Introduction to Quantum Error Correction

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Why Quantum Error-Correction ?

- State Preparation errors
- Gate errors
- Errors due to environment (Open Systems)

1 The Bit-flip code

Error : |0> becomes |1> with some probability due to the effect of environment

The Bit-Flip Code

- Lets find a way to correct it

$|0\rangle$ - $|000\rangle$ $|1\rangle$-$|111\rangle$

- A general state in this encoding would be

$$a|000\rangle + b|111\rangle$$

The Bit-Flip code

Possible single qubit bit-flip errors

$$a|000\rangle + b|111\rangle \longrightarrow a|001\rangle + b|110\rangle$$
$$a|000\rangle + b|111\rangle \longrightarrow a|100\rangle + b|011\rangle$$
$$a|000\rangle + b|111\rangle \longrightarrow a|010\rangle + b|101\rangle$$

The Bit-Flip code

To correct the errors we need to identify where the error has occurred -Syndrome diagnosis

We cannot measure in the computational basis since that will destroy our state.

The Bit-Flip code

Think about these operators,

$$\boxed{S_1 = Z_1 Z_2, \qquad S_2 = Z_2 Z_3.}$$

The general state a|000> + b|111> is an Eigenstate with +1 eigenvalue of the above operators

The Bit-Flip code

But the error affected states are not +1 eigenstates of all the stabilizers.

- **Error on qubit 1:** $|\psi^{(1)}\rangle = X_1(\alpha\,|000\rangle + \beta\,|111\rangle) = \alpha\,|100\rangle + \beta\,|011\rangle$.

$$S_1\,|\psi^{(1)}\rangle = Z_1Z_2(\alpha\,|100\rangle + \beta\,|011\rangle) = (-\alpha)\,|100\rangle + (-\beta)\,|011\rangle = -\,|\psi^{(1)}\rangle\,,$$
$$S_2\,|\psi^{(1)}\rangle = Z_2Z_3(\alpha\,|100\rangle + \beta\,|011\rangle) = (+\alpha)\,|100\rangle + (+\beta)\,|011\rangle = +\,|\psi^{(1)}\rangle\,.$$

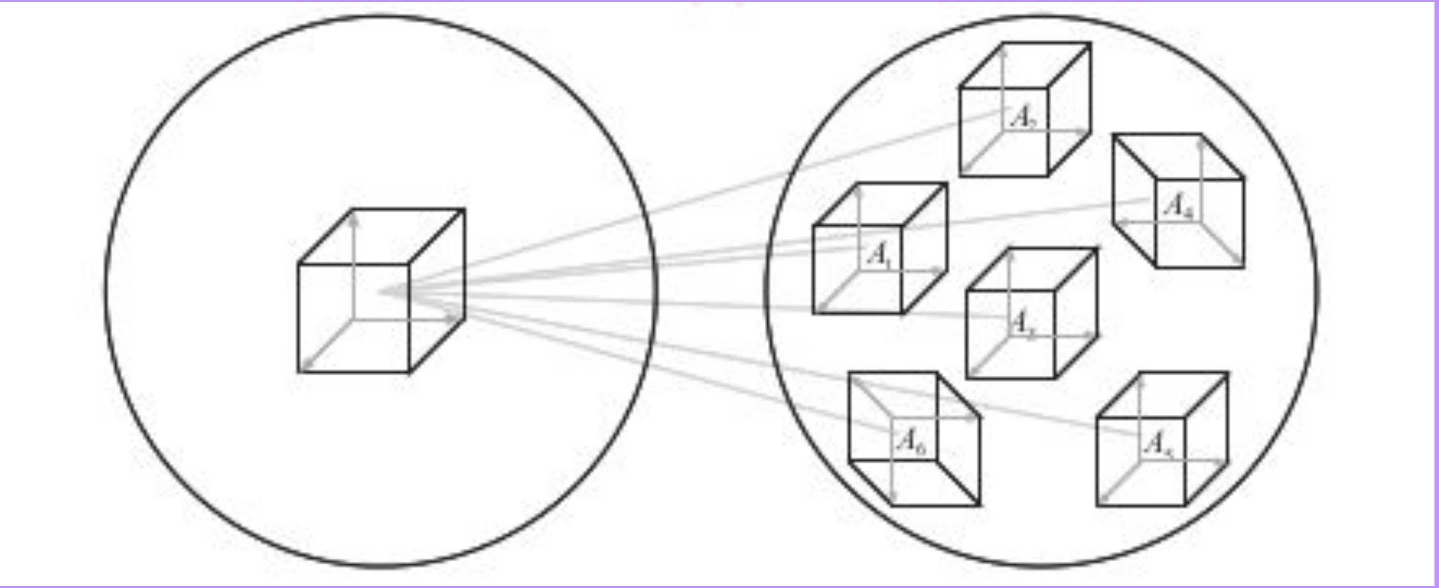
- **Error on qubit 2:** $|\psi^{(2)}\rangle = \alpha\,|010\rangle + \beta\,|101\rangle$.

$$S_1\,|\psi^{(2)}\rangle = Z_1Z_2(\alpha\,|010\rangle + \beta\,|101\rangle) = (-\alpha)\,|010\rangle + (-\beta)\,|101\rangle = -\,|\psi^{(2)}\rangle\,,$$
$$S_2\,|\psi^{(2)}\rangle = Z_2Z_3(\alpha\,|010\rangle + \beta\,|101\rangle) = (-\alpha)\,|010\rangle + (-\beta)\,|101\rangle = -\,|\psi^{(2)}\rangle\,.$$

- **Error on qubit 3:** $|\psi^{(3)}\rangle = \alpha|001\rangle + \beta|110\rangle$.

$$S_1|\psi^{(3)}\rangle = Z_1Z_2(\alpha|001\rangle + \beta|110\rangle) = (+\alpha)|001\rangle + (+\beta)|110\rangle = +|\psi^{(3)}\rangle,$$
$$S_2|\psi^{(3)}\rangle = Z_2Z_3(\alpha|001\rangle + \beta|110\rangle) = (-\alpha)|001\rangle + (-\beta)|110\rangle = -|\psi^{(3)}\rangle.$$

The errors are taking our codespace vectors from +1 eigenspace to -1 eigenspace.


A_1
A_4
A_6

The Bit-Flip code

- Based on which stabilizer gave a -1 result we can identify on which location the error has occurred i.e on which qubit location the error has occurred

- The error correction can be done by applying an X on the location

The Bit-Flip code

- Right now we have found a way to correct bit-flip errors on a single qubit out of our three qubit encoding.

- Lets try to do something more powerful next time

Shor’s 9 Qubit Code

- Right now we have found a way to correct bit-flip errors on a single qubit out of our three qubit encoding.

- Lets try to do something more powerful next time

Shor's 9 Qubit Code

- Our encoding in this qubit is given by,

$$\boxed{\begin{aligned} |0_L\rangle &= \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle), \\ |1_L\rangle &= \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle). \end{aligned}}$$

Shor’s 9 Qubit Code

- Our errors in this code are

1. Bit-Flip Error on any qubit
2. Phase-Flip Error
3. Both Bit-flip and Phase flip error

Shor’s 9 Qubit Code

- The useful operators/stabilizers in this code are

Name	Operator
g_1	$ZZIIIIIII$
g_2	$IZZIIIIII$
g_3	$IIIZZIIII$
g_4	$IIIIZZIII$
g_5	$IIIIIIZZI$
g_6	$IIIIIIIZZ$
g_7	$XXXXXXIII$
g_8	$IIIXXXXXX$
$\bar{Z}$	$XXXXXXXXX$
$\bar{X}$	$ZZZZZZZZZ$

Shor’s 9 Qubit Code

- The code-space is as expected a positive eigenspace of all stabilizers

Name	Operator
g_1	$Z\,Z\,I\,I\,I\,I\,I\,I\,I$
g_2	$I\,Z\,Z\,I\,I\,I\,I\,I\,I$
g_3	$I\,I\,I\,Z\,Z\,I\,I\,I\,I$
g_4	$I\,I\,I\,I\,Z\,Z\,I\,I\,I$
g_5	$I\,I\,I\,I\,I\,I\,Z\,Z\,I$
g_6	$I\,I\,I\,I\,I\,I\,I\,Z\,Z$
g_7	$X\,X\,X\,X\,X\,X\,I\,I\,I$
g_8	$I\,I\,I\,X\,X\,X\,X\,X\,X$
$\bar{Z}$	$X\,X\,X\,X\,X\,X\,X\,X\,X$
$\bar{X}$	$Z\,Z\,Z\,Z\,Z\,Z\,Z\,Z\,Z$

Shor’s 9 Qubit Code

- The errors again as expected will take the codespace to the -1 eigenspace of some stabilizers

Shor's 9 Qubit Code

Error X_2 (in first block):

Stabilizer	Effect	Eigenvalue
Z_1Z_2	touches qubit 2 → anticommutes	−1
Z_2Z_3	touches qubit 2 → anticommutes	−1
others ($Z_4Z_5, \ldots$)	commute	+1
$X_1 \ldots X_6, X_4 \ldots X_9$	commute	+1

So the two Z-type stabilizers in that block give **−1**, uniquely identifying the flipped qubit.

Shor’s 9 Qubit Code

Error: X_7

Stabilizer	Eigenvalue
Z_1Z_2	$+1$
Z_2Z_3	$+1$
Z_4Z_5	$+1$
Z_5Z_6	$+1$
Z_7Z_8	-1
Z_8Z_9	$+1$
$X_1X_2X_3X_4X_5X_6$	$+1$
$X_4X_5X_6X_7X_8X_9$	$+1$

Shor's 9 Qubit Code

Block with Z error	$X_1 \ldots X_6$	$X_4 \ldots X_9$
(1,2,3)	−1	+1
(4,5,6)	−1	−1
(7,8,9)	+1	−1

Shor’s 9 Qubit Code

- Once detected we can again correct X errors by applying X_i on the correct location and Z errors by applying Z_i Z_i+1 Z_i+2.

- Both Bit-flip and phase flip errors (XZ) error can be corrected doing it step-wise

Shor’s 9 Qubit Code

- Any Unitary U can be written as,

  $E = I/2 + aX + bY + cZ$

- E1 $|\psi\rangle$ can thus be written as a superposition of four terms, $|\psi\rangle$, X1 $|\psi\rangle$, Z1 $|\psi\rangle$, X1 Z1 $|\psi\rangle$.

Shor’s 9 Qubit Code

Every error U is broken down to either X error or Z error or XZ errors

- Once we correct these three we can correct any arbitrary unitary error.