CS 70 Discrete Mathematics and Probability Theory
Summer 2021

HW 7

1 A Bit of Everything

Suppose that $X_0, X_1, \ldots$ is a Markov chain with finite state space $S = \{1, 2, \ldots, n\}$, where $n > 2$, and transition matrix P. Suppose further that

$$\begin{aligned} P(1,i) &= \frac{1}{n} \quad \text{for all states } i \text{ and} \\ P(j, j-1) &= 1 \quad \text{for all states } j \neq 1, \end{aligned}$$

with $P(i, j) = 0$ everywhere else.

(a) Prove that this Markov chain is irreducible and aperiodic.

(b) Suppose you start at state 1. What is the distribution of T, where T is the number of transitions until you leave state 1 for the first time?

(c) Again starting from state 1, what is the expected number of transitions until you reach state n for the first time?

(d) Again starting from state 1, what is the probability you reach state n before you reach state 2?

(e) Compute the stationary distribution of this Markov chain.

(f) Suppose now you start in state n. What is the expected number of transitions until you return to state n for the first time?

Solution:

(a) For any two states i and j, we can consider the path $(i, i-1, \ldots, 2, 1, j)$, which has nonzero probability of occurring. Thus, this chain is irreducible. To see that it is aperiodic, observe that $d(1) = 1$, as we have self-loop from state 1 to itself.

(b) At any given transition, we leave state 1 with probability with probability $\frac{n-1}{n}$, independently of any previous transition. Thus, the distribution is Geometric, with parameter $\frac{n-1}{n}$.

(c) Suppose that $\beta(i)$ is the expected number of transitions necessary to reach state n for the first time, starting from state i. We have the following first step equations:

$$\begin{aligned} \beta(1) &= 1 + \sum_{j=1}^{n} \frac{1}{n} \beta(j), \\ \beta(i) &= 1 + \beta(i-1) \quad \text{for } 1 < i < n, \text{ and} \\ \beta(n) &= 0. \end{aligned}$$

We can simplify the second recurrence to

$$\beta(i) = i - 1 + \beta(1) \quad \text{for } 1 < i < n.$$

Substituting this simplified recurrence into the first equation, we get that

$$\beta(1) = 1 + \frac{1}{n}\sum_{i=1}^{n-1}(i - 1 + \beta(1)) = 1 + \frac{1}{n}\sum_{i=1}^{n-1}(i-1) + \frac{1}{n}\sum_{i=1}^{n-1}\beta(1) = 1 + \frac{(n-2)(n-1)}{2n} + \frac{n-1}{n}\beta(1),$$

which we can solve to get that

$$\beta(1) = \boxed{n + \frac{1}{2}(n-1)(n-2)}.$$

(d) Suppose that $\alpha(i)$ is the probability that we reach state n before we reach state 2, starting from state i. One immediate observation we can make is that from any state i in $\{2,\ldots,n-1\}$, we are guaranteed to see state 2 before state n, as we can only take the path $(i, i-1, \ldots, 2, 1)$. Hence, $\alpha(i) = 0$ if $i \in \{2,\ldots,n-1\}$. Moreover, $\alpha(n) = 1$, so

$$\alpha(1) = \sum_{i=1}^{n}\frac{1}{n}\alpha(i) = \frac{1}{n}\alpha(1) + \frac{1}{n},$$

hence $\alpha(1) = \boxed{\frac{1}{n-1}}$.

(e) We have the balance equations

$$\begin{aligned}\pi(i) &= \frac{1}{n}\pi(1) + \pi(i+1) \quad \text{if } i \neq n, \text{ and}\\ \pi(n) &= \frac{1}{n}\pi(1).\end{aligned}$$

We can collapse the first recurrence to

$$\pi(i) = \frac{n-i}{n}\pi(1) + \pi(n) = \frac{n-i+1}{n}\pi(1),$$

so we can express each stationary probability in terms of the stationary probability of state 1. We can finish by using the normalization equation:

$$\pi(1) + \pi(2) + \cdots + \pi(n) = 1 \implies \frac{1}{n}\pi(1)\sum_{i=1}^{n} n - i + 1 = 1.$$

The last sum can be rearranged to be the sum of the integers from 1 up to n, so we get that

$$\pi(1) = \frac{2}{n+1} \implies \pi = \boxed{\frac{2}{n(n+1)}\begin{bmatrix} n & n-1 & \cdots & 1\end{bmatrix}}.$$

(f) This is the reciprocal of the stationary probability of state n, which is $\boxed{\frac{n(n+1)}{2}}$.

2 Boba in a Straw

Imagine that Jonathan is drinking milk tea and he has a very short straw: it has enough room to fit two boba (see figure).

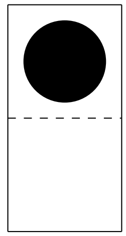

Figure 1: A straw with one boba currently inside. The straw only has enough room to fit two boba.

Here is a formal description of the drinking process: We model the straw as having two "components" (the top component and the bottom component). At any given time, a component can contain nothing, or one boba. As Jonathan drinks from the straw, the following happens every second:

1. The contents of the top component enter Jonathan's mouth.
2. The contents of the bottom component move to the top component.
3. With probability p, a new boba enters the bottom component; otherwise the bottom component is now empty.

Help Jonathan evaluate the consequences of his incessant drinking!

(a) Draw the Markov chain that models this process, and show that it is both irreducible and aperiodic.

(b) At the very start, the straw starts off completely empty. What is the expected number of seconds that elapse before the straw is completely filled with boba for the first time? [Write down the equations; you do not have to solve them.]

(c) Consider a slight variant of the previous part: now the straw is narrower at the bottom than at the top. This affects the drinking speed: if either (i) a new boba is about to enter the bottom component or (ii) a boba from the bottom component is about to move to the top component, then the action takes two seconds. If both (i) and (ii) are about to happen, then the action takes three seconds. Otherwise, the action takes one second. Under these conditions, answer the previous part again. [Write down the equations; you do not have to solve them.]

(d) Jonathan was annoyed by the straw so he bought a fresh new straw (same as the straw from Figure 1). What is the long-run average rate of Jonathan's calorie consumption? (Each boba is roughly 10 calories.)

(e) What is the long-run average number of boba which can be found inside the straw? [Maybe you should first think about the long-run distribution of the number of boba.]

(f) What is the long run probability that the amount of boba in the straw doesn't change from one second to the next?

Solution:

(a) We model the straw as a four-state Markov chain. The states are $\{(0,0),(0,1),(1,0),(1,1)\}$, where the first component of a state represents whether the top component is empty (0) or full (1); similarly, the second component represents whether the bottom component is empty or full. See Figure 2. This chain is irreducible as we can get from any state to any other with the cycle $(0,0) \to (0,1) \to (1,0) \to (0,1) \to (1,1) \to (1,0) \to (0,0)$. Furthermore, this chain contains a self-loop, so it is aperiodic.

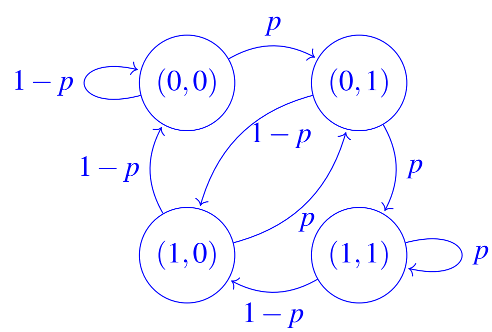

Figure 2: Transition diagram for the Markov chain.

(b) We set up the hitting time equations. Let T denote the time it takes to reach state $(1,1)$, i.e. $T = \min\{n > 0 : X_n = (1,1)\}$. Let $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot \mid X_0 = i]$ denote the expectation starting from state i (for convenience of notation). The hitting-time equations are

$$\begin{aligned}
\mathbb{E}_{(0,0)}[T] &= 1 + (1-p)\,\mathbb{E}_{(0,0)}[T] + p\,\mathbb{E}_{(0,1)}[T], \\
\mathbb{E}_{(0,1)}[T] &= 1 + (1-p)\,\mathbb{E}_{(1,0)}[T] + p\,\mathbb{E}_{(1,1)}[T], \\
\mathbb{E}_{(1,0)}[T] &= 1 + (1-p)\,\mathbb{E}_{(0,0)}[T] + p\,\mathbb{E}_{(0,1)}[T], \\
\mathbb{E}_{(1,1)}[T] &= 0.
\end{aligned}$$

The question did not ask you to solve the equations. If you solved the equations anyway and would like to check your work, the hitting time is $\mathbb{E}_{(0,0)}[T] = (1+p)/p^2$.

(c) The new hitting-time equations are

$$\begin{aligned}
\mathbb{E}_{(0,0)}[T] &= (1-p)(1+\mathbb{E}_{(0,0)}[T]) + p(2+\mathbb{E}_{(0,1)}[T]), \\
\mathbb{E}_{(0,1)}[T] &= (1-p)(2+\mathbb{E}_{(1,0)}[T]) + p(3+\mathbb{E}_{(1,1)}[T]), \\
\mathbb{E}_{(1,0)}[T] &= (1-p)(1+\mathbb{E}_{(0,0)}[T]) + p(2+\mathbb{E}_{(0,1)}[T]), \\
\mathbb{E}_{(1,1)}[T] &= 0.
\end{aligned}$$

You did not have to solve the equations, but to get a sense for what the solution is like, solving the equations and plugging in $p = 1/2$ yields (after some tedious algebra) $\mathbb{E}_{(0,0)}[T] = 11$.

(d) This part is actually more straightforward than it might initially seem: the average rate at which Jonathan consumes boba must equal the average rate at which boba enters the straw, which is p per second. Hence, his long-run average calorie consumption rate is $10p$ per second.

(e) We compute the stationary distribution. The balance equations are

$$\begin{aligned}
\pi(0,0) &= (1-p)\pi(0,0) + (1-p)\pi(1,0), \\
\pi(0,1) &= p\pi(0,0) + p\pi(1,0), \\
\pi(1,0) &= (1-p)\pi(0,1) + (1-p)\pi(1,1), \\
\pi(1,1) &= p\pi(0,1) + p\pi(1,1).
\end{aligned}$$

Expressing everything in terms of $\pi(0,0)$, we find

$$\begin{aligned}
\pi(0,1) = \pi(1,0) &= \frac{p}{1-p}\pi(0,0), \\
\pi(1,1) &= \frac{p^2}{(1-p)^2}\pi(0,0).
\end{aligned}$$

From the normalization condition we have

$$\pi(0,0)\left(1 + \frac{2p}{1-p} + \frac{p^2}{(1-p)^2}\right) = 1,$$

so $\pi(0,0) = (1-p)^2$. Hence, the stationary distribution is

$$\begin{aligned}
\pi(0,0) &= (1-p)^2, \\
\pi(0,1) = \pi(1,0) &= p(1-p), \\
\pi(1,1) &= p^2.
\end{aligned}$$

In states $(0,1)$ and $(1,0)$, there is one boba in the straw; in state $(1,1)$, there are two boba in the straw. Therefore, the long-run average number of boba in the straw is

$$\pi(0,1) + \pi(1,0) + 2\pi(1,1) = 2p(1-p) + 2p^2 = 2p.$$

Alternate Solution: The goal of the question was to have you work through the balance equations, but there is a simple solution. Observe that at any given time after at least two seconds have passed, each component has probability p of being filled with boba. Therefore, the number of boba in the straw is like a binomial distribution with 2 independent trials and success probability p, so the average number of boba in the straw is $2p$.

(f) The long run probability that the amount of boba doesn't change is the probability that either (a) we are in state $(0,1)$ and transition to $(1,0)$, (b) we are in state $(1,0)$ and transition to $(0,1)$, (c) we are in state $(1,1)$ and transition to $(1,1)$, or (d) we are in state $(0,0)$ and transition to $(0,0)$. In the long run, the probability we are in a particular state is given by the stationary distribution, so we have

$$\begin{aligned}
\mathbb{P}[(0,0) \to (0,0)] &= \pi(0,0)\mathbb{P}[X_{n+1} = (0,0) \mid X_n = (0,0)] = (1-p)^3 \\
\mathbb{P}[(0,1) \to (1,0)] &= \pi(0,1)\mathbb{P}[X_{n+1} = (1,0) \mid X_n = (0,1)] = p(1-p)^2 \\
\mathbb{P}[(1,0) \to (0,1)] &= \pi(1,0)\mathbb{P}[X_{n+1} = (0,1) \mid X_n = (1,0)] = p^2(1-p) \\
\mathbb{P}[(1,1) \to (1,1)] &= \pi(1,1)\mathbb{P}[X_{n+1} = (1,1) \mid X_n = (1,1)] = p^3.
\end{aligned}$$

Thus, our overall probability is $\boxed{p^3 + p^2(1-p) + p(1-p)^2 + (1-p)^3}$.