Algorithms for Programming Contests - Week 5

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Flow Network

Definition (Flow network)

A *flow network* is a directed graph $G = (V, E)$, where each edge (u, v) is assigned a nonnegative *capacity* $c(u, v) \geq 0$ and there are two designated vertices, the *source* s and the *target* t.
W.l.o.g., we will only consider flow networks without antiparallel edges, i.e. if $(u, v) \in E$, then $(v, u) \notin E$.

Definition (Flow)

For a given flow network $G = (V, E)$ with capacity function c, a *flow* is a function $f \colon E \to \mathbb{R}$ satisfying

$$\begin{aligned} \forall (u, v) \in E \colon \quad & 0 \leq f(u, v) \leq c(u, v) \\ \forall u \in V \setminus \{s, t\} \colon \quad & \sum_{\{v \colon (v,u) \in E\}} f(v, u) = \sum_{\{v \colon (u,v) \in E\}} f(u, v) \end{aligned}$$

Maximum Flow Problem

Definition (Flow value)

The *value* $|f|$ of a flow f is defined as

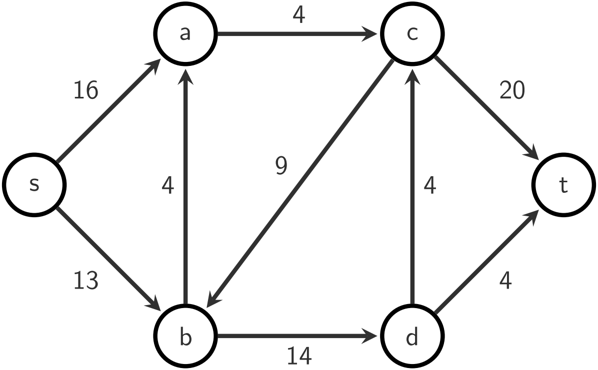
$$|f| = \sum_{\{v\colon (s,v)\in E\}} f(s,v) - \sum_{\{v\colon (v,s)\in E\}} f(v,s)$$

Definition (Maximum Flow Problem)

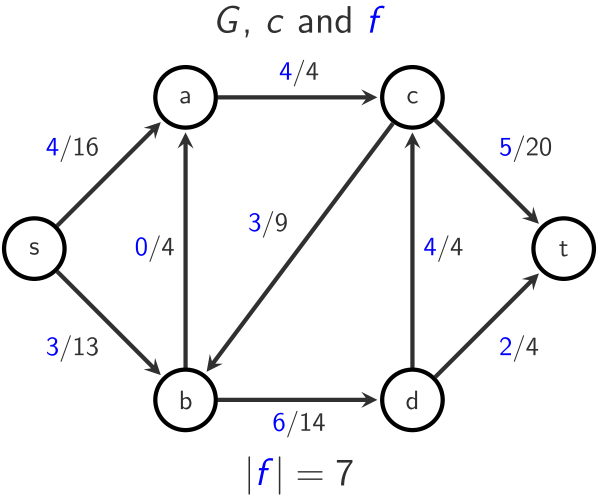
For a given flow network G with source s and target t, what is a flow f with maximal value $|f|$ over all flows?

Example: Flow network

G and c



Example: Flow network with flow



Reductions to maximum flow

Hints

- Minimum flow: send no flow at all.
- Multiple sources/sinks: add super-source/sink and edges with infinite capacity to other sources/sinks.
- Sources/sinks with supply constraints: add super-source/sink with edges to sources/sinks with corresponding capacity.
- Demands for sinks: check value of maximum flow.
- Vertex capacities: Split up vertex with edge of that capacity in between.
- Antiparallel edges: Insert vertex in between one edge (or use multi-edges when necessary).
- Undirected edges: Convert to two antiparallel directed edges.

Max-flow min-cut

Definition (Cut)

For a given flow network G with source s and target s, a *cut* $C = (S, T)$ is a partititon of V into two subsets S and T such that $s \in S$ and $t \in T$. The *capacity* $c(S, T)$ of a cut (S, T) is defined as

$$c(S, T) = \sum_{(u,v) \in (S \times T) \cap E} c(u, v)$$

Theorem (Min-cut max-flow theorem)

The maximum value $|f|$ over all flows f is equal to the minimum capacity $c(S, T)$ over all cuts (S, T).

Residual network

Definition (Residual capacity and residual network)

For a given flow network G and a flow f, and a pair of vertices $u, v \in V$, the *residual capacity* $c_f(u, v)$ is defined by

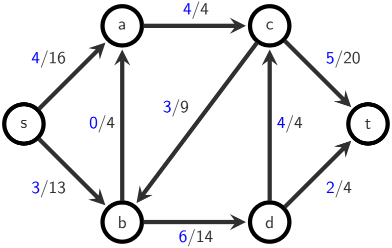
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$

The *residual network* of G induced by f is $G_f = (V, E_f)$, where

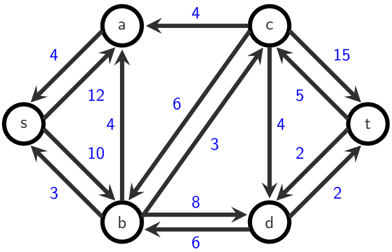
$$E_f = \{(u,v) \in V \times V \colon c_f(u,v) > 0\}$$

Example: Residual network

G, c and f



G_f and c_f



Augmenting flow

Definition (Augmenting flows)

If f is a flow in a flow network G and f' is a flow in the corresponding residual network G_f, then the *augmentation* $f \uparrow f'$ of f by f' is a flow of G defined as

$$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases}$$

Lemma

If f is a flow in a flow network G and f' is a flow in the corresponding residual network G_f, then $f \uparrow f'$ is also a flow in G and

$$|f \uparrow f'| = |f| + |f'|$$

Augmenting path

Definition (Augmenting path)

Given a flow network G and a flow f, an *augmenting path* p is a simple path in the residual network G_f from s to t.
The *residual capacity* of an augmenting path p is given by

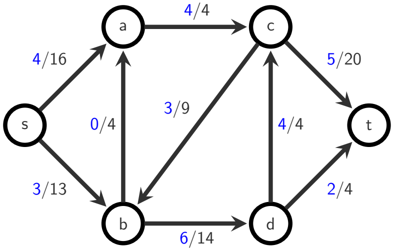
$$c_f(p) = \min\{c_f(u,v) \colon (u,v) \text{ is on } p\}$$

The flow f_p of an augmenting path p in G_f is defined as

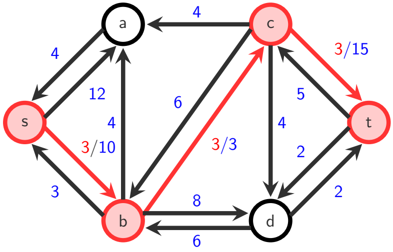
$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is on } p \\ 0 & \text{otherwise} \end{cases}$$

Example: Augmenting path

G, c and f

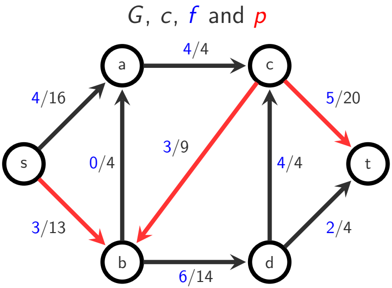


G_f, c_f, p and f_p



$$p = \text{sbct}$$
$$c_f(p) = 3$$

Example: Augmenting flow



$$p = \text{sbct}$$

$$c_f(p) = 3$$

Augmenting path algorithm (Ford-Fulkerson algorithm)

Algorithm 1 Ford-Fulkerson algorithm

```
▷ Initial flow is 0 (f ← 0)
for (u, v) ∈ E do
    f(u, v) ← 0
end for
while there exists a path p from s to t in the residual network G_f do
    ▷ Augment f by f_p (f ← f ↑ f_p)
    c_f(p) ← min{c_f(u, v): (u, v) is on p}
    for each edge (u, v) in p do
        if (u, v) ∈ E then
            f(u, v) ← f(u, v) + c_f(p)
        else
            f(v, u) ← f(v, u) − c_f(p)
        end if
    end for
end while
```

Analysis

How to decide with augmenting path to choose?

- Any path (with DFS): Ford-Fulkerson algorithm. Complexity $\mathcal{O}(|E|\,U)$ with integer capacities, where U is the value of the maximum flow. Possibly non-terminating for irrational capacities.
- Shortest path by number of edges (with BFS): Edmonds-Karp algorithm. Complexity $\mathcal{O}(|V|\,|E|^2)$.
- All shortest paths (blocking flows): Dinic's algorithm. Complexity $\mathcal{O}(|V|^2\,|E|)$.

Blocking flow

Definition (Level graph)

Given a residual network $G_f = (V, E_f)$, let $d_{G_f}(s, v)$ be the length of the shortest path from s to v in G_f (by number of edges).
The *level graph* of G_f is the graph $G_L = (V, E_L, c_L)$, where

$$E_L = \{(u, v) \in E_f : d_{G_f}(s, v) = d_{G_f}(s, u) + 1\}$$

$$c_L(u, v) = \begin{cases} c_f(u, v) & \text{if } (u, v) \in E_L \\ 0 & \text{otherwise} \end{cases}$$

Definition (Blocking flow)

A *blocking flow* in the level graph G_L is a flow f such that every path from s to t in G_L contains a saturated edge, i.e., an edge (u, v) with $f(u, v) = c_L(u, v)$.

Dinic's algorithm

Algorithm 2 Dinic's algorithm

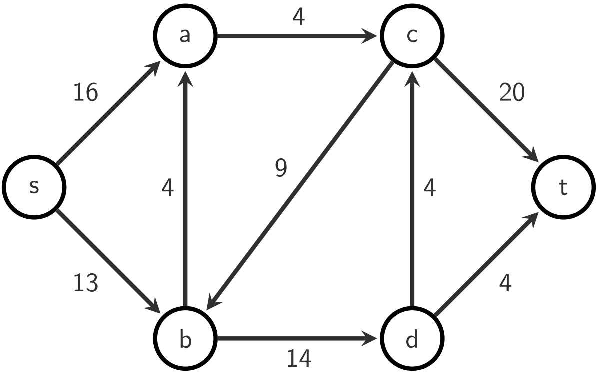
$f \leftarrow 0$
while there exists a blocking flow f' in G_L with $|f'| > 0$ **do**
    $f \leftarrow f \uparrow f'$
end while

Finding a blocking flow

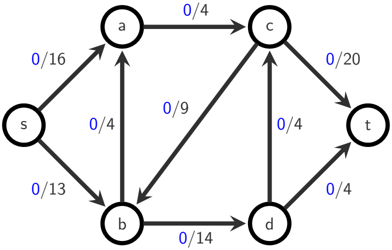
Algorithm 3 Finding blocking flows via DFS

$f' \leftarrow 0$; $p \leftarrow s$; $u \leftarrow s$
while $u \neq t$ **do**
  while there is an edge $(u, v) \in E_L$ with $f'(u, v) < c_L(u, v)$ **do**
    $p \leftarrow pv$
    $u \leftarrow v$
  end while
  if $u = t$ **then**
    $f' \leftarrow f' \uparrow f_p$; $p \leftarrow s$; $u \leftarrow s$
  else if $u = s$ **then**
    return f'
  else
    let (v, w) be the last edge on p; delete w from p
    delete (v, w) from E_L; $u \leftarrow v$
  end if
end while

Dinic's algorithm (example)



G, c and f



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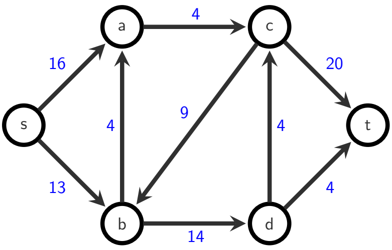
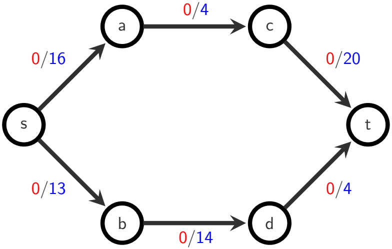


G_L, c_L and f'



v Vertex currently explored

v Vertex on current path

Edge on augmenting path

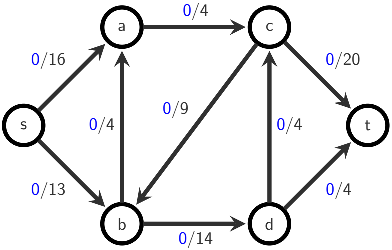
Edge on current path

Saturated/deleted edge

Current operation:

Find blocking flow

G, c and f



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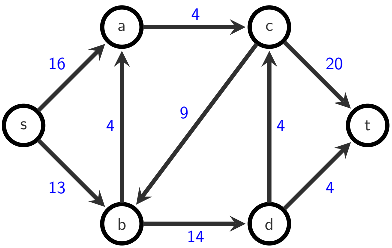

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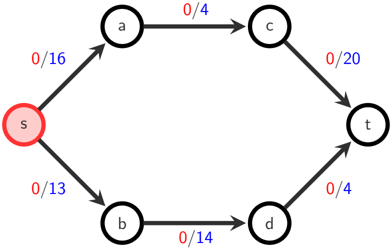

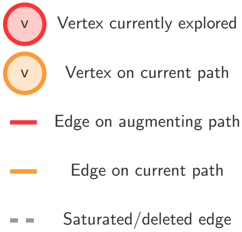

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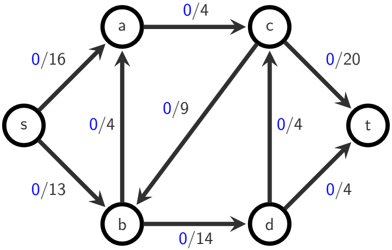
Find augmenting path

G, c and f



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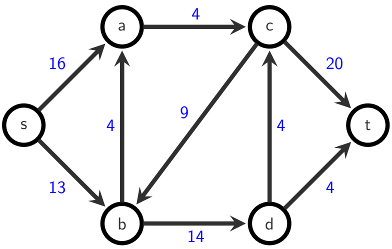

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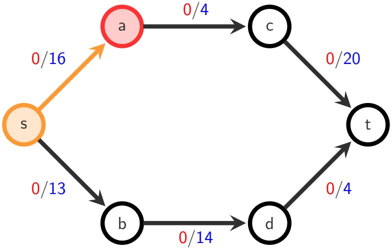

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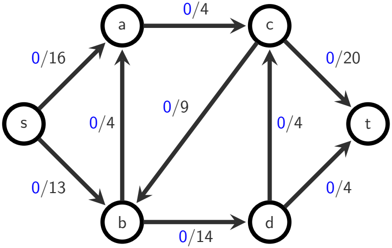

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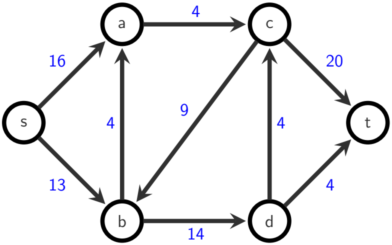

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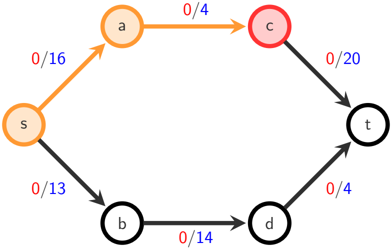

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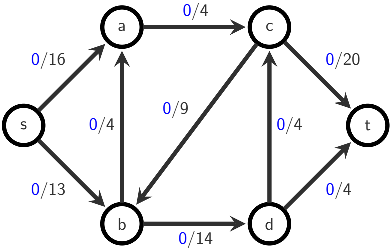

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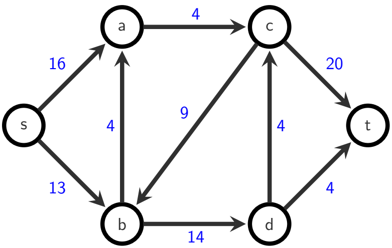

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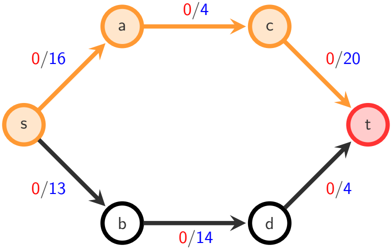

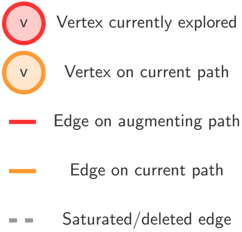

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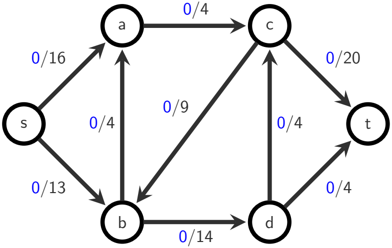
Find augmenting path

G, c and f



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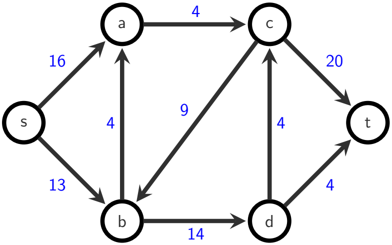

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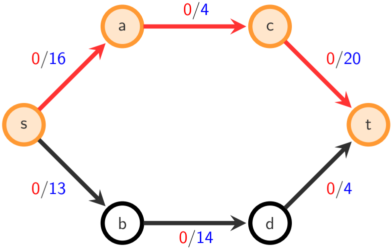

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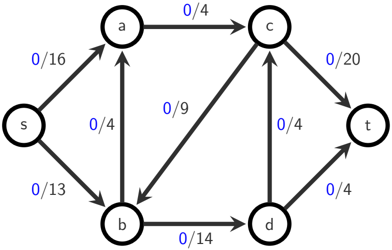
Augment f' by f_p

G, c and f



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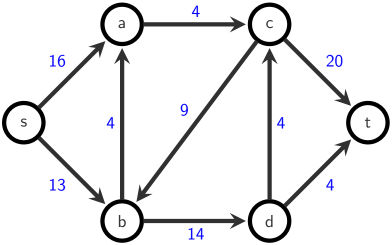

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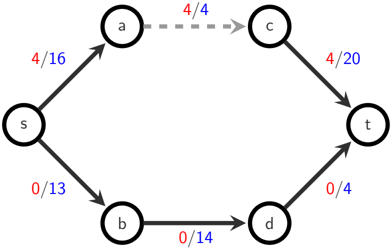

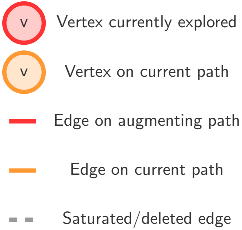

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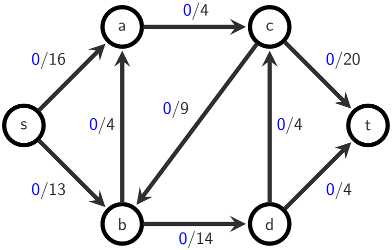

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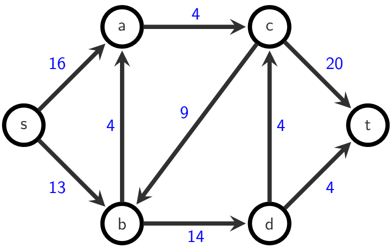

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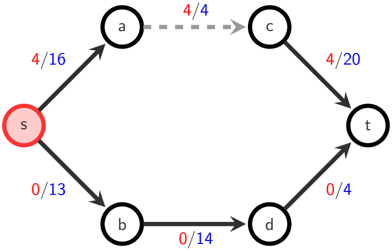
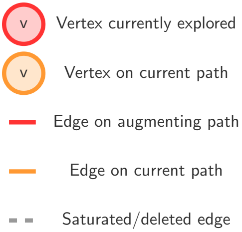
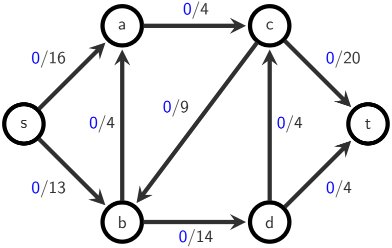




Current operation:

Find augmenting path

G, c and f



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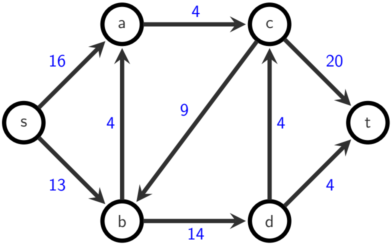

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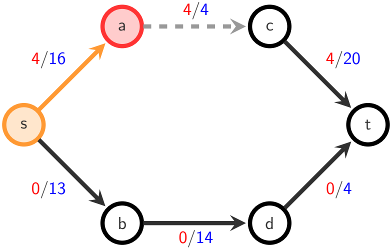

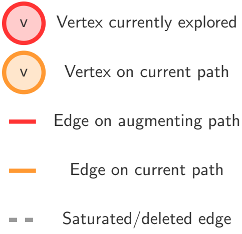

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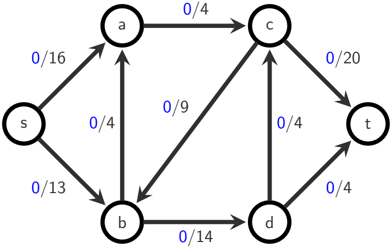

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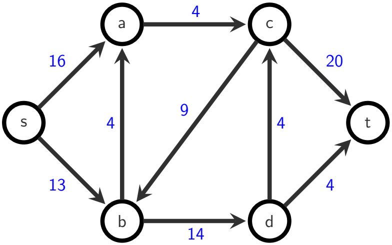

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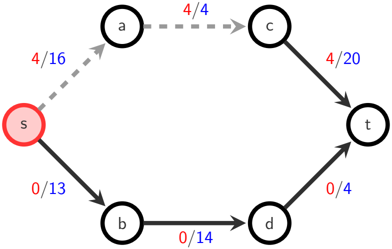

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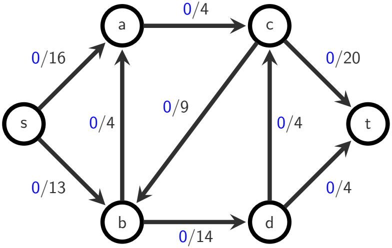

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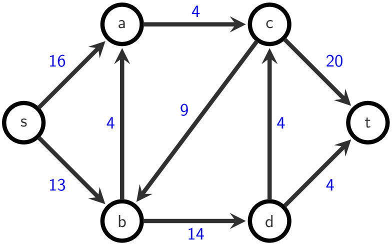

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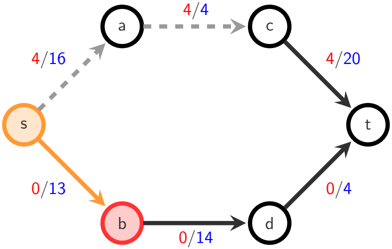

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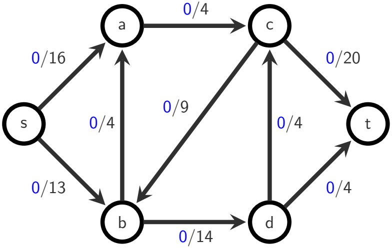

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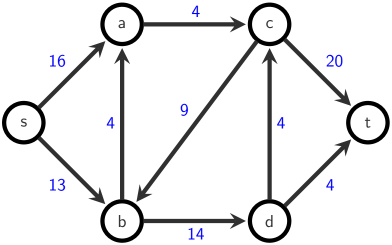

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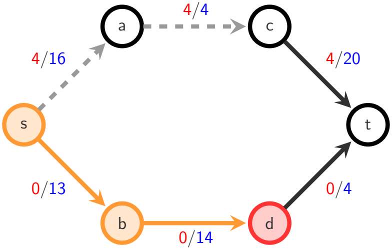

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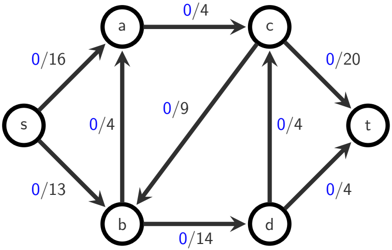
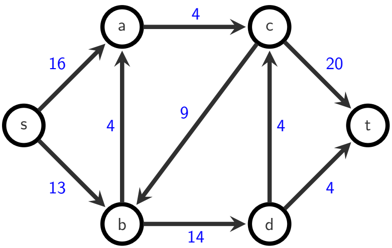
Find augmenting path

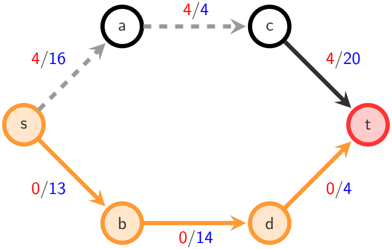
G, c and f

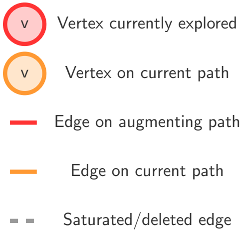


G_f and c_f



G_L, c_L and f'

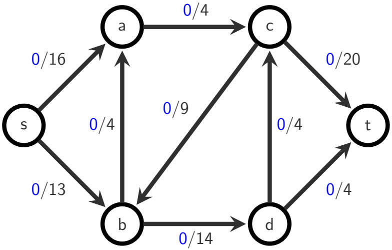




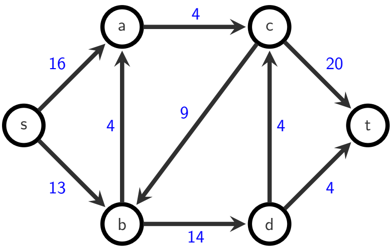
Current operation:

Find augmenting path

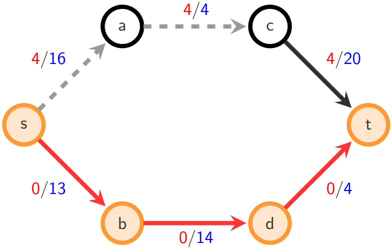
G, c and f



G_f and c_f



G_L, c_L and f'

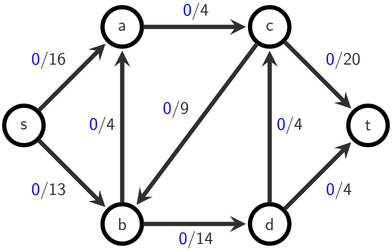


- v Vertex currently explored
- v Vertex on current path
- Edge on augmenting path
- Edge on current path
- Saturated/deleted edge

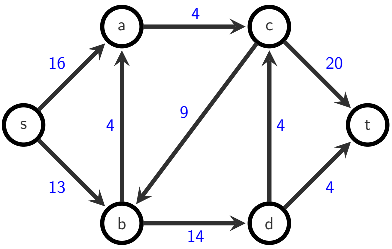
Current operation:

Augment f' by f_p

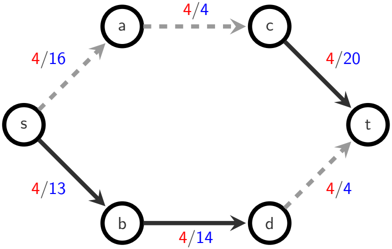
G, c and f

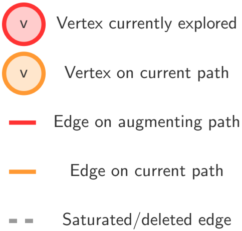


G_f and c_f



G_L, c_L and f'

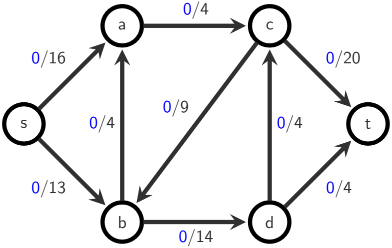




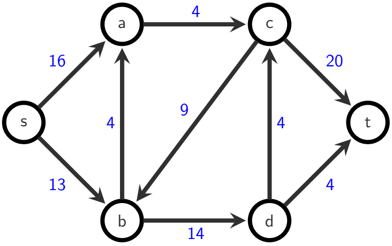
Current operation:

Augment f' by f_p

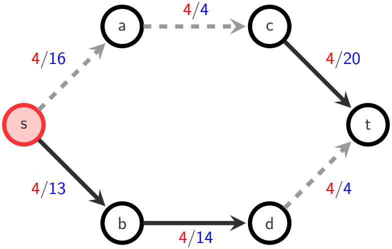
G, c and f



G_f and c_f



G_L, c_L and f'



Vertex currently explored

Vertex on current path

Edge on augmenting path

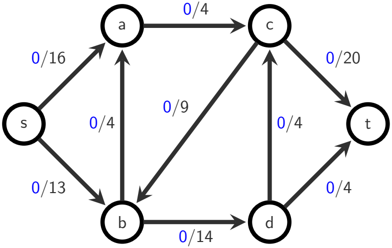
Edge on current path

Saturated/deleted edge

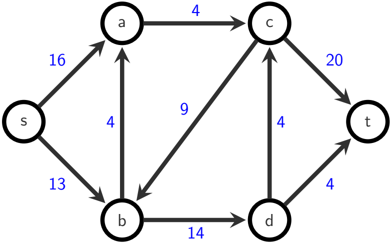
Current operation:

Find augmenting path

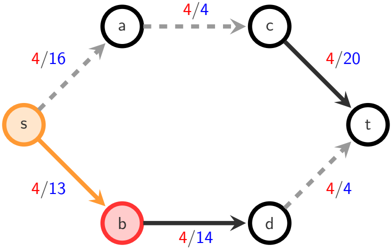
G, c and f



G_f and c_f



G_L, c_L and f'



Vertex currently explored

Vertex on current path

Edge on augmenting path

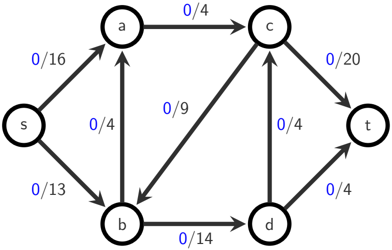
Edge on current path

Saturated/deleted edge

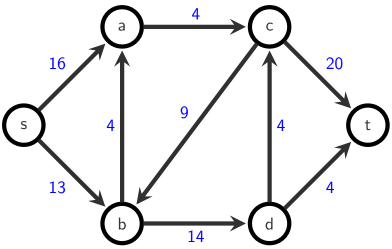
Current operation:

Find augmenting path

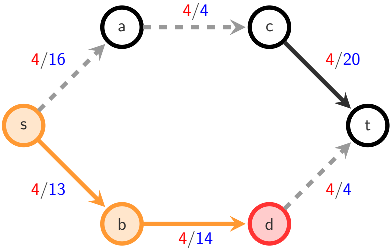
G, c and f



G_f and c_f



G_L, c_L and f'



Vertex currently explored

Vertex on current path

Edge on augmenting path

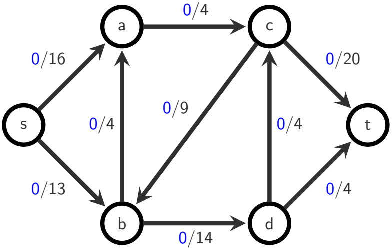
Edge on current path

Saturated/deleted edge

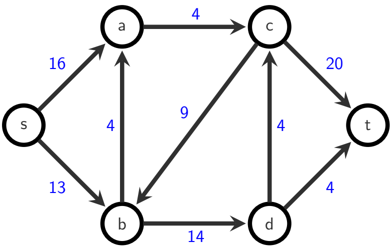
Current operation:

Find augmenting path

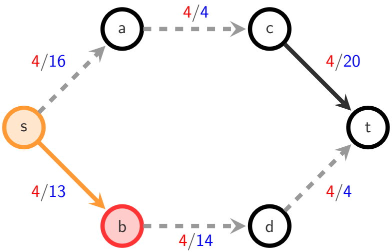
G, c and f

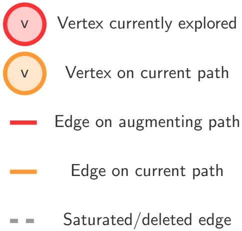


G_f and c_f



G_L, c_L and f'

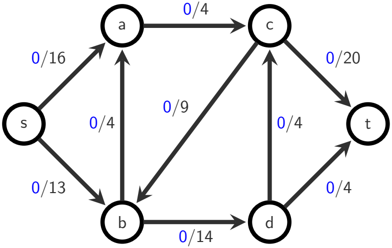




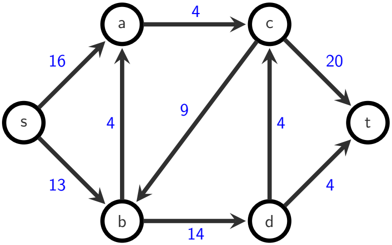
Current operation:

Find augmenting path

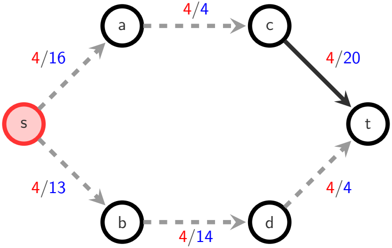
G, c and f

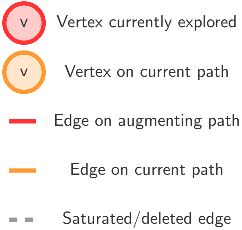


G_f and c_f



G_L, c_L and f'

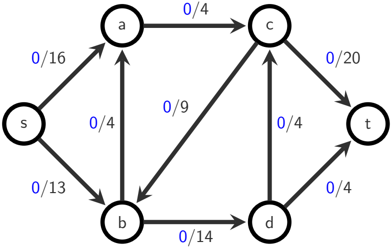




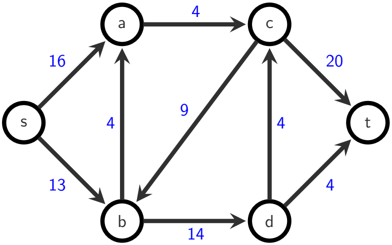
Current operation:

Find augmenting path

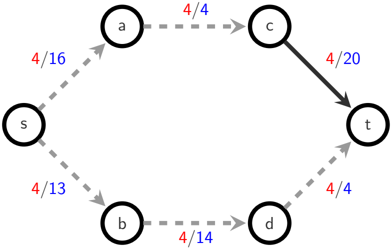
G, c and f



G_f and c_f



G_L, c_L and f'



Vertex currently explored

Vertex on current path

Edge on augmenting path

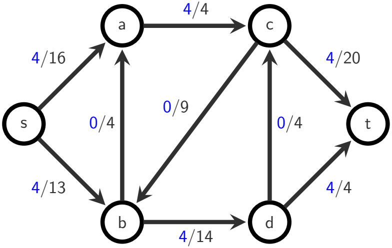
Edge on current path

Saturated/deleted edge

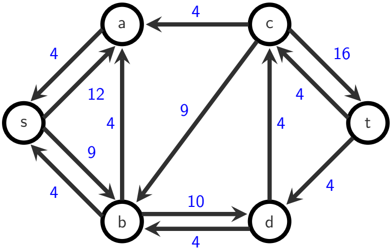
Current operation:

Augment f by f'

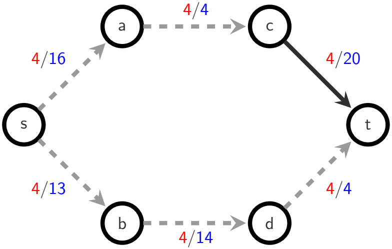
G, c and f



G_f and c_f



G_L, c_L and f'



v Vertex currently explored

v Vertex on current path

Edge on augmenting path

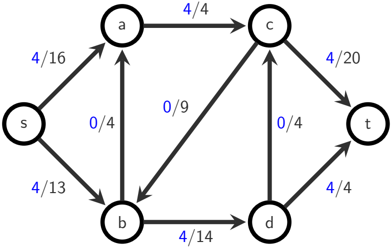
Edge on current path

Saturated/deleted edge

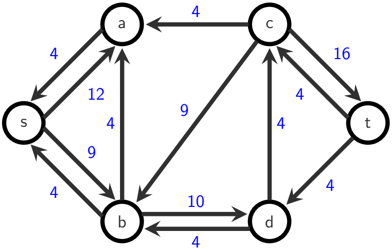
Current operation:

Augment f by f'

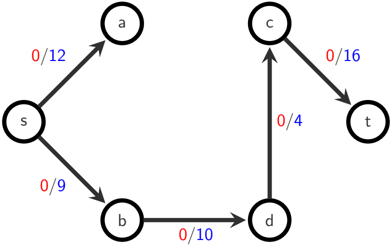
G, c and f



G_f and c_f



G_L, c_L and f'



Vertex currently explored

Vertex on current path

Edge on augmenting path

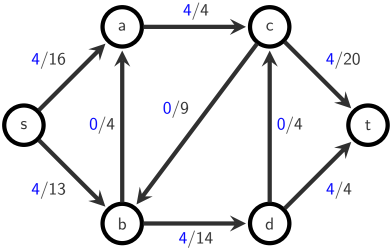
Edge on current path

Saturated/deleted edge

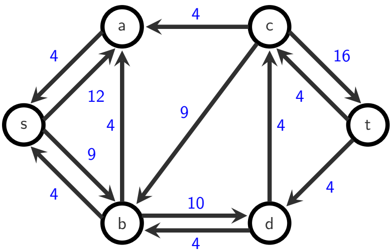
Current operation:

Find blocking flow

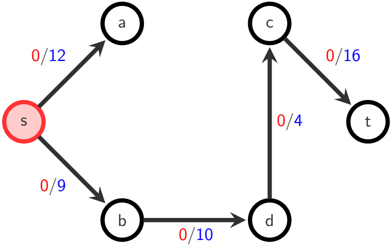
G, c and f

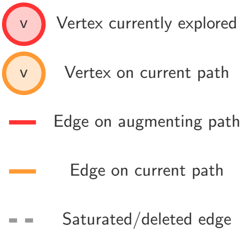


G_f and c_f



G_L, c_L and f'

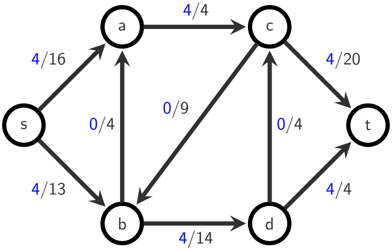




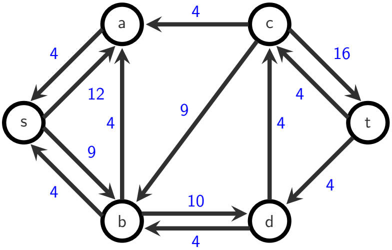
Current operation:

Find augmenting path

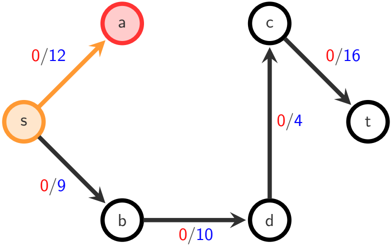
G, c and f



G_f and c_f



G_L, c_L and f'



v Vertex currently explored

v Vertex on current path

Edge on augmenting path

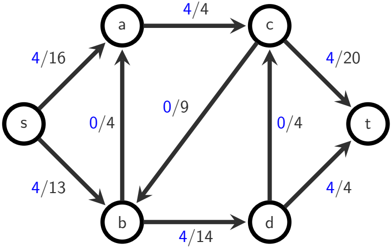
Edge on current path

Saturated/deleted edge

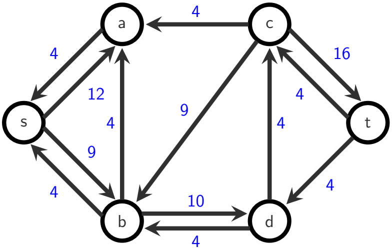
Current operation:

Find augmenting path

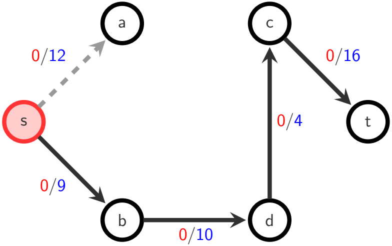
G, c and f



G_f and c_f



G_L, c_L and f'



Vertex currently explored

Vertex on current path

Edge on augmenting path

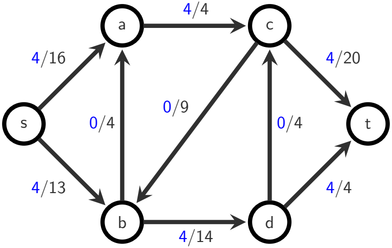
Edge on current path

Saturated/deleted edge

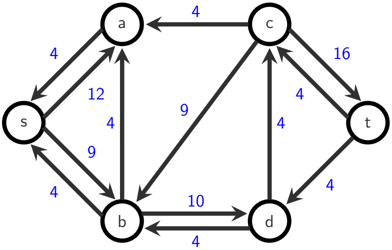
Current operation:

Find augmenting path

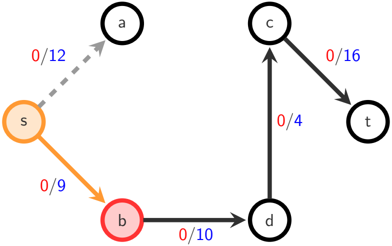
G, c and f

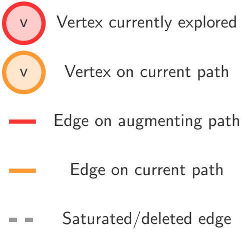


G_f and c_f



G_L, c_L and f'

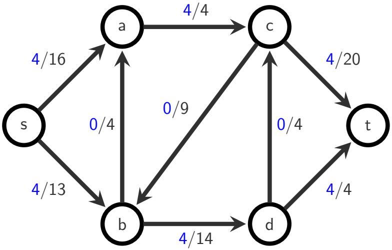




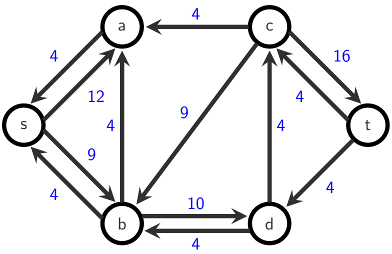
Current operation:

Find augmenting path

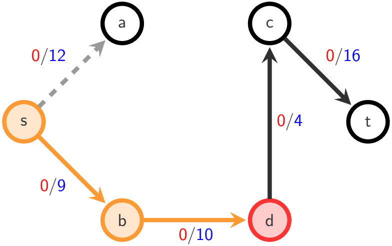
G, c and f



G_f and c_f



G_L, c_L and f'



v Vertex currently explored

v Vertex on current path

Edge on augmenting path

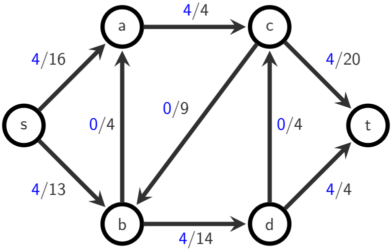
Edge on current path

Saturated/deleted edge

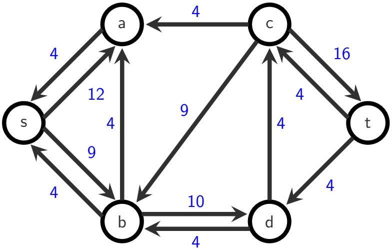
Current operation:

Find augmenting path

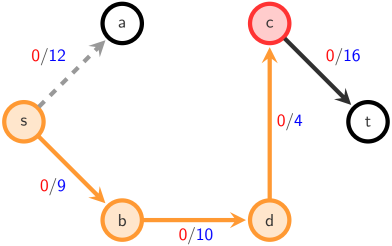
G, c and f

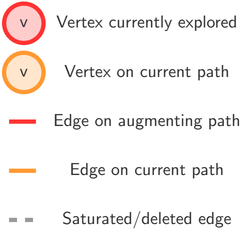


G_f and c_f



G_L, c_L and f'

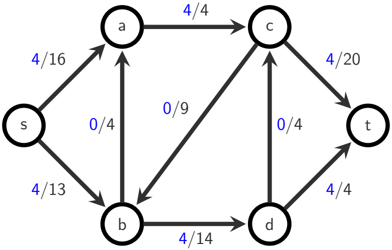




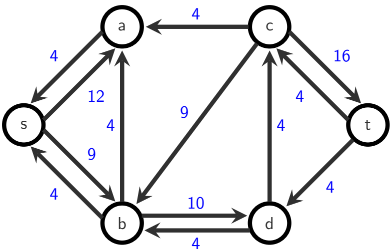
Current operation:

Find augmenting path

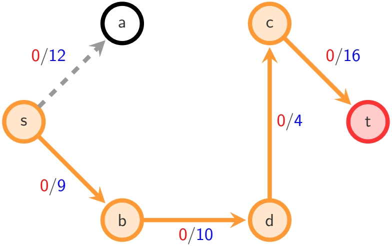
G, c and f



G_f and c_f



G_L, c_L and f'



v Vertex currently explored

v Vertex on current path

Edge on augmenting path

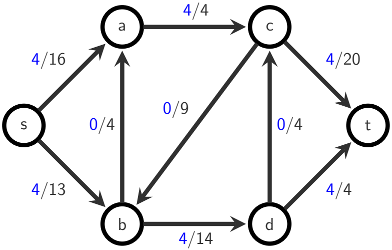
Edge on current path

Saturated/deleted edge

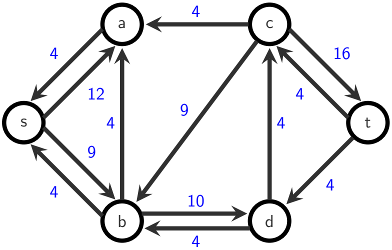
Current operation:

Find augmenting path

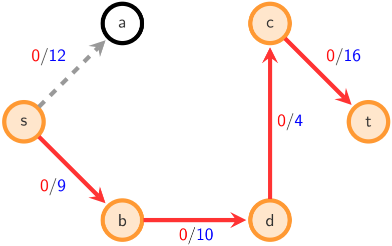
G, c and f



G_f and c_f



G_L, c_L and f'



v Vertex currently explored

v Vertex on current path

Edge on augmenting path

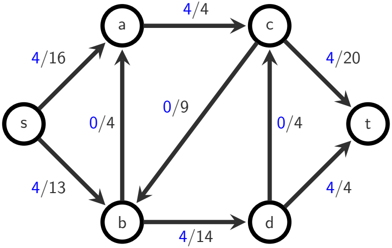
Edge on current path

Saturated/deleted edge

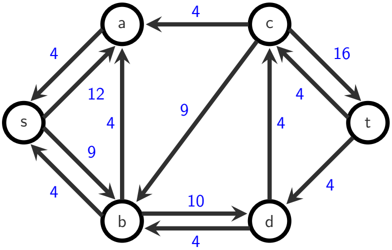
Current operation:

Augment f' by f_p

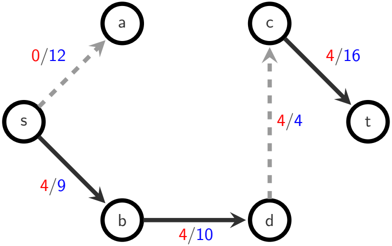
G, c and f

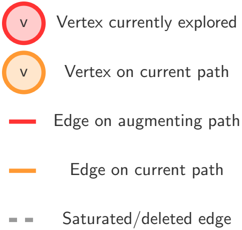


G_f and c_f



G_L, c_L and f'

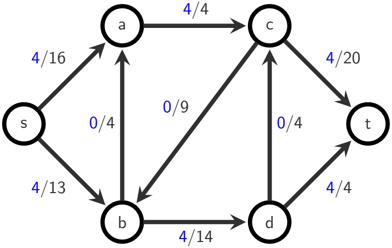




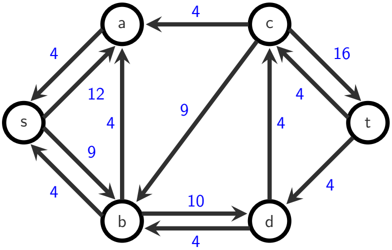
Current operation:

Augment f' by f_p

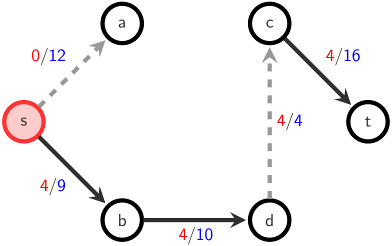
G, c and f

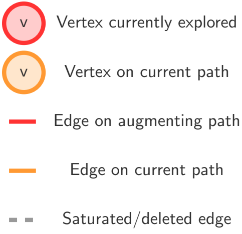


G_f and c_f



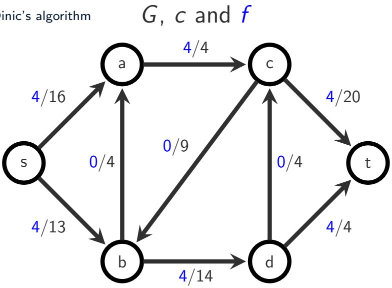
G_L, c_L and f'

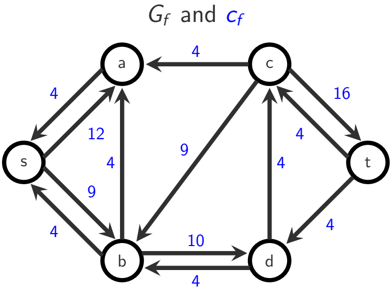


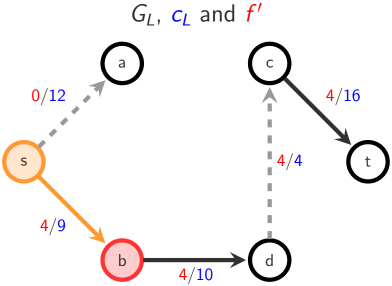


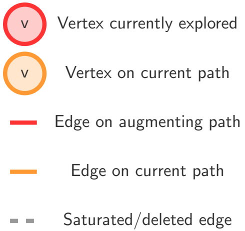
Current operation:

Find augmenting path





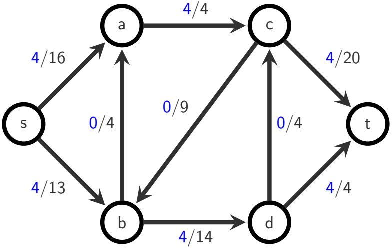




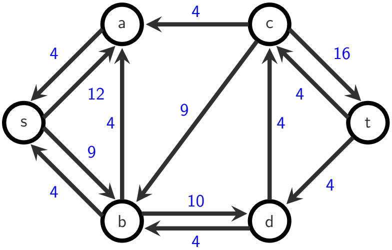
Current operation:

Find augmenting path

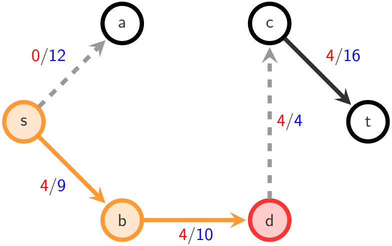
G, c and f

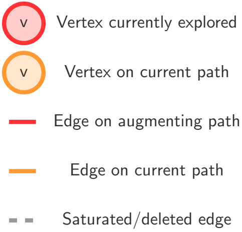


G_f and c_f



G_L, c_L and f'

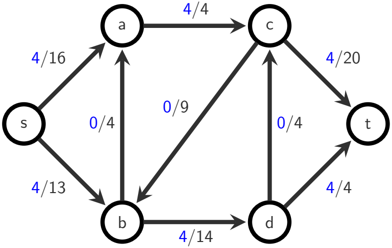




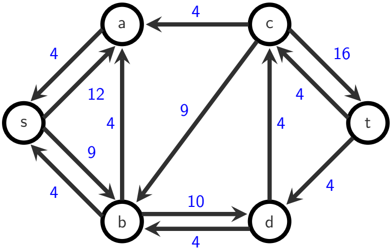
Current operation:

Find augmenting path

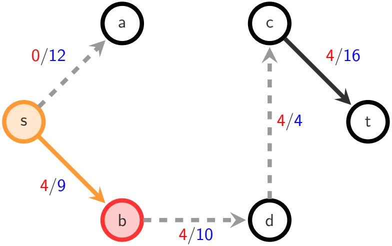
G, c and f



G_f and c_f



G_L, c_L and f'



v Vertex currently explored

v Vertex on current path

Edge on augmenting path

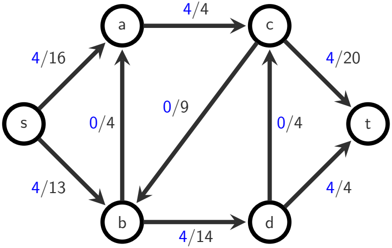
Edge on current path

Saturated/deleted edge

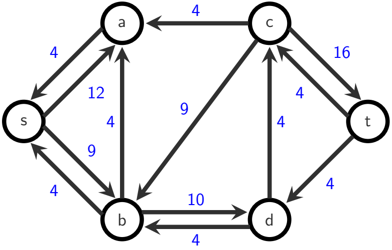
Current operation:

Find augmenting path

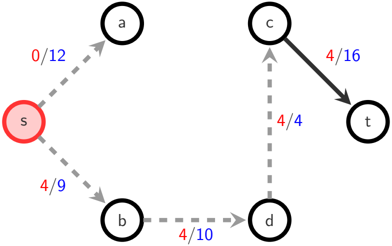
G, c and f



G_f and c_f



G_L, c_L and f'



Vertex currently explored

Vertex on current path

Edge on augmenting path

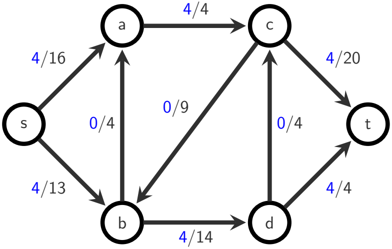
Edge on current path

Saturated/deleted edge

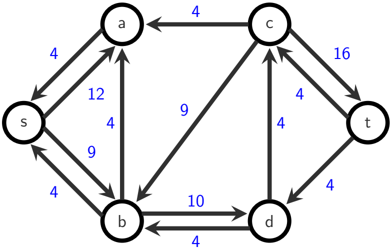
Current operation:

Find augmenting path

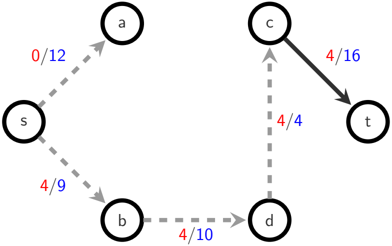
G, c and f

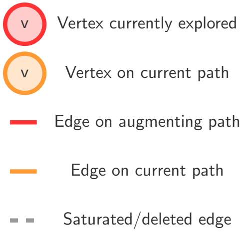


G_f and c_f



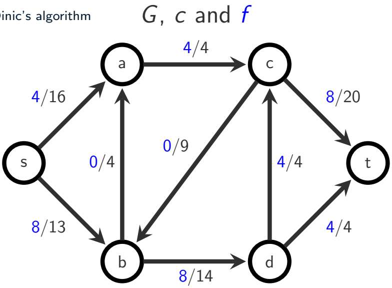
G_L, c_L and f'

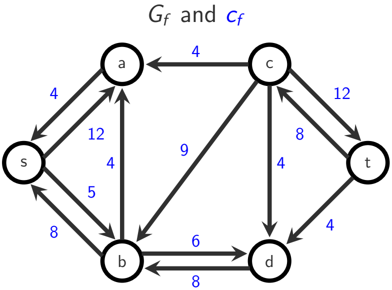


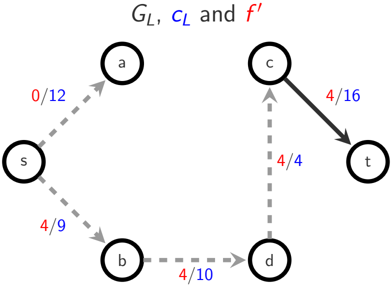


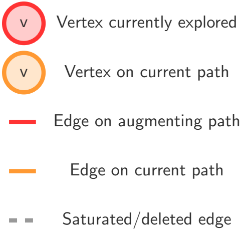
Current operation:

Augment f by f'



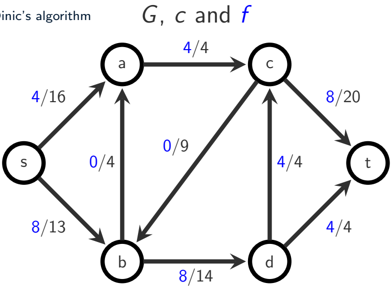


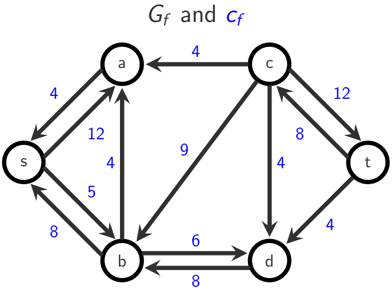


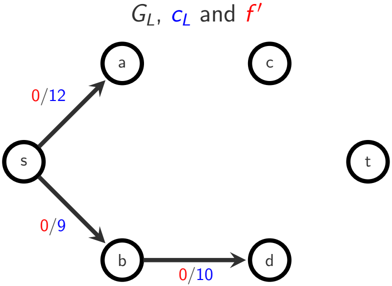


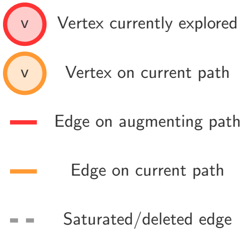
Current operation:

Augment f by f'





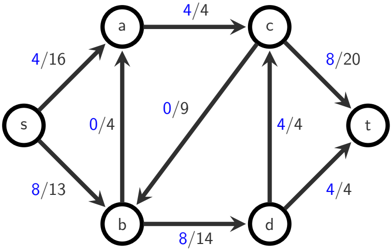




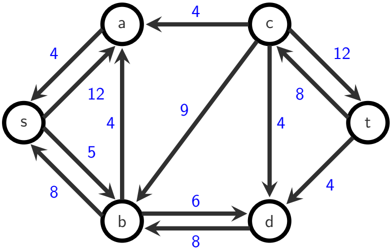
Current operation:

Find blocking flow

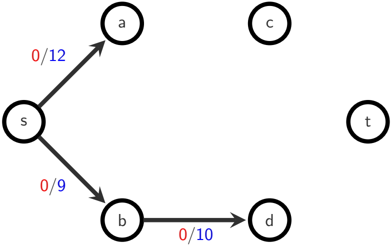
G, c and f



G_f and c_f



G_L, c_L and f'



Vertex currently explored

Vertex on current path

Edge on augmenting path

Edge on current path

Saturated/deleted edge

No path from s to t:

Maximum flow found

Push-relabel algorithms: Preflow

The class of *push-relable* algorithms for maximum flow work by maintaining a *preflow* and pushing it along edges, while (re-)labeling vertices to determine where flow can be pushed.

Definition (Preflow)

For a given flow network $G = (V, E)$ with capacity function c, a *preflow* is a function $f : E \to \mathbb{R}$ satisfying

$$\forall (u, v) \in E: \quad 0 \leq f(u, v) \leq c(u, v)$$
$$\forall u \in V \setminus \{s, t\}: \quad \sum_{\{v:\ (v,u) \in E\}} f(v, u) - \sum_{\{v:\ (u,v) \in E\}} f(u, v) \geq 0$$

Push-relabel algorithms: Excess flow and height labels

Definition (Excess flow)

For a given flow network G and a preflow f, the *excess flow* $e(u)$ of a vertex u is given by

$$e(v) = \sum_{\{v\colon (v,u)\in E\}} f(v,u) - \sum_{\{v\colon (u,v)\in E\}} f(u,v)$$

A vertex $u \in V \setminus \{s,t\}$ is said to be *overflowing* if $e(u) > 0$.

Definition (Height function)

For a given flow network G and a flow f, a function $h\colon V \to \mathbb{N}$ is a *height function* if $h(s) = |V|$, $h(t) = 0$ and $h(u) \leq h(v) + 1$ for every residual edge $(u,v) \in E_f$.

Push and relabel operations

Algorithm 4 Push operation

▷ Applies to $(u, v) \in E_f$ when u is overflowing and $h(u) = h(v) + 1$
$\Delta_f(u, v) \leftarrow \min(e(u), c_f(u, v))$
if $(u, v) \in E$ **then**
  $f(u, v) \leftarrow f(u, v) + \Delta_f(u, v)$
else
  $f(v, u) \leftarrow f(v, u) - \Delta_f(u, v)$
end if
$e(u) \leftarrow e(u) - \Delta_f(u, v)$
$e(v) \leftarrow e(v) + \Delta_f(u, v)$

Algorithm 5 Relabel operation

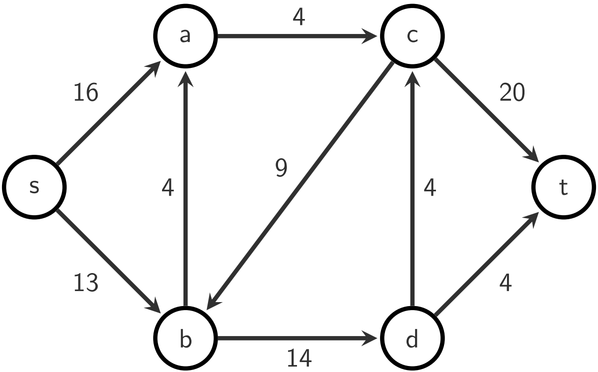
▷ Applies to u when u is overflowing and $h(u) \leq h(v)$ for all $(u, v) \in E_f$
$h(u) \leftarrow 1 + \min\{h(v) \colon (u, v) \in E_f\}$

Push-relable algorithm (Goldberg-Tarjan algorithm)

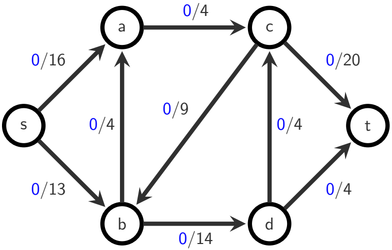
Algorithm 6 Push-relable algorithm

for each vertex $v \in V$ **do**
    $h(v) \leftarrow 0$; $e(v) \leftarrow 0$
end for
for $(u, v) \in E$ **do**
    $f(u, v) \leftarrow 0$
end for
$h(s) \leftarrow |V|$
for each vertex $v \in sE$ **do**
    $f(s, v) \leftarrow c(s, v)$
    $e(v) \leftarrow e(v) + c(s, v)$
    $e(s) \leftarrow e(s) - c(s, v)$
end for
while there is an applicable push or relabel operation **do**
    select an applicable push or relabel operation and perform it
end while

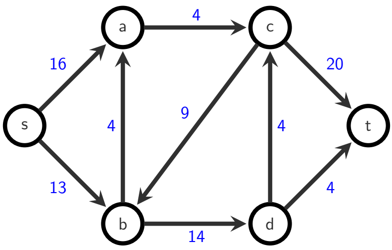
Push-relabel algorithm (example)



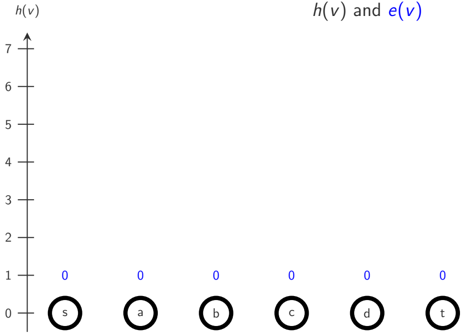
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



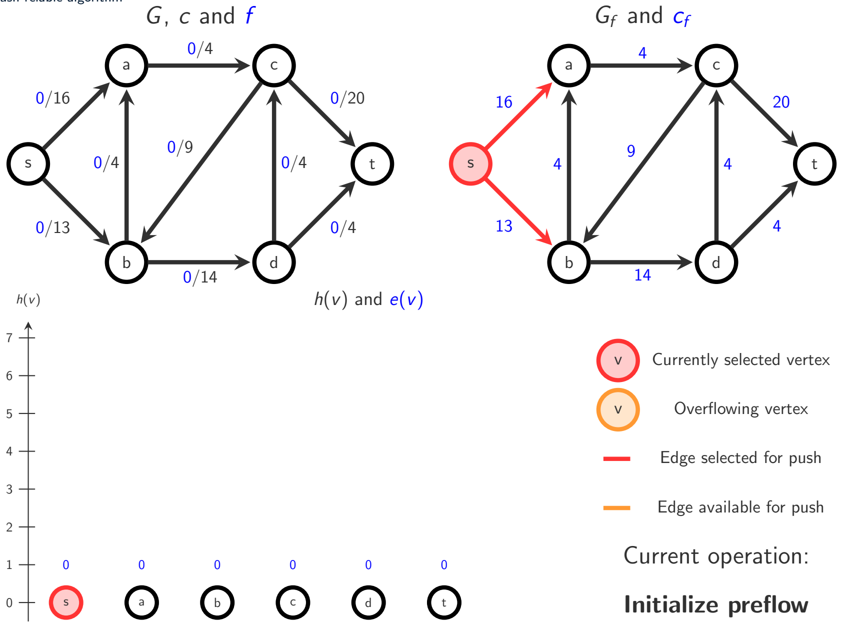
v Currently selected vertex

v Overflowing vertex

Edge selected for push

Edge available for push

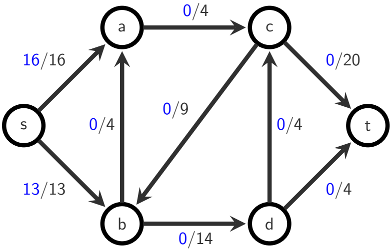
Current operation:



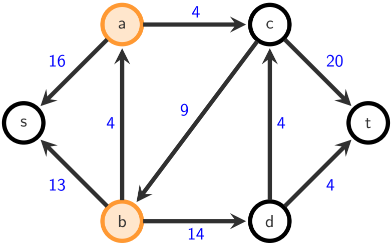
Current operation:

Initialize preflow

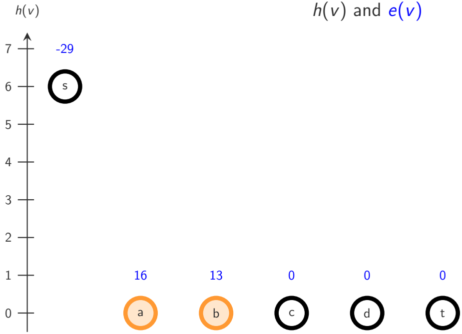
G, c and f

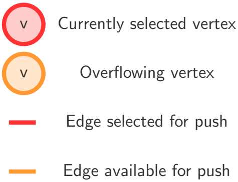


G_f and c_f



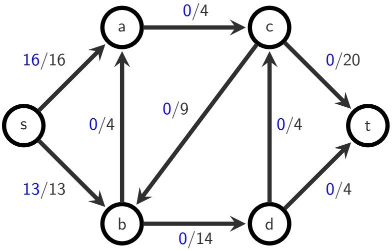
$h(v)$ and $e(v)$



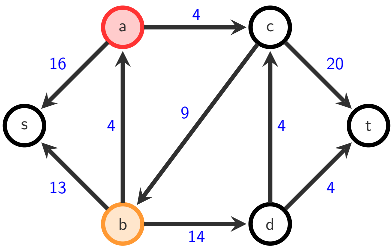


Current operation:

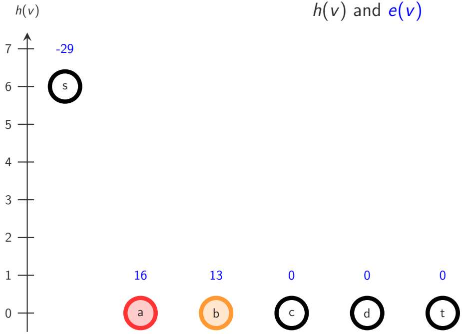
G, c and f

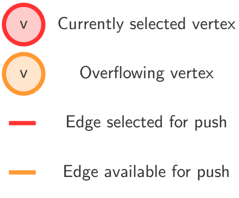


G_f and c_f



$h(v)$ and $e(v)$

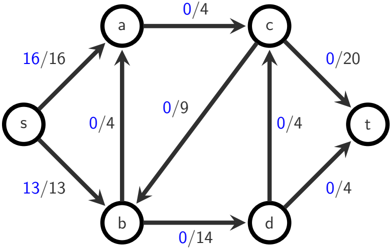




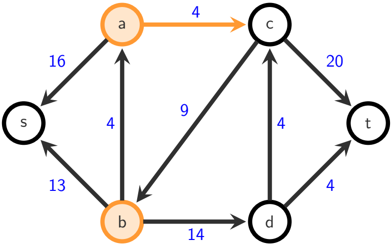
Current operation:

Relabel a

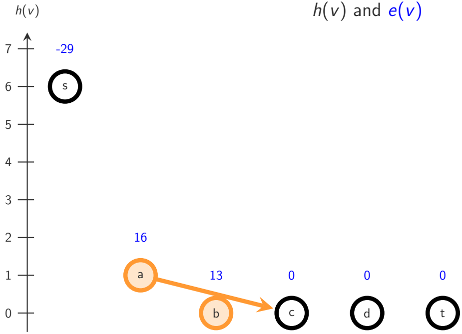
G, c and f

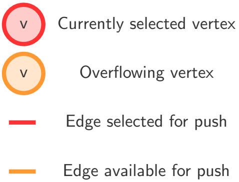


G_f and c_f



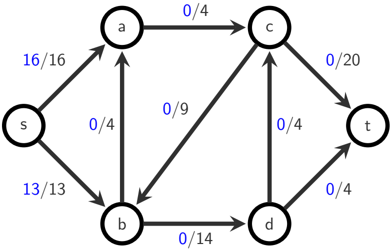
$h(v)$ and $e(v)$



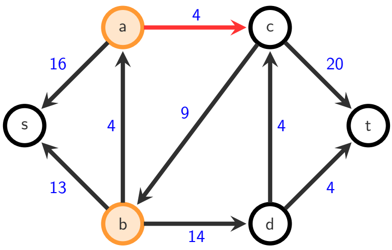


Current operation:

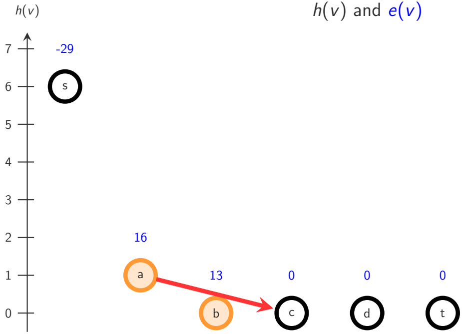
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



Currently selected vertex

Overflowing vertex

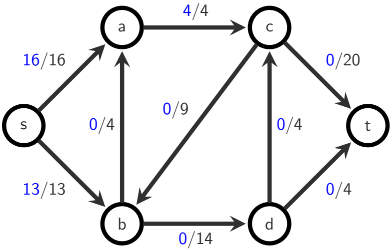
Edge selected for push

Edge available for push

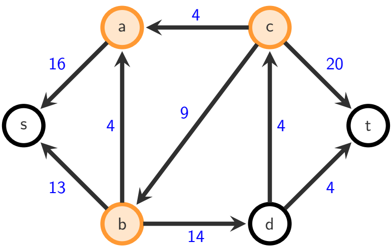
Current operation:

Push from a to c

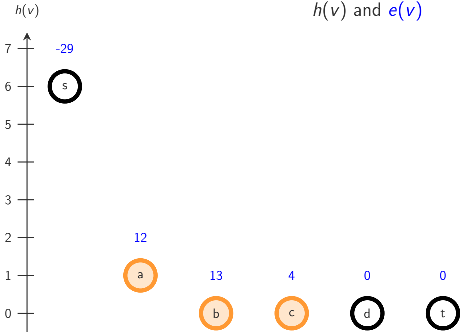
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



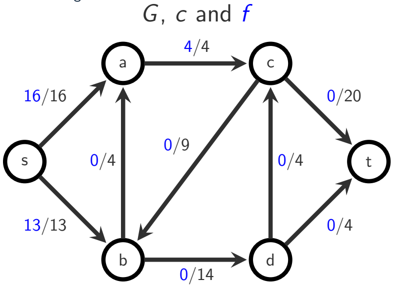
Currently selected vertex

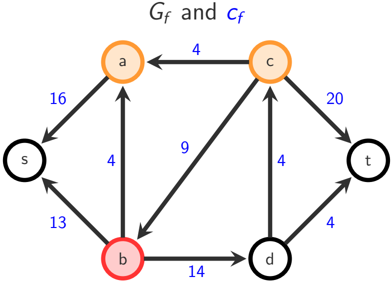
Overflowing vertex

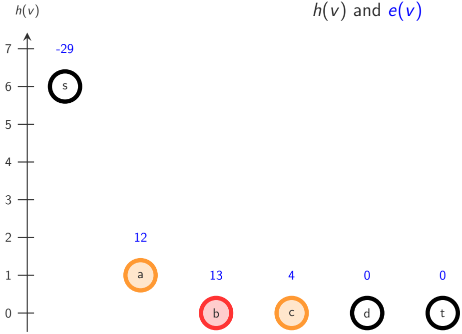
Edge selected for push

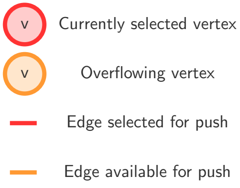
Edge available for push

Current operation:



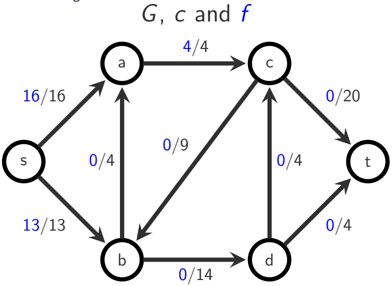


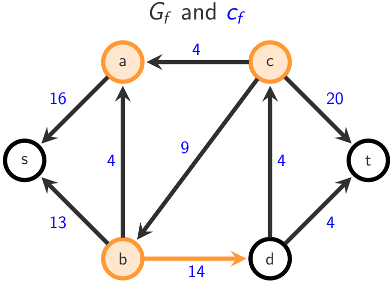


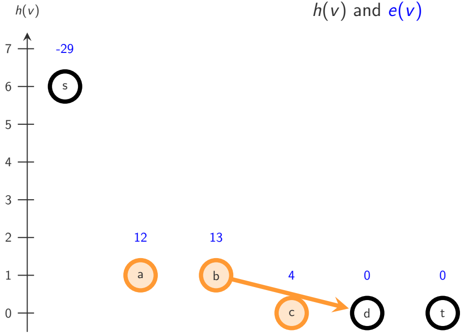


Current operation:

Relabel b







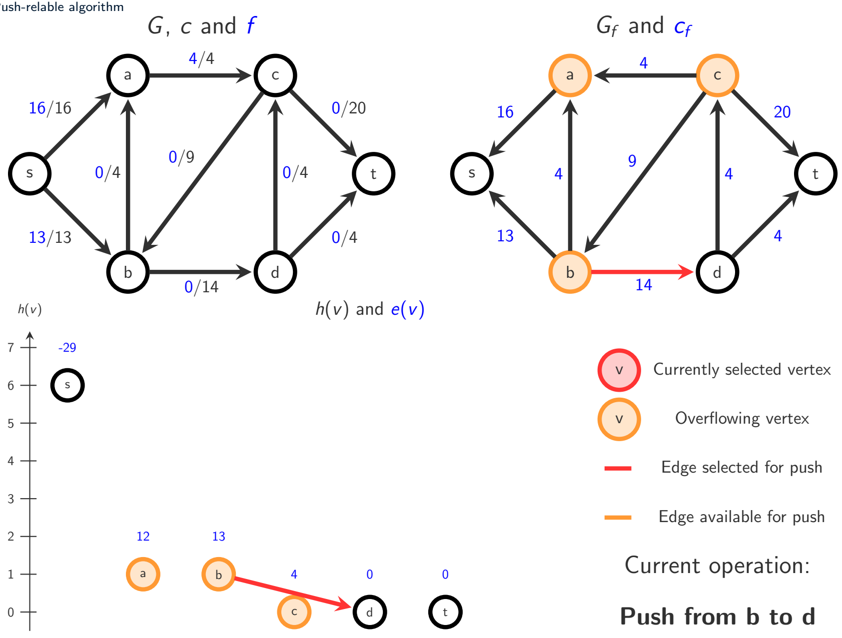
v Currently selected vertex

v Overflowing vertex

Edge selected for push

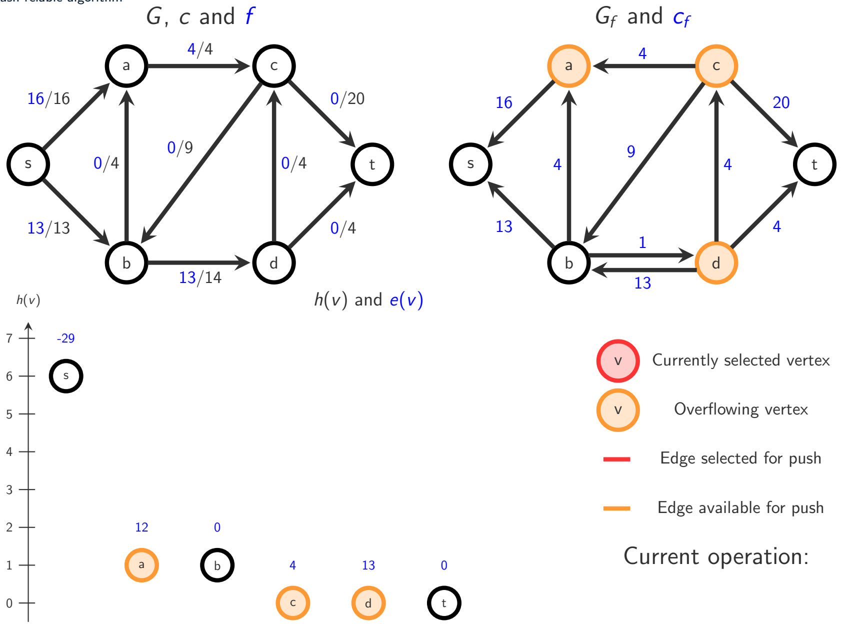
Edge available for push

Current operation:



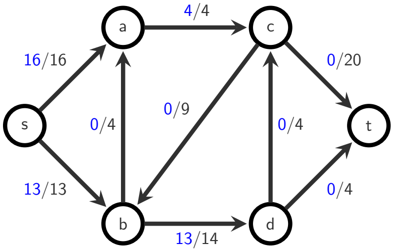
Current operation:

Push from b to d

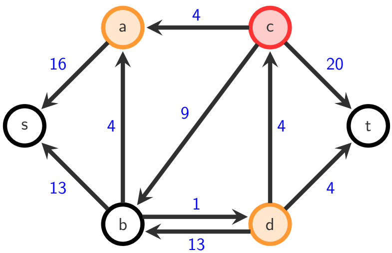


Current operation:

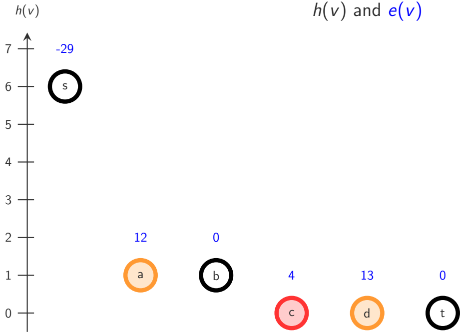
G, c and f

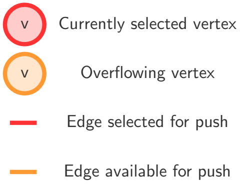


G_f and c_f



$h(v)$ and $e(v)$

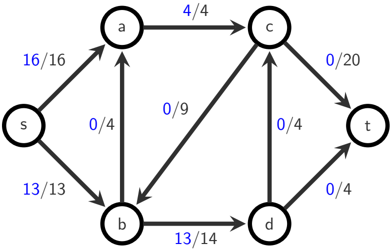




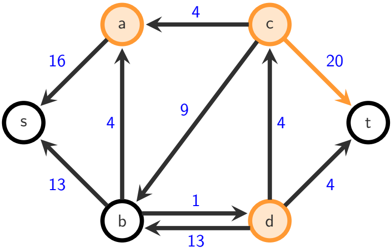
Current operation:

Relabel c

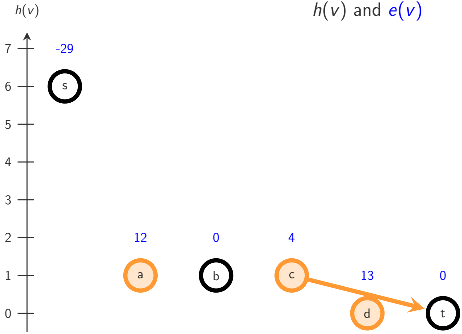
G, c and f

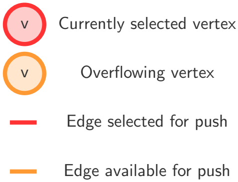


G_f and c_f



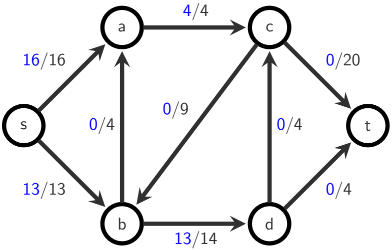
$h(v)$ and $e(v)$



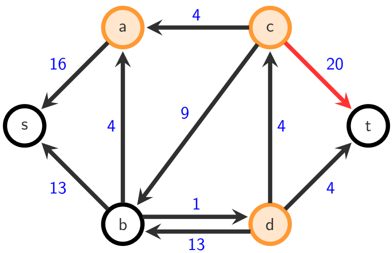


Current operation:

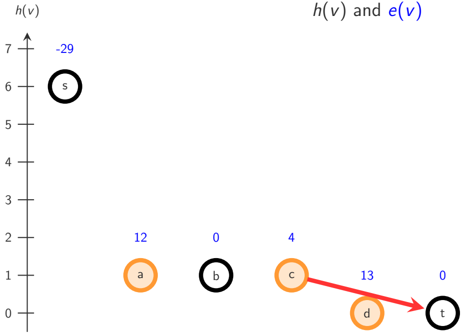
G, c and f

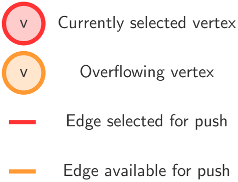


G_f and c_f



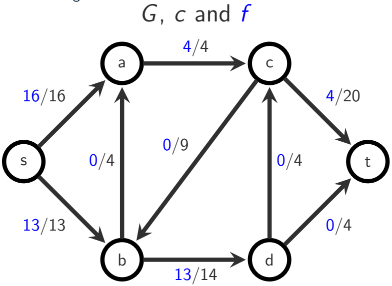
$h(v)$ and $e(v)$

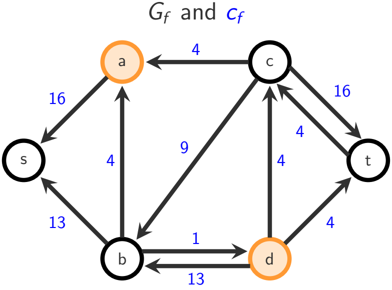


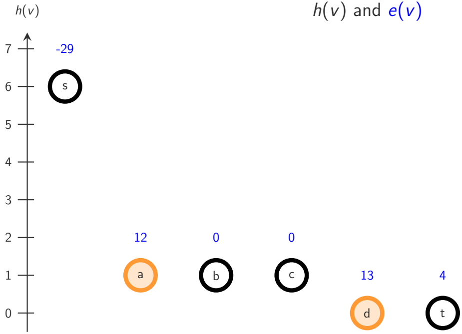


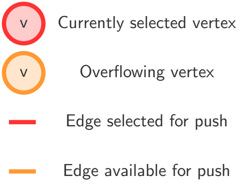
Current operation:

Push from c to t



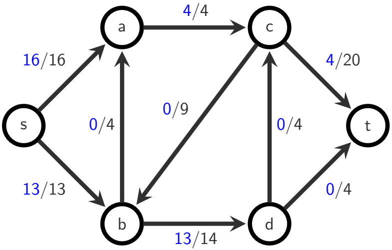




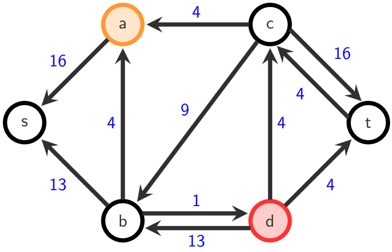


Current operation:

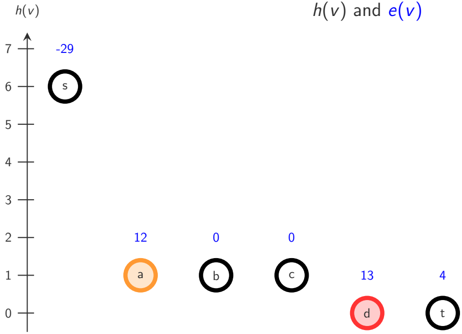
G, c and f

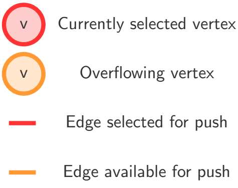


G_f and c_f



$h(v)$ and $e(v)$

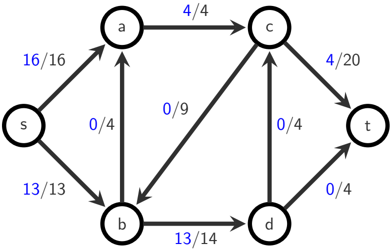




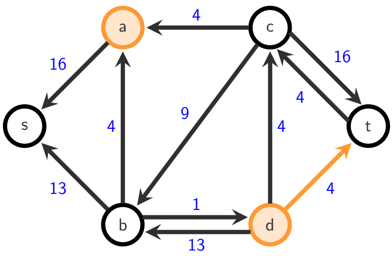
Current operation:

Relabel d

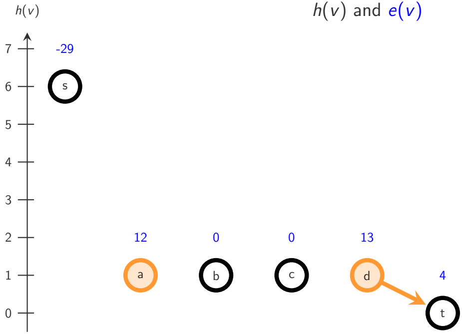
G, c and f

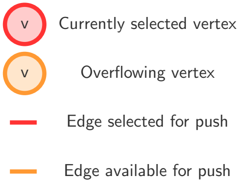


G_f and c_f



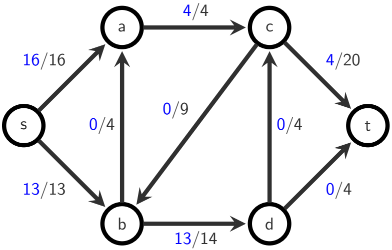
$h(v)$ and $e(v)$



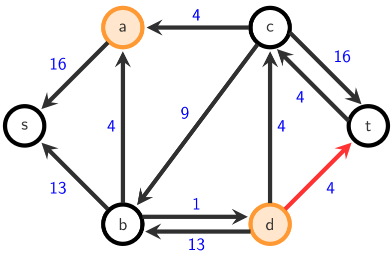


Current operation:

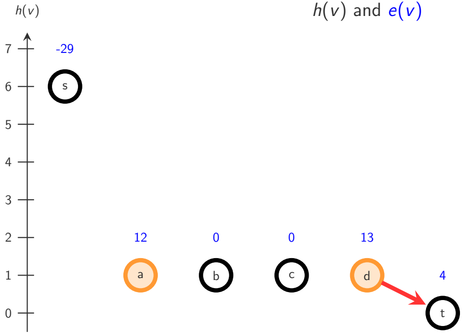
G, c and f

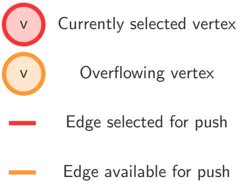


G_f and c_f



$h(v)$ and $e(v)$

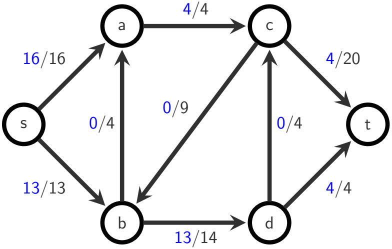




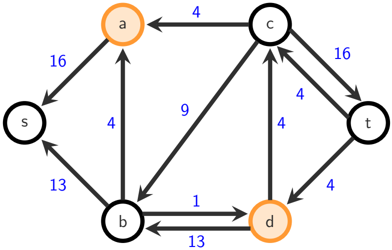
Current operation:

Push from d to t

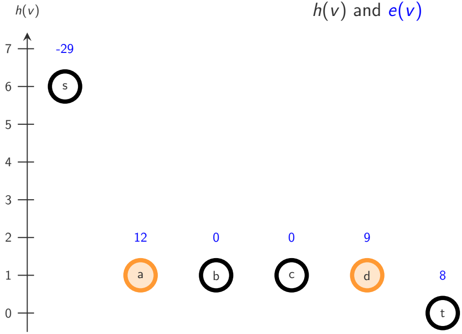
G, c and f

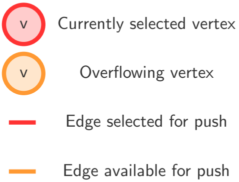


G_f and c_f

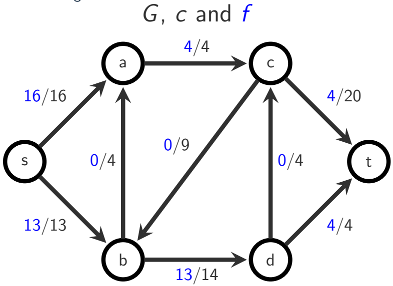


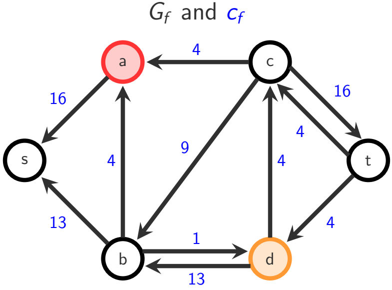
$h(v)$ and $e(v)$

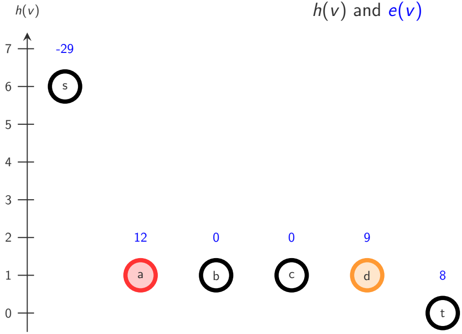


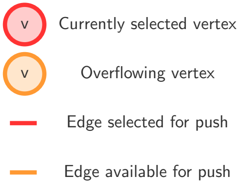


Current operation:





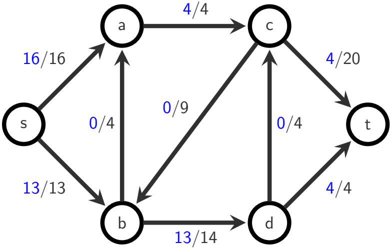




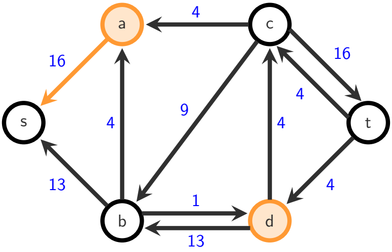
Current operation:

Relabel a

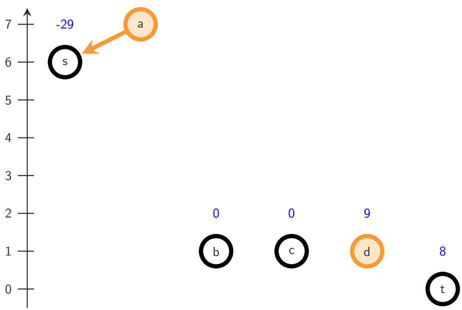
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



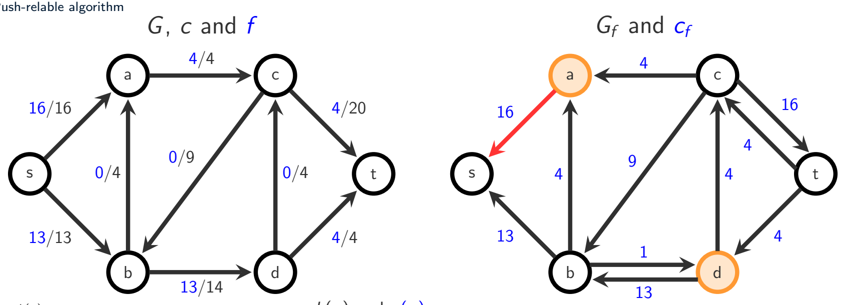
Currently selected vertex

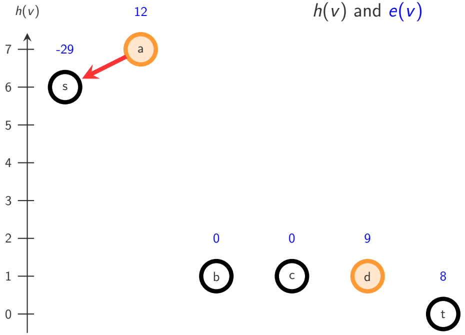
Overflowing vertex

Edge selected for push

Edge available for push

Current operation:





Currently selected vertex

Overflowing vertex

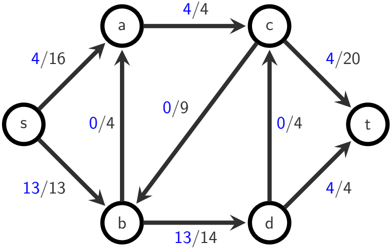
Edge selected for push

Edge available for push

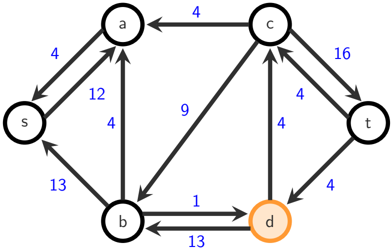
Current operation:

Push from a to s

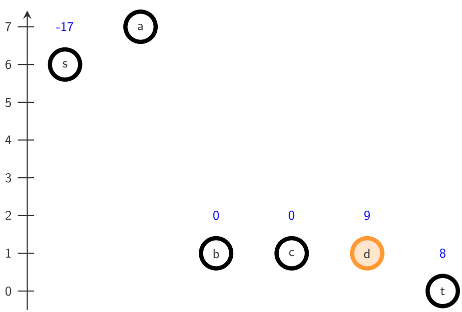
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



Currently selected vertex

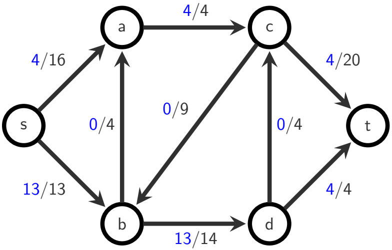
Overflowing vertex

Edge selected for push

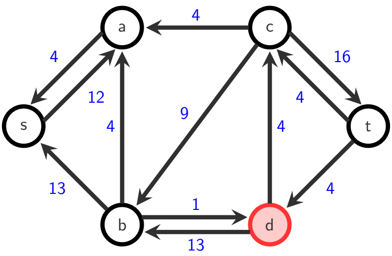
Edge available for push

Current operation:

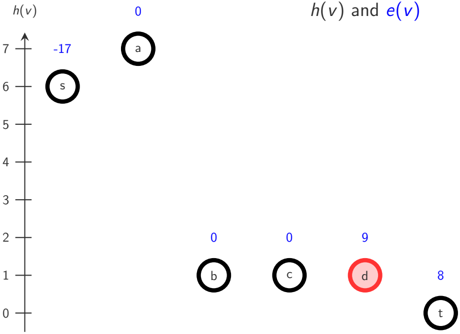
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



v Currently selected vertex

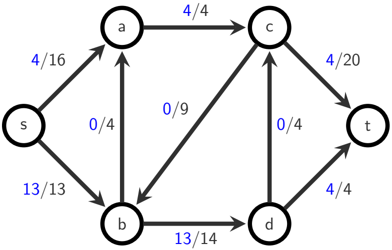
v Overflowing vertex

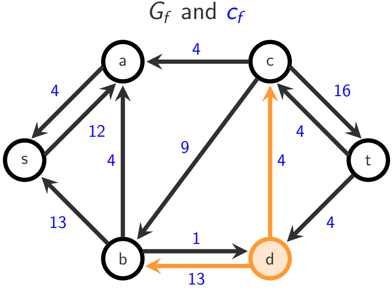
Edge selected for push

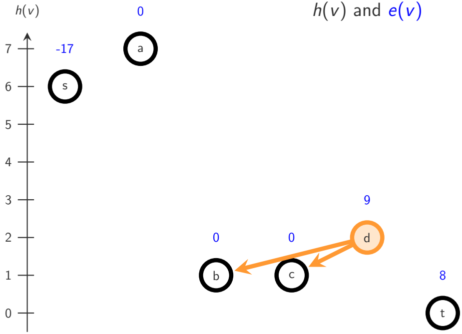
Edge available for push

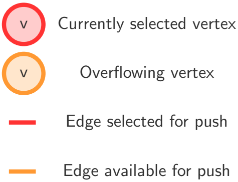
Current operation:

Relabel d



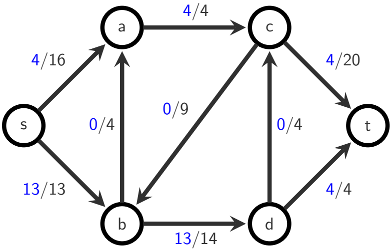




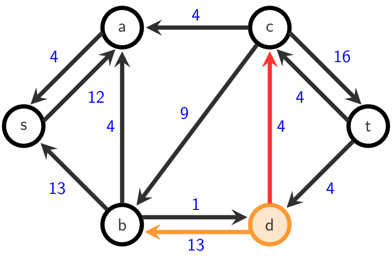


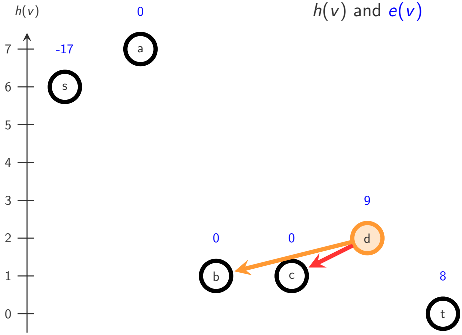
Current operation:

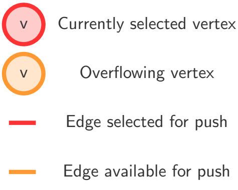
G, c and f



G_f and c_f

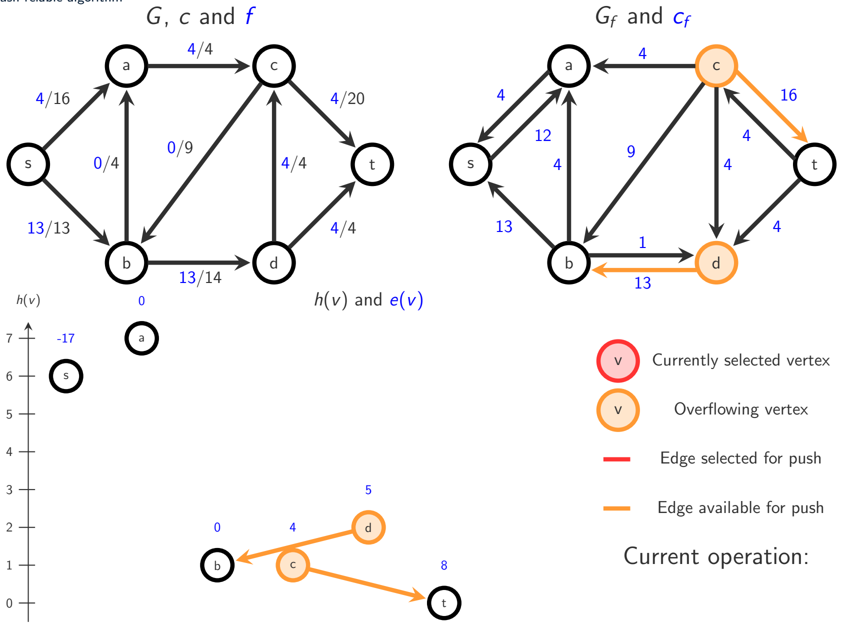




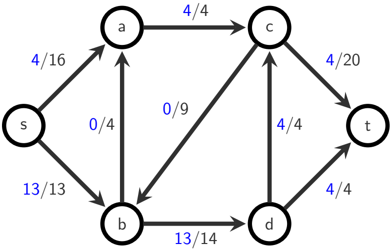


Current operation:

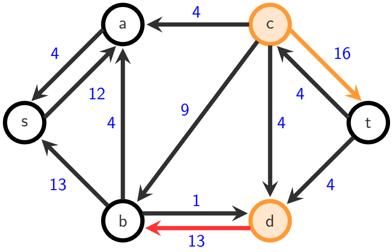
Push from d to c


G, c and f
4/4
4/16
4/20
0/9
0/4
4/4
13/13
4/4
13/14
Gf and cf
4
16
4
12
4
9
4
4
13
1
4
13
h(v) and e(v)
h(v)
-17
0
5
0
4
8
Currently selected vertex
Overflowing vertex
Edge selected for push
Edge available for push
Current operation:

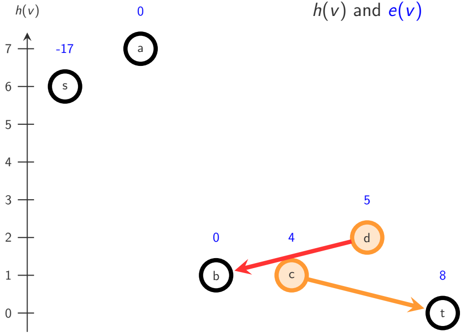
G, c and f

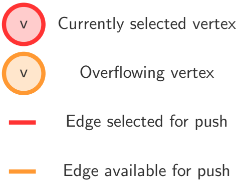


G_f and c_f



$h(v)$ and $e(v)$

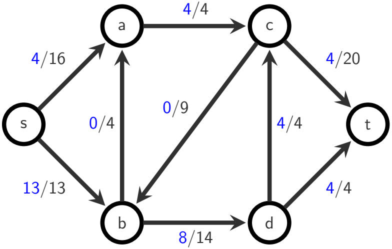




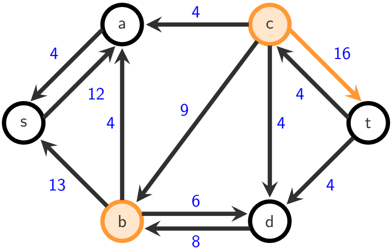
Current operation:

Push from d to b

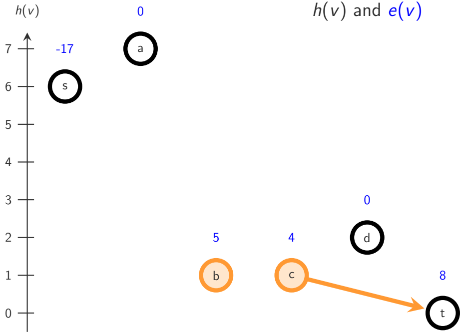
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



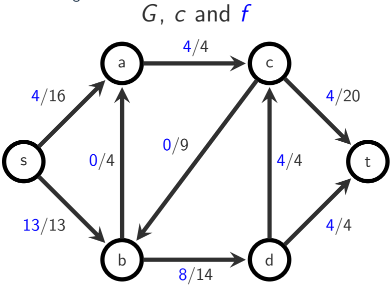
Currently selected vertex

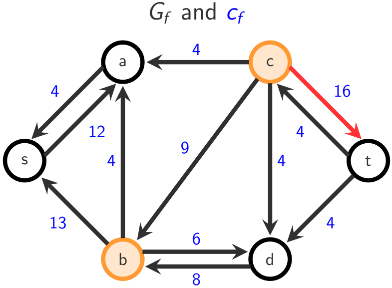
Overflowing vertex

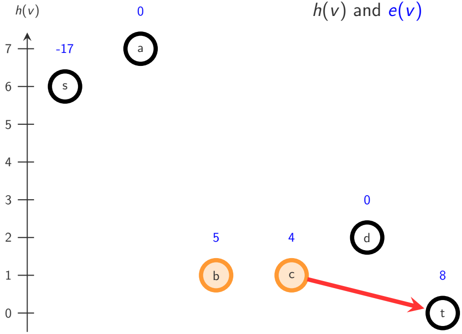
Edge selected for push

Edge available for push

Current operation:







Currently selected vertex

Overflowing vertex

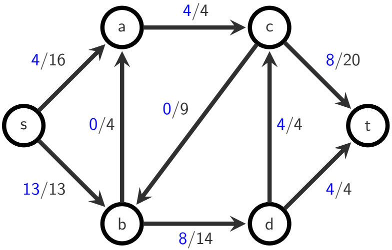
Edge selected for push

Edge available for push

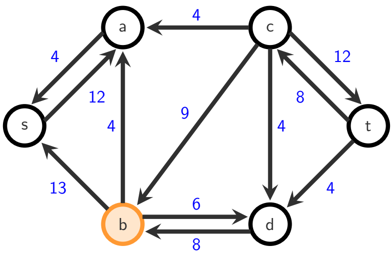
Current operation:

Push from c to t

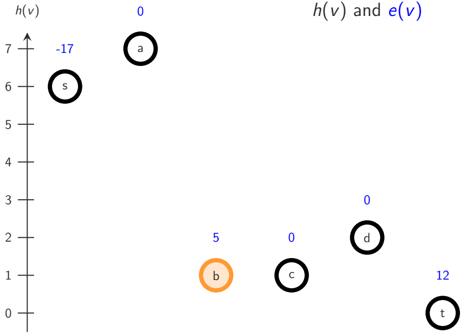
G, c and f

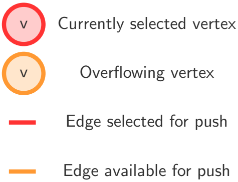


G_f and c_f

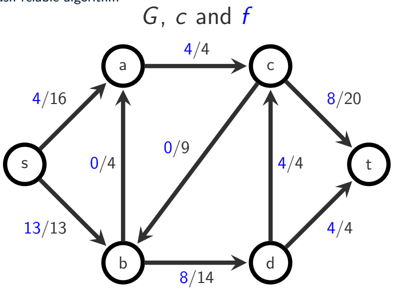


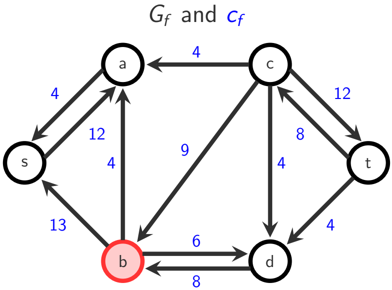
$h(v)$ and $e(v)$

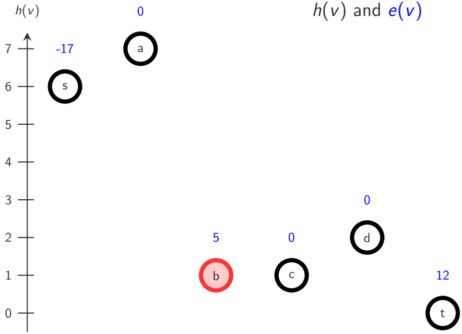




Current operation:







Currently selected vertex

Overflowing vertex

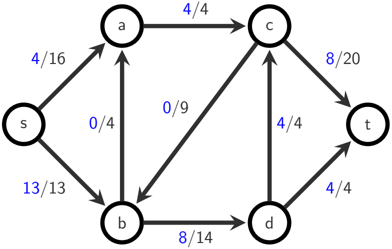
Edge selected for push

Edge available for push

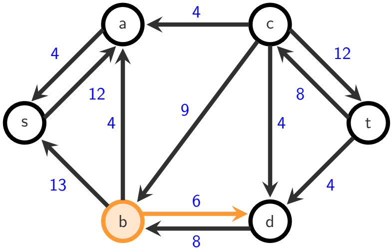
Current operation:

Relabel b

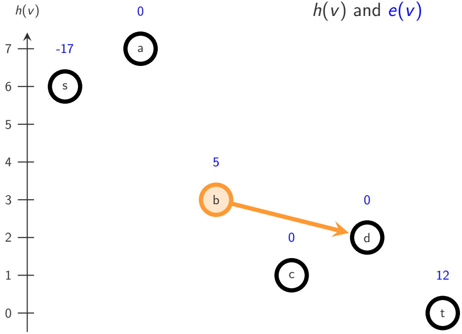
G, c and f

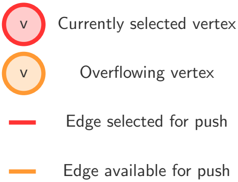


G_f and c_f



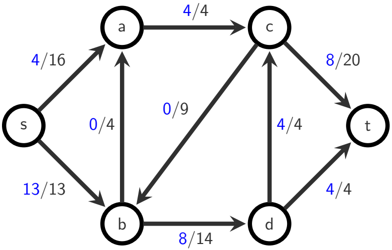
$h(v)$ and $e(v)$



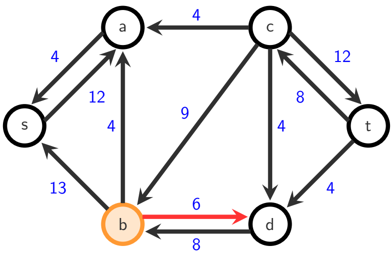


Current operation:

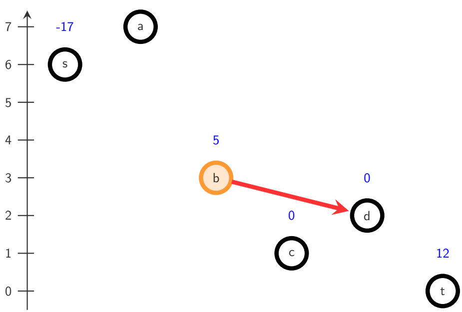
G, c and f

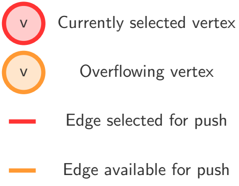


G_f and c_f



$h(v)$ and $e(v)$

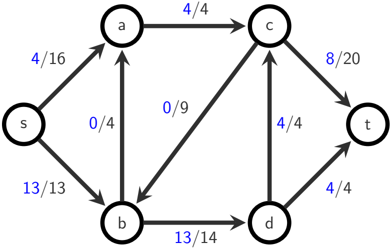




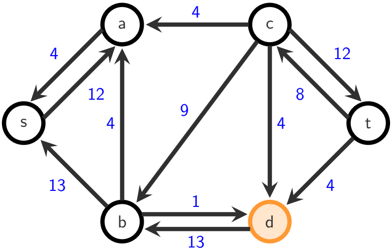
Current operation:

Push from b to d

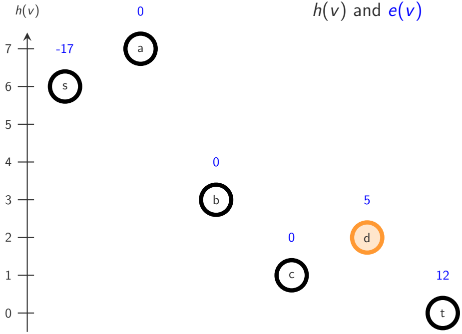
G, c and f

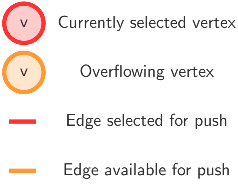


G_f and c_f



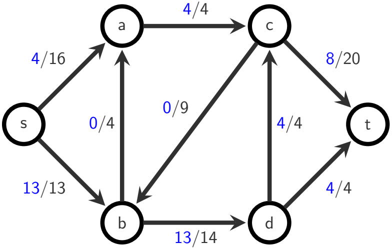
$h(v)$ and $e(v)$



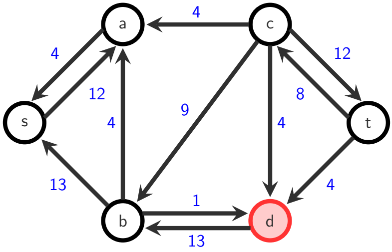


Current operation:

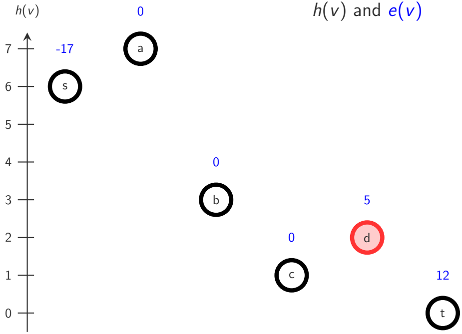
G, c and f



G_f and c_f



$h(v)$ and $e(v)$

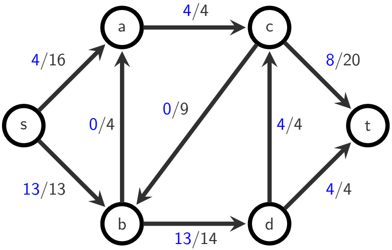


- v Currently selected vertex
- v Overflowing vertex
- Edge selected for push
- Edge available for push

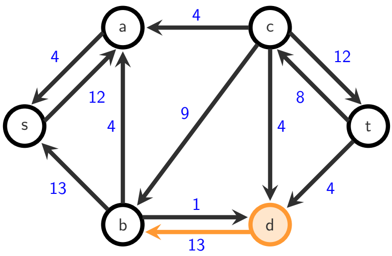
Current operation:

Relabel d

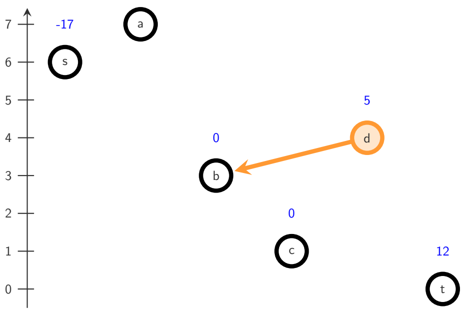
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



v Currently selected vertex

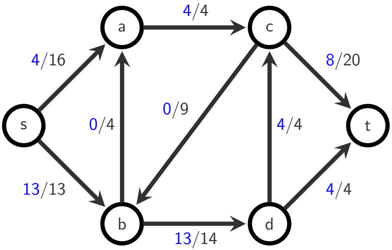
v Overflowing vertex

Edge selected for push

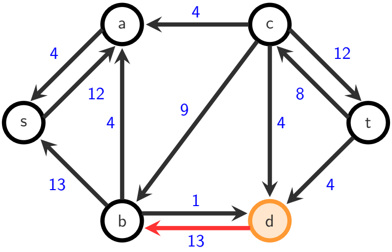
Edge available for push

Current operation:

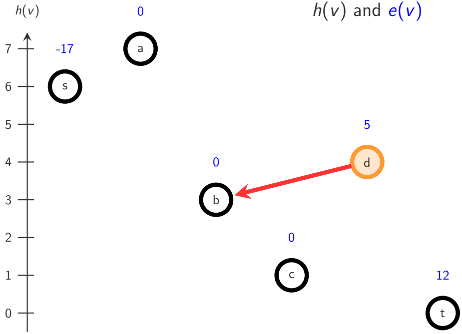
G, c and f

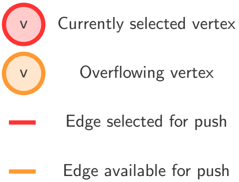


G_f and c_f



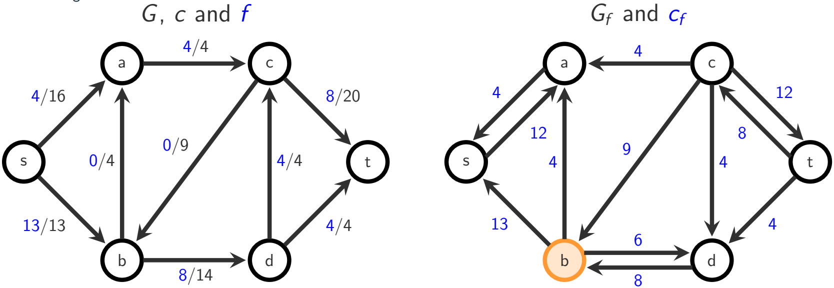
$h(v)$ and $e(v)$

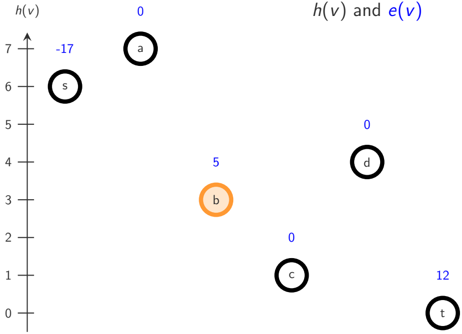


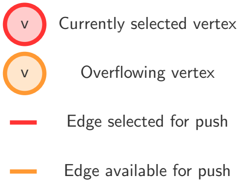


Current operation:

Push from d to b

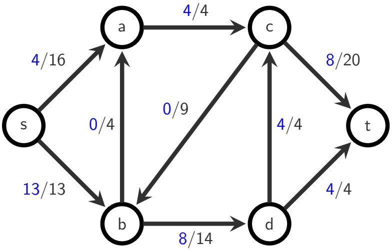




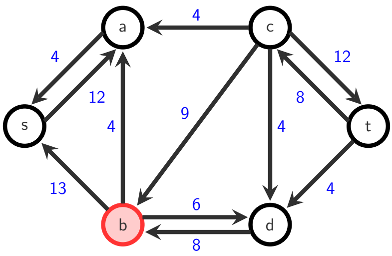


Current operation:

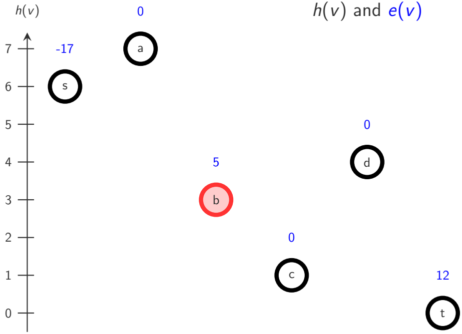
G, c and f

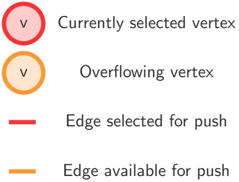


G_f and c_f



$h(v)$ and $e(v)$

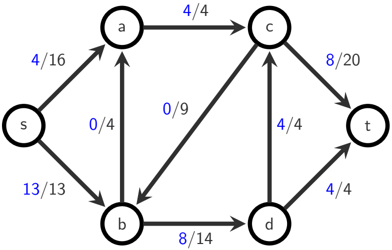




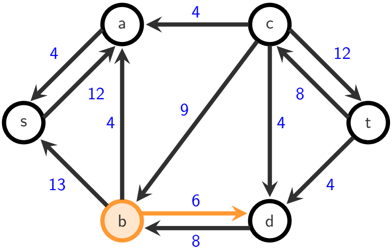
Current operation:

Relabel b

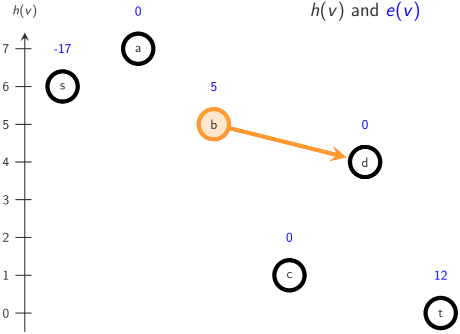
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



Currently selected vertex

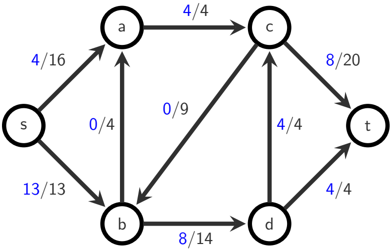
Overflowing vertex

Edge selected for push

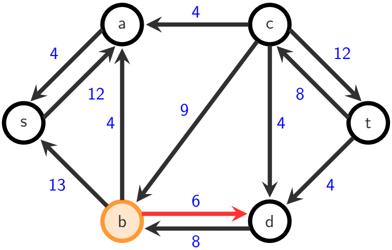
Edge available for push

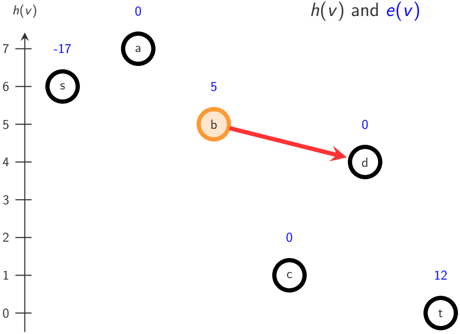
Current operation:

G, c and f



G_f and c_f





v Currently selected vertex

v Overflowing vertex

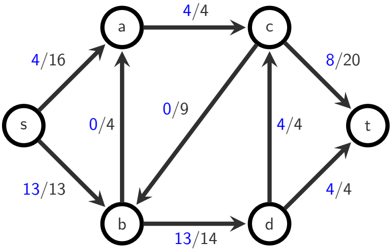
Edge selected for push

Edge available for push

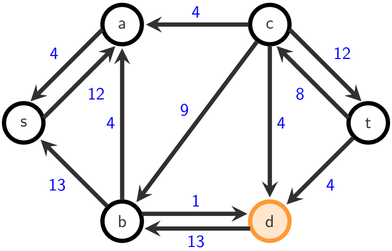
Current operation:

Push from b to d

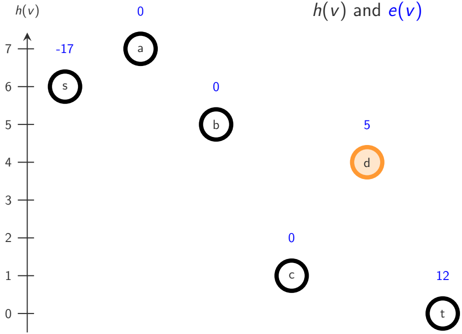
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



v Currently selected vertex

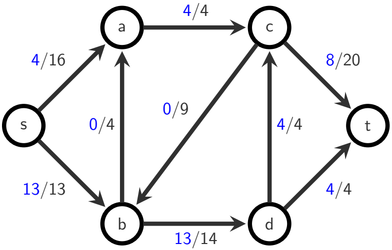
v Overflowing vertex

Edge selected for push

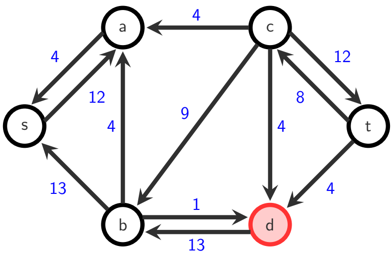
Edge available for push

Current operation:

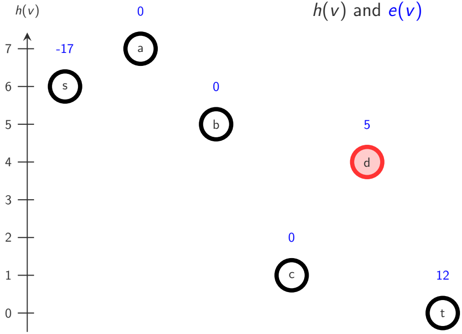
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



Currently selected vertex

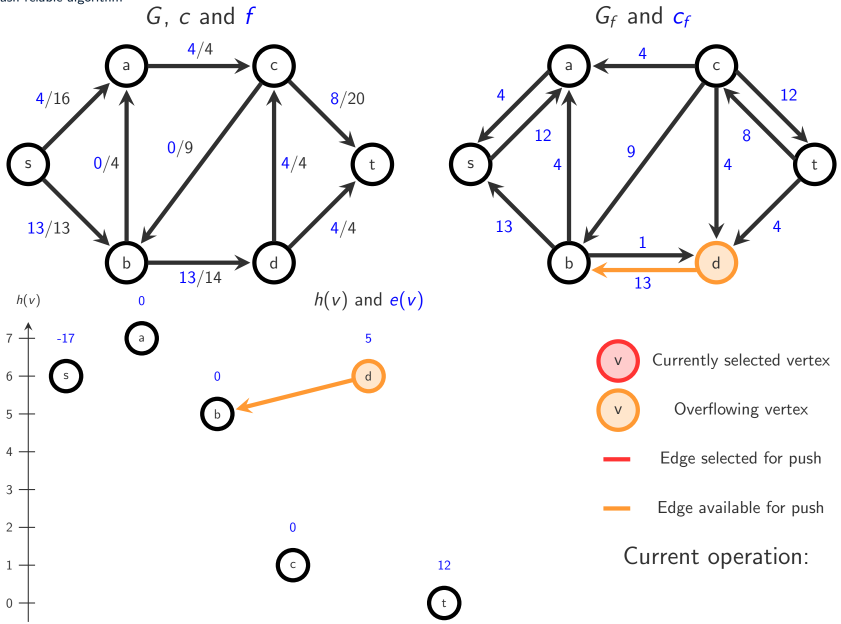
Overflowing vertex

Edge selected for push

Edge available for push

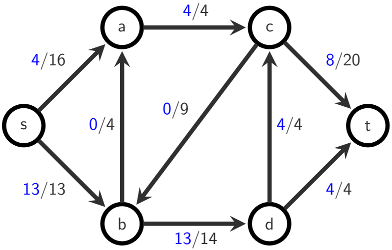
Current operation:

Relabel d

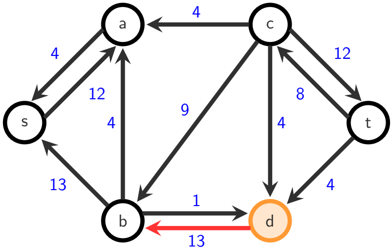


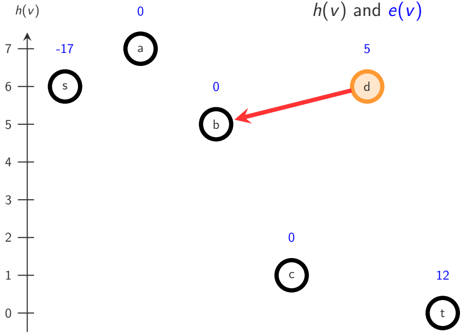
Current operation:

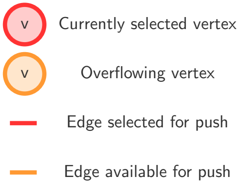
G, c and f



G_f and c_f



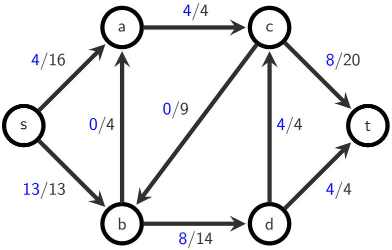




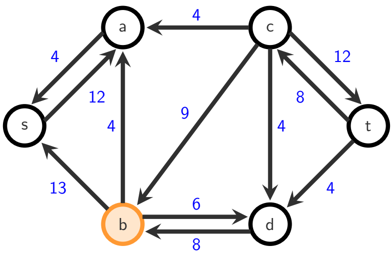
Current operation:

Push from d to b

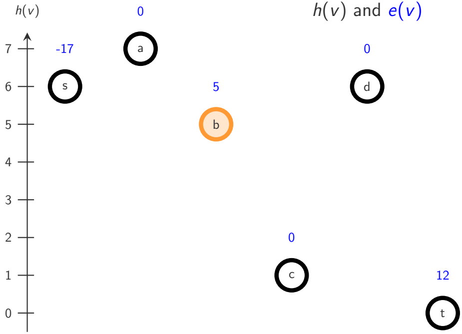
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



v Currently selected vertex

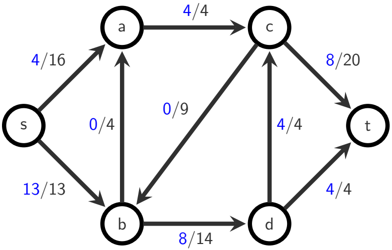
v Overflowing vertex

Edge selected for push

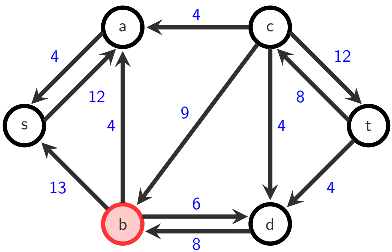
Edge available for push

Current operation:

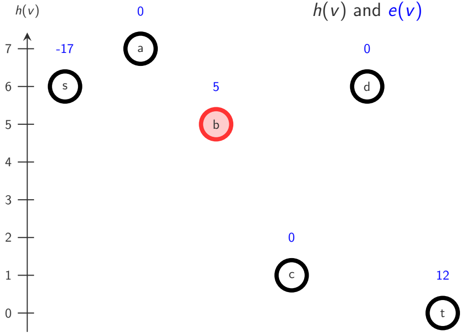
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



Currently selected vertex

Overflowing vertex

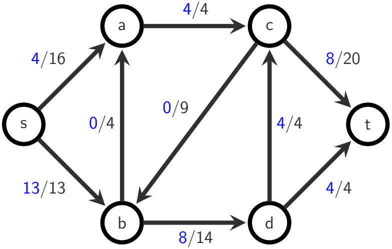
Edge selected for push

Edge available for push

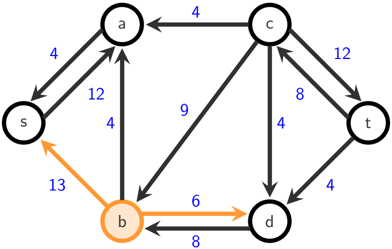
Current operation:

Relabel b

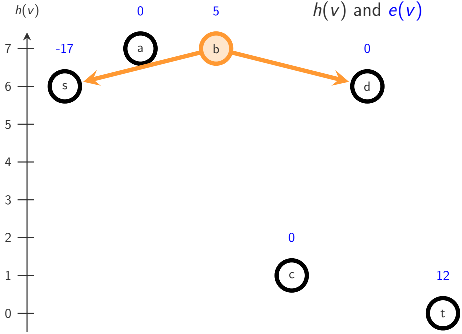
G, c and f

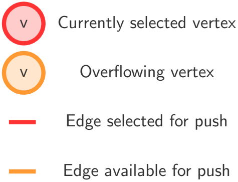


G_f and c_f



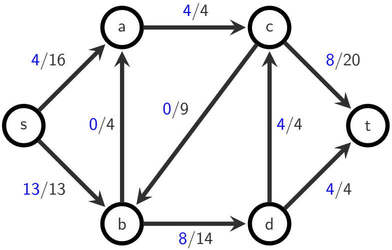
$h(v)$ and $e(v)$



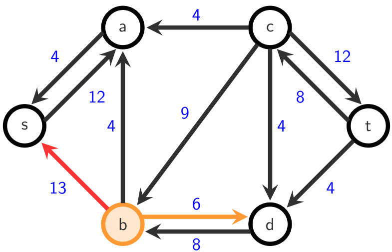


Current operation:

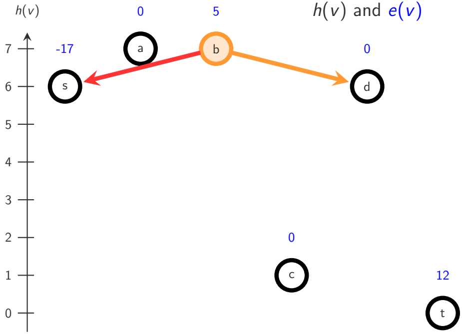
G, c and f

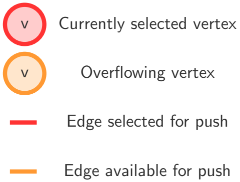


G_f and c_f



$h(v)$ and $e(v)$

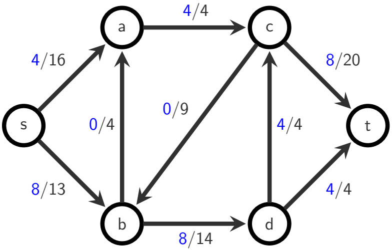




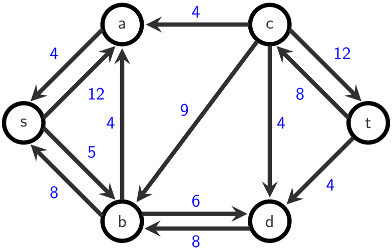
Current operation:

Push from b to s

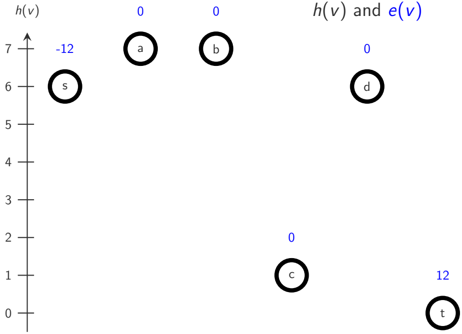
G, c and f



G_f and c_f



$h(v)$ and $e(v)$



Currently selected vertex

Overflowing vertex

Edge selected for push

Edge available for push

No overflowing vertex:

Maximum flow found

Analysis

Order for choosing next operation?

- Any order (e.g. with stack): Goldberg-Tarjan algorithm, $\mathcal{O}(|V|^2 |E|)$.
- FIFO (with a queue): $\mathcal{O}(|V|^3)$.
- Highest label (with buckets): $\mathcal{O}(|V|^2 \sqrt{|E|})$.

Keep list of overflowing vertices in appropriate data structure and update accordingly after each operation!

Heuristics for the push-relable algorithm

Two-phase algorithm

- In first phase, only push/relabel vertices with $h(v) < |V|$.
- Does not compute complete flow, but value of maximum flow at t.
- Remaining excess flow may be pushed back to s in second phase.

Initial labeling heuristic

- Compute initial heights as minimal distance to t by backwards BFS, computing $h(v) \leftarrow d_G(v, t)$.
- Avoids unnecessary initial relabeling operations.
- Can also compute labeling for second phase with $h(v) \leftarrow d_{G_f}(v, s)$.

Heuristics for the push-relable algorithm

Gap heuristic

- After each relabeling, check if there is a height k with $0 < k < |V|$ such that there is no vertex v with $h(v) = k$ (keep a count array).
- If yes, all vertices u with $k < h(u) < |V|$ are disconnected from t in G_f and can be disregarded (set $h(u) \leftarrow |V|$).
- One of the most efficient heuristics, crucial for improving the performance.

Further reading

Several improved algorithms available:

- Orlin: Max flows in $\mathcal{O}(nm)$ time, or better, 2013: $\mathcal{O}(|V|\,|E|)$
- Sidford and Lee: Path-Finding Methods for Linear Programming, 2014: $\widetilde{\mathcal{O}}(|E|\sqrt{|V|}\log^{\mathcal{O}(1)} U)$ (where $\widetilde{\mathcal{O}}$ hides polylog$(|V|, |E|)$).

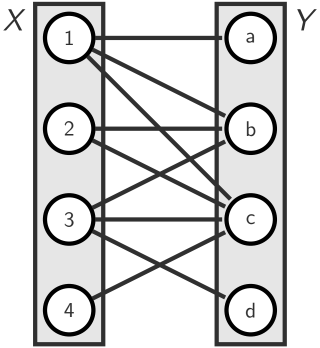
Additional literature on flow problems and algorithms:

- T. H. Cormen et al.: Introduction to Algorithms. MIT press, 2009.
- R. Ahuja, T. Magnanti and J. B. Orlin: Network Flows: Theory, Algorithms and Applications. Prentice Hall, 1993.
- B. Korte, J. Vygen: Combinatorial Optimization: Theory and Algorithms. Springer, 2012.

Application: Bipartite Matching

Definition (Bipartite Matching / Maximum Matching Problem)

Given two disjoint sets of vertices X and Y and a set of edges $E \subseteq X \times Y$, a *matching* $M \subseteq E$ is a subset of edges such that each node of $X \cup Y$ appears in at most one edge of M. The *maximum matching problem* is finding a matching M such that $|M|$ is maximal.



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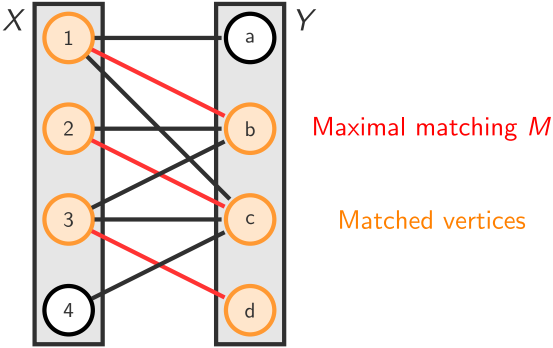

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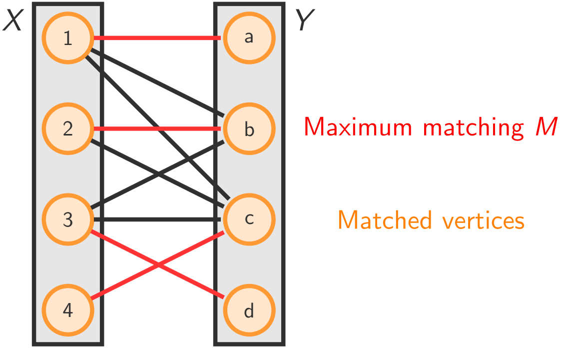
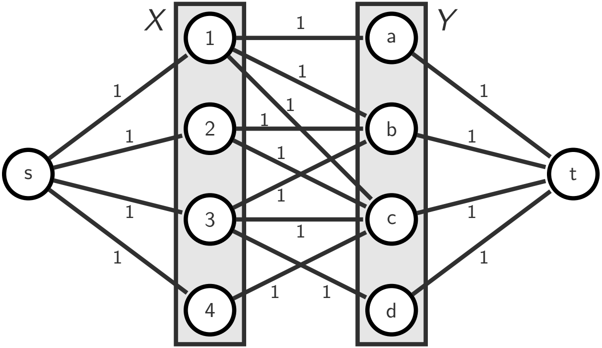


Application: Bipartite Matching

Matching problem as a flow problem

Given a bipartite matching problem, construct flow network $G = (V, E')$ with $V = \{s, t\} \cup X \cup Y$ and $E' = E \cup (\{s\} \times X) \cup (Y \times \{t\})$ and $c(e) = 1$ for all $e \in E'$. Then the value of the maximum flow is equal to the size of the maximum matching.



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