Problem 1 (4= 2+2 points) We know that if $M_{D\times D}$ is real and symmetric matrix then M can be factored as

$$M = U\Lambda U^T$$

Where $U = [u_1, u_2, \cdots, u_D]$ are the eigenvectors of M. Eigen vectors u_i are usually normalized in that case U is orthonormal i.e U^{-1} =U^T and

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_D \end{bmatrix}$$

contains eignevalues along the diagonal.
Let $X_c = U_{N\times D}\Sigma_{D\times D}V^T_{D\times D}$ be reduced SVD of our centrlize matrix of size $X_{C_{N\times D}}$ where $N \geq D$.

1. Prove that V contains the eigen vector of $X_c^T X_C$.
2. Prove $\Sigma^2 = \Sigma\,\Sigma$ contains eigen values of $X_c^T X_c$ along the diagonal.

Similarly U contains the eigen vectors of $X_c X_c^T$ but you need not to prove this part.

Problem 2 (4 = 1 + 1 + 2 points) Show that if k_1 and k_2 are mercer kernels then

$$k(\boldsymbol{x}, \boldsymbol{x}^{'}) = k_1(\boldsymbol{x}, \boldsymbol{x}^{'}) + k_2(\boldsymbol{x}, \boldsymbol{x}^{'})$$

is also mercer a kernel where $\boldsymbol{x}, \boldsymbol{x}^{'} \in R^d$. Prove all the 3 conditions of mercel kernel.
Problem 3.(4 points.) Prove that RBF kernal $k(x, x^{'}) = \exp(-\frac{\|x-x^{'}\|}{2\sigma^2})$, maps the D dimentional input space $\mathcal{X}$ into the surface of an unit hypersphere(Infact it is infinite dimentional hypersurface but no need to show infinite part). Note for any point z on unit hyper sphere $\|z\|_2^2 = 1$.