Algorithms for Programming Contests - Week 12

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Geometry

Classical geometry

- Points as (x, y) coordinates.
- Lines represented by two points, or e.g. with Hesse normal form.

Problems

- Special cases: e.g. meet of parallel vs. non-parallel lines
- No simple systematic representation of all objects and operations.

Towards projective geometry

Idea of projective geometry

- Represent all objects uniformly through (higher-dimensional) vectors.
- Construct new objects with vector products.
- Represent transformations through matrix multiplications.

Projective geometry

Homogeneus coordinates

- Drawing plane is embedded into $\mathbb{R}^3$ at $z = 1$.
- Homogenization:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \lambda \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \lambda \in \mathbb{R} \setminus \{0\}$$

- Identify scalar multiples as the same point.
- Normalization/Dehomogenization:

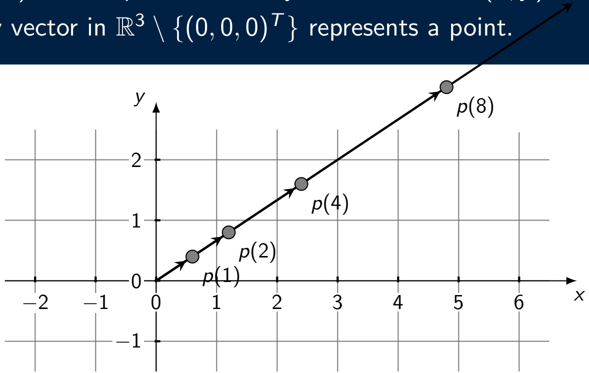
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} x/z \\ y/z \\ 1 \end{pmatrix} \to \begin{pmatrix} x/z \\ y/z \end{pmatrix}$$

- What happens when $z = 0$?

Projective geometry

Points at infinity

- Consider $p(t) = (x \cdot t, y \cdot t, 1)^T \sim (x, y, {}^1\!/_t)^T$.
- We have $\lim_{t \to \infty} p(t) = (x, y, 0)^T$.
- $(x, y, 0)^T$ is the point at infinity in the direction (x, y).
- Every vector in $\mathbb{R}^3 \setminus \{(0,0,0)^T\}$ represents a point.



Projective geometry

Lines in projective geometry

- Every vector $l = (x, y, z)^T \neq (0, 0, 0)^T$ also represents a line.
- Line given by intersection of plane through $(0, 0, 0)^T$ where l is a normal with drawing plane.
- Scalar multiples represent the same line.

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Lines at infinity

- One line not contained in $\mathbb{R}^2$: $(0, 0, 1)^T$.
- Contains all the points at infinity.

Operations in projective space

Point-on-line

- A point p is on a line l iff the vectors are orthogonal.
- Vectors p and l orthogonal iff *dot/scalar product* $p \cdot l = 0$.

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Line through two points

- Line through points p and q has to be orthogonal to both vectors.
- Line given by *cross product* $p \times q$.

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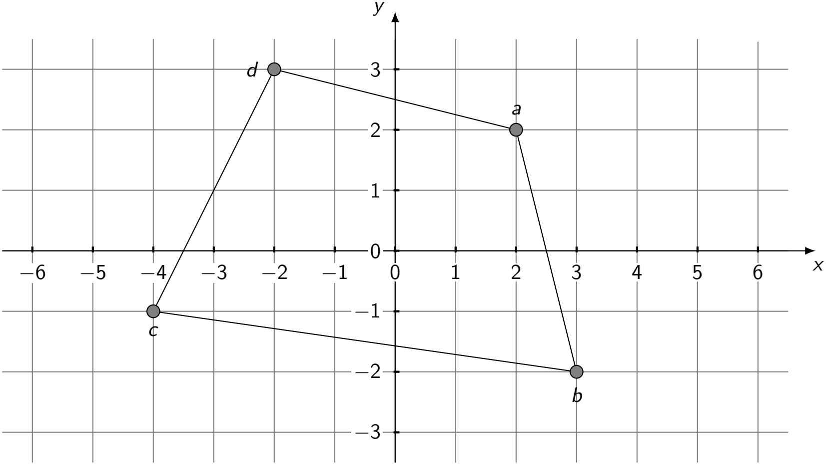
Intersection of two lines

- Intersection point of lines l and m has to be on l and m.
- Orthogonal to both: point given by $l \times m$.

Example: intersection of diagonals



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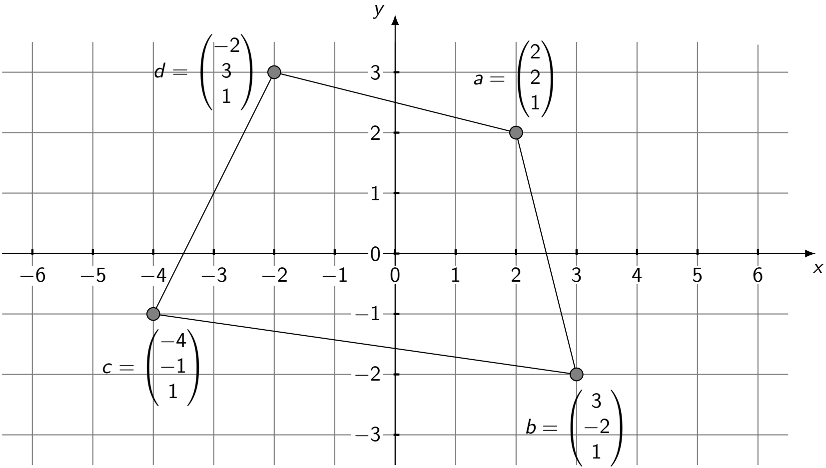

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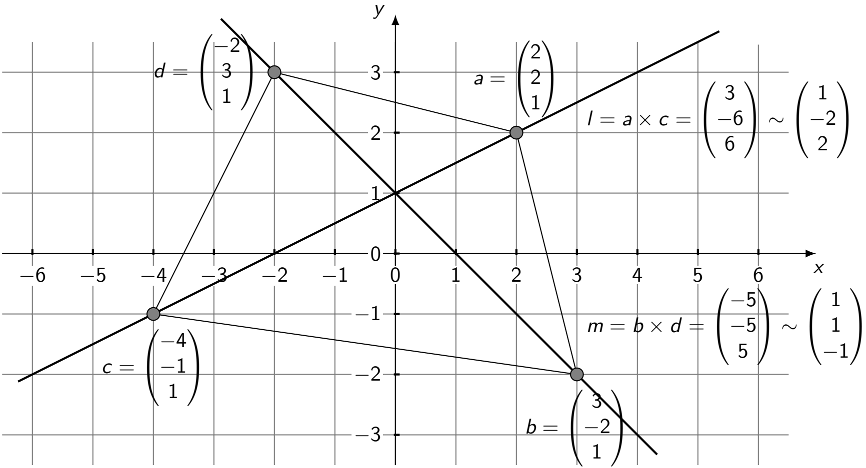

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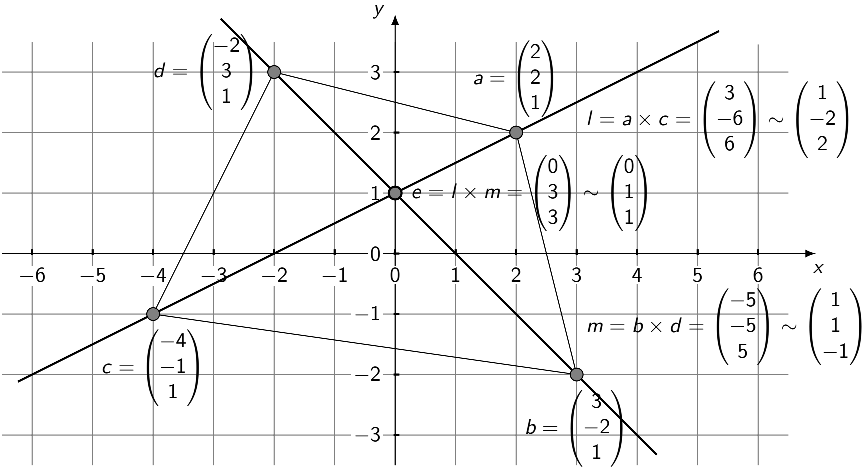
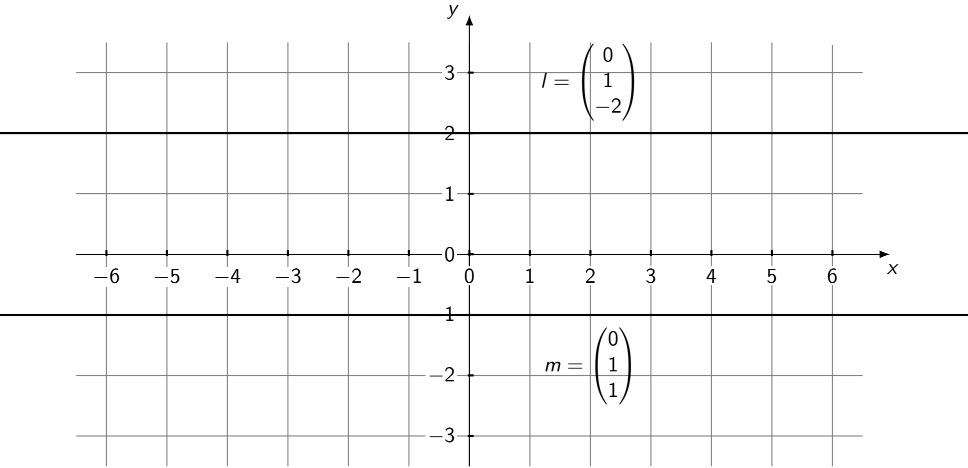


$$e = l \times m = (a \times c) \times (b \times d)$$

Points at infinity revisited

What happens if we take the intersection of two parallel lines?



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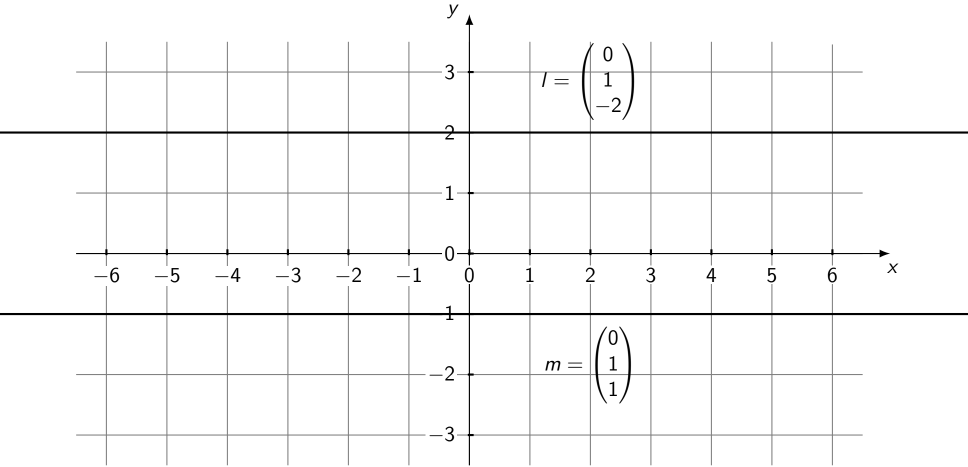

$$p = l \times m = (3, 0, 0)^T \sim (1, 0, 0)^T \sim (-1, 0, 0)^T$$

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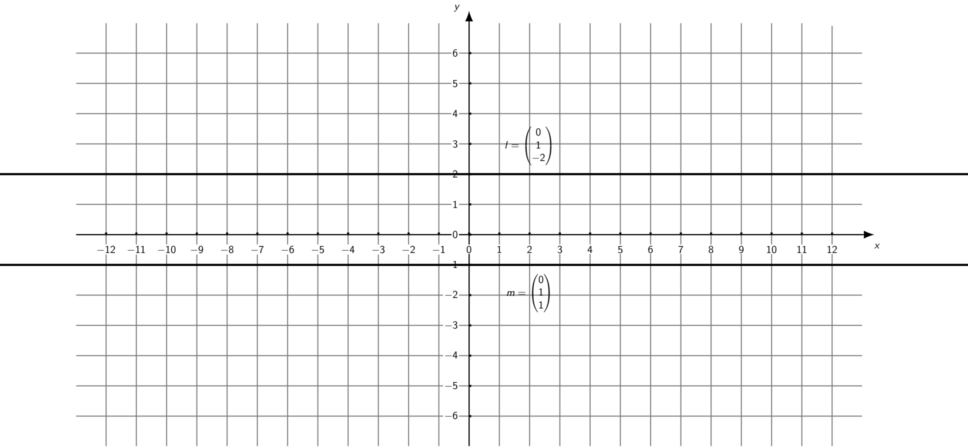

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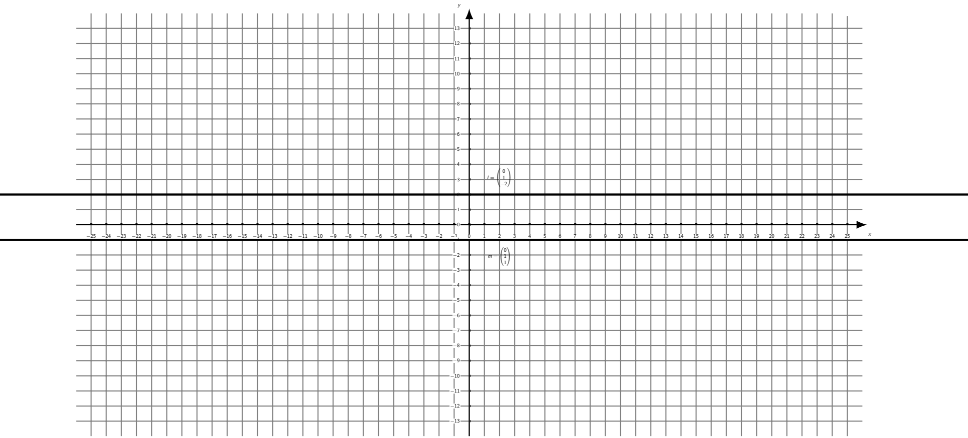

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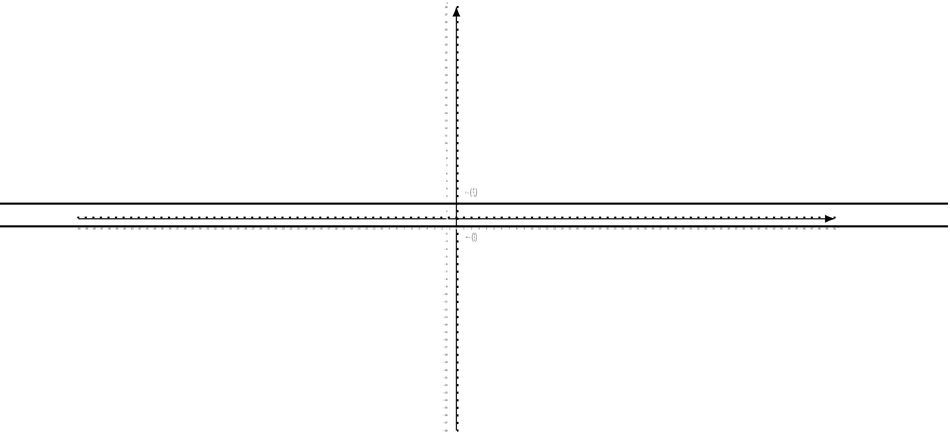

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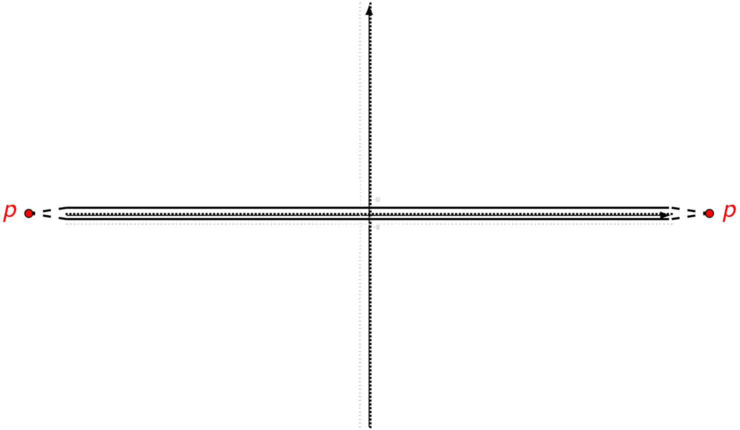

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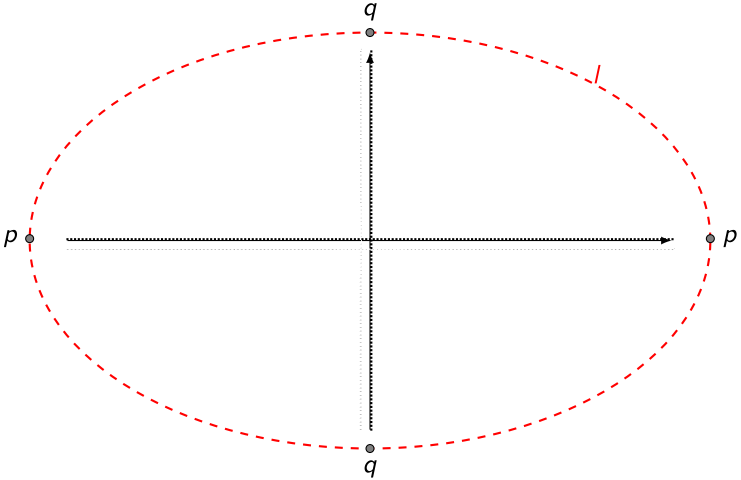
Points at infinity revisited

Connection of points at infinity gives line at infinity:



$$l = p \times q = (1,0,0)^T \times (0,1,0)^T = (0,0,1)^T$$

Constructions with points at infinity

Parallel

- Line m parallel to l through a point p.
- Infinite point in direction of l: $q = l \times (0, 0, 1)^T$. Then $m = p \times q$.

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- Line m perpendicular to l through a point p.
- Infinite point in direction of l: $q = l \times (0,0,1)^T = (x,y,0)^T$.
- Orthogonal direction to q: $q^\perp = (y,-x,0)^T$. Then $m = p \times q^\perp$.

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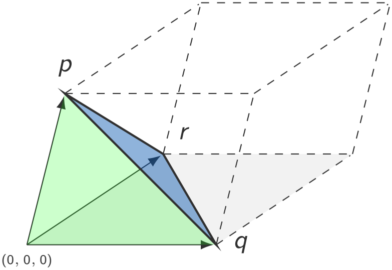
Projection

- Projection of point p on line l.
- Compute perpendicular m, then $r = m \times l$.

Area of a triangle

- Find area of the triangle given by three points p, q and r.
- First normalize p, q, r (scale to $z = 1$).
- Obtain matrix $M = \begin{pmatrix} p & q & r \end{pmatrix}$.
- Area is then $\frac{1}{2} |\det M| = \frac{1}{2} |p \cdot (q \times r)|$.
- $p \cdot (q \times r) = (p \times q) \cdot r$ is called the *triple product* of p, q, r.
- Gives value of counter-clockwise function: $\mathrm{CCW}(p, q, r) = p \cdot (q \times r)$.

Area of a triangle



- Volume of parallelepiped spanned by p, q, r: $V_{\text{par}} = |p \cdot (q \times r)|$
- Volume of tetrahedron spanned by p, q, r: $V_{\text{tetra}} = \frac{1}{6} V_{\text{par}}$.
- Volume of tetrahedron spanned by p, q, r: $V_{\text{tetra}} = \frac{1}{3} A_{\text{triangle}} \cdot h$, where A_{triangle} is area of base triangle (p, q, r) and h is the height.
- $h = 1$ by normalization $\Rightarrow A_{\text{triangle}} = \frac{1}{2} |p \cdot (q \times r)|$.

Transformations in Euclidean and projective space

Rotations

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Translations

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

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Problem: concatenation of operations

In projective space

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Affine transformations

Transformations preserving parallelity

- Shifting
- Scaling
- Rotating
- Mirroring
- Shearing
- Combinations of all these operations

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix},\ \det(M) \neq 0 \qquad \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto M \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

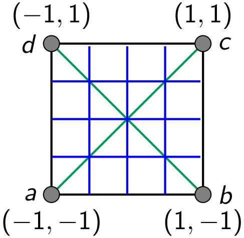
Projective transformations

Arbitrary projective transformation

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},\ \det(M) \neq 0 \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto M \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- M and λM for $\lambda \neq 0$ represent the same transformation.
- All affine transformations and perspective distortions.
- Collinearity and point-on-line property is preserved.
- Every transformation preserving collinearity is projective.
- Uniquely determined by four non-collinear points and their images.

Example: projective transformation



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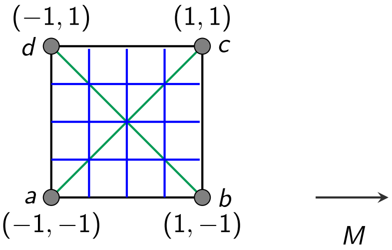
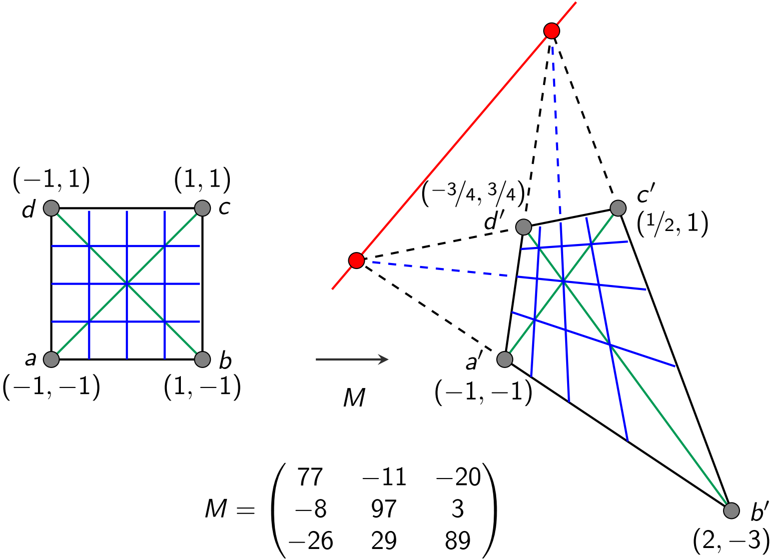


$$M = \begin{pmatrix} 77 & -11 & -20 \\ -8 & 97 & 3 \\ -26 & 29 & 89 \end{pmatrix}$$

Example: projective transformation



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Computing projective transformations

Computing M

Given four points a, b, c, d and their respective images a', b', c', d' by a projective transformation, what is a matrix M for this transformation?

Assumption: no three points of a, b, c, d or a', b', c', d' are collinear.

Computing projective transformations

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Assumption: no three points of a, b, c, d or a', b', c', d' are collinear.

Solve linear equation system to find M and $\lambda_a, \lambda_b, \lambda_c, \lambda_d$ such that:

$$M \cdot a = \lambda_a a' \qquad M \cdot b = \lambda_b b' \qquad M \cdot c = \lambda_c c' \qquad M \cdot d = \lambda_d d'$$

M unique up to scalar multiple: one λ can be fixed.

Solve linear equation system with $9 + 3 = 12$ variables and 12 equations.

Computing projective transformations

Fix $\lambda_d = 1$, and consider the following special case:

$$a = (1,0,0)^T \qquad b = (0,1,0)^T \qquad c = (0,0,1)^T \qquad d = (1,1,1)^T$$

$$M \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} = \lambda_a a' \quad M \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = \lambda_b b' \quad M \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} = \lambda_c c' \quad M \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = d'$$

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$$M = \begin{pmatrix} | & | & | \\ \lambda_a a' & \lambda_b b' & \lambda_c c' \\ | & | & | \end{pmatrix} \qquad \begin{pmatrix} | & | & | \\ a' & b' & c' \\ | & | & | \end{pmatrix} \cdot \begin{pmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \end{pmatrix} = d'$$

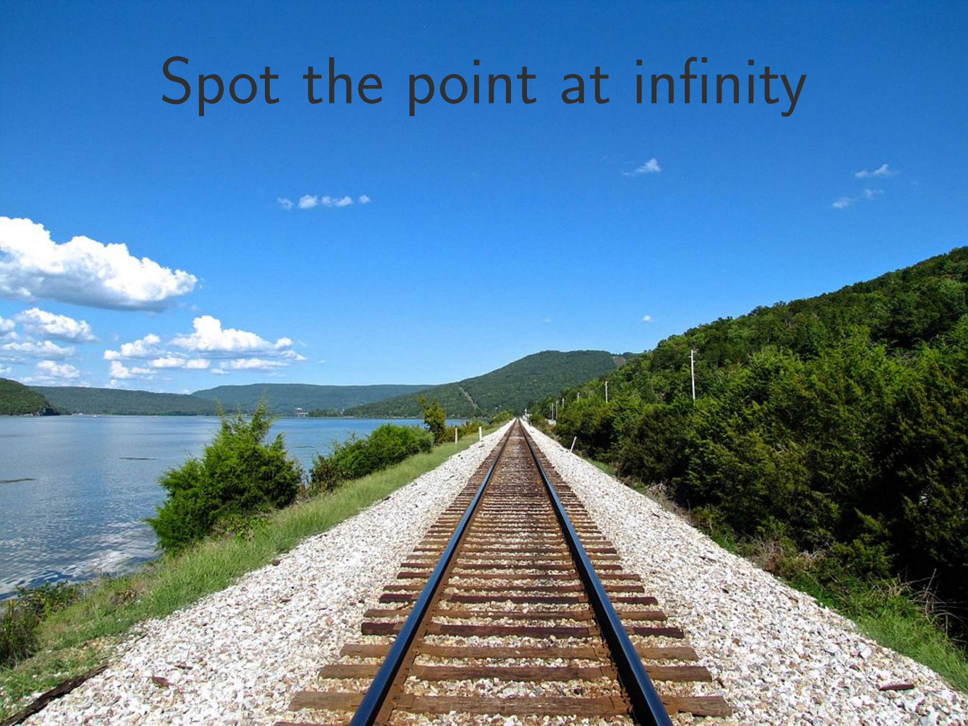
Only need to invert single 3×3 matrix to obtain $\lambda_a, \lambda_b, \lambda_c$ and then M.

Computing projective transformations

General case:

$$
\begin{tikzcd}
a, b, c, d \arrow[rr, bend right, "M_1 \cdot M_2^{-1}"'] & \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \arrow[l, bend right, "M_2"'] \arrow[r, bend left, "M_1"] & a', b', c', d'
\end{tikzcd}
$$

Spot the point at infinity



More projective geometry

There is a lot more

- Circles, conic sections.
- Measurements of lengths and angles.
- Cross-ratios and projective invariants.
- Higher-dimensional projective geometry.
- $\Rightarrow$ lectures “Geometriekalküle” and “Projective Geometry” by Prof. Richter-Gebert.