Linear Degrees of Freedom of the MIMO X-Channel with Delayed CSIT

David T.H. Kao and A. Salman Avestimehr
University of Southern California, Los Angeles, CA

***Abstract*—We establish the sum degrees of freedom of the multiple-input multiple-output X-channel with delayed Channel State Information at the transmitters (CSIT), assuming linear coding strategies at the transmitters. The converse is based on developing a novel rank-ratio inequality that upper bounds the ratio between the dimensions of received linear subspaces at the two receivers. The achievability is based on a three-phase strategy that optimally exploits delayed CSIT in each phase.**

I. Introduction

The availability of channel state information at transmitters (CSIT) enables considerable capacity gains in wireless networks. However, due to overhead in acquiring channel state information, instantaneous CSIT may be an unrealistic requirement. A more realistic assumption in a fast fading scenario is to assume that CSIT is acquired with some delay.

Surprisingly, in [1] it was shown that, in the multiple-input multiple-output broadcast (MIMO BC) channel, outdated CSIT can still be quite useful and provide capacity gains. Since then, the impact of delayed CSIT has been explored for many networks. For example, in [2]–[6], it was shown that delayed CSIT can also be useful for interference management in various network configurations by developing novel transmission strategies that dynamically adapt to the past receptions at the receivers to better manage the interference.

In this work, we study the impact of delayed CSIT in the X-channel, which is a canonical setting for information-theoretic study of interference management in wireless networks. This channel consists of two transmitters causing interference at two receivers, and each transmitter aims to communicate intended messages to both receivers. For the case of the X-channel with single-antenna nodes, a novel converse was recently derived in [7], [8], confirming that the sum DoF of $\frac{6}{5}$ originally achieved in [9] is the maximum achievable by any linear strategy. For the case of the multiple-input multiple-output X-channel (MIMO XC) with delayed CSIT, the scheme that achieves the highest sum DoF to date is the three-phase retrospective interference alignment scheme proposed in [10] for symmetric antenna configurations, i.e. M (N) antennas per transmitter (receiver). The general DoF-optimality of this approach remains unsolved, even assuming linear strategies.

This paper establishes the sum degrees of freedom of the MIMO XC with delayed CSIT, assuming linear coding strategies at the transmitters. We generalize the converse of [8] to the MIMO XC with delayed CSIT, by establishing an inequality for the ratio between dimensions of received signal subspaces at the two receivers. This rank-ratio inequality results from a combination of two bounds. The first is a cooperative bound, which assumes transmitters share message information. The second bound focuses instead on how effectively distributed transmitters can exploit delayed CSIT.

We further define a class of transmission strategies that achieves the linear sum DoF upper bound, for all antenna configurations. Our strategies consist of three phases. Transmissions during Phase 1 contain only message symbols intended for Receiver 1, and transmissions during Phase 2 contain only symbols intended for Receiver 2. The number of symbols sent from each transmitter during Phases 1 and 2 is dictated by a function of the maximum rank-ratios. During Phase 3, both transmitters use delayed CSIT to send linear combinations of past transmissions such that each receiver receives a superposition of desired message data and known interference. To maximize the number of Phase 3 transmissions (and thus the sum DoF), our strategy exploits delayed CSIT during Phases 1 and 2, but only at the transmitter with fewer antennas.

In [12], we provide further analyses, as well as results on linear DoF regions of MIMO XC with delayed CSIT.

II. Problem Statement and Main Result

The X-channel is a four-node network containing two transmitters and two receivers, where each transmitter has an independent message for each receiver. This paper focuses on the multiple-input multiple-output X-channel (MIMO XC), where Transmitter j and Receiver i have M_j and N_i antennas respectively, with $i, j \in \{1, 2\}$. Without loss of generality, we assume $M_1 \geq M_2$. An example of a MIMO XC is shown in Figure 1. The channel output at Receiver i is

$$\vec{\mathbf{y}}_i[t] = \sum_{j=1}^{2} \mathbf{G}_{ij}[t]\vec{\mathbf{x}}_j[t] + \vec{\mathbf{z}}_i[t],$$

where $\vec{\mathbf{x}}_j[t]$ and $\vec{\mathbf{y}}_i[t]$ are the (vector) input of the j-th transmitter and output of i-th receiver respectively at time t, $\vec{\mathbf{z}}_i[t] \sim \mathcal{CN}(0, \mathbb{I}_{N_i})$ is an additive white Gaussian noise vector, and $\mathbf{G}_{ij}[t]$ the fading channel matrix between the j-th transmitter and i-th receiver. The $N_i \times M_j$ channel matrix $\mathbf{G}_{ij}[t]$ is drawn from a continuous complex distribution, i.i.d. across time. We denote the m-th element of $\vec{\mathbf{x}}_i[t]$ as $\mathbf{x}_i^m[t]$ and the (n, m)-th element of $\mathbf{G}_{ij}[t]$ as $\mathbf{g}_{ij}^{nm}[t]$. Additionally, we denote the set of channel matrices up until time T as $\boldsymbol{\mathcal{G}}^T \triangleq \{\mathbf{G}_{ij}[t] : \ i, j \in \{1, 2\}, t \in \{1, \ldots, T\}\}$. Since we assume delayed channel state information at the transmitters,

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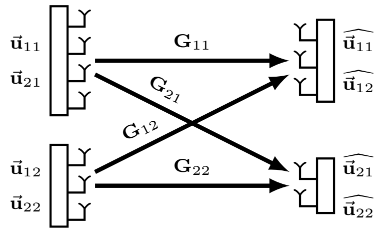

Fig. 1. A MIMO X-channel where $M_1 = 4$, $M_2 = 3$, $N_1 = 3$, and $N_2 = 2$.

at time t, transmitters know $\mathcal{G}^{t-1}$ whereas receivers know $\mathcal{G}^t$.[1]

Each transmitter is subject to an average power constraint evaluated over a block of length T, i.e., let $\Sigma_j[t] \triangleq \mathrm{E}[\vec{\mathbf{x}}_j[t]\vec{\mathbf{x}}_j[t]^\dagger]$ with $\dagger$ representing the Hermitian transpose, and $\frac{1}{T}\sum_t \mathrm{tr}\,\Sigma_j[t] \leq P$.

We restrict ourselves to linear coding strategies as defined in [7], [8], [11], in which DoF simply represents the dimension of the linear subspace of transmitted signals. More specifically, consider a communication scheme with blocklength T, in which Transmitter j wishes to transmit a vector $\vec{\mathbf{u}}_{ij} \in \mathbb{C}^{m_{ij}^{(T)}}$ of $m_{ij}^{(T)} \in \mathbb{N}$ information symbols to Receiver i. These information symbols are then modulated with precoding matrices$\mathbf{V}_{ij}[t] \in \mathbb{C}^{M_j \times m_{ij}^{(T)}}$ at times $t = 1, 2, \ldots, T$. Note that, due to the delayed CSIT, the precoding matrix $\mathbf{V}_{ij}[t]$ can only depend upon the outcome of $\mathcal{G}^{t-1}$:

$$V_{ij}[t] = f_{ijt}^{(T)}\left(\mathcal{G}^{t-1}\right), \tag{1}$$

where $f_{ijt}^{(T)}(\cdot)$ is the precoding function that Transmitter j uses to choose the precoding matrix $V_{ij}[t]$ for symbols for Receiver i at time t. For notational simplicity, we denote the collection of precoding functions used by Transmitter j as $f_j^{(T)} = \left\{f_{1jt}^{(T)}, f_{2jt}^{(T)}\right\}_{t=1,\ldots,T}$.

Based on this linear precoding, Transmitter j will then send $\vec{\mathbf{x}}_j[t] = \mathbf{V}_{1j}[t]\vec{\mathbf{u}}_{1j} + \mathbf{V}_{2j}[t]\vec{\mathbf{u}}_{2j}$ at time t, and Receiver i receives at time t

$$\begin{aligned}\vec{\mathbf{y}}_i[t] = \mathbf{G}_{i1}[t]\left(\mathbf{V}_{11}[t]\vec{\mathbf{u}}_{11} + \mathbf{V}_{21}[t]\vec{\mathbf{u}}_{21}\right)\\ + \mathbf{G}_{i2}[t]\left(\mathbf{V}_{12}[t]\vec{\mathbf{u}}_{12} + \mathbf{V}_{22}[t]\vec{\mathbf{u}}_{22}\right) + \vec{\mathbf{z}}_i[t].\end{aligned}$$

We denote by $\mathbf{V}_{ij}^T \in \mathbb{C}^{(TM_2)\times \mathbf{m}_{ij}^{(T)}}$ the overall precoding matrix of Transmitter j for Receiver i and $\mathbf{G}_{ij}^T$ the $TN_i \times TM_j$ block diagonal matrix, such that

$$\mathbf{G}_{ij}^t \triangleq \begin{bmatrix}\mathbf{G}_{ij}[1] & 0 & \ldots & 0\\ 0 & \mathbf{G}_{ij}[2] & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \mathbf{G}_{ij}[t]\end{bmatrix}, \quad \mathbf{V}_{ij}^t \triangleq \begin{bmatrix}\mathbf{V}_{ij}[1]\\ \mathbf{V}_{ij}[2]\\ \vdots\\ \mathbf{V}_{ij}[t]\end{bmatrix}.$$

Based on the above setting, the received signal at Receiver i ($i \in \{1, 2\}$) after the T time steps will be

$$\vec{\mathbf{y}}_i^T = \mathbf{G}_{i1}^T\left(\mathbf{V}_{11}^T\vec{\mathbf{u}}_{11} + \mathbf{V}_{21}^T\vec{\mathbf{u}}_{21}\right) + \mathbf{G}_{i2}^T\left(\mathbf{V}_{12}^T\vec{\mathbf{u}}_{12} + \mathbf{V}_{22}^T\vec{\mathbf{u}}_{22}\right) + \vec{\mathbf{z}}_i^T.$$

With respect to decoding $\vec{\mathbf{u}}_{ij}$ at Receiver i, we let $i' \triangleq 3 - i$ and $j' \triangleq 3 - j$, and notice that the corresponding interference subspace at Receiver i is

$$\boldsymbol{\mathcal{I}}_{ij} = \mathrm{colspan}\left[\mathbf{G}_{ij}^T\mathbf{V}_{i'j}^T \quad \mathbf{G}_{ij'}^T\mathbf{V}_{ij'}^T \quad \mathbf{G}_{ij'}^T\mathbf{V}_{i'j'}^T\right],$$

where $\mathrm{colspan}(\cdot)$ of a matrix is the space spanned its columns.

Let $\boldsymbol{\mathcal{I}}_{ij}^c = \mathbb{C}^{TN_i} \setminus \boldsymbol{\mathcal{I}}_{ij}$ denote the orthogonal complement of $\boldsymbol{\mathcal{I}}_{ij}$. Then, for asymtotically high transmit powers (i.e., ignoring noise), the decodability of information symbols from Transmitter j to Receiver i corresponds to the constraints that the image of $\mathrm{colspan}\left(\mathbf{G}_{ij}^T\mathbf{V}_{ij}^T\right)$ on $\boldsymbol{\mathcal{I}}_{ij}^c$ has dimension $m_{ij}^{(T)}$:

$$\dim\left(\mathrm{Proj}_{\boldsymbol{\mathcal{I}}_{ij}^c}\mathrm{colspan}\left(\mathbf{G}_{ij}^T\mathbf{V}_{ij}^T\right)\right) = \mathrm{rk}\left[\mathbf{V}_{ij}^T\right] = m_{ij}^{(T)}. \tag{2}$$

As proven in [8], satisfying (2) for all $i, j \in \{1, 2\}$ is equivalent to satisfying $\mathrm{rk}[\mathbf{V}_{ij}^T] = m_{ij}^{(T)}$ and

$$\begin{aligned}&\mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_{i1}^T] + \mathrm{rk}[\mathbf{G}_{i2}^T\mathbf{V}_{i2}^T] + \mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_{i'1}^T \quad \mathbf{G}_{i2}^T\mathbf{V}_{i'2}^T]\\ &= \mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_{i1}^T \quad \mathbf{G}_{i2}^T\mathbf{V}_{i2}^T \quad \mathbf{G}_{i1}^T\mathbf{V}_{i'1}^T \quad \mathbf{G}_{i2}^T\mathbf{V}_{i'2}^T].\end{aligned} \tag{3}$$

Based on this setting, we now define the linear sum DoF for the MIMO XC.

Definition 1. The DoF four-tuple $(d_{11}, d_{12}, d_{11}, d_{22})$ is linearly achievable if there exists a sequence of linear encoding strategies with blocklength $T = 1, 2, \ldots$, such that for every T, the message sizes $(m_{11}^{(T)}, m_{12}^{(T)}, m_{21}^{(T)}, m_{22}^{(T)})$, precoding functions $f_1^{(T)}, f_2^{(T)}$, and corresponding precoding matrices $(\mathbf{V}_{11}^T, \mathbf{V}_{12}^T, \mathbf{V}_{21}^T, \mathbf{V}_{22}^T)$, satisfy the condition given in (2) with probability 1, and for all $i, j \in \{1, 2\}$,

$$d_{ij} = \lim_{T\to\infty}\frac{m_{ij}^{(T)}}{T}. \tag{4}$$

We also define the linear DoF region $\mathcal{D}_{\mathsf{lin}}$ as the closure of the set of all achievable 4-tuples $(d_{11}, d_{12}, d_{11}, d_{22})$. Furthermore, the linear sum DoF ($\mathsf{DoF}_{\text{L-sum}}$) is then defined as follows:

$$\begin{aligned}\mathsf{DoF}_{\text{L-sum}} \triangleq \ &\text{maximize} \quad d_{11} + d_{12} + d_{21} + d_{22},\\ &\text{subject to} \quad (d_{11}, d_{12}, d_{11}, d_{22}) \in \mathcal{D}_{\mathsf{lin}}.\end{aligned} \tag{5}$$

Let $Q_{ij} \triangleq \max\{M_j, N_{i'}\}$, and let the parameter Γ_i be defined as in (6). Using these, we now state the main result:

Theorem 1. *For the MIMO X-Channel with delayed CSIT, the linear sum degrees of freedom is*

$$\mathsf{DoF}_{\text{L-sum}} = \begin{cases}\frac{\Gamma_1\Gamma_2(N_1+N_2)-\Gamma_1N_2-\Gamma_2N_1}{\Gamma_1\Gamma_2-1} & \text{if } M_1 + M_2 \geq \max\{N_1, N_2\}\\ M_1 + M_2 & \text{otherwise}\end{cases}, \tag{7}$$

where Γ_1 and Γ_2 are as defined in (6).

The main ingredient of the converse for Theorem 1 is the following rank-ratio inequality:

Lemma 2 (Maximum Rank-Ratio)**.** *For any linear coding strategy with precoding functions $f_1^{(T)}$ and $f_2^{(T)}$ with corresponding precoding matrices $\mathbf{V}_{i1}^T$ and $\mathbf{V}_{i2}^T$ as defined in (1) and Γ_i defined as in (6),*

$$\frac{\mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_{i1}^T \quad \mathbf{G}_{i2}^T\mathbf{V}_{i2}^T]}{\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_{i1}^T \quad \mathbf{G}_{i'2}^T\mathbf{V}_{i2}^T]} \overset{a.s.}{\leq} \Gamma_i. \tag{8}$$

[1]For i.i.d. channel fades, a delay of one time slot is sufficient to model any finite delay length.

$$\Gamma_i \triangleq \max\left\{\min\left\{\frac{M_1+M_2}{N_{i'}}, \frac{N_i+N_{i'}}{N_{i'}}, \frac{Q_{i1}N_{i'}+Q_{i2}\left(N_{i'}+N_i\right)}{N_{i'}(Q_{i2}+N_{i'})}\right\}, 1\right\}. \tag{6}$$

III. Converse for Theorem 1

The main ingredient of the converse is the rank-ratio inequality stated in Lemma 2. In this section, we first prove Lemma 2 before proceeding with the linear sum DoF converse.

A. Proof of Lemma 2

To prove Lemma 2, we derive two upper bounds on the rank-ratio between the outputs at Receivers i and i'. Evaluating the min of the two bounds yields exactly the definition given in (6). The first bound adds cooperation between the two transmitters, whereas the second focuses on the impact of non-cooperating transmitters. Before deriving these two bounds, we first state the following lemma whose proof is omitted due to space constraints, and can be found in Appendix A of [12].

Lemma 3. *Consider a 2-user MIMO BC with delayed CSIT and M antennas at the transmitter, N_1 antennas at Receiver 1, and N_2 antennas at Receiver 2. Let $i \in \{1,2\}$ and $i' = 3-i$, and $Q_i \triangleq \max\{M, N_i'\}$. The rank-ratio is bounded as*

$$\frac{\mathrm{rk}[\mathbf{G}_i^T\mathbf{V}^T]}{\mathrm{rk}[\mathbf{G}_{i'}^T\mathbf{V}^T]} \overset{a.s.}{\leq} \frac{\min\{Q_i, N_i+N_{i'}\}}{N_{i'}}. \tag{9}$$

Bound 1: To arrive at the first bound, we allow the MIMO XC transmitters to share messages, emulating a single transmitter with M_1+M_2 antennas, and using Lemma 3, we find

$$\begin{aligned}
&\frac{\mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_1^T \quad \mathbf{G}_{i2}^T\mathbf{V}_2^T]}{\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_1^T \quad \mathbf{G}_{i'2}^T\mathbf{V}_2^T]} \\
&\quad \overset{a.s.}{\leq} \frac{\min\{\max\{M_1+M_2, N_i'\}, N_1+N_2\}}{N_i'} \\
&\quad = \max\left\{\min\left\{\frac{M_1+M_2}{N_i'}, \frac{N_1+N_2}{N_i'}\right\}, 1\right\}.
\end{aligned} \tag{10}$$

Bound 2: To construct the second bound, we require the following inequality.

Lemma 4. *Consider a MIMO XC with M_1 and M_2 antennas at Transmitter 1 and 2, and N_1 and N_2 antennas at Receivers 1 and 2, respectively. WLOG assume $M_1 \geq M_2$. Let $i \in \{1,2\}$, $i' \triangleq 3-i$, and $Q_{ij} \triangleq \max\{M_j, N_i'\}$. Then we have*

$$\begin{aligned}
&\frac{\mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_1^T \quad \mathbf{G}_{i2}^T\mathbf{V}_2^T]}{N_i} - \frac{\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_1^T \quad \mathbf{G}_{i'2}^T\mathbf{V}_2^T]}{N_{i'}} \\
&\leq \frac{1}{N_i}\left(\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_1^T \quad \mathbf{G}_{i'2}^T\mathbf{V}_2^T] - \mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_1^T]\right. \\
&\qquad \left. + \mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_1^T \quad \mathbf{G}_{i'2}^T\mathbf{V}_2^T] - \mathrm{rk}[\mathbf{G}_{i'2}^T\mathbf{V}_2^T]\right).
\end{aligned} \tag{11}$$

Sketch of Proof The full proof of Lemma 4 may be found in Appendix B of [12]. We outline the major steps here:

1) Each term on the left side of (11) is decomposed in time into a telescoping series of normalized rank increases, e.g., $\frac{\mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_1^T \quad \mathbf{G}_{i2}^T\mathbf{V}_2^T]}{N_i} = \sum_{t=1}^T \frac{\Delta_i^t}{N_i}$, where $\Delta_i^t = \mathrm{rk}[\mathbf{G}_{i1}^t\mathbf{V}_1^t \quad \mathbf{G}_{i2}^t\mathbf{V}_2^t] - \mathrm{rk}[\mathbf{G}_{i1}^{t-1}\mathbf{V}_1^{t-1} \quad \mathbf{G}_{i2}^{t-1}\mathbf{V}_2^{t-1}]$.
2) We then focus on, for each time t, the difference $\frac{\Delta_i^t}{N_i} - \frac{\Delta_{i'}^t}{N_{i'}}$. We observe that the positive term is at most one (i.e., $\Delta_i^t \leq N_i$), so the difference is positive only if $\Delta_{i'}^t < N_{i'}$.
3) We observe that an increase in the difference occurs if a sufficiently large subspace of each transmitter's transmission at time t is already within the span of Receiver i''s past receptions. Due to continuity of the channel distribution, we argue that this is almost surely the only scenario where an increase occurs, i.e., actively precoding using delayed CSIT is the only method to cause a non-zero difference.
4) Finally, we take the result of the previous step, which applies for a single time t, and consolidate all such subspace coding opportunities into a single expression evaluated over the entire length-T block. □

Now using Lemma 3 (steps (a) and (c)), submodularity of rank (step (b)), and monotonicity of rank (step (d)), we find

$$\begin{aligned}
&\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_1^T] + \mathrm{rk}[\mathbf{G}_{i'2}^T\mathbf{V}_2^T] \\
&\overset{(a)}{\geq} \frac{N_{i'}}{Q_{i1}}\mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_1^T] + \frac{N_{i'}}{Q_{i2}}\mathrm{rk}[\mathbf{G}_{i2}^T\mathbf{V}_2^T] \\
&\overset{(b)}{\geq} \left(\frac{N_{i'}}{Q_{i1}} - \frac{N_{i'}}{Q_{i2}}\right)\mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_1^T] + \frac{N_{i'}}{Q_{i2}}\mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_1^T \ \mathbf{G}_{i2}^T\mathbf{V}_2^T] \\
&\overset{(c)}{\geq} \frac{N_{i'}}{Q_{i2}}\mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_1^T \ \mathbf{G}_{i2}^T\mathbf{V}_2^T] - \left(\frac{Q_{i1}}{Q_{i2}} - 1\right)\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_1^T] \\
&\overset{(d)}{\geq} \frac{N_{i'}}{Q_{i2}}\mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_1^T \ \mathbf{G}_{i2}^T\mathbf{V}_2^T] - \left(\frac{Q_{i1}}{Q_{i2}} - 1\right)\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_1^T \ \mathbf{G}_{i'2}^T\mathbf{V}_2^T].
\end{aligned} \tag{12}$$

Scaling (11) by N_i and summing with (12), and then manipulating the resulting inequality yields

$$\frac{\mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_1^T \quad \mathbf{G}_{i2}^T\mathbf{V}_2^T]}{\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_1^T \quad \mathbf{G}_{i'2}^T\mathbf{V}_2^T]} \leq \frac{Q_{i1}N_{i'}+Q_{i2}\left(N_{i'}+N_i\right)}{N_{i'}(Q_{i2}+N_{i'})}. \tag{13}$$

Taking the minimum of (10) and (13) yields (6), as desired.

B. Converse for Theorem 1

Recall that linearly achieving a DoF tuple $(d_{11}, d_{12}, d_{21}, d_{22})$ implies existence of sequences of message dimensions $\{m_{11}^{(T)}, m_{12}^{(T)}, m_{21}^{(T)}, m_{22}^{(T)}\}_{T=1}^{\infty}$, such that $\lim_{T\to\infty}\frac{m_{ij}^{(T)}}{T} = d_{ij}$. Now consider the following weighted sum of the message dimensions (again letting $i' = 3-i$):

$$\begin{aligned}
&m_{i1}^{(T)} + m_{i2}^{(T)} + \Gamma_i(m_{i'1}^{(T)} + m_{i'2}^{(T)}) \\
&\quad \overset{a.s.}{=} \mathrm{rk}[\mathbf{V}_{i1}^T] + \mathrm{rk}[\mathbf{V}_{i2}^T] + \Gamma_i(\mathrm{rk}[\mathbf{V}_{i'1}^T] + \mathrm{rk}[\mathbf{V}_{i'2}^T])
\end{aligned}$$

$$
\begin{aligned}
&\overset{a.s.}{=} \mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_{i1}^T] + \mathrm{rk}[\mathbf{G}_{i2}^T\mathbf{V}_{i2}] \\
&\quad + \Gamma_i(\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_{i'1}] + \mathrm{rk}[\mathbf{G}_{i'2}^T\mathbf{V}_{i'2}]) \\
&\overset{(a)}{=} \mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_{i1}^T \quad \mathbf{G}_{i2}^T\mathbf{V}_{i2} \quad \mathbf{G}_{i1}^T\mathbf{V}_{i'1} \quad \mathbf{G}_{i2}^T\mathbf{V}_{i'2}] \\
&\quad - \mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_{i'1} \quad \mathbf{G}_{i2}^T\mathbf{V}_{i'2}] - \Gamma_i\,\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_{i1} \quad \mathbf{G}_{i'2}^T\mathbf{V}_{i2}] \\
&\quad + \Gamma_i\,\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_{i1}^T \quad \mathbf{G}_{i'2}^T\mathbf{V}_{i2} \quad \mathbf{G}_{i'1}^T\mathbf{V}_{i'1} \quad \mathbf{G}_{i'2}^T\mathbf{V}_{i'2}] \\
&\overset{(b)}{\leq} \mathrm{rk}[\mathbf{G}_{i1}^T\mathbf{V}_{i1}^T \quad \mathbf{G}_{i2}^T\mathbf{V}_{i2}] - \Gamma_i\,\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_{i1} \quad \mathbf{G}_{i'2}^T\mathbf{V}_{i2}] \\
&\quad + \Gamma_i\,\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_{i1}^T \quad \mathbf{G}_{i'2}^T\mathbf{V}_{i2} \quad \mathbf{G}_{i'1}^T\mathbf{V}_{i'1} \quad \mathbf{G}_{i'2}^T\mathbf{V}_{i'2}] \\
&\overset{(c)}{\leq} \Gamma_i\,\mathrm{rk}[\mathbf{G}_{i'1}^T\mathbf{V}_{i1}^T \quad \mathbf{G}_{i'2}^T\mathbf{V}_{i2} \quad \mathbf{G}_{i'1}^T\mathbf{V}_{i'1} \quad \mathbf{G}_{i'2}^T\mathbf{V}_{i'2}] \\
&\leq T\Gamma_i \min\{N_{i'}, M_1 + M_2\}, \qquad (14)
\end{aligned}
$$

where (a) is due to the decodability condition (3), (b) is due to submodularity of rank, and (c) is due to Lemma 2.

Normalizing by T, as $T \to \infty$ and evaluating for $i = 1, 2$, we have two weighted sum linear sum DoF bounds:

$$d_{11} + d_{12} + \Gamma_1(d_{21} + d_{22}) \leq \Gamma_1 \min\{N_2, M_1 + M_2\}, \quad (15)$$

$$d_{21} + d_{22} + \Gamma_2(d_{11} + d_{12}) \leq \Gamma_2 \min\{N_1, M_1 + M_2\}. \quad (16)$$

Now we consider two cases. In the first case, when $M_1+M_2 \leq N_{i'}$ for either $i = 1$ or $i = 2$, then evaluating (6) and either (15) or (16) depending on the value of i we find $\Gamma_i = 1$ and $d_{11} + d_{12} + d_{21} + d_{22} \leq M_1 + M_2$. In the second case, when $M_1 + M_2 \geq N_{i'}$ for both $i = 1$ and $i = 2$, we find

$$
\begin{aligned}
\sum_{i,j\in\{1,2\}} d_{ij} &\overset{(a)}{=} \frac{\Gamma_2 - 1}{\Gamma_1\Gamma_2 - 1}(d_{11} + d_{12} + \Gamma_1 d_{21} + \Gamma_1 d_{22}) \\
&\quad + \frac{\Gamma_1 - 1}{\Gamma_1\Gamma_2 - 1}(\Gamma_2 d_{11} + \Gamma_2 d_{12} + d_{21} + d_{22}) \\
&\overset{(b)}{\leq} \frac{(\Gamma_2 - 1)\Gamma_1 N_2 + (\Gamma_1 - 1)\Gamma_2 N_1}{\Gamma_1\Gamma_2 - 1} \\
&= \frac{\Gamma_1\Gamma_2(N_1 + N_2) - \Gamma_1 N_2 - \Gamma_2 N_1}{\Gamma_1\Gamma_2 - 1}, \qquad (17)
\end{aligned}
$$

where in step (a) we factored the sum DoF into two parts, and in step (b) we applied both (15) and (16).

IV. Transmission Strategy

We now present a transmission strategy that, for all antenna configurations, achieves the sum DoF in (7). We first describe the general structure of the three-phase strategy and how delayed CSIT is used to shape the input of Transmitter 2 during Phases 1 and 2 and inputs of both transmitters during Phase 3. Then, we outline the steps necessary to prove achievability of the sum DoF given in (7). Due to limited space, we are unable to evaluate the necessary terms for all antenna configuration regimes, and instead present a representative case. A complete analysis for all regimes may be found in [12].

A. General Description

Our transmission strategy consists of 3 phases. During Phase i ($i \in \{1, 2\}$), symbols desired by Receiver i are sent by both transmitters. In Phase 3, transmitters employ a retrospective alignment approach and each sends linear combinations of past transmissions that both receivers simultaneously desire.

Phases 1 and 2 are further divided into transmission rounds. Symbols used to generate transmissions within each round are independent of other rounds. We let κ_1 and κ_2 denote the number of rounds in Phase 1 and 2 respectively. Later, we describe how κ_1 and κ_2 are chosen to maximize sum DoF.

Before describing the structure of transmissions in each phase, we point out the two key points of our strategy:

1) For each round in Phase i ($i \in \{1, 2\}$), the number of channel uses, the numbers of symbols at Transmitter 1, and number of symbols at Transmitter 2 are all functions of Γ_i and the antenna configuration.
2) We exploit delayed CSIT in Phases 1 and 2. The method used only requires delayed CSIT at Transmitter 2 (assuming $M_1 \geq M_2$).

Phases 1 and 2: Recall that Phase i is divided into κ_i rounds ($i \in \{1, 2\}$). We denote with S_i the number of channel uses per round, and continue the convention $i' = 3 - i$.

If $\Gamma_i \leq \frac{M_1}{N_{i'}}$, our strategy requires Transmitter 2 be silent in Phase i. In this case, a round consists of a single channel use ($S_i = 1$), and Transmitter 1 broadcasts $\min\{M_1, N_1 + N_2\}$ symbols per round, each on a different antenna.

On the other hand, if $\Gamma_i N_{i'} > M_1$, let $\xi_i \in \mathbb{N}$ be the smallest positive integer such that S_i, defined as

$$S_i = \frac{M_2 \xi_i}{\min\{[\Gamma_i N_{i'} - M_1]_+, M_2\}}, \qquad (18)$$

is an integer. Transmitter 1 will send $M_1 S_i$ symbols per round, whereas Transmitter 2 will send $M_2 \xi_i$.

Every channel use, Transmitter 1 sends M_1 new symbols, each on a different antenna. During the first ξ_i channel uses, Transmitter 2 sends a new symbol on each antenna.

After the initial ξ_i channel uses, the image of Transmitter 2's symbols at Receiver i' almost surely lies within a subspace of dimension $\xi_i \min\{N_{i'}, M_2\}$. Transmitter 2 uses delayed CSIT to identify the subspace of the transmit signal that spans the image. Transmitter 2 sends precoded equations on different antennas for the rest of the round, using linear basis elements of this subspace as precoding vectors.

At the end of each round, for both $\Gamma_i \leq \frac{M_1}{N_{i'}}$ and $\Gamma_i > \frac{M_1}{N_{i'}}$, each transmitter uses delayed CSIT to identify as many linearly independent equations of symbols it transmitted that 1) Receiver i' can (linearly) reconstruct from its received signal, and 2) Receiver i cannot reconstruct. Receiver i sees such equations as useful information, whereas Receiver i' sees these as interference that can be cancelled.

These equations are buffered at each transmitter for use in Phase 3. The number of such equations is dependent on the antenna configuration, so to simplify explanation of Phase 3, we denote as λ_i the total number of such equations per round during Phase i, buffered by either Transmitter 1 or 2.

Before continuing on to Phase 3 we stress that application of delayed CSIT at Transmitter 2 is a key innovation of our strategy. Because using delayed CSIT can reduce the dimension of the image of Transmitter 2 symbols at Receiver i' (e.g., when $M_2 > N_{i'}$), more equations in terms of only Transmitter 1's symbols are buffered, and λ_i is increased.

Phase 3: Recall that the number of equations that were buffered during Phase i was $\kappa_i \lambda_i$. Moreover, any linear combination of equations buffered during Phase i is interference that can be canceled by Receiver i'.

During each channel use of Phase 3, the two transmitters send a combined total of $\min\{M_1 + M_2, N_1\}$ equations buffered during Phase 1 on different antennas, and $\min\{M_1 + M_2, N_2\}$ equations buffered during Phase 2 on different antennas; this requires certain antennas to transmit a scaled sum of two equations buffered during Phases 1 and 2.

Each receiver (i) uses what it received during Phase i' and knowledge of the channel state to cancel undesired equations. Consequently, as long as buffers are not empty, during each channel use of Phase 3, Receiver i gains $\min\{M_1 + M_2, N_i\}$ new linear equations that describe its desired messages.

Maximizing efficiency of Phase 3 requires κ_1 and κ_2 satisfy

$$\frac{\kappa_1 \lambda_1}{\min\{M_1 + M_2, N_1\}} = \frac{\kappa_2 \lambda_2}{\min\{M_1 + M_2, N_2\}} \in \mathbb{Z}_+. \quad (19)$$

If λ_i is zero, we let $\kappa_i = 1$. Otherwise, we set κ_1 and κ_2 equal to the smallest non-negative integers satisfying (19).

B. On Computing Achieved Sum DoF

To prove that our transmission strategy achieves the sum DoF stated in (7), we must first verify that all symbols transmitted during Phase i are decodable at Receiver i ($i \in \{1, 2\}$). To verify this, we claim that Receiver i can almost surely decode all of the symbols from each round if and only if

$$\Lambda_i \leq S_i \min\{M_1 + M_2, N_i\} + \lambda_i, \quad (20)$$

where Λ_i is the number of symbols per round of Phase i:

$$\Lambda_i = \begin{cases} \min\{M_1, N_1 + N_2\} & \text{if } \Gamma_i \leq \frac{M_1}{N_{i'}} \\ M_1 S_i + M_2 \xi_i & \text{if } \Gamma_i > \frac{M_1}{N_{i'}} \end{cases}. \quad (21)$$

The rationale for this claim follows from counting received linearly independent equations, noting that the probability of rank deficient channels from continuous distribution is zero.

After verifying decodability, we must then evaluate the sum DoF. Assuming that (19) is satisfied, we have:

$$\underline{\mathsf{DoF}} = \frac{\kappa_1 \Lambda_1 + \kappa_2 \Lambda_2}{\kappa_1 S_1 + \kappa_2 S_2 + \frac{\kappa_1 \lambda_1}{\min\{M_1 + M_2, N_1\}}} \quad (22)$$

C. Sum DoF of a Representative Case

Proving (20) is satisfied and evaluating (22) for all antenna configurations requires evaluating Γ_i, S_i, ξ_i and computing λ_i for many regimes. We address all regimes in [12], but here we do so for the single case of symmetric configurations (i.e., $M_1 = M_2 = M$, $N_1 = N_2 = N$) where $\frac{1}{2} < \frac{N}{M} < 1$. Because we restrict the example to the symmetric case, the resulting achieved DoF is readily compared to that achieved in [10].

By evaluating (6), (18), and (21) (and exploiting symmtery of the configuration), we find that $\Gamma = \frac{3M}{M+N}$, $S = \frac{\xi(M+N)}{2N-M}$, and $\Lambda = \xi \frac{3MN)}{2N-M}$. The rank of each receiver's output at the end of a round is $NS = \frac{\xi N(M+N)}{(2N-M)}$. The image of Transmitter 1's symbols at Receiver i' almost surely spans the full received space, whereas the image of Transmitter 2's symbols at Receiver i' are contained within a $N\xi$-dimensional subspace. Therefore, only Transmitter 1 will buffer equations and the number of such equations per round is

$$\lambda = NS - N\xi = \xi \frac{N(M+N) - N(2N-M)}{2N-M}.$$

Evaluating the RHS of (20) we find

$$S \min\{2M, N\} + \lambda = \xi \frac{3MN}{2N-M} = \Lambda,$$

i.e., all symbols from each round are almost surely decodable. Choosing $\kappa_1 = \kappa_2 = \kappa$ yields the achieved DoF

$$\underline{\mathsf{DoF}} = \frac{2\kappa\Lambda}{2\kappa S + \frac{\kappa\lambda}{\min\{2M,N\}}} = \frac{6MN}{4M+N}. \quad (23)$$

This is equal to (7) evaluated for $\Gamma = \frac{3M}{M+N}$. One can also show that $\frac{6MN}{4M+N}$ is strictly greater than the sum DoF of $\frac{2(M+2N)N}{M+4N}$ achieved in [10] for the regime $\frac{1}{2} < \frac{N}{M} < 1$.

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