicate persistence at t = 50

Conclusions

- Demonstrated utility of Moment Closure for understanding persistence of disease in stochastic SIRS model
- Extended application of Moment Closure to networks case
- Good qualitative agreement between analytic results and simulations
- Quantitative disagreement in parameter regime where assumptions fail

Further Questions

- For what types of network is our Moment Closure approximation more or less accurate?
- Are there further analytical tools to correct for when our approximations fail?

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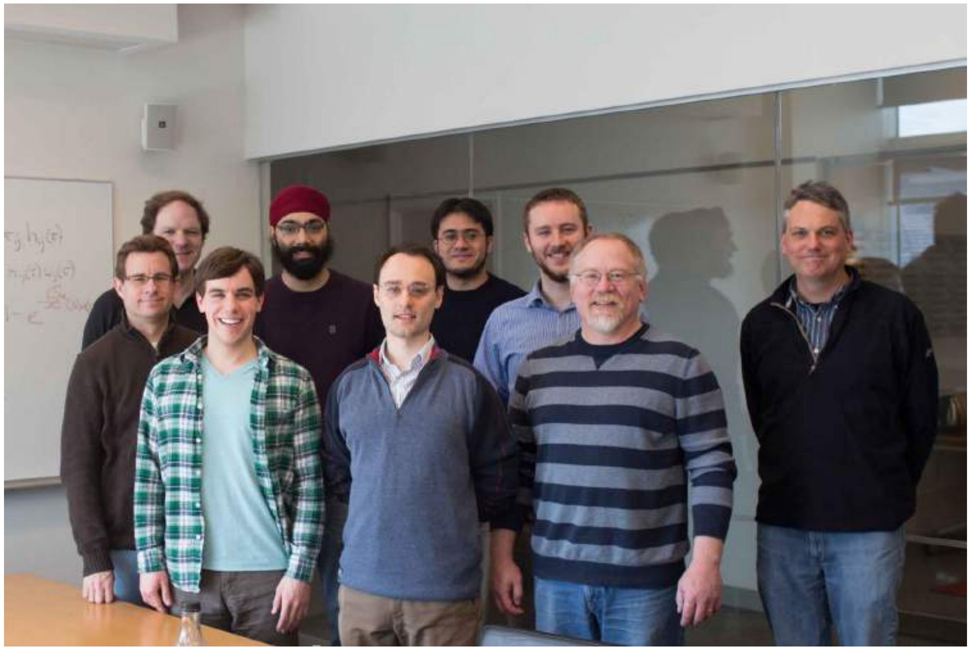

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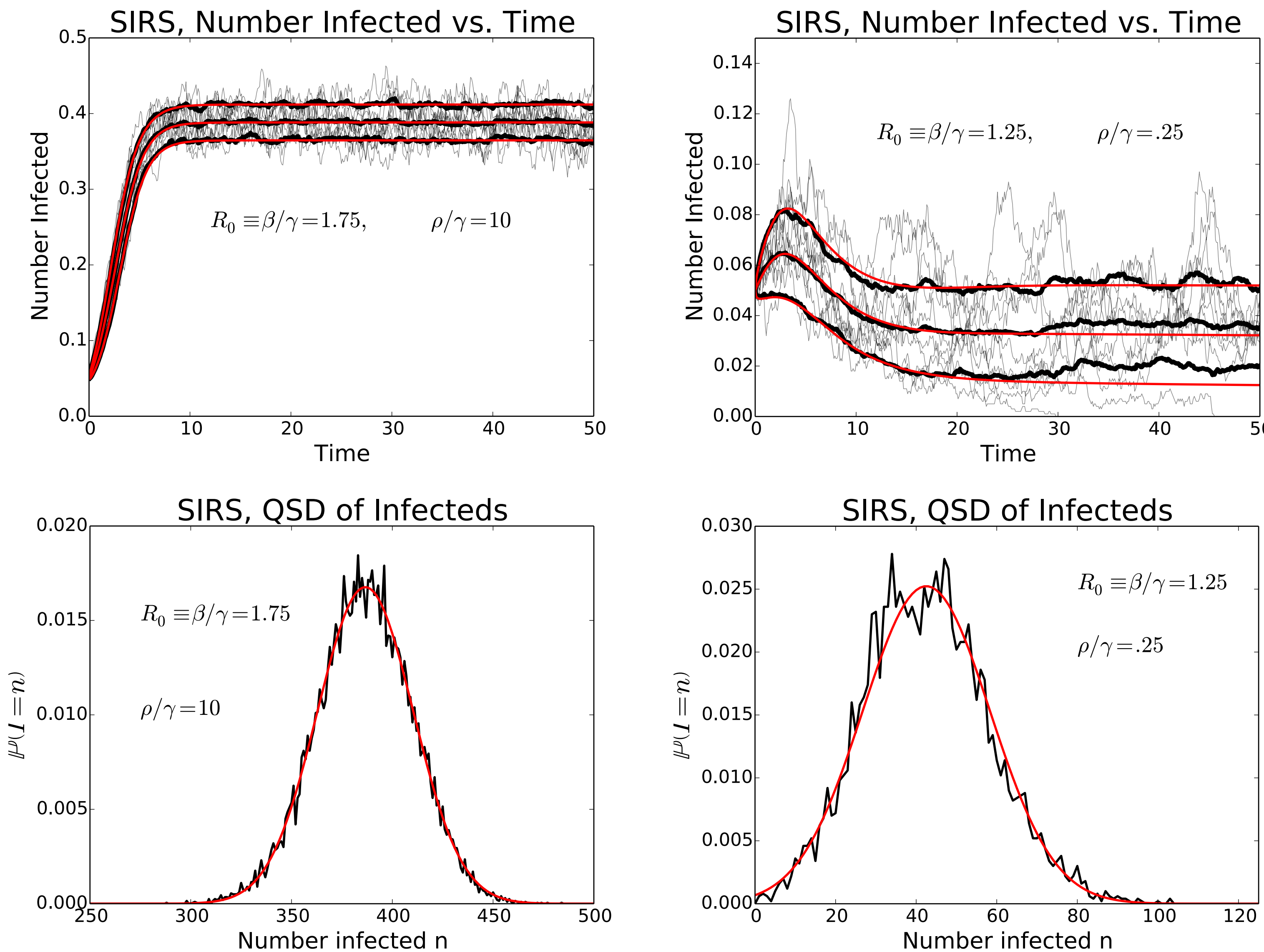
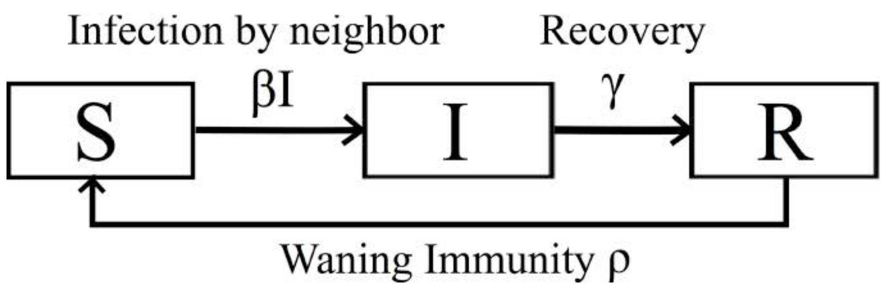
Moment Closure



Stochastic SIRS Model

SIRS Model



Define transitions in stochastic model

Event	Transition	Rate
$S+I\longrightarrow I+I$	$(m,n)\longrightarrow(m-1,n+1)$	$\beta mn/N$
$I\longrightarrow R$	$(m,n)\longrightarrow(m,n-1)$	γm
$R\longrightarrow S$	$(m,n)\longrightarrow(m+1,n-1)$	$\rho\,(N-m-n)$

Kolmogorov Forward Equation

$$\frac{\partial}{\partial t}p_{m,n}=\beta/N(m+1)(n-1)p_{m+1,n-1}(t)+\gamma(n+1)p_{m,n+1}(t)+\rho(N-(m-1)-n)p_{m-1,n}(t)$$
$$-(\beta mn/N+\gamma n+\rho(N-m-n))p_{m,n}(t)$$

Moment Closure

- Condition on no extinction → KFE for QSD
- Extra nonlinear term appears, proportional to extinction rate
- Assume it is small (for analytical tractability)

$$\gamma q_{\cdot,1}(t)\, q_{m,n}(t) \longrightarrow 0$$

- Define Probability Generating Function (PGF)

$$P(x, y, t) \equiv \sum_{m,n=0}^{\infty} q_{m,n}(t)\, x^m y^n$$

- Change variables in KFE to obtain PDE for PGF

$$\frac{\partial}{\partial t} P(x,y,t) = \beta(y^2 - xy)\frac{\partial^2 P}{\partial x \partial y} + \gamma(1-y)\frac{\partial P}{\partial y} + \rho(x-1)\left(N - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)P$$

Moment Closure

- Obtain PDE for *Cumulant* Generating Function (CGF)
- Assume Gaussian form of probability distribution, restricts CGF to quadratic form

$$K(\theta, \phi, t) = \mu_x \theta + \mu_y \phi + \sigma_{xy} \theta \phi + \frac{1}{2}\sigma_x^2 \theta^2 + \frac{1}{2}\sigma_y^2 \phi^2$$

- Obtain set of ODEs for all moments:

$$\frac{\partial \mu_x}{\partial t} = -\beta/N \mu_x \mu_y + \rho(N - \mu_x - \mu_y) - \beta/N \sigma_{xy}$$

$$\frac{\partial \mu_y}{\partial t} = \beta/N \mu_x \mu_y - \gamma \mu_y + \beta/N \sigma_{xy}$$

$$\frac{\partial \sigma_{xy}}{\partial t} = -\beta/N \mu_x \mu_y - (\beta/N + \gamma + \rho)\sigma_{xy} + \beta/N(\mu_x \sigma_{xy} - \mu_y \sigma_{xy} + \mu_y \sigma_x^2 - \mu_x \sigma_y^2) - \rho \sigma_y^2$$

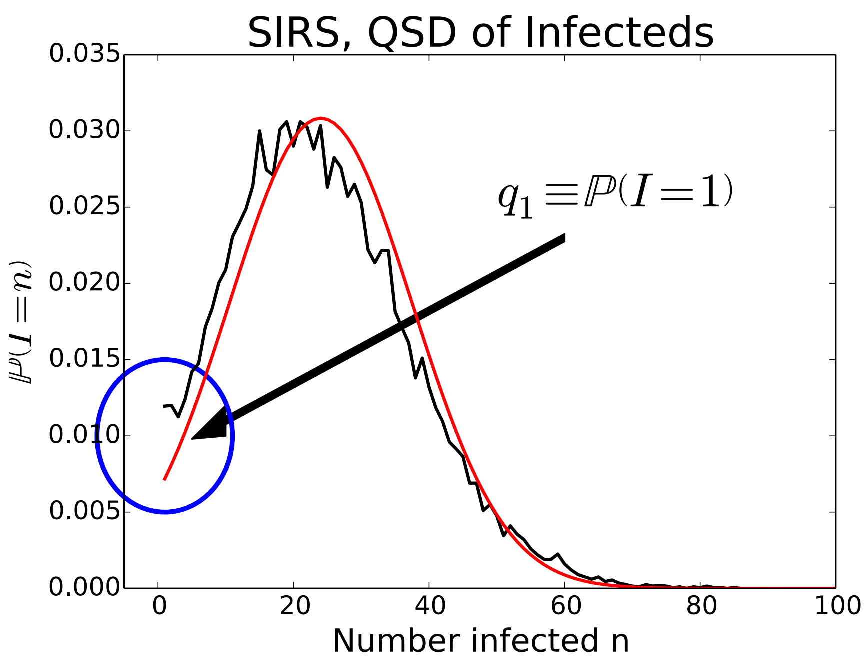
$$\frac{\partial}{\partial t}\sigma_x^2 = \beta/N \mu_x \mu_y + \beta/N \sigma_{xy} + \rho(N - \mu_x - \mu_y) - 2\beta/N(\mu_x \sigma_{xy} + \mu_y \sigma_x^2) - 2\rho(\sigma_{xy} + \sigma_x^2)$$

$$\frac{\partial}{\partial t}\sigma_y^2 = \beta/N \mu_x \mu_y + \beta/N \sigma_{xy} + \gamma \mu_y + 2\beta/N(\mu_y \sigma_{xy} + \mu_x \sigma_y^2) - 2\gamma \sigma_y^2$$

Moment Closure Assumptions

- Small rate of extinction q_1
  - Fails as $\mu_I \downarrow$ and $\sigma_I \uparrow$
  - Especially tricky for low μ_I – here we expect rapid extinction
  - Poor estimation of extinction rate when extinction rate is high
  - Is there a better analytical approximation in this regime?

- Gaussian assumption
  - Does not account for skew
  - Does not account for I > 0
  - "Renormalized Gaussian"

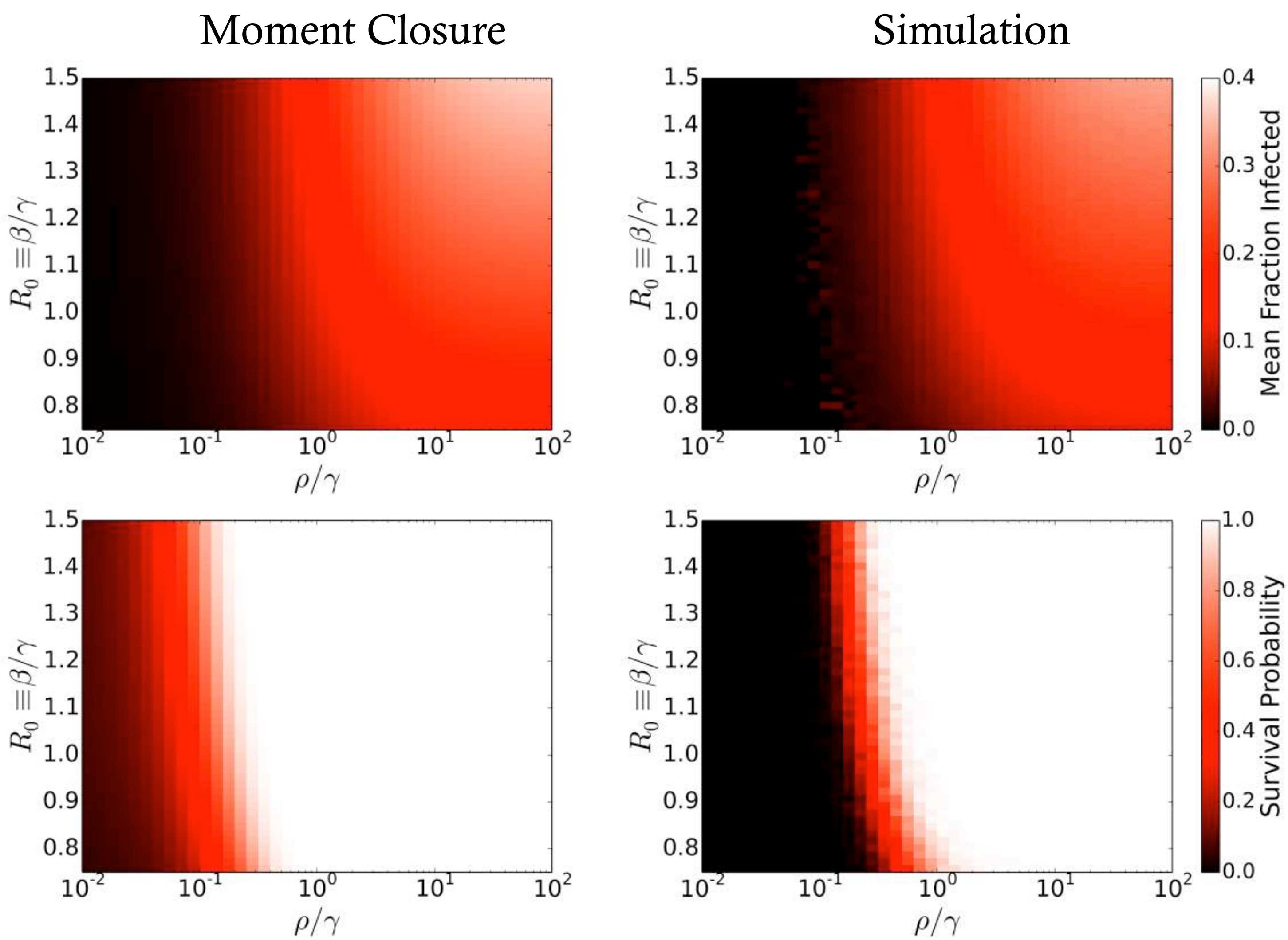


Moment Closure Assumptions

- Poor estimation of extinction rate when extinction rate is high

Mean Endemic Level

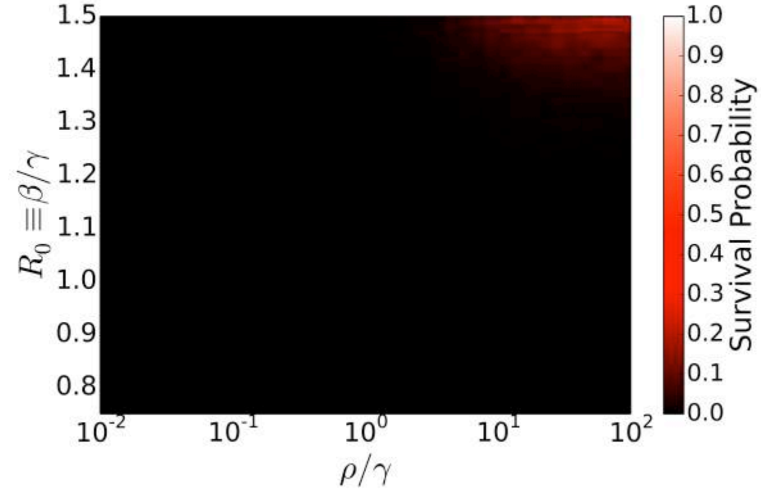
Survival Probability



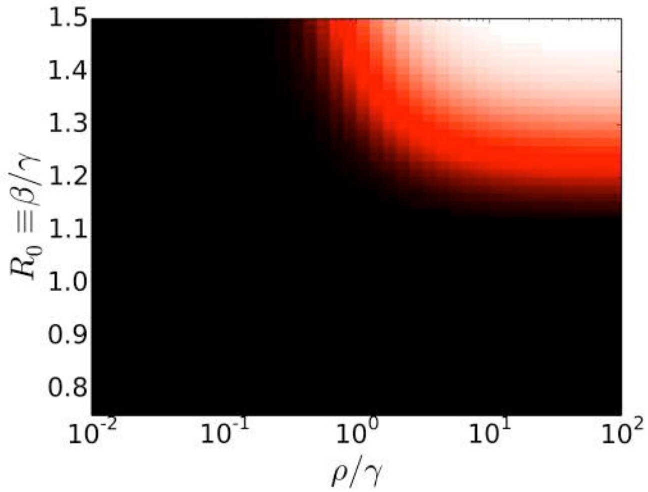
Other Types of Graphs

- Many different graphs with same degree distribution
- HMFT insufficient to describe
- What other graph properties affect disease persistence?

Random Regular Graph
<k> = 4



Moment Closure



Grid Lattice, <k>=4