An Axiomatic Theory of Fairness in Network Resource Allocation

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***Abstract*— Five axioms for fairness measures in resource allocation are presented. A family of fairness measures satisfying the axioms is then constructed. Special cases of this family include α-fairness, Jain's index, and entropy. Properties of fairness measures satisfying the axioms are proven, including Schur-concavity. Among the engineering implications is a generalized Jain's index that tunes the resolution of fairness measure, a new understanding of α-fair utility functions, and an interpretation of "larger α is more fair". Finally, an alternative set of axioms is developed to capture system efficiency and feasibility constraints. Comparison with several other works of axiomatization in information and economics is made and connections with Rawls' theory of justice explored.**

I. Quantifying Fairness

Given a vector $\mathbf{x} \in \mathbb{R}^n_+$, where x_i is the resource allocated to user i, *how fair* is it?

One approach to quantify the degree of fairness associated with $\mathbf{x}$ is through a fairness measure, which is a function f that maps $\mathbf{x}$ into a real number. These measures are sometimes referred to as diversity indices in statistics. Various fairness measures have been proposed throughout the years, e.g., in [1]–[6]. These range from simple ones, e.g., the ratio between the smallest and the largest entries of $\mathbf{x}$, to more sophisticated functions, e.g., Jain's index and the entropy function. Some of these fairness measures map $\mathbf{x}$ to normalized ranges between 0 and 1, where 0 denotes the minimum fairness, 1 denotes the maximum fairness (often corresponding to an $\mathbf{x}$ where all x_i are the same) and a larger value indicates more fairness. For example, min-max ratio [1] is given by the maximum ratio of any two user's resource allocation, while Jain's index [3] computes a normalized square mean. How are these fairness measures are related? Is one measure "better" than any other? What other measures of fairness may be useful?

An alternative approach that has gained attention in the networking research community since [7], [8] is the optimization-theoretic approach of α-fairness and the associated utility maximization. Given a set of feasible allocations, a maximizer of the α-fair utility function satisfies the definition of α-fairness. Two well-known examples are as follows: a maximizer of the log utility function ($\alpha = 1$) is proportionally fair, and a maximizer of the α-fair utility function as $\alpha \to \infty$ is max-min fair. More recently, α-fair utility functions have been connected to divergence measures [9], and in [10], [11], the parameter α was viewed as a fairness measure in the sense that a fairer allocation is one that is the maximizer of an α-fair utility function with larger α — although the exact role of α in trading-off fairness and throughput can sometimes be surprising [12]. While it is often held that $\alpha \to \infty$ is more fair than $\alpha = 1$, which is in turn more fair than $\alpha = 0$, it remains unclear what it means to say, for example, that $\alpha = 3$ is more fair than $\alpha = 2$.

Clearly, these two approaches for quantifying fairness are different. On the one hand, α-fair utility functions are continuous and strictly increasing in each entry of $\mathbf{x}$, thus its maximization results in Pareto optimal resource allocations. On the other hand, scale-invariant fairness measures (ones that map $\mathbf{x}$ to the same value as a normalized $\mathbf{x}$) are unaffected by the magnitude of $\mathbf{x}$, and an allocation that does not use all the resources can be as fair as one that does. Can the two approaches be unified?

To address the above questions, we develop an axiomatic approach to fairness measures. We show that a set of five axioms, each of which simple and intuitive, can lead to a useful family of fairness measures. The axioms are: the axiom of continuity, of homogeneity, of saturation, of partition, and of starvation. Starting with these five axioms, we can *generate* a family of fairness measures from generator functions g: any increasing and continuous functions that lead to a well-defined "mean" function (i.e., from any Kolmogorov-Nagumo function [16]). For example, using power functions with exponent β as the generator function, we derive a unique family of fairness measures f_β that include all of the following as special cases, depending on the choice of β: Jain's index, maximum or minimum ratio, entropy, and α-fair utility, and reveals new fairness measures corresponding to other ranges of β.

In particular, for $\beta \leq 0$, well-known fairness measures (e.g., Jain's index and entropy) are special cases of our construction, and we generalize Jain's index to provide a flexible tradeoff between "resolution" and "strictness" of the fairness measure. For $\beta \geq 0$, α-fair utility functions can be factorized as the product of two components: our fairness measure with $\beta = \alpha$ and a function of the total throughput that captures the scale, or efficiency, of $\mathbf{x}$. Such a factorization also quantifies a tradeoff between fairness and efficiency in achieving Pareto dominance with the maximum possible α, and facilitates a clearer understanding of what it means to say that a larger α is "more fair" for general $\alpha \in [0, \infty)$.

The axiomatic construction of fairness measures also illuminates their engineering implications. Any fairness measure

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satisfying the five axioms can be proven to have many properties quantifying common beliefs about fairness, including Schur-concavity [14]. Our fairness measures presents a new ordering of Lorenz curves [4], [13], different from the Gini coefficient.

The development of an axiomatic theory of fairness takes another turn towards the end of the paper. By removing the Axiom of Homogeneity, we propose an alternative set of four axioms, which allows *efficiency* of resource allocation be *jointly captured* in the fairness measure. We show how this alternative system connects with constrained optimization based resource allocation, where magnitude matters due to the feasibility constraint and an objective function that favors efficiency. The fairness measures constructed out of the reduced set of axioms is a generalization of that out of the original set of axioms.

Axiomatization of key notions in bargaining, collaboration, and information has been carried out over the past 60 years, and we will compare and contrast with several of them. Finally, we will explore the potential connections with an axiomatic quantification of Rawls' theory of justice in political philosophy.

Variable	Meaning
$\mathbf{x}$	Resource allocation vector of length n
$\mathbf{x}^{\uparrow}$	Sorted vector with smallest element being first
$w(\mathbf{x})$	Sum of all elements of $\mathbf{x}$
$f(\cdot), f_\beta(\cdot)$	Fairness measure (of parameter β)
$g(\cdot)$	Generator function
s_i	Positive weights for weighted mean
$\mathbf{1}_n$	Vector of all ones of length n
$\mathbf{x} \succeq \mathbf{y}$	Vector $\mathbf{x}$ majorizes vector $\mathbf{y}$
β	Parameter for power function $g(y) = y^\beta$
$U_\alpha(\cdot)$	α-fair utility with parameter α
$H(\cdot)$	Shannon entropy function
L_x	Lorenz curve of a vector $\mathbf{x}$
$J(\cdot)$	Jain's index
$\Phi_\lambda(\cdot)$	Our multicriteria (fairness and efficiency) utility function
$F(\cdot), F_{\beta,\lambda}(\cdot)$	Fairness measure from an alternative set of axioms

TABLE I

MAIN NOTATION

II. AXIOMS

Let $\mathbf{x}$ be a resource allocation vector with n non-negative elements. A fairness measure $f(\mathbf{x})$ is a mapping from $\mathbf{x}$ to a real number, i.e., $f : \mathbb{R}^n_+ \to \mathbb{R}$, for all integer $n \geq 1$. We introduce the following set of axioms about f, whose explanations and implications are provided after the statement of each axiom.

1) *Axiom of Continuity.* Fairness measure $f(\mathbf{x})$ is continuous on $\mathbb{R}^n_+$ for all integer $n \geq 1$.

Axioms 1 is intuitive: A slight change in resource allocation shows up as a slight change in the fairness measure.

2) *Axiom of Homogeneity.* Fairness measure $f(\mathbf{x})$ is a homogeneous function of degree 0:

$$f(\mathbf{x}) = f(t \cdot \mathbf{x}), \quad \forall\ t > 0. \tag{1}$$

Without loss of generality, for a single user, we take $|f(x_1)| = 1$ for all $x_1 > 0$, i.e., fairness is a constant for $n = 1$.

The Axiom of Homogeneity says that the fairness measure is independent of the unit of measurement or absolute magnitude of the resource allocation. For an optimization formulation of resource allocation, the fairness measure $f(\mathbf{x})$ alone cannot be used as the objective function if efficiency (which depends on magnitude $\sum_i x_i$) is to be captured. In Section VI, we will connect this fairness measure with an efficiency measure in α-fair utility function. In Section VII, we will remove the Axiom of of Homogeneity and propose an alternative set of axioms, which make measure $f(\mathbf{x})$ dependent on both magnitude and distribution of $\mathbf{x}$, thus capturing fairness and efficiency at the same time.

3) *Axiom of Saturation.* Even allocation's fairness value is independent of number of users as the number of users becomes large, i.e.,

$$\lim_{n\to\infty} \frac{f(\mathbf{1}_{n+1})}{f(\mathbf{1}_n)} = 1. \tag{2}$$

This axiom is a technical condition used to ensure *uniqueness* of the fairness measure and invariance under change of variable by fixing a scaling. For example, suppose $f(\mathbf{x})$ is a fairness measure satisfying all axioms (with respect to a mean function $g(y)$) except Axiom 3. It is easy to see that by making a logarithmic change of variables, fairness measure $\log f(\mathbf{x})$ also satisfies all axioms, respect to a mean function $e^{g(y)}$, other than Axiom 3.

4) *Axiom of Partition.* Consider an arbitrary partition of vector $\mathbf{x}$ into two segments $\mathbf{x} = \left[\mathbf{x}^1, \mathbf{x}^2\right]$. There exists a continuous and strictly monotonic *generator function* $g(y)$ such that

$$f(\mathbf{x}) = f\left(w(\mathbf{x}^1), w(\mathbf{x}^2)\right) \cdot g^{-1}\left(\sum_{i=1}^{2} s_i \cdot g\left(f(\mathbf{x}^i)\right)\right). \tag{3}$$

The following function h generated by $g(y)$ qualifies as a *mean* function [15] of $\{f(\mathbf{x}^i), \forall i\}$:

$$h = g^{-1}\left(\sum_{i=1}^{2} s_i \cdot g\left(f(\mathbf{x}^i)\right)\right), \tag{4}$$

with positive weights satisfying $\sum_i s_i = 1$. $w(\mathbf{x}^1)$ and $w(\mathbf{x}^2)$ denote the sum of resource vectors $\mathbf{x}^1$ and $\mathbf{x}^2$ respectively.

None of Axioms 1–3 concerns the *construction* of fairness measure as the number of users varies. A hierarchical construction of fairness is defined in Axiom 4, which allows us to derive a fairness measure $f : \mathbb{R}^n_+ \to \mathbb{R}$ of n users recursively (with respect to a generator function $g(y)$) from lower dimensions, $f : \mathbb{R}^k_+ \to \mathbb{R}$ and $f : \mathbb{R}^{n-k}_+ \to \mathbb{R}$ for integer $0 < k < n$. Since there are more than one way to partition a vector $\mathbf{x}$, Axiom 4 must be *well-defined*. It means that the construction is independent of the partition and generates the same fairness measure for any choice of $0 < k < n$.

Initial:	$\sum_{i=1}^{n} x_i$	
Level 1:	$\sum_{i=1}^{k} x_i$	$\sum_{i=k+1}^{n} x_i$
Level 2:	$x_1, \ldots, x_k$	$x_{k+1}, \ldots, x_n$

TABLE II

ILLUSTRATION OF THE HIERARCHICAL COMPUTATION OF FAIRNESS.

The recursive computation is illustrated by a two-level representation in Table II. Let $\mathbf{x}^1 = [x_1, \ldots, x_k]$ and $\mathbf{x}^2 = [x_{k+1}, \ldots, x_n]$. The computation is performed as follows. At level 1, since the total resource is divided into two chunks, $w(\mathbf{x}^1)$ and $w(\mathbf{x}^1)$, fairness across the chunks obtained in this level is measured by $f\left(w(\mathbf{x}^1), w(\mathbf{x}^2)\right)$. At level 2, the two chunks of resources are further allocated to k and $n-k$ users, achieving fairness $f(\mathbf{x}^1)$ and $f(\mathbf{x}^2)$, respectively. To compute overall fairness of the resource allocation $\mathbf{x} = [x_1, x_2, \ldots, x_n]$, we combine the fairness obtained in the two levels using a multiplication in equation (3).

As we consider a continuous and strictly increasing generator function $g(y)$, the function (4) is a mean value [15] for $\{f(\mathbf{x}^i), \forall i\}$, which represents the average fairness of individual parts of $\mathbf{x}$. The set of generator functions giving rise to the same fairness measures may not unique, e.g., logarithm and power functions. The simplest case is when g is identity and $s_i = 1/n$ for all i. A natural choice of the weight s_i in (3) is to choose the value proportional to the sum resource of vector $\mathbf{x}^i$. More generally, we will consider the following weights

$$s_i = \frac{w^\rho(\mathbf{x}^i)}{\sum_j w^\rho(\mathbf{x}^j)}, \ \forall i \tag{5}$$

where $\rho \geq 0$ is an arbitrary exponent. When $\rho = 0$, weights in (5) are equal and lead to an un-weighted mean in Axiom 4. As shown in Section IV, the parameter ρ can be chosen such that the hierarchical computation is independent of partition as stated in Axiom 4. As a special case of Axiom 4, if we denote the resource allocation at level 1 by a vector $\mathbf{z} = [w(\mathbf{x}^1), w(\mathbf{x}^2)]$ and if the resource allocation at level 2 are equal $\mathbf{x}^1 = \mathbf{x}^2 = \mathbf{y}$, it is straight forward to verify that Axioms 2 and 4 imply

$$\begin{aligned} f(\mathbf{y} \otimes \mathbf{z}) &= f(\mathbf{y} \cdot w(\mathbf{z})) \cdot g^{-1}\left(\sum_i s_i \cdot g\left(f(\mathbf{z})\right)\right) \\ &= f(\mathbf{y}) \cdot f(\mathbf{z}), \end{aligned} \tag{6}$$

where $\otimes$ is the direct product of two vectors. We later show in Section VII, an extension of equation (6) gives an alternative way of stating Axiom 4 and leads to a set of more general axioms on fairness.

5) *Axiom of Starvation* For $n = 2$ users, we have $f(1, 0) \leq f(1, 1)$, i.e., starvation is no more fair than equal distribution.

Axiom 5 is the only axiom that involves a *value* statement on fairness: starvation is less fair than equal distribution for two users. This intuition also holds for all existing fairness measures, e.g., various, spread, deviation, max-min ratio, Jain's index, α-fair utility, and entropy. In our work, Axiom 5 specifies an increasing direction of fairness and is used to ensure uniqueness of $f(\mathbf{x})$. $f(1, 0) \leq f(1, 1)$ also implies that a larger value of $f(\cdot)$ means larger fairness.

By definition, axioms are true, as long as they are consistent. They should also be non-redundant. However, not all sets of axioms are useful: unifying known notions, discovering new measures and properties, and providing useful insights. We first demonstrate the following existence (the axioms are consistent) and uniqueness results.

Theorem 1: (*Existence.*) There exists a fairness measure $f(\mathbf{x})$ satisfying Axioms 1–5.

Theorem 2: (*Uniqueness.*) Given a generator function g, the resulting $f(\mathbf{x})$ satisfying Axioms 1–5 is unique.

III. Properties of Fairness Measures

We first prove an intuitive corollary from the five axioms that will be useful for the rest of the presentation.

Corollary 1: (Symmetry.) A fairness measure satisfying Axioms 1–5 is symmetric over $\mathbf{x}$:

$$f(x_1, x_2, \ldots, x_n) = f(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), \tag{7}$$

where $i_1, \ldots, i_n$ is an arbitrary permutation of indices $1, \ldots, n$.

The symmetry property shows that the fairness measure $f(\mathbf{x})$ satisfying Axioms 1–5 is irrelevant of labeling of users.

We now connect of our axiomatic theory to a line of work on measuring statistical dispersion by vector majorization, including the popular Gini coefficient [22]. Majorization [14] is a partial order over vectors to study whether the elements of vector $\mathbf{x}$ are less spread out than the elements of vector $\mathbf{y}$. We say that $\mathbf{x}$ is majorized by $\mathbf{y}$, and we write $\mathbf{x} \preceq \mathbf{y}$, if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ (always satisfied due to Axiom 2) and

$$\sum_{i=1}^{d} x_i^{\uparrow} \leq \sum_{i=1}^{d} y_i^{\uparrow}, \text{ for } d = 1, \ldots, n, \tag{8}$$

where $x_i^{\uparrow}$ and $y_i^{\uparrow}$ are the ith elements of $\mathbf{x}^{\uparrow}$ and $\mathbf{y}^{\uparrow}$, sorted in ascending order. According to this definition, among the vectors with the same sum of elements, one with the equal elements is the most majorizing vector.

Intuitively, $\mathbf{x} \preceq \mathbf{y}$ can be interpreted as $\mathbf{y}$ being a fairer allocation than $\mathbf{x}$. It is a classical result [14] that $\mathbf{x}$ is majorized by $\mathbf{y}$, if and only if, from $\mathbf{x}$ we can produce $\mathbf{y}$ by a finite sequence of Robin Hood operations.[1]

Majorization alone cannot be used to define a fairness measure since it is a partial order and fails to compare vectors in certain cases. Still, if resource allocation $\mathbf{x}$ is majorized by $\mathbf{y}$, it is desirable to have a fairness measure f such that $f(\mathbf{x}) \leq f(\mathbf{y})$. A function satisfying this property is known as Schur-concave. In statistics and economics, many measures of statistical dispersion or diversity are known to be Schur-concave, e.g., Gini Coefficient and Robin Hood Ratio [22], and we show our fairness measure also is Schur-concave. More discussions on the relationship of our axioms and other economics theories are provided in Section VIII.

Theorem 3: (*Schur-concavity.*) A fairness measure satisfying Axioms 1–5 is Schur-concave:

$$f(\mathbf{x}) \leq f(\mathbf{y}), \text{ if } \mathbf{x} \preceq \mathbf{y}. \tag{9}$$

Next we present several properties of fairness measures satisfying the axioms, whose proofs rely on Schur-concavity.

Corollary 2: (Equal-allocation is most fair.) A fairness measure $f(\mathbf{x})$ satisfying Axioms 1–5 is maximized by equal-resource allocations, i.e.,

$$f(\mathbf{1}_n) = \max_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}). \tag{10}$$

Corollary 3: (Equal-allocation fairness value is independent of g.) The fairness achieved by equal-resource allocations $\mathbf{1}_n$ is independent of the choice of generator function g, i.e.,

$$f(\mathbf{1}_n) = n^r \cdot f(1), \tag{11}$$

where r is a constant exponent.

Corollary 4: (Constant tax reduces fairness.) If a fixed amount $c > 0$ of the resource is subtracted from each user (i.e., $x_i - c$ for all i), the resulting fairness measure decreases

$$f(\mathbf{x} - c \cdot \mathbf{1}_n) \leq f(\mathbf{x}), \quad \forall c > 0, \tag{12}$$

where $c > 0$ must be small enough such that all elements of $\mathbf{x} - c \cdot \mathbf{1}_n$ are positive.

Corollary 5: (Inactive users contribute no fairness.) When a fairness measure $f(\mathbf{x})$ satisfying Axioms 1–5 is generated by by $\rho > 0$ in 5, Adding or removing users with zero resources does not change fairness:

$$f(\mathbf{x}, \mathbf{0}_n) = f(\mathbf{x}), \quad \forall n \geq 1. \tag{13}$$

[1]In a Robin Hood operation, we replace two elements x_i and $x_j < x_i$ with $x_i - \epsilon$ and $x_j + \epsilon$, respectively, for some $\epsilon \in (0, x_i - x_j)$. In other words, we take from the rich (x_i), and give to the poor (x_j).

IV. A Family of Fairness Measures

A. Constructing Fairness Measures

From any function $g(y)$ satisfying the condition in Axiom 4, we can generate a unique $f(\mathbf{x})$. Such an $f(\mathbf{x})$ is a well-defined fairness measure if it also satisfies Axioms 1–5. We then refer to the corresponding $g(y)$ as the generator of the fairness measure.

Definition 1: Function $g(y)$ is a generator if there exists a $f(\mathbf{x})$ satisfying Axioms 1–5 with respect to $g(y)$.

We note, however, that different generator functions may generate the same fairness measure. Although it is difficult to find the entire set of generators $g(y)$, we have found that many forms of $g(y)$ functions (e.g., logarithm, polynomial, exponential, and their combinations) result in fairness measures equivalent to those generated by the family of power functions. It remains to be determined if all fairness measures satisfying Axioms 1–5 can be generated by power functions.

In this section, we consider power functions, $g(y) = |y|^\beta$, parameterized by β and derive the resulting family of fairness measures, which indeed satisfy all the axioms. The absolute value ensures that $g(y)$ is non-increasing over $\mathbb{R}_+$ for $\beta \geq 0$, and over $\mathbb{R}_-$ for $\beta < 0$. From here on, we replace Equation (3) in Axiom 4 by

$$f(\mathbf{x}^1, \mathbf{x}^2) = f\left(w(\mathbf{x}^1), w(\mathbf{x}^2)\right) \cdot \left(\sum_{i=1}^{2} s_i \cdot f^\beta(\mathbf{x}^i)\right)^{\frac{1}{\beta}},$$

where the weights s_i are given by (5).

Theorem 4: (*Fairness measures generated by power functions*) For power mean ($g(y) = |y|^\beta$ with parameter β), Axioms 1–5 define a unique family of fairness measures as follows

$$f(\mathbf{x}) = \left[\sum_{i=1}^{n} \left(\frac{x_i}{\sum_j x_j}\right)^{1-\beta r}\right]^{\frac{1}{\beta}}, \text{ for } \beta r \leq 1 \tag{14}$$

$$f(\mathbf{x}) = -\left[\sum_{i=1}^{n} \left(\frac{x_i}{\sum_j x_j}\right)^{1-\beta r}\right]^{\frac{1}{\beta}}, \text{ for } \beta r \geq 1, \tag{15}$$

where $r = \frac{1-\rho}{\beta}$ is a constant exponent that depends on the two parameters: ρ in the weights of the weighted sum and β in the definition of the power generator function. It determines the growth rate of maximum fairness as population size n increases:

$$f(\mathbf{1}_n) = n^r \cdot f(1). \tag{16}$$

For different values of parameter β, the fairness measures derived above are equivalent up to a constant exponent r:

$$f_{\beta,r}(\mathbf{x}) = \left[f_{\beta r,1}\right]^r(\mathbf{x}), \tag{17}$$

if we denote $f_{\beta,r}$ as the fairness measure with parameters β and r. According to Theorem 1, r determines the growth rate of maximum fairness as population size n increases. From a user's perspective, her perception of maximum fairness is independent of the population size of the system. We will now use a unified representation of the constructed fairness measures:

$$f_\beta(\mathbf{x}) = \text{sign}(1-\beta) \cdot \left[\sum_{i=1}^{n} \left(\frac{x_i}{\sum_j x_j} \right)^{1-\beta} \right]^{\frac{1}{\beta}}. \qquad (18)$$

We summarize the special cases in Table III, where β sweeps from $-\infty$ to ∞ and $H(\cdot)$ denotes the entropy function. For some values of β, the corresponding mean function h has a standard name, and for some, known approaches to measure fairness are recovered, while for $\beta \in (0, -1)$ and $\beta \in (-1, -\infty)$, new fairness measures are revealed. For example, when $\beta = -1$ (i.e., harmonic mean is used in Axiom 4), we get Jain's index $J(\mathbf{x}) = f(\mathbf{x})/n$. For $0 < \beta < 1$ and $\beta > 1$, we obtain the part of the α-fair utility functions that is related to fairness, as we will show in Section VI that α-fair utility functions are equal to the product of our fairness measure and a function of total throughput for any $\beta = \alpha \geq 0$. The mapping from generators $g(y)$ to fairness measures $f(\mathbf{x})$ is illustrated in Figure 1. It remains open to show whether there are functions not of the form in (18) that satisfy the five axioms. Our trials using other generator functions g, such as logarithmic, ended up with the same form of f.

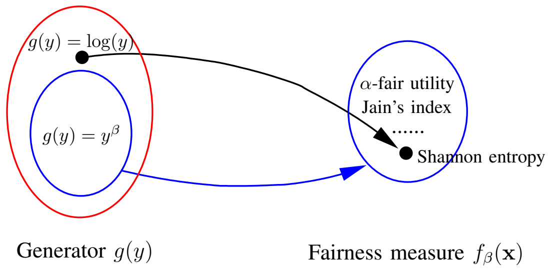


Fig. 1. The mapping from generators $g(y)$ to fairness measures $f(\mathbf{x})$.

Our axiomatic theory provides a unification of many existing fairness indexes and utilities. In particular, for $\beta < 1$, we recover the measured-based approach in [1]–[6] which define fairness as a direct measure of fairness by mapping an arbitrary resource allocation vector to a real number. For $\beta > 0$, we get the utility-based approach in [7], [8], [10]–[12], which quantify fairness for the maximizers of the α-fair utility functions. Notice that for $\beta > 1$ the measures are in fact negative, however this is actually a result of requirement in Axiom 5 that the measure be monotonically *increasing* towards fairer solutions. As an illustration, for a fixed resource allocation vector $\mathbf{x} = [1, 2, 5]$, we plot fairness $f(\mathbf{x})$ for different values of β in Figure 2.

B. Further Properties

The fairness measures f_β in (18) corresponding to the generator function $g(y) = |y|^\beta$ satisfy a number of properties, which give interesting engineering implications. In addition to those proven in Section III, a number of properties of

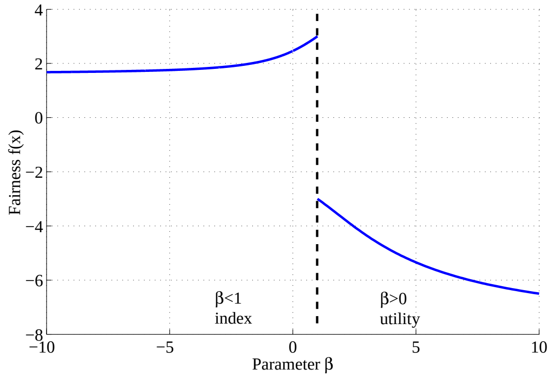


Fig. 2. Plot of fairness $f_\beta(\mathbf{x})$ for different values of β: $\beta > 0$ recovers the utility-based approach, and $\beta < 0$ recovers the index-based approach.

the fairness measures f_β in (18) constructed by the generator function $g(y) = y^\beta$ are proven.

Corollary 6: (Fairness value bounds the number of inactive users.) The fairness measures in (18) also count the number of inactive users in the system. When $f_\beta < 0$, $f(\mathbf{x}) \to -\infty$ if any user is assigned zero resource. When $f > 0$,

$$\text{Number of users with zero resource} \leq n - f(\mathbf{x}). \qquad (19)$$

Corollary 7: (Fairness value bounds the maximum resource to a user.) The fairness measures in (18) bounds the maximum resource to a user when $f > 0$:

$$\text{Maximum resource to a user} \geq \frac{\sum_i x_i}{f(\mathbf{x})}. \qquad (20)$$

Corollary 8: (Perturbation of fairness value from slight change in a users resource.) If we increase resource allocation to user i by a small amount ϵ, while not changing other users' allocation, the fairness measures in (18) increases if and only if $x_i < \bar{x} = \left(\frac{\sum_j x_j}{\sum_j x_j^{1-\beta}} \right)^{\frac{1}{\beta}}$ and $0 < \epsilon < \bar{x} - x_i$.

This corollary implies that x_f serves as a threshold for identifying poor and rich users, since assigning an additional ϵ amount of resource to user i improves fairness if $x_i < x_f$, while the same assignment reduces fairness if $x_i > x_f$.

Corollary 9: (Box constraints of resources allocation bound fairness value.) If a resource allocation $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ satisfies box-constraints, i.e., $x_{min} \leq x_i \leq x_{max}$ for all i, the fairness measures in (18) is lower bounded by a constant that only depends on β, x_{min}, x_{max}:

$$f(\mathbf{x}) \geq \text{sign}(1-\beta) \cdot \frac{(\mu\Gamma^{1-\beta} + 1 - \mu)^{\frac{1}{\beta}}}{(\mu\Gamma + 1 - \mu)^{\frac{1}{\beta}-1}}, \qquad (21)$$

where $\Gamma = \frac{x_{max}}{x_{min}}$ and $\mu = \frac{\Gamma - \Gamma^{1-\beta} - \beta(\Gamma-1)}{\beta(\Gamma-1)(\Gamma^{1-\beta}-1)}$. The bound is tight when a μ fraction of users have $x_i = x_{max}$ and the remaining $1-\mu$ fraction of users have $x_i = x_{min}$.

These results provide possible interpretations and applications of the family of fairness measures. Through Corollary 5, by specifying a level of fairness, we can limit the number of starved users in a system. Corollary 6 implies that $\bar{x}$ serves

Value of β	Type of Mean	Our Fairness Measure	Known Names
$\beta \to \infty$	maximum	$-\max_i \left\{ \frac{\sum_i x_i}{x_i} \right\}$	Max ratio
$\beta \in (1, \infty)$		$-\left[(1-\beta) U_{\alpha=\beta} \left(\frac{\mathbf{x}}{w(\mathbf{x})} \right) \right]^{\frac{1}{\beta}}$	α-fair utility
$\beta \in (0, 1)$		$\left[(1-\beta) U_{\alpha=\beta} \left(\frac{\mathbf{x}}{w(\mathbf{x})} \right) \right]^{\frac{1}{\beta}}$	α-fair utility
$\beta \to 0$	geometric	$e^{H\left(\frac{\mathbf{x}}{w(\mathbf{x})} \right)}$	Entropy
$\beta \in (0, -1)$		$\left[\sum_{i=1}^n \left(\frac{x_i}{w(\mathbf{x})} \right)^{1-\beta r} \right]^{\frac{1}{\beta}}$	No name
$\beta = -1$	harmonic	$\frac{(\sum_i x_i)^2}{\sum_i {x_i}^2} = n \cdot J(\mathbf{x})$	Jain's index
$\beta \in (-1, -\infty)$		$\left[\sum_{i=1}^n \left(\frac{x_i}{w(\mathbf{x})} \right)^{1-\beta r} \right]^{\frac{1}{\beta}}$	No name
$\beta \to -\infty$	minimum	$\min_i \left\{ \frac{\sum_i x_i}{x_i} \right\}$	Min ratio

TABLE III

PREVIOUS RESULTS ARE RECOVERED AS SPECIAL CASES OF OUR AXIOMATIC CONSTRUCTION. FOR $\beta \in (0, -1)$ AND $\beta \in (-1, -\infty)$, NEW FAIRNESS MEASURES OF GENERALIZED JAIN'S INDEX ARE REVEALED.

as a threshold for identifying "poor" and "rich" users, since assigning additional resources to user i improves fairness if $x_i < \bar{x}$, and reduces fairness if $x_i > \bar{x}$. Additionally, this provides intuition into threshold methods for allocating resources serially. Finally, Corollary 9 provides us with a lower bound on evaluation of the fairness measure, with which one might better benchmark empirical values.

V. IMPLICATION 1: GENERALIZING JAIN'S INDEX

When $\beta = -1$ (i.e., harmonic mean is used in Axiom 4), we get a scalar multiple of the widely used Jain's index $J(\mathbf{x}) = \frac{1}{n} f(\mathbf{x})$. Upon inspection of (18) and the specific cases noted in Table III, we note that any $(0, -\infty) \cup \beta \in (0, 1)$ the range of fairness measure $f_\beta(\mathbf{x})$ lies between 1 and n. Equivalently, we can say that the fairness *per user* resides in the interval $\left[\frac{1}{n}, 1\right]$. When the limit as $\beta \to 0$ is considered, the resulting fairness measure can also be shown to have this property. Because $f_\beta(\mathbf{x})$ for $\beta < 1$ has this characteristic, we refer to this subclass of our family of fairness measures as the generalization of Jain's index.

Definition 2: $J_\beta(\mathbf{x}) = \frac{1}{n} f_\beta(\mathbf{x})$ is a generalized Jain's index parameterized by $\beta \leq 1$.

The common properties of our fairness index proven in Section III and IV carry over to this generalized Jain's index. For $\beta = -1$, $J_{-1}(\mathbf{x})$ reduces to the original Jain's index. The parameter β determines the choice of generator function $g(y) = y^\beta$ in Axiom 4. We use $f_\beta(\mathbf{x})$ to denote the fairness measures in (18), parameterized by β. we first prove that, for a given resource allocation $\mathbf{x}$, fairness $f_\beta(\mathbf{x})$ is monotonic as $\beta \to 1$. Its engineering implication is discussed next.

Theorem 5: (*Monotonicity with respect to β.*) The fairness measures in (18) is negative and decreasing for $\beta \in (1, \infty)$, and positive and increasing for $\beta \in (-\infty, 1)$:

$$\frac{\partial f_\beta(\mathbf{x})}{\partial \beta} \leq 0 \text{ for } \beta \in (1, \infty), \tag{22}$$

$$\frac{\partial f_\beta(\mathbf{x})}{\partial \beta} \geq 0 \text{ for } \beta \in (-\infty, 1). \tag{23}$$

As $\beta \to 1$, f point-wise converges to constant values:

$$\lim_{\beta \uparrow 1} f_\beta(\mathbf{x}) = n \text{ and } \lim_{\beta \downarrow 1} f_\beta(\mathbf{x}) = -n. \tag{24}$$

The monotonicity of fairness measures $f_\beta(\mathbf{x})$ on $\beta \in (-\infty, 1)$ gives an engineering interpretation of β. Figure 3 plots fairness $f_\beta(\theta, 1-\theta)$ for resource allocation $\mathbf{x} = [\theta, 1-\theta]$ and different choices of $\beta = \{-4.0, -2.5, -1.0, 0.5\}$. The vertical bars in the figure represent the level sets of function f, for values $f_\beta(\theta_i, 1-\theta_i) = \frac{i}{10}\left(f_{max} - f_{min}\right), i = 1, 2, \ldots, 9$. For fixed resource allocations, since f increases as β approaches 1, the level sets of f are pushed toward the region with small θ (i.e., the low-fairness region), resulting in a steeper incline in the region. In the extreme case of $\beta = 1$, all level set boundaries align with the y-axis in the plot. The fairness measure f point-wise converges to step functions $f_\beta(\theta, 1-\theta) = 2$. Therefore, parameter β characterizes the shape of the fairness measures: a smaller value of $|1-\beta|$ (i.e., β closer to 1) causes the level sets to be condensed in the low-fairness region.

Since the fairness measure must still evaluate to a number between 1 and n here, the monotonicity and resulting shift in granularity of the fairness measure associated with varying β suggests differences in evaluating unfairness. At one extreme, $\beta \to 1$ any solution where no user receives an allocation of zero is fairest. On the other hand, as $\beta \to -\infty$ the relationship between $f_\beta(\mathbf{x})$ and θ becomes linear, suggesting a stricter concept of fairness – for the same allocation, as $\beta \to -\infty$ more fairness is lost. Therefore, the parameter β can tune the generalization of Jain's index f for different tradeoffs

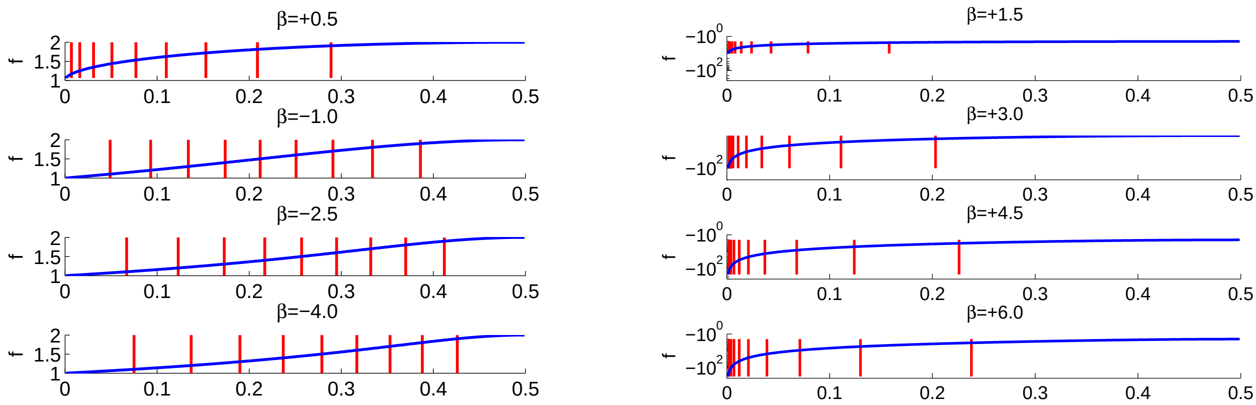


Fig. 3. Plot of the fairness measure $f_\beta(\theta, 1-\theta)$ against θ, for resource allocation $\mathbf{x} = [\theta, 1-\theta]$ and different choices of $\beta = \{-4.0, -2.5, -1.0, 0.5\}$ and $\beta = \{1.5, 3.0, 4.5, 6.0\}$, respectively. It can be observed that $f_\beta(\theta, 1-\theta)$ is monotonic as $\beta \to 1$. Further, smaller values of $|1-\beta|$ results in a steeper incline over small θ, i.e., the low-fairness region.

between the resolution and the strictness of fairness measure. If the fairness measure f is used for classifying different resource allocations, a larger β is desirable, since it gives more quantization levels in low-fairness region and provides finer granularity control for unfair resource. On the other hand, if the fairness measure f is used as an objective function, a smaller β is desirable, since it has a steeper incline in the low-fairness region and give more incentive for the system to operate in the high-fairness region.

VI. Implication 2: Understanding α-Fairness

Due to Axiom 2, the Axiom of Homogeneity, our fairness measures only express desirability over the $(n-1)$-dimension subspace orthogonal to the $\mathbf{1}_n$ vector. Hence, they do not capture any notion of efficiency of an allocation. The component of resource vectors along the vector $\mathbf{1}_n$ describes another quantity used to classify the efficiency of an allocation, as a function of the sum $w(\mathbf{x})$ of resources.

We focus in this section on the widely applied α-fair utility function:

$$\sum_i U_\alpha(x_i), \text{ where } U_\alpha(x) = \begin{cases} \frac{x^{1-\alpha}}{1-\alpha} & \alpha \geq 0, \ \alpha \neq 1 \\ \log(x) & \alpha = 1 \end{cases}. \tag{25}$$

We first show that the α-fairness network utility function can be *factored* into two components: one corresponding to the family of fairness measures we constructed and one corresponding to efficiency. We then demonstrate that, for a fixed α, the factorization can be viewed as a single point on the optimal tradeoff curve between fairness and efficiency. Furthermore, this particular point is one where maximum emphasis is placed on fairness while maintaining Pareto optimality of the allocation. This allows us to quantitatively interpret the belief of "larger α is more fair" *across all* $\alpha \geq 0$.

A. *Factorization of α-fair Utility Function*

Re-arranging the terms of the equation in Table III, we have

$$\begin{aligned} U_{\alpha=\beta}(\mathbf{x}) &= \frac{1}{1-\beta} |f_\beta(\mathbf{x})|^\beta \left(\sum_i x_i\right)^{1-\beta} \\ &= |f_\beta(\mathbf{x})|^\beta \cdot U_\beta\left(\sum_i x_i\right), \end{aligned} \tag{26}$$

where $U_\beta\left(\sum_i x_i\right)$ is the one-dimensional version of the α-fair utility function with $\alpha = \beta$. For $\beta \to 1$, it is easy to show that our fairness measure $f_\beta(\mathbf{x})$, multiplied by a function of throughput $\sum_i x_i$, equals α-fair utility function with $\alpha = 1$. Similarly, for $\beta \to \infty$, it equals α-fair utility function as $\alpha \to \infty$. Therefore, Equation (26) also holds for proportional fairness at $\alpha = 1$ and max-min fairness at $\alpha \to \infty$.

Equation (26) demonstrates that the α-fair utility functions can be factorized as the product of two components: a fairness measure, $|f_\beta(\mathbf{x})|^\beta$, and an efficiency measure, $U_\beta\left(\sum_i x_i\right)$. The fairness measure $|f_\beta(\mathbf{x})|^\beta$ only depends on the normalized distribution, $\mathbf{x}/(\sum_i x_i)$, of resources (due to Axiom 2), while the efficiency measure is a function of the sum resource $\sum_i x_i$.

Allocation:	$\mathbf{x}$	
	$\swarrow$	$\searrow$
Factorize:	$\mathbf{x}/\sum_i x_i$	$\sum_i x_i$
	$\downarrow$	$\downarrow$
Measure:	$f_\beta\left(\mathbf{x}/\sum_i x_i\right)$	$U_\beta\left(\sum_i x_i\right)$
	$\searrow$	$\swarrow$
Combine:	$U_{\alpha=\beta}(\mathbf{x})$	

TABLE IV
Illustration of the factorization of the α-fair utility functions into a fairness component of the normalized resource distribution and a efficiency component of the sum resource.

The factorization of α-fair utility functions is illustrated in

Table IV and decouples the two components to tackle issues such as fairness-efficiency tradeoff and feasibility of $\mathbf{x}$ under a given constraint set. For example, it helps to explain the counter-intuitive throughput behavior in [12]: an allocation vector that maximizes the α-fair utility with a larger α may not be less efficient, because the α-fair utility incorporates both fairness and efficiency at the same time. When maximizing an α-fair utility, any improvement in efficiency for less fairness loss is also desirable, because these values are coupled in a way determined by the parameter $\alpha = \beta$ and the shape of the constraint set.

The additional efficiency component in (26) can skew the optimizer (i.e., the resource allocation resulting from α-fair utility maximization) away from an equal distribution. For this to happen there must exist an allocation that is feasible (within the constraint set of realizable allocations) with a large enough gain in efficiency over all equal distribution allocations. Hence, the magnitude of this skewing depends on the fairness parameter ($\alpha = \beta$), the constraint set of $\mathbf{x}$, and the relative importance of fairness and efficiency.

Guided by the product form of (26), we consider a scalarization of the maximization of the two objectives: fairness and efficiency:

$$\Phi_\lambda(\mathbf{x}) = \lambda \ell\left(f_\beta\left(\mathbf{x}\right)\right) + \ell\left(\sum_i x_i\right), \tag{27}$$

where $\beta \in (0,1) \cup (1,\infty)$ is fixed, $\lambda \in [0,\infty)$ absorbs the exponent β in the fairness component of (26) and is a weight specifying the relative emphasis placed on the fairness, and

$$\ell(y) = \text{sign}(y)\log(|y|). \tag{28}$$

The use of the log function later recovers the product in the factorization of (26) from the sum in the scalarized (27).

B. What Does "Larger α is More Fair" Mean?

It is commonly believed that larger α is more fair, but it is not exactly clear what this statement means for general $\alpha \in [0,\infty]$. Guided by the factorization above and the axiomatic construction of fairness measures, we provide two interpretation of this statement that justify it from the viewpoints of Pareto optimality and geometry of the constraint set.

An allocation vector $\mathbf{x}$ is said to be Pareto dominated by $\mathbf{y}$ if $x_i \le y_i$ for all i and $x_i < y_i$ for at least some i. An allocation is called Pareto optimal if it is not Pareto dominated by any other feasible allocation. If the relative emphasis on efficiency is sufficiently high, Pareto optimality of the solution can be maintained. To preserve Pareto optimality, we require that if $\mathbf{y}$ Pareto dominates $\mathbf{x}$, then $\Phi_\lambda(\mathbf{y}) > \Phi_\lambda(\mathbf{x})$.

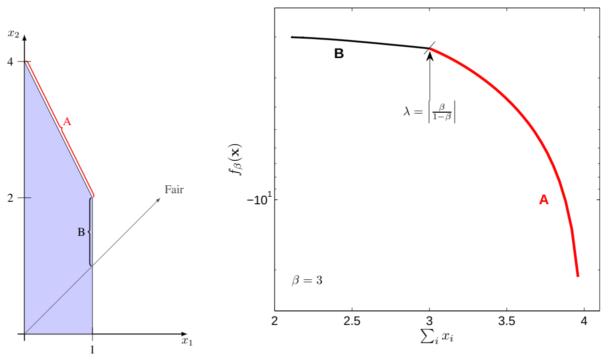
Theorem 6: (*Preserving Pareto optimality.*) The necessary and sufficient condition on λ such that $\Phi_\lambda(\mathbf{y}) > \Phi_\lambda(\mathbf{x})$ if $\mathbf{y}$ Pareto dominates $\mathbf{x}$ is

$$\lambda \le \left|\frac{\beta}{1-\beta}\right|. \tag{29}$$

Consider the set of maximizers of (27) for λ in the range in Theorem 6:

$$\mathbb{P} = \left\{\mathbf{x} : \mathbf{x} = \arg\max_{\mathbf{x}\in\mathbb{R}} \Phi_\lambda(\mathbf{x}),\ \forall\lambda \le \left|\frac{\beta}{1-\beta}\right|\right\}. \tag{30}$$

When weight $\lambda = 0$, the corresponding points in $\mathbb{P}$ is most efficient. When weight $\lambda = \left|\frac{\beta}{1-\beta}\right|$, it can be shown that the factorization in (26) becomes the same as (27). Therefore, α-fairness corresponds to the solution of an optimization that places the *maximum emphasis* on the fairness measure parameterized by $\beta = \alpha$ while *preserving Pareto optimality*. Allocations in $\mathbb{P}$ corresponding to other values of λ achieve a tradeoff between fairness and efficiency, while Pareto optimality is preserved.



(a) (b)

Fig. 4. (a) Feasible region (i.e., the constraint set of the utility maximization problem) where overemphasis of fairness violates Pareto dominance, and (b) its fairness-efficiency tradeoff for $\beta = 3$. Region A corresponds to Pareto optimal solutions. Region B is when the condition of Theorem 6 is violated, and solutions are more fair, but no longer Pareto optimal.

Figure 4(b) illustrates an optimal fairness-efficiency tradeoff curve $\left\{\left[f_\beta(\mathbf{x}), \Sigma_i x_i\right],\ \forall\mathbf{x} = \arg\max_{\mathbf{x}\in\mathbb{R}} \Phi_\lambda(\mathbf{x}),\ \forall\lambda\right\}$ corresponding to the constraint set shown in Figure 4(a). The set of optimizers $\mathbb{P}$ in (30), which is obtained by maximizing Pareto optimal utilities (27), is shown by curve A in Figure 4(b).

We just demonstrated the factorization (26) is an extreme point on the tradeoff curve between fairness and efficiency for fixed $\beta = \alpha$. What happens when α becomes bigger?

We denote by $\bigtriangledown_{\mathbf{x}}$ the gradient operator with respect to the vector $\mathbf{x}$. For a differentiable function, we use the standard inner product ($\langle \mathbf{x}, \mathbf{y}\rangle = \sum_i x_i y_i$) between the gradient of the function and a normalized vector to denote the directional derivative of the function.

Theorem 7: (*Monotonicity of fairness-efficiency reward ratio.*) Let allocation $\mathbf{x}$ be given. Define $\boldsymbol{\eta} = \frac{1}{n}\mathbf{1}_n - \frac{\mathbf{x}}{\sum x_i}$ as the vector pointing from the allocation to the nearest fairness maximizing solution. Then the fairness-efficiency reward ratio:

$$\frac{\left\langle \bigtriangledown_{\mathbf{x}} U_{\alpha=\beta}(\mathbf{x}), \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}\right\rangle}{\left\langle \bigtriangledown_{\mathbf{x}} U_{\alpha=\beta}(\mathbf{x}), \frac{\mathbf{1}_n}{\|\mathbf{1}_n\|}\right\rangle}, \tag{31}$$

is non-decreasing with α, i.e., higher α gives a greater relative reward for fairer solutions.

The choice of direction $\boldsymbol{\eta}$ is a direct result of Axiom 2 and Corollary 2, which together imply that $\boldsymbol{\eta}$ is the direction that most increases fairness and is orthogonal to increases in efficiency. Figure 5 illustrates Theorem 7 by showing the two directions: $\boldsymbol{\eta}$ (which points to the nearest fairness maximizing solution) and $\mathbf{1}$ (which points to the direction for maximizing total throughput).

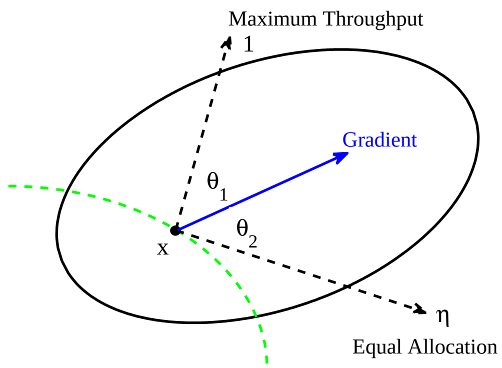


Fig. 5. This figure illustrates the two directions in Theorem 7: $\boldsymbol{\eta}$ (which points to the nearest fairness maximizing solution) and $\mathbf{1}$ (which points to the direction for maximizing total throughput).

An increase in either fairness or efficiency is a "desirable" outcome. The choice of α dictates exactly *how desirable* one objective is relative to the other (for a fixed allocation). Theorem 7 states that, with a larger α, there is a larger component of the utility function gradient in the direction of fairer solutions, relative to the component in the direction of more efficiency. Economically, this could be seen as decreasing the pricing of fairer allocations relative to a fixed price on allocations of increased throughput. Notice, however, that comparison is in terms of the ratio between these two gradient components rather than the magnitude of the gradient, and both fairness and efficiency may increase simultaneously.

This result provides a justification for the belief that larger α is "more fair", not just for $\alpha \in \{0, 1, \infty\}$, but for any $\alpha \in [0, \infty)$. Figure 6 depicts how this ratio increases with $\alpha = \beta$ for some examples allocations.

VII. A New Set of Four Axioms

Given a set of useful axioms, it is important to ask if other useful axiomatic systems are possible. By removing or modifying some of the five axioms here — for example, Axiom 2 that decouples the concern on efficiency from fairness — what kind of fairness measures would result? Could an alternative set of axioms lead to the construction of fairness measures that do not automatically decouple from the notions of efficiency and feasibility of resource allocation?

In this section, we propose a set of alternative axioms, which includes Axioms 1–5 as a special case. Let $F : \mathbb{R}_+^n \to \mathbb{R}$ be a general fairness measure satisfying four axioms as follows.

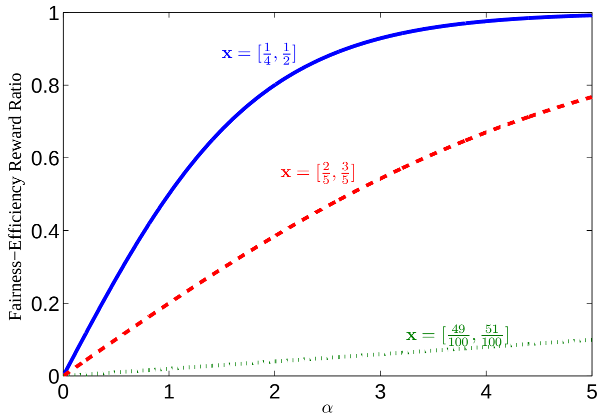


Fig. 6. Monotonic behavior of the ratio (31) as a function of α. Three fixed allocations are considered, and solutions that are already more fair have a lower ratio.

1′) *Axiom of Continuity.* Fairness measure $F(\mathbf{x})$ is continuous on $\mathbb{R}_+^n$ for all integer $n \geq 1$.

2′) *Axiom of Saturation.* Fairness measure $f(\mathbf{x})$ of equal resource allocations eventually becomes independent of the number of users:

$$\lim_{n\to\infty} \frac{F(\mathbf{1}_{n+1})}{F(\mathbf{1}_n)} = 1. \tag{32}$$

3′) *Axiom of Partition.* We define a *generalized direct product* of a vector $\mathbf{x} = [x_1, x_2]$ with two equal-weight vectors $\mathbf{y^1}, \mathbf{y^2}$ (satisfying $w(\mathbf{y^1}) = w(\mathbf{y^2})$) by:

$$\mathbf{x} \odot \{\mathbf{y^1}, \mathbf{y^2}\} = [x_1\mathbf{y^1}, x_2\mathbf{y^2}]. \tag{33}$$

There exists a continuous and strictly monotonic *generator function* $g(y)$ such that

$$F(\mathbf{x} \odot \{\mathbf{y^1}, \mathbf{y^2}\}) = F(\mathbf{x}) \cdot g^{-1}\left(\sum_{i=1}^{2} s_i \cdot g\left(F(\mathbf{y}^i)\right)\right), \tag{34}$$

where $\sum_i s_i = 1$ are positive weights.

4′) *Axiom of Starvation* For $n = 2$ users, we have $f(1, 0) \leq f(1, 1)$, i.e., starvation is less fair than equal distribution.

Since the Axiom 2 of Homogeneity is removed, fairness measure $F(\mathbf{x})$ may depend on the absolute magnitude of resource vector $\mathbf{x}$. Using Axiom 3′, we can prove that $F(\mathbf{x})$ is a homogeneous function of real degree. Furthermore, the two sets of axioms are equivalent, if the order of homogeneity is zero. This means that the new axiomatic system is more general than the original one.

Theorem 8: (*Existence and Uniqueness.*) For each generator $g(y)$, there exists a unique fairness measure $F(\mathbf{x})$ satisfying Axioms $1' - 4'$. We have,

$$F(\mathbf{x}) = f(\mathbf{x}) \cdot \left(\sum_i x_i\right)^{\frac{1}{\lambda}}, \tag{35}$$

where $\frac{1}{\lambda} \in \mathbb{R}$ is the degree of homogeneity and $f(\mathbf{x})$ is a fairness measure satisfying Axioms 1–5 with respect to the same generator $g(y)$.

While it is easy to verify that some properties, like that of symmetry, in Section III also hold for fairness measure $F(\mathbf{x})$, some properties of fairness measures satisfying Axioms 1–5 are lost in the generalization. In particular, the new fairness measure may not be Schur-concave, and equal allocation may not be fairness-maximizing. For instance, we can verify that $F(1,1) < F(0.2,10)$ for $\beta = 2$ and $\frac{1}{\lambda} = 10$.

When power generators $g(y) = |y|^\beta$ are considered, from Axioms $1'$–$4'$ we can derive fairness measure $F_{\beta,\lambda}(\mathbf{x})$, which is parameterized by both λ and β,

$$F_{\beta,\lambda}(\mathbf{x}) = f_\beta(\mathbf{x}) \cdot \left(\sum_i x_i\right)^{\frac{1}{\lambda}}. \quad (36)$$

This unifies our results in Sections IV-VI: Generalized Jain's index is a special case of $F_{\beta,\lambda}(\mathbf{x})$ for $1/\lambda = 0$ and $\beta < 1$; fairness measure $f_\beta(\mathbf{x})$ is a subclass of $F_{\beta,\lambda}(\mathbf{x})$ for $\lambda = 0$; and α-utility is obtained for $1/\lambda = \beta/(1-\beta)$ and $\beta > 0$ by comparing (36) and (27). The degree of homogeneity $1/\lambda$ determines how $F_{\beta,\lambda}(\mathbf{x})$ scales as throughput increases. The decomposition of fairness and efficiency in Section VI is now an immediate consequence from Axioms $1' - 4'$.

There is a useful connection with the characterization of α-fair utility function in the last section. The absolute value $|\lambda|$ is exactly equivalent to the parameter used for defining the utility function (27) in Section VI.B. From Theorem 6, we can conclude that fairness measure $F_{\beta,\lambda}(\mathbf{x})$ is Pareto optimal if and only if

$$\frac{1}{|\lambda|} \geq \left|\frac{1-\beta}{\beta}\right|. \quad (37)$$

For every β, there is a minimum degree of homogeneity such that Pareto optimality can be achieved. When inequality (37) is not satisfied, $F_{\beta,\lambda}(\mathbf{x})$ loses Pareto optimality and produces less throughput-efficient solutions if it is used as an objective function in utility optimization. Fairness measures with small degree of homogeneity $1/\lambda$ are more suitable for computing index values of fairness.

The degree of homogeneity of a fairness measure satisfying Axioms $1'$–$4'$ parameterizes a tradeoff between the concept of fairness and efficiency. Moreover, when power functions are used as generating functions, the degree of homogeneity is exactly equivalent to $\frac{1}{\lambda}$ in (27). Therefore, the intuition behind our result on a maximum $|\lambda|$ (minimum degree of homogeneity) to ensure Pareto optimality can be extended to the general optimization-theoretic approach to fairness, i.e., for a fairness measure F generated from any g, there is a minimum degree of homogeneity $\frac{1}{\lambda}$ to produce a Pareto optimal solution. In the way our first set of axioms generalized Jain's index and revealed new fairness measures with desirable properties, the second set of axioms offers a rich family of objective functions.

VIII. Related Axiomatic Theories

Reducing metrics of important notions to be the inevitable consequence of simple statements is a practice commonly found in mathematical way of thinking. Here we compare and contrast with several axiomatic theories in information, economics, and political philosophy, most of which are quantitative and one qualitative.

A. Renyi Entropy

Renyi entropy is a family of functionals for quantifying the uncertainty or randomness of generalized probability distributions [16]. Renyi entropy is derived from a set of five axioms as follows:

1) Symmetry.
2) Continuity.
3) Normalization.
4) Additivity.
5) Mean-value property.

Comparing Renyi's axioms to ours, we notice that the Axiom of Continuity and the Axiom of Normalization are equivalent to our Axiom 1 and 2, respectively. The Axiom of Symmetry is redundant as proven in our Corollary 1. Next, the Axiom of Additivity and Axiom of Mean-value are replaced by our Axiom 4, which quantify the notion of fairness. More precisely, the Axiom of Additivity can be directly derived from our Axiom 4 as in Equation (6). The Axiom of Mean-value, which states that the entropy of the union of two incomplete distributions is the weighted mean value of the entropies of the two distributions, plays a role similar to our recursive construction in Axiom 4. The key difference is that in our Axiom 4, fairness of a vector is given by the product of a "global" fairness across partitions, and a weighted mean of "local" fairness within each partition. For $\beta \leq 1$, it is straightforward to verify that our fairness measure satisfying the alternative set of axioms includes Renyi entropy as a special case. If we choose $\beta = 1-\alpha$ and $\lambda = -1$ in our fairness measure $F_{\beta,\lambda}$, then $F_{\beta,\lambda}$ equals to the exponential of Renyi entropy with parameter α, i.e.,

$$F_{\beta=1-\alpha,\lambda=-1}(\mathbf{x}) = e^{h_\alpha(\mathbf{x})}. \quad (38)$$

Our proof for existence and uniqueness of fairness measures in Theorems 1-2 extends those of Renyi entropy [16] and Shannon entropy [17]: A key step in the proofs is to show that any function satisfying the (fairness or entropy) axioms is an *additive number-theoretical function* [16]. For fairness we show that $\log f(\mathbf{1}_n)$ is an additive number-theoretical function over integer $n \geq 1$, i.e.,

$$\log f(\mathbf{1}_{mn}) = \log f(\mathbf{1}_n) + \log f(\mathbf{1}_m), \quad (39)$$
$$\lim_{n\to\infty} \log f(\mathbf{1}_{n+1}) - \log f(\mathbf{1}_n) = 0. \quad (40)$$

where $\mathbf{1}_n$ is an all-one resource allocation vector of length n. Conditions (39) and (40) follow from the Axiom of Partition and the Axiom of Saturation, respectively. Using the result in [16] we have $\log f(\mathbf{1}_n) = r \log n$ which, together with the Axiom of Homogeneity, leads to an explicit expression of fairness measure $f(\mathbf{x})$ for the case of two users and resource vectors of rational numbers. The result is then generalized to resource vectors of real numbers using the Axiom of Continuity, and to arbitrary number of users based on Axiom of Partition. The Axiom of Saturation and the Axiom of Starvation are used to ensure uniqueness of fairness measures.

Renyi entropy has been further extended by others since the 1960s, and this work is in some sense a (different) generalization of Renyi entropy: first along the line of general λ and then along the line of $\beta > 1$.

B. Lorenz Curve

Schur-concavity of fairness measures proven in Theorem 3 is a critical property for rationalizing fairness measures (e.g. Gini Coefficient) by establishing orderings on the set of Lorenz curves [24]. For a resource allocation vector $\mathbf{x}$, its Lorenz curve L_x is defined by

$$L_x(d) = \sum_{i=1}^{d} x_i^{\uparrow}, \tag{41}$$

where $x_i^{\uparrow}$ is the ith elements of $\mathbf{x}^{\uparrow}$, sorted in non-decreasing order. The Lorenz curve L_x is a graphical representation of the distribution of $\mathbf{x}$, and has been shown to be a very useful tool for analyzing the social welfare distributions and relative income differences in economics.

In [24], an axiomatic characterization of Lorenz curve orderings is proposed based on a set of four axioms:

1) Order. The ordering is transitive and complete.
2) Dominance. The ordering is Shur-concavity.
3) Continuity.
4) Independence.

C. Cooperative Economic Theories

In economic theory, a number of theories have been developed to study the collective decisions of groups. Many of these theories have also been uniquely associated with sets of axioms [19]. While applications of these theories range from political theory to voting methods, here we focus on two well-known axiomatic constructions: the the Nash bargaining solution [20] and the Shapley value [21].

The Nash bargaining solution satisfies a set of four axioms:

1) Invariance to affine transformation.
2) Pareto optimality.
3) Independence of irrelevant alternatives (IIA).
4) Symmetry.

Symmetry is shown as a corollary in our theory, and Pareto optimality not imposed as an axiom given our focus on fairness. Nash's axiom of IIA contributes most to the uniqueness of his solution, and it is also most considered a value statement. It has been shown by many others, that replacing IIA with other value statements may result in solution classes different from the bargaining solution.

Note that given a feasible region of individual utilities, the Nash bargaining solution is equivalent to a maximization of the proportion fairness utility function; thus, the Nash bargaining solution can be derived from a special case within our axiomatic framework. Solutions derived from other special cases may, and in fact some cases do, coincide with solutions resulting from replacement of IIA, which justifies the study of *classes* of fairness measures.

Another well known solution concept in economic study of groups is the Shapley value [21]. In a coalitional form game, individual decide whether or not to form coalitions in order to increase the maximum utility of the group, while ensuring that their share of the group utility is maximized. The Shapley value concerns an operator that maps the structure of the game, to a set of allocations of utility of the overall group. Given a coalitional game, four axioms uniquely define the Shapley value as the solution concept:

1) Efficiency/Pareto Optimality.
2) Symmetry.
3) Dummy.
4) Additivity.

Again Pareto optimality is included as an axiom. Although the structure of a coalitional game is a bit different than a simple division of resources, some parallels are apparent. For instance, the Dummy axiom refers to a scenario where an individual does not increase any groups utility by joining it. In this case, the Shapley value assigns a utility of zero to that individual. This is similar to Corollary 4, where an inactive user has no impact on the measure. Shapley's axiom of additivity provides a method of building up a single coalitional game with potentially many individuals, from small games, which may have only two players. This is similar to our Axiom 4, wherein the fairness measure itself is recursively constructed from the fairness attained by subsets of the overall allocation.

In both Nash bargaining solution and Shapley value, efficiency is an axiom in defining the solution. The first family of fairness measures, f, is confined to homogeneous functions of degree zero, and the resulting measures are an integral part of the α-fairness utility function. One might suspect that it is possible to extend our axiomatic structure to the optimization theoretic fairness approach by relaxing the Axiom of Homogeneity to homogeneous functions of arbitrary degree. This is indeed the case as developed by the previous section.

D. Rawls' Theory of Justice and Distributive Fairness

In political philosophy, the work of John Rawls has been both influential and provacative since the original publication [23]. The arguments posed by Rawls are based on two fundamental *principles* (axioms stated in English),

1) Each person is to have an equal right to the most extensive scheme of equal basic liberties compatible with a similar scheme of liberties for others.
2) Social and economic inequalities should be arranged so that they are both (a) to the greatest benefit of the least advantaged persons, and (b) attached to offices and positions open to all under conditions of equality of opportunity.

While not intended to add one more angle of critique against or defense for Rawls theory of justice, our results can be viewed as an axiomatic quantification of part of his theory, and a generalization of other part. Rawls' Principle 1 can now be captured as a theorem (not an axiom) that says any fairness measures satisfying the 5 axioms will satisfy the following property: add an equal amount of resource to each user (representing the basic resource needs compatible with each others demands) will only increase fairness value. Rawls' Principle 2 is on max-min fairness, and now it is axiomatically constructed as a special case of a continuum of generalized notions trading off fairness with efficiency.

IX. Concluding Remarks

An axiomatic approach to the fundamental concepts of fairness illuminates many issues in resource allocation. This paper is far from the end of axiomatic *theories* of fairness. One way to re-examine axioms is to refute their corollaries in the context of network resource allocation. Another approach is to expand the data structure, an extremely simple one (given a vector, look for a scalar-valued function), that we started with. Perhaps the fairness measure should be a function dependent on the feasible region of allocations. Making some x_i bigger should be called more fair if the resulting $\mathbf{x}$ is bigger in all coordinates, i.e., those contributing to the overall efficiency should "fairly" receive more resources. Furthermore, we have assumed that the resource x is infinitely divisible and has no user-specific or time-dependent values, a simplification that clearly is inadequate in many scenarios. Finally, we have ignored the process of resource allocation, such as the issues of who makes the allocation and can it be made with autonomity of users. Time and time again, centralized systems with unchecked power claims to achieve fairness maximizing allocation and actually produces extreme unfairness. We have also ignored the long-term evolution of the system where users react to the allocation at any given time, which further impacts the long term efficiency, fairness, and stability of the system.

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Appendix

A. Proof of Theorems 1 and 2

We first show that fairness achieved by equal-resource allocations $\mathbf{1}_n$ is independent of the choice of $g(y)$. Without loss of generality, we assume that $f(1) = 1$.

Lemma 1: To satisfy Axioms 1–5, fairness achieved by equal-resource allocations $\mathbf{1}_n$ is given by

$$f(\mathbf{1}_n) = n^r \cdot f(1), \quad \forall\, n \geq 1, \tag{42}$$

where r is a constant exponent.

Proof: Applying Axiom 4 to resource allocation vector $\mathbf{1}_{mn}$ with integers $m, n \geq 1$, we have

$$\begin{aligned} f(\mathbf{1}_{mn}) &= f(\underbrace{\mathbf{1}_m, \ldots, \mathbf{1}_m}_{n \text{ segments}}) \\ &= f(\underbrace{m, \ldots, m}_{n \text{ numbers}}) \cdot g^{-1}\left(\sum_{i=1}^{n} s_i \cdot g\left(f(\mathbf{1}_m)\right)\right) \\ &= f(\mathbf{1}_n) \cdot g^{-1}\left(g\left(f(\mathbf{1}_m)\right)\right) \\ &= f(\mathbf{1}_n) \cdot f(\mathbf{1}_m) \end{aligned} \tag{43}$$

where the second step follows from the Axiom 2 by letting $t = 1/m$ and the fact that $\sum_i s_i = 1$. Equation (43) shows that

$\log f(\mathbf{1}_{mn})$ is an additive number-theoretical function [18], i.e.,

$$\log f(\mathbf{1}_{mn}) = \log f(\mathbf{1}_n) + \log f(\mathbf{1}_m) \tag{44}$$

Further, from Axiom 3, we derive

$$\lim_{n\to\infty} [\log f(\mathbf{1}_{n+1}) - \log f(\mathbf{1}_n)] = 0 \tag{45}$$

Using the result in [18], equation (44) and (45) implies that $\log f(\mathbf{1}_n)$ must be a logarithmic function. We have

$$\log f(\mathbf{1}_n) = r \log n, \tag{46}$$

where r is a real constant. This is exactly (42) after taking an exponential on both sides. ■

Now, we use (42) to derive an expression for the fairness measure deductively, starting from $n = 2$ users. Let x_1 and x_2 be two rational numbers, such that $x_1 = \frac{a_1}{b_1}$ and $x_2 = \frac{a_2}{b_2}$ for some positive integers a_1, b_1, a_2, b_2. Using Axiom 4 and Lemma 1, we have

$$\begin{aligned}
&(a_1b_2 + a_2b_1)^r \\
&\quad = f(\mathbf{1}_{a_1b_2+a_2b_1}) \\
&\quad = f(\mathbf{1}_{a_1b_2}, \mathbf{1}_{a_2b_1}) \\
&\quad = f(a_1b_2, a_2b_1) \cdot g^{-1}\left(s_1 g\left(f(\mathbf{1}_{a_1b_2})\right) + s_2 g\left(f(\mathbf{1}_{a_2b_1})\right)\right) \\
&\quad = f(a_1b_2, a_2b_1) \cdot g^{-1}\left(s_1 g\left(a_1^r b_2^r\right) + s_2 g\left(a_2^r b_1^r\right)\right)
\end{aligned} \tag{47}$$

Applying Axiom 2 to (47) with $t = b_1b_2$, we have

$$\begin{aligned}
f(x_1, x_2) &= f\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) \\
&= f(a_1b_2, a_2b_1) \\
&= \frac{(a_1b_2 + a_2b_1)^r}{g^{-1}\left(s_1 g\left(a_1^r b_2^r\right) + s_2 g\left(a_2^r b_1^r\right)\right)}
\end{aligned} \tag{48}$$

For a given function $g(y)$, equation (48) defines fairness measure $f(x_1, x_2)$ for two users for rational vector $[x_1, x_2]$. When vector $[x_1, x_2]$ is real, by Axiom 1, fairness measure $f(x_1, x_2)$ is uniquely determined by a sequence of rational allocation vectors, whose limit is $[x_1, x_2]$. Therefore, equation (48) uniquely defines fairness measure $f(x_1, x_2)$ for arbitrary real numbers x_1, x_2.

Suppose that we have an expression for the fairness measure $f(x_1, \ldots, x_k)$ with $k \geq 2$ users. To derive $f(x_1, \ldots, x_k, x_{k+1})$ for $k+1$ users, we use Axiom 4 to obtain the following:

$$\begin{aligned}
&f(x_1, \ldots, x_k, x_{k+1}) \\
&= f\Big(\sum_{i=1}^{k} x_i, x_{k+1}\Big) \cdot g^{-1}\left(s_1 g\left(f(x_1, \ldots, x_k)\right) + s_2 g\left(f(x_{k+1})\right)\right) \\
&= f\Big(\sum_{i=1}^{k} x_i, x_{k+1}\Big) \cdot g^{-1}\left(s_1 g\left(f(x_1, \ldots, x_k)\right) + s_2 g\left(1\right)\right)
\end{aligned} \tag{49}$$

By induction, equations (48) and (49) together defines fairness measure for all integer $n \geq 1$, when the mean function $g(y)$ is given. If the resulting fairness measure satisfies Axioms 1–5, it must be unique according to equations (48) and (49). This proves the uniqueness in Theorem 2.

To prove the existence in Theorem 1, we show that there exists a mean function $g(y)$, such that the resulting fairness index in (48) and (49) satisfies Axioms 1–5. We choose $g(y) = \log(y)$ and proportional weights (i.e. $\rho = 1$) in (5). From (48), we derive

$$\begin{aligned}
f(x_1, x_2) &= \frac{(a_1b_2 + a_2b_1)^r}{g^{-1}\left(s_1 g\left(a_1^r b_2^r\right) + s_2 g\left(a_2^r b_1^r\right)\right)} \\
&= \frac{(a_1b_2 + a_2b_1)^r}{(a_1b_2)^{rs_1}(a_2b_1)^{rs_2}} \\
&= \frac{(x_1 + x_2)^r}{x_1^{\frac{rx_1}{x_1+x_2}} x_2^{\frac{rx_2}{x_1+x_2}}}.
\end{aligned} \tag{50}$$

Let $u_k = \sum_{i=1}^{k} x_i$ be the sum of the first k elements in vector $[x_1, \ldots, x_k, x_{k+1}]$. Then, using (49) inductively, we obtain

$$\begin{aligned}
&f(x_1, \ldots, x_k, x_{k+1}) \\
&\quad = f\Big(\sum_{i=1}^{k} x_i, x_{k+1}\Big) \cdot g^{-1}\left(s_1 g\left(f(x_1, \ldots, x_k)\right) + s_2 g\left(1\right)\right) \\
&\quad = f\Big(\sum_{i=1}^{k} x_i, x_{k+1}\Big) \cdot f^{s_1}(x_1, \ldots, x_k) \\
&\quad = \frac{(u_k + x_{k+1})^r}{(u_k)^{\frac{ru_k}{u_k+x_{k+1}}} x_{k+1}^{\frac{rx_{k+1}}{u_k+x_{k+1}}}} \cdot \left[\frac{(u_k)^r}{\prod_{i=1}^{k} x_i^{\frac{rx_i}{u_k}}}\right]^{\frac{u_k}{u_k+x_{k+1}}} \\
&\quad = \frac{(u_k + x_{k+1})^r}{x_{k+1}^{\frac{x_{k+1}}{u_k+x_{k+1}}} \cdot \prod_{i=1}^{k} x_i^{\frac{rx_i}{u_k+x_{k+1}}}}
\end{aligned} \tag{51}$$

By rearranging the terms in (51), we obtain a fairness measure generated by logarithmic function $g(y) = \log(y)$ and proportional weights:

$$f(x_1, \ldots, x_k, x_{k+1}) = \left(\sum_{i=1}^{k+1} x_i\right)^r \cdot \prod_{i=1}^{k+1} x_i^{\frac{-rx_i}{u_{k+1}}}, \tag{52}$$

where $u_{k+1} = \sum_{i=1}^{k+1} x_i$ is the sum of all elements.

We need to prove that the fairness measure in (52) satisfies Axioms 1–5. It is easy to see that Axioms 1–3 are satisfied by the fairness measure in (52). To verify Axiom 4, we consider partitioning a resource allocation vector $\mathbf{x}$ of n users into two segments: $\mathbf{x}^1 = [x_1, \ldots, x_k]$ and $\mathbf{x}^1 = [x_{k+1}, \ldots, x_n]$ for arbitrary $0 < k < n$. Let $u_{n-k} = u_n - u_k$. From (52), we conclude that

$$\begin{aligned}
&f(x_1, \ldots, x_n) \\
&= \left(\sum_{i=1}^{n} x_i\right)^r \cdot \prod_{i=1}^{n} x_i^{\frac{-rx_i}{u_n}}, \\
&= \frac{(u_k + u_{n-k})^r}{u_k^{r\frac{u_k}{u_n}} u_{n-k}^{r\frac{u_n-u_k}{u_n}}} \cdot \left[\frac{u_k^r}{\prod_{i=1}^{k} x_i^{\frac{x_i}{u_k}}}\right]^{\frac{u_k}{u_n}} \cdot \left[\frac{u_{n-k}^r}{\prod_{i=k+1}^{n} x_i^{\frac{x_i}{u_{n-k}}}}\right]^{\frac{u_{n-k}}{u_n}} \\
&= f(u_k, u_{n-k}) \cdot e^{\frac{u_k}{u_n}\log f(\mathbf{x}^1) + \frac{u_{n-k}}{u_n}\log f(\mathbf{x}^2)} \\
&= f\Big(\sum_{i=1}^{k} x_i, \sum_{i=k+1}^{n} x_i\Big) \cdot g^{-1}\left(\sum_{i=1}^{2} s_i \cdot g\left(f(\mathbf{x}^i)\right)\right),
\end{aligned} \tag{53}$$

where weights $s_1 = \frac{u_k}{u_n}$ and $s_2 = \frac{u_{n-k}}{u_n}$ are proportional to the sum resource in each segment. This shows that the fairness measure in (52) is irrelevant to partition.

To verify Axiom 5, we consider an allocation vector $\mathbf{x} = [\theta, 1-\theta]$ and compute its fairness measure as follows

$$f(\theta, 1-\theta) = \frac{1}{\theta^{r\theta}(1-\theta)^{r(1-\theta)}}. \tag{54}$$

Taking a logarithm on both sides, we have

$$\log f(\theta, 1-\theta) = r\left[\theta \log\frac{1}{\theta} + (1-\theta)\log\frac{1}{1-\theta}\right]. \tag{55}$$

Since the right hand side of (55) is the entropy function, we conclude that $f(\theta, 1-\theta)$ is monotonically increasing for $\theta \in [0, \frac{1}{2}]$ and monotonically decreasing for $\theta \in [\frac{1}{2}, 1]$. Therefore, the fairness measure in (52) satisfies Axioms 1–5.

B. Proof of Corollary 1

For $n = 2$ users, symmetry follows directly from equation (48) in Appendix A, i.e.,

$$f(x_1, x_2) = f(x_2, x_1),\ \forall x_1, x_2 \geq 0. \tag{56}$$

Assume symmetry holds for n users. Let $\mathbf{x} = [x_1, \ldots, x_n, x_{n+1}]$ be a resource allocation vector and $i_1, \ldots, i_n, i_{n+1}$ be an arbitrary permutation of the indices $1, \ldots, n, n+1$. When $i_{n+1} > 1$, applying Axiom 4, we can use equation (57) to show that

$$f(x_{i_1}, \ldots, x_{i_n}, x_{i_{n+1}}) = f(x_1, \ldots, x_n, x_{n+1}). \tag{58}$$

When $i_{n+1} = 1$, using the same technique, we have

$$\begin{aligned} f(x_{i_1}, \ldots, x_{i_n}, x_{i_{n+1}}) &= f(x_{i_1}, \ldots, x_{i_{n+1}}, x_{i_n}) \\ &= f(x_1, \ldots, x_n, x_{n+1}). \end{aligned} \tag{59}$$

Then symmetry also holds for $n+1$ users.

C. Proof of Theorem 3

Because vector $\mathbf{x}$ is majorized by vector $\mathbf{y}$, if and only if, from $\mathbf{x}$ we can produce $\mathbf{y}$ by a finite sequence of Robin Hood operations [14], where we replace two elements x_i and $x_j < x_i$ with $x_i - \epsilon$ and $x_j + \epsilon$, respectively, for some $\epsilon \in (0, x_i - x_j)$, it is necessary and sufficient to show that such an Robin Hood operation always improves a fairness measure defined by Axioms 1–5.

Toward this end, we consider partitioning a resource allocation vector $\mathbf{x}$ of n users into two segments: $\mathbf{x}^1 = [x_i, x_j]$ and $\mathbf{x}^2 = [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n]$. Let $\mathbf{y} = [\mathbf{y}^1, \mathbf{x}^2]$ where $\mathbf{y}^1 = [x_i - \epsilon, x_j + \epsilon]$ be the vector obtained from $\mathbf{x}^1$ by the Robin Hood operation. Using Axiom 4, we have

$$\begin{aligned} &\frac{f(\mathbf{x})}{f(\mathbf{y})} \\ &= \frac{f(\mathbf{x}^1, \mathbf{x}^2)}{f(\mathbf{y}^1, \mathbf{x}^2)} \\ &= \frac{f(x_i + x_j, \sum_{k\neq i,j} x_k) \cdot g^{-1}\left(s_1 g\left(f(\mathbf{x}^1)\right) + s_2 g\left(f(\mathbf{x}^2)\right)\right)}{f(x_i + x_j, \sum_{k\neq i,j} x_k) \cdot g^{-1}\left(s_1 g\left(f(\mathbf{y}^1)\right) + s_2 g\left(f(\mathbf{x}^2)\right)\right)} \\ &= \frac{g^{-1}\left(s_1 g\left(f(x_i, x_j)\right) + s_2 g\left(f(\mathbf{x}^2)\right)\right)}{g^{-1}\left(s_1 g\left(f(x_i - \epsilon, x_j + \epsilon)\right) + s_2 g\left(f(\mathbf{x}^2)\right)\right)} \\ &\leq 1, \end{aligned}$$

where the last step follows form the monotonicity of g and the monotonicity of fairness measure with two-users in Axiom 5, i.e.,

$$f(x_i, x_j) \leq f(x_i - \epsilon, x_j + \epsilon). \tag{60}$$

Therefore, if $\mathbf{x}$ is majorized by $\mathbf{y}$, then we have $f(\mathbf{x}) \leq f(\mathbf{y})$. The fairness measure is Schur-concave.

D. Proof of Corollary 2

The proof for Corollary 2 is straightforward, because among the vectors with the same sum of elements, one with the equal elements is the most majorizing vector. Let $\sum_{i=1}^n x_i = n$ (which is always satisfied due to Axiom 2). The sum of the d smallest elements satisfies

$$\begin{aligned} \textstyle\sum_{i=1}^d x_i^{\uparrow} &= n\frac{\sum_{i=1}^d x_i^{\uparrow}}{\sum_{i=1}^n x_i^{\uparrow}} \\ &\leq n\frac{d}{n} \\ &\leq d. \end{aligned} \tag{61}$$

Then, $\mathbf{x} \preceq \mathbf{1}_n$ implies $f(\mathbf{x}) \leq f(\mathbf{1}_n)$, for any resource allocation vector $\mathbf{x}$.

E. Proof of Corollary 3

This corollary is a direct result of Equation (46) in the proof of Theorem 1, in Appendix A.

F. Proof of Corollary 4

Due to Schur-concavity in Theorem 3, it is sufficient to prove that collecting fixed-tax leads to a more majorizing allocation vector. From Axiom 2, we consider a vector $\mathbf{y} = t\left(\mathbf{x} - c\cdot\mathbf{1}_n\right)$, which achieves the same fairness as $\mathbf{x} - c\cdot\mathbf{1}_n$, i.e.,

$$f(\mathbf{x} - c\cdot\mathbf{1}_n) = f(t\left(\mathbf{x} - c\cdot\mathbf{1}_n\right)) \tag{62}$$

where $t = \frac{\sum_i x_i}{\sum x_i - nc}$, such that

$$\sum_{i=1}^n x_i = \sum_{i=1}^n t(x_i - c) = \sum_{i=1}^n y_i. \tag{63}$$

$$
\begin{aligned}
f(x_{i_1},\ldots,x_{i_n},x_{i_{n+1}}) &= f\left(\sum_{j=1}^{n} x_{i_j}, x_{i_{n+1}}\right)\cdot g^{-1}\left(s_1\cdot g\left(f(x_{i_1},\ldots,x_{i_n})\right)+s_2 g\left(f(x_{i_{n+1}})\right)\right)\\
&= f\left(\sum_{j=1}^{n} x_{i_j}, x_{i_{n+1}}\right)\cdot g^{-1}\left(s_1\cdot g\left(f(x_1,\ldots,x_{i_{n+1}-1},x_{i_{n+1}+1},\ldots,x_{n+1})\right)+s_2 g\left(f(x_{i_{n+1}})\right)\right)\\
&= f(x_1,\ldots,x_{i_{n+1}-1},x_{i_{n+1}+1},\ldots,x_{n+1},x_{i_{n+1}})\\
&= f\left(\sum_{j=1}^{i_{n+1}-1} x_j, \sum_{j=i_{n+1}}^{n+1} x_j\right)\cdot g^{-1}\left(s_1\cdot g\left(f(x_1,\ldots,x_{i_{n+1}-1})\right)+s_2 g\left(f(x_{i_{n+1}+1},\ldots,x_{n+1},x_{i_{n+1}})\right)\right)\\
&= f\left(\sum_{j=1}^{i_{n+1}-1} x_j, \sum_{j=i_{n+1}}^{n+1} x_j\right)\cdot g^{-1}\left(s_1\cdot g\left(f(x_1,\ldots,x_{i_{n+1}-1})\right)+s_2 g\left(f(x_{i_{n+1}},x_{i_{n+1}+1},\ldots,x_{n+1})\right)\right)\\
&= f(x_1,\ldots,x_{i_{n+1}-1},x_{i_{n+1}},x_{i_{n+1}+1},\ldots,x_{n+1}) \qquad (57)
\end{aligned}
$$

Then, for any integer $1 \le d \le n$ we have

$$
\begin{aligned}
\textstyle\sum_{i=1}^{d} y_i^{\uparrow} &= \sum_{i=1}^{d} t(x_i^{\uparrow}-c)\\
&= \frac{\sum_{i=1}^{d} x_i^{\uparrow} - dc}{\sum_{i=1}^{n} x_i - nc}\sum_{i=1}^{n} x_i\\
&\le \frac{\sum_{i=1}^{d} x_i^{\uparrow} - nc\frac{\sum_{i=1}^{d} x_i^{\uparrow}}{\sum_{i=1}^{n} x_i}}{\sum_{i=1}^{n} x_i - nc}\sum_{i=1}^{n} x_i\\
&= \sum_{i=1}^{d} x_i^{\uparrow}.
\end{aligned}
$$

where the third step following from the following inequality

$$
\frac{\sum_{i=1}^{d} x_i^{\uparrow}}{\sum_{i=1}^{n} x_i} \le \frac{d}{n}. \qquad (64)
$$

We have $\mathbf{x} \succeq \mathbf{y}$, which implies $f(\mathbf{x}) \ge f(\mathbf{y}) = f(\mathbf{x} - c\cdot\mathbf{1})$.

G. Proof of Corollary 5

Let $\mathbf{x}$ be an arbitrary resource allocation vector and $t > 0$ be a positive number. From Axiom 1, we have

$$
\begin{aligned}
&f(\mathbf{x},\mathbf{0}_n)\\
&= \lim_{t\to\infty} f(\mathbf{x}, \frac{1}{t}\mathbf{1}_n)\\
&= \lim_{t\to\infty} f(\sum_i x_i, \frac{n}{t})\cdot g^{-1}\left(\frac{(\sum_i x_i)^{\rho} g\left(f(\mathbf{x})\right)}{(\sum_i x_i)^{\rho} + \left(\frac{n}{t}\right)^{\rho}}\right.\\
&\qquad \left. + \frac{\left(\frac{n}{t}\right)^{\rho} g\left(f(\mathbf{1}_n)\right)}{(\sum_i x_i)^{\rho} + \left(\frac{n}{t}\right)^{\rho}}\right)\\
&= \lim_{t\to\infty} f(\sum_i x_i, \frac{n}{t})\cdot g^{-1}\left(g\left(f(\mathbf{x}_n)\right)\right)\\
&= f(\mathbf{x}),
\end{aligned}
$$

where the third step follows from Axiom 4.

H. Proof of Theorem 4

Without loss of generality, we assume that $f(1) = 1$. First, we plug into equations (48) and (49) power mean $g(y) = y^{\beta}$ with weights generated by arbitrary ρ. Equation (48) gives the fairness measure for two users:

$$
\begin{aligned}
f(x_1,x_2) &= \frac{(a_1b_2+a_2b_1)^r}{g^{-1}\left(s_1 g\left(a_1^r b_2^r\right)+s_2 g\left(a_2^r b_1^r\right)\right)}\\
&= \frac{(a_1b_2+a_2b_1)^r}{\left(s_1(a_1b_2)^{\beta r}+s_2(a_2b_1)^{\beta r}\right)^{\frac{1}{\beta}}}\\
&= \frac{(a_1b_2+a_2b_1)^r\left((a_1b_2)^{\rho}+(a_2b_1)^{\rho}\right)^{\frac{1}{\beta}}}{\left((a_1b_2)^{\rho+\beta r}+(a_2b_1)^{\rho+\beta r}\right)^{\frac{1}{\beta}}}\\
&= \frac{(x_1+x_2)^r\left(x_1^{\rho}+x_2^{\rho}\right)^{\frac{1}{\beta}}}{\left(x_1^{\rho+\beta r}+x_2^{\rho+\beta r}\right)^{\frac{1}{\beta}}}. \qquad (65)
\end{aligned}
$$

To derive the fairness measure for three users, we consider two different partitions of the resource allocation vector $[x_1,x_2,x_3]$ as $[x_1,x_2],[x_3]$ and $[x_1],[x_2,x_3]$. Using (49), we obtain two equivalent form of the fairness measure in (66) and (67).

As in Axiom 4, the fairness measure is irrelevant to partition. Hence, equations (66) and (67) should be equivalent for all $x_1,x_2,x_3 \ge 0$. Comparing the terms in (66) and (67), we must have $r = 0$ or $\rho + \beta r = 1$. When $r = 0$, it is easy to see that $f(x_1,x_2,x_3) = 1$ is constant. This case is trivial. We conclude that the fairness measure must have the following form

$$
f(x_1,x_2,x_3) = \frac{\left(\sum_{i=1}^{3} x_i^{1-\beta r}\right)^{\frac{1}{\beta}}}{\left(\sum_{i=1}^{3} x_i\right)^{\frac{1}{\beta}-r}}, \qquad (68)
$$

where $r = \frac{1-\rho}{\beta}$ is a proper exponent. Let $u_k = \sum_{i=1}^{k} x_i$ be the sum of the first k elements in vector $[x_1,\ldots,x_k,x_{k+1}]$.

$$f(x_1,x_2,x_3) = f(x_1+x_2,x_3)\cdot g^{-1}\left(s_1 g\left(f(x_1,x_2)\right)+s_2 g\left(1\right)\right)$$

$$= \frac{(x_1+x_2+x_3)^r\left((x_1+x_2)^\rho+x_3^\rho\right)^{\frac{1}{\beta}}}{\left((x_1+x_2)^{\rho+\beta r}+x_3^{\rho+\beta r}\right)^{\frac{1}{\beta}}}\cdot\left[\frac{\frac{(x_1+x_2)^{\rho+\beta r}(x_1^\rho+x_2^\rho)}{x_1^{\rho+\beta r}+x_2^{\rho+\beta r}}+x_3^\rho}{(x_1+x_2)^\rho+x_3^\rho}\right]^{\frac{1}{\beta}}$$

$$= \frac{(x_1+x_2+x_3)^r}{\left((x_1+x_2)^{\rho+\beta r}+x_3^{\rho+\beta r}\right)^{\frac{1}{\beta}}}\cdot\left[\frac{(x_1+x_2)^{\rho+\beta r}(x_1^\rho+x_2^\rho)}{x_1^{\rho+\beta r}+x_2^{\rho+\beta r}}+x_3^\rho\right]^{\frac{1}{\beta}} \tag{66}$$

$$f(x_1,x_2,x_3) = f(x_1,x_2+x_3)\cdot g^{-1}\left(s_1 g\left(f(1)\right)+s_2 g\left(x_2,x_3\right)\right)$$

$$= \frac{(x_1+x_2+x_3)^r\left(x_1^\rho+(x_2+x_3)^\rho\right)^{\frac{1}{\beta}}}{\left(x_1^{\rho+\beta r}+(x_2+x_3)^{\rho+\beta r}\right)^{\frac{1}{\beta}}}\cdot\left[\frac{\frac{(x_2+x_3)^{\rho+\beta r}(x_2^\rho+x_3^\rho)}{x_2^{\rho+\beta r}+x_3^{\rho+\beta r}}+x_1^\rho}{(x_2+x_3)^\rho+x_1^\rho}\right]^{\frac{1}{\beta}}$$

$$= \frac{(x_1+x_2+x_3)^r}{\left(x_1^{\rho+\beta r}+(x_2+x_3)^{\rho+\beta r}\right)^{\frac{1}{\beta}}}\cdot\left[\frac{(x_2+x_3)^{\rho+\beta r}(x_2^\rho+x_3^\rho)}{x_2^{\rho+\beta r}+x_3^{\rho+\beta r}}+x_1^\rho\right]^{\frac{1}{\beta}} \tag{67}$$

Then, using (49) inductively, we obtain

$$\begin{aligned}
&f(x_1,\ldots,x_k,x_{k+1})\\
&= f(\sum_{i=1}^k x_i, x_{k+1})\cdot g^{-1}\left(s_1 g\left(f(x_1,\ldots,x_k)\right)+s_2 g\left(1\right)\right)\\
&= \frac{\left(u_k^{1-\beta r}+x_{k+1}^{1-\beta r}\right)^{\frac{1}{\beta}}}{(u_k+x_{k+1})^{\frac{1}{\beta}-r}}\cdot\left[\frac{u_k^\rho\frac{\sum_{i=1}^k x_i^{1-\beta r}}{u_k^{1-\beta r}}+x_{k+1}^\rho}{u_k^\rho+x_{k+1}^\rho}\right]^{\frac{1}{\beta}}\\
&= \frac{\left(u_k^{1-\beta r}+x_{k+1}^{1-\beta r}\right)^{\frac{1}{\beta}}}{(u_k+x_{k+1})^{\frac{1}{\beta}-r}}\cdot\left[\frac{\sum_{i=1}^k x_i^{1-\beta r}+x_{k+1}^{1-\beta r}}{u_k^{1-\beta r}+x_{k+1}^{1-\beta r}}\right]^{\frac{1}{\beta}}\\
&= \frac{\left(\sum_{i=1}^{k+1} x_i^{1-\beta r}\right)^{\frac{1}{\beta}}}{\left(\sum_{i=1}^{k+1} x_i\right)^{\frac{1}{\beta}-r}},
\end{aligned} \tag{69}$$

which is exactly equation (14) in Theorem 4.

We still need to prove that the fairness measure in (69) satisfies Axioms 1–5. It is easy to see that Axioms 1–3 are satisfied by the fairness measure in (69). To verify Axiom 4, we consider partitioning a resource allocation vector $\mathbf{x}$ of n users into two segments: $\mathbf{x}^1=[x_1,\ldots,x_k]$ and $\mathbf{x}^1=[x_{k+1},\ldots,x_n]$ for arbitrary $0<k<n$. Let $u_{n-k}=u_n-u_k$

From (69), we conclude

$$\begin{aligned}
&f(x_1,\ldots,x_n)\\
&= \frac{\left(\sum_{i=1}^n x_i^{1-\beta r}\right)^{\frac{1}{\beta}}}{\left(\sum_{i=1}^n x_i\right)^{\frac{1}{\beta}-r}}\\
&= \frac{\left(u_k^{1-\beta r}+u_{n-k}^{1-\beta r}\right)^{\frac{1}{\beta}}}{\left(\sum_{i=1}^n x_i\right)^{\frac{1}{\beta}-r}}\cdot\left[\frac{\sum_{i=1}^n x_i^{1-\beta r}}{u_k^\rho+u_{n-k}^\rho}\right]^{\frac{1}{\beta}}\\
&= f(u_k,u_{n-k})\cdot\left[\frac{u_k^\rho\frac{\sum_{i=1}^k x_i^{1-\beta r}}{u_k^{1-\beta r}}+u_{n-k}^\rho\frac{\sum_{i=k+1}^n x_i^{1-\beta r}}{u_{n-k}^{1-\beta r}}}{u_k^\rho+u_{n-k}^\rho}\right]^{\frac{1}{\beta}}\\
&= f(\sum_{i=1}^k x_i, \sum_{i=k+1}^n x_i)\cdot g^{-1}\left(\sum_{i=1}^2 s_i\cdot g\left(f(\mathbf{x}^i)\right)\right),
\end{aligned} \tag{70}$$

where weights $s_1=\frac{u_k^\rho}{u_k^\rho+u_{n-k}^\rho}$ and $s_2=\frac{u_{n-k}^\rho}{u_k^\rho+u_{n-k}^\rho}$ are proportional to some power of the sum resource in each segment. This proves that the fairness measure in (69) is irrelevant to partition.

To verify Axiom 5, we consider an allocation vector $\mathbf{x}=[\theta,1-\theta]$ and compute its fairness measure as follows

$$f(\theta,1-\theta)=\left[\theta^{1-\beta r}+(1-\theta)^{1-\beta r}\right]^{\frac{1}{\beta}}. \tag{71}$$

It is easy to verify that when $1-\beta r>0$, the fairness measure $f(\theta,1-\theta)$ is increasing for $\theta\in[0,\frac{1}{2}]$ and decreasing for $\theta\in[\frac{1}{2},1]$. Axiom 5 is satisfied given $1-\beta r>0$.

Putting all conditions in the proof together, we conclude that, when $\rho=1-\beta r>0$, the fairness measure given by (69) is positive and satisfies Axioms 1–5. Similarly, when $\rho=1-\beta r<0$, the fairness measure given by (14) is negative. The proof for this case is the same and not repeated here.

I. Proof of Corollary 6 & 7

When $f < 0$ is negative, it is easy to show that $f(\mathbf{x}) \to -\infty$ if $x_i \to 0$. When $f > 0$, suppose that k users are inactive. From equation (42) and Corollaries 1 and 3, we have

$$f(\mathbf{x}) \le f(\mathbf{1}_{n-k}) = n - k. \tag{72}$$

which gives $k \le n - f(\mathbf{x})$. Further, since the number of active users $n-k$ is upper bounded by $f(\mathbf{x})$, the maximum resource is lower bounded by $\sum_i x_i / f(\mathbf{x})$.

J. Proof of Corollary 8

Let $k(\mathbf{x}) = \sum_{i=1}^n \left(\frac{x_i}{\sum_j x_j}\right)^{1-\beta}$ be an auxiliary function, such that

$$f(\mathbf{x}) = \text{sign}(1-\beta) \cdot k^{\frac{1}{\beta}}(\mathbf{x}). \tag{73}$$

Since $f(\mathbf{x})$ is differentiable, we have

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = \frac{1}{\beta} k^{\frac{1}{\beta}-1}(\mathbf{x}) \cdot \frac{|1-\beta|}{(\sum_j x_j)^{1-\beta}} \left[x_i^{-\beta} - \frac{\sum_j x_j^{1-\beta}}{\sum_j x_j} \right]$$

Because $k(\mathbf{x}) > 0$ is positive, $\frac{\partial f(\mathbf{x})}{\partial x_i}$ has a single root at

$$x_i = \bar{x} = \left(\frac{\sum_j x_j}{\sum_j x_j^{1-\beta}} \right)^{\frac{1}{\beta}}. \tag{74}$$

It is straightforward to show that for any $\beta \neq 1$, we have

$$\frac{\partial f(\mathbf{x})}{\partial x_i} > 0, \text{ if } x_i > \bar{x} \text{ and } \frac{\partial f(\mathbf{x})}{\partial x_i} < 0, \text{ if } x_i < \bar{x}$$

Therefore, when $x_j \ \forall j \neq i$ are fixed, $f(\mathbf{x})$ is maximized by $x_i = \bar{x}$.

K. Proof of Corollary 9

To derive an lower bound on $f(\mathbf{x})$ under the box constraints $x_{min} \le x_i \le x_{max} \ \forall i$, we first argue that $f(\mathbf{x})$ is minimized only if users are assigned resource x_{min} or x_{max}. Using the box constraints and Corollary 6, we have

$$\begin{aligned} \bar{x} &= \left(\frac{\sum_j x_j}{\sum_j x_j^{1-\beta}} \right)^{\frac{1}{\beta}} \\ &= \left(\sum_i \frac{x_i}{\sum_j x_j} \cdot x_i^{-\beta} \right)^{-\frac{1}{\beta}} \\ &\ge \left(\sum_i \frac{x_i}{\sum_j x_j} \cdot x_{min}^{-\beta} \right)^{-\frac{1}{\beta}} \\ &= x_{min}. \end{aligned} \tag{75}$$

Similarly, we can show

$$\bar{x} \le x_{max}. \tag{76}$$

According to Axiom 4, $f(\mathbf{x})$ is increasing on $x_i \in [x_{min}, \bar{x}]$ and decreasing on $x_i \in [\bar{x}, x_{max}]$. Hence, $f(\mathbf{x})$ is minimized only if all x_i take the boundary values in the box constraints, i.e.,

$$x_i = x_{min} \text{ or } x_i = x_{max}. \tag{77}$$

Let $\Gamma = \frac{x_{max}}{x_{min}}$ and μ be fraction of users who receive x_{max}. By relaxing the constraint $\mu \in \left\{\frac{i}{n}, \forall i\right\}$ to $\mu \in [0,1]$, we derive an lower bound on $f(\mathbf{x})$ as follows

$$\begin{aligned} &\min_{x_i \in [x_{min}, x_{max}], \forall i} f(\mathbf{x}) \\ &= \min_{\mu \in \left\{\frac{i}{n}, \forall i\right\}} \text{sign}(1-\beta) \cdot n \left[\frac{\mu \Gamma^{1-\beta} + (1-\mu)}{(\mu\Gamma + 1 - \mu)^{1-\beta}} \right]^{\frac{1}{\beta}} \\ &\ge \min_{\mu \in [0,1]} \text{sign}(1-\beta) \cdot n \left[\frac{\mu \Gamma^{1-\beta} + (1-\mu)}{(\mu\Gamma + 1 - \mu)^{1-\beta}} \right]^{\frac{1}{\beta}}. \end{aligned} \tag{78}$$

To find the minimizer in the last optimization problem above, we first recognize that at the two boundary points $\mu = 0$ and $\mu = 1$ (i.e. all users receive the same amount of resource), $f(\mathbf{x}) = n$ achieves its maximum value. Therefore, the minimum value is achieved by some $\mu \in (0,1)$. If μ^* is the minimizer of (78), it is necessary that the first order derivative of the right hand side of (78) is zero, i.e.,

$$\frac{\partial \left[\frac{\mu \Gamma^{1-\beta} + (1-\mu)}{(\mu\Gamma + 1 - \mu)^{1-\beta}} \right]}{\partial \mu} = 0. \tag{79}$$

Soling the above equation, we obtain

$$(\Gamma - 1)(1-\beta) \left[\left(\Gamma^{1-\beta} - 1\right) \mu + 1 \right] = \left(\Gamma^{1-\beta} - 1\right) \left[(\Gamma - 1)\mu + 1 \right].$$

Because this equation is a linear in μ, its root μ^* is the unique minimizer of (78):

$$\mu^* = \frac{\Gamma - \Gamma^{1-\beta} - \beta(\Gamma - 1)}{\beta(\Gamma - 1)(\Gamma^{1-\beta} - 1)}. \tag{80}$$

The lower bound in Corollary 7 follows by plugging μ^* into (78).

L. Proof of Theorem 5

We first prove the monotonicity of $f_\beta(\mathbf{x})$ for $\beta \in (-\infty, 0)$. Consider two different values $0 > \beta_1 \ge \beta_2$. We define the a function $\phi(y) = y^{\frac{\beta_2}{\beta_1}}$ for $y \in \mathbb{R}_+$. Since $\beta_2/\beta_1 \ge 1$, the function $\phi(y)$ is convex in y. Therefore, we have

$$\begin{aligned} f_{\beta_2}(\mathbf{x}) &= \left[\sum_{i=1}^n \left(\frac{x_i}{\sum_j x_j} \right)^{1-\beta_2} \right]^{\frac{1}{\beta_2}} \\ &= \left[\sum_{i=1}^n \frac{x_i}{\sum_j x_j} \cdot \phi\left(\left(\frac{x_i}{\sum_j x_j} \right)^{-\beta_1} \right) \right]^{\frac{1}{\beta_2}} \\ &\le \left[\phi\left(\sum_{i=1^n} \frac{x_i}{\sum_j x_j} \left(\frac{x_i}{\sum_j x_j} \right)^{-\beta_1} \right) \right]^{\frac{1}{\beta_2}} \\ &= \left[\phi\left(\sum_{i=1}^n \left(\frac{x_i}{\sum_j x_j} \right)^{1-\beta_1} \right) \right]^{\frac{1}{\beta_2}} \\ &= \left[\sum_{i=1}^n \left(\frac{x_i}{\sum_j x_j} \right)^{1-\beta_1} \right]^{\frac{1}{\beta_1}} \\ &= f_{\beta_2}(\mathbf{x}), \end{aligned} \tag{81}$$

where the third step follows from Jensen's inequality and $\beta_2 < 0$. This shows that $f_\beta(\mathbf{x})$ is increasing on $(-\infty, 0)$.

For $\beta \in (0,1)$, we consider $1 > \beta_1 \geq \beta_2 > 0$. The function $\phi(y) = y^{\frac{\beta_2}{\beta_1}}$ becomes concave. We have

$$\begin{aligned} f_{\beta_2}(\mathbf{x}) &= \left[\sum_{i=1}^{n}\left(\frac{x_i}{\sum_j x_j}\right)^{1-\beta_2}\right]^{\frac{1}{\beta_2}} \\ &= \left[\sum_{i=1}^{n}\frac{x_i}{\sum_j x_j}\cdot\phi\left(\left(\frac{x_i}{\sum_j x_j}\right)^{-\beta_1}\right)\right]^{\frac{1}{\beta_2}} \\ &\leq \left[\phi\left(\sum_{i=1^n}\frac{x_i}{\sum_j x_j}\left(\frac{x_i}{\sum_j x_j}\right)^{-\beta_1}\right)\right]^{\frac{1}{\beta_2}} \\ &= f_{\beta_2}(\mathbf{x}). \end{aligned} \tag{82}$$

where the third step follows from Jensen's inequality and $\beta_2 > 0$. Therefore, $f_\beta(\mathbf{x})$ is increasing on $(0,1)$.

For $\beta \in (1,\infty)$, we consider $\beta_1 \geq \beta_2 > 1$. The function $\phi(y) = y^{\frac{\beta_2}{\beta_1}}$ is concave. We have

$$\begin{aligned} f_{\beta_2}(\mathbf{x}) &- \left[\sum_{i=1}^{n}\left(\frac{x_i}{\sum_j x_j}\right)^{1-\beta_2}\right]^{\frac{1}{\beta_2}} \\ &= -\left[\sum_{i=1}^{n}\frac{x_i}{\sum_j x_j}\cdot\phi\left(\left(\frac{x_i}{\sum_j x_j}\right)^{-\beta_1}\right)\right]^{\frac{1}{\beta_2}} \\ &\geq -\left[\phi\left(\sum_{i=1^n}\frac{x_i}{\sum_j x_j}\left(\frac{x_i}{\sum_j x_j}\right)^{-\beta_1}\right)\right]^{\frac{1}{\beta_2}} \\ &= f_{\beta_2}(\mathbf{x}). \end{aligned} \tag{83}$$

where the third step follows from Jensen's inequality and $\beta_2 > 0$. Therefore, $f_\beta(\mathbf{x})$ is decreasing on $(1,\infty)$. This completes the proof of Theorem 5.

M. Proof of Theorem 6

We first assume $\beta > 1$ (which implies $f_\beta(\cdot) < 0$) and show that the condition $\lambda \leq \left|\frac{\beta}{1-\beta}\right|$ is necessary and sufficient for preserving Pareto optimality. The case where $\beta < 1$ can be shown using a completely analogous proof.

To show that the condition $\lambda \leq \left|\frac{\beta}{1-\beta}\right|$ is sufficient, we consider an allocation $\mathbf{x}$ and a vector $\boldsymbol{\gamma}$ such that $\gamma_i \geq 0$ for all i and $\sum_i \gamma_i = \sum_i x_i$. Clearly, $\mathbf{x}' = \mathbf{x} + \delta\boldsymbol{\gamma}$ Pareto dominates $\mathbf{x}$ for $\delta > 0$. We now consider the difference between the function (27) evaluated for these two allocations. First assume $\beta > 1$, which implies $f_\beta(\cdot) < 0$, and

$$\begin{aligned} &\Phi_\lambda(\mathbf{x}') - \Phi_\lambda(\mathbf{x}) \\ &\quad= \lambda\left(\ell\left(f_\beta\left(\mathbf{x}'\right)\right) - \ell\left(f_\beta\left(\mathbf{x}\right)\right)\right) + \ell\left(\sum_i x_i'\right) - \ell\left(\sum_i x_i\right) \\ &\quad= -\lambda\left(\log\left|f_\beta\left(\mathbf{x}'\right)\right| - \log\left|f_\beta\left(\mathbf{x}\right)\right|\right) + \log\left((1+\delta)\sum_i x_i\right) \\ &\qquad\quad - \log\left(\sum_i x_i\right) \\ &\quad= -\lambda\left(\log\left|f_\beta\left(\mathbf{x}'\right)\right| - \log\left|f_\beta\left(\mathbf{x}\right)\right|\right) + \log\left(1+\delta\right). \end{aligned} \tag{84}$$

If $\mathbf{x}'$ is also more fair than $\mathbf{x}$, then showing

$$-\lambda\left(\log\left|f_\beta\left(\mathbf{x}'\right)\right| - \log\left|f_\beta\left(\mathbf{x}'\right)\right|\right) > 0 \tag{85}$$

is trivial, and the difference between the objective evaluated at the two allocations is strictly positive. Therefore, we consider the case where $\mathbf{x}'$ is less fair.

Continuing from (84) and applying the definition in (18) yields

$$\begin{aligned} &\Phi_\lambda(\mathbf{x}') - \Phi_\lambda(\mathbf{x}) \\ &\quad= -\lambda\log\left(\left[\sum_{i=1}^{n}\left(\frac{x_i'}{\sum_j x_j'}\right)^{1-\beta}\right]^{\frac{1}{\beta}}\right) + \log\left(1+\delta\right) \\ &\qquad + \lambda\log\left(\left[\sum_{i=1}^{n}\left(\frac{x_i}{\sum_j x_j}\right)^{1-\beta}\right]^{\frac{1}{\beta}}\right) \\ &\quad= -\frac{\lambda}{\beta}\log\left(\frac{\sum_{i=1}^{n}\left(x_i'\right)^{1-\beta}}{\sum_{i=1}^{n}\left(x_i\right)^{1-\beta}}\right) + \left(1-\lambda\frac{\beta-1}{\beta}\right)\log\left(1+\delta\right). \end{aligned} \tag{86}$$

Because $x_i' \geq x_i$ for all i, we know that for $\beta > 1$, $(x_i')^{1-\beta} \leq (x_i)^{1-\beta}$, which implies

$$-\frac{\lambda}{\beta}\log\left(\frac{\sum_{i=1}^{n}\left(x_i'\right)^{1-\beta}}{\sum_{i=1}^{n}\left(x_i\right)^{1-\beta}}\right) > 0. \tag{87}$$

Consequently, for the entire difference to be positive, it is sufficient that

$$1 - \lambda\frac{\beta-1}{\beta} \geq 0, \tag{88}$$

or, equivalently,

$$\lambda \leq \frac{\beta}{\beta-1}. \tag{89}$$

Next, we prove that the condition $\lambda \leq \left|\frac{\beta}{1-\beta}\right|$ is necessary. Suppose $\beta > 1$ and $\lambda > \left|\frac{\beta}{1-\beta}\right|$. We show that there exists two vectors $\mathbf{x}$ and $\mathbf{x}'$, such that $\Phi_\lambda(\mathbf{x}') - \Phi_\lambda(\mathbf{x}) < 0$, while $\mathbf{x}'$ Pareto dominates $\mathbf{x}$.

Consider an allocation $\mathbf{x}$ of length $n+1$, such that $x_i = 1$ for $i = 1,\ldots,n$ and $x_{n+1} = n$. Clearly, $\mathbf{x}$ is Pareto dominated by another vector $\mathbf{x}'$, where $x_i' = x_i$ for $i = 1,\ldots,n$ and

$x'_{n+1} = x_i + \delta\left(\sum_i x_i\right)$, for some positive $\delta > 0$. From the last step of (86), we have

$$
\begin{aligned}
&\Phi_\lambda(\mathbf{x}') - \Phi_\lambda(\mathbf{x}) \\
&\quad = -\frac{\lambda}{\beta}\log\left(\frac{\sum_{i=1}^{n+1}(x'_i)^{1-\beta}}{\sum_{i=1}^{n+1}(x_i)^{1-\beta}}\right) + \left(1-\lambda\frac{\beta-1}{\beta}\right)\log(1+\delta) \\
&\quad = -\frac{\lambda}{\beta}\log\left(\frac{n+(n+2n\delta)^{1-\beta}}{n+n^{1-\beta}}\right) + \left(1-\lambda\frac{\beta-1}{\beta}\right)\log(1+\delta) \\
&\quad \le -\frac{\lambda}{\beta}\log\left(\frac{n}{n+n^{1-\beta}}\right) + \left(1-\lambda\frac{\beta-1}{\beta}\right)\log(1+\delta) \\
&\quad = -\frac{\lambda}{\beta}\log\left(1+n^{-\beta}\right) + \left(1-\lambda\frac{\beta-1}{\beta}\right)\log(1+\delta)
\end{aligned}
$$

It is straight forward to verify that $\Phi_\lambda(\mathbf{x}') - \Phi_\lambda(\mathbf{x}) < 0$, if we set

$$\delta = \frac{1}{2}\left[\left(1+n^{-\beta}\right)^{\frac{\lambda}{\lambda(\beta-1)-\beta}}\right] > 0. \tag{90}$$

As a result, the condition (29) of the theorem is sufficient and necessary for ensuring Pareto optimality of the solution.

N. Proof of Theorem 7

From the definition of α-fair utility, we compute the numerator and denominator:

$$
\begin{aligned}
\left\langle \nabla_{\mathbf{x}} U_{\alpha=\beta}(\mathbf{x}), \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}\right\rangle &= \sum_i x_i^{-\beta}\frac{\frac{1}{N} - \frac{x_i}{\sum_i x_i}}{\sqrt{\frac{\|\mathbf{x}\|^2}{(\sum_i x_i)^2} - \frac{1}{N}}} \\
&= \frac{1}{\sqrt{\frac{\|\mathbf{x}\|^2 N}{(\sum_i x_i)^2} - 1}} \cdot \frac{1}{\sqrt{N}} \\
&\quad \cdot \sum_i x_i^{-\beta}\left(1 - \frac{x_i}{\sum_j x_j}N\right),
\end{aligned} \tag{91}
$$

and

$$\left\langle \nabla_{\mathbf{x}} U_{\alpha=\beta}(\mathbf{x}), \frac{\mathbf{1}}{\|\mathbf{1}\|}\right\rangle = \sum_i x_i^{-\beta}\frac{1}{\sqrt{N}} \tag{92}$$

$$= \frac{1}{\sqrt{N}}\sum_i x_i^{-\beta}. \tag{93}$$

Notice that both values are positive. The ratio between these then is

$$\frac{\left\langle \nabla_{\mathbf{x}} U_{\alpha=\beta}(\mathbf{x}), \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}\right\rangle}{\left\langle \nabla_{\mathbf{x}} U_{\alpha=\beta}(\mathbf{x}), \frac{\mathbf{1}}{\|\mathbf{1}\|}\right\rangle} = \frac{1}{\sqrt{\frac{\|\mathbf{x}\|^2 N}{(\sum_i x_i)^2} - 1}}\left(1 - \frac{\sum_i \frac{x_i}{\sum_j x_j} x_i^{-\beta}}{\sum_i \frac{1}{N} x_i^{-\beta}}\right). \tag{94}$$

It is easily shown that the factor out fron is strictly positive. The only component that varies with β is the ratio between two weighted averages of the same vector with different weights:

$$\frac{\sum_i \frac{x_i}{\sum_j x_j} x_i^{-\beta}}{\sum_i \frac{1}{N} x_i^{-\beta}}. \tag{95}$$

That average in the numerator places more weight ($\frac{x_i}{\sum_j x_j} > \frac{1}{N}$) on elements that decrease more rapidly (or increase more slowly for the case $x_i < 1$) with β, implies that the overall numerator decreases more rapidly (or increases more slowly) than the denominator. Therefore, (95) is monotonically non-increasing, and Theorem 7 is true.

O. Proof of Theorem 8

To prove Theorem 8, we need to show that if $F(\mathbf{x})$ satisfies Axioms 1′-4′, its normalization

$$f(\mathbf{x}) = F(\mathbf{x}) \cdot \left(\sum_i x_i\right)^{-\frac{1}{\lambda}} \tag{96}$$

is a fairness measure satisfying Axioms 1-5.

The continuity of $f(\mathbf{x})$ follows directly from that of $F(\mathbf{x})$ in Axioms 1′. Let $z > 0$ be a positive real number and $\mathbf{y}$ be a vector of arbitrary length. To prove homogeneity, we make use of Axioms 3′:

$$
\begin{aligned}
&F(z \cdot [\mathbf{y}, \mathbf{y}]) \\
&\quad = F(z,z) \cdot g^{-1}\left(s_1 \cdot g(F(\mathbf{y})) + s_2 \cdot g(F(\mathbf{y}))\right) \\
&\quad = F(z,z) \cdot F(\mathbf{y}) \\
&\quad = F(1,1) \cdot g^{-1}\left(s_1 \cdot g(F(z)) + s_2 \cdot g(F(z))\right) \cdot F(\mathbf{y}) \\
&\quad = F(1,1) \cdot F(\mathbf{y}) \cdot F(z)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&F(z\mathbf{y}, z\mathbf{y}) \\
&\quad = F(1,1) \cdot g^{-1}\left(s_1 \cdot g(F(z\mathbf{y})) + s_2 \cdot g(F(z\mathbf{y}))\right) \\
&\quad = F(1,1) \cdot F(z\mathbf{y})
\end{aligned} \tag{97}
$$

Comparing the above two equations, we have

$$F(z\mathbf{y}) = F(z) \cdot F(\mathbf{y}). \tag{98}$$

When $\mathbf{y}$ is a scalar, using the result in [18], equation (98) implies that $\log F(z) = \frac{1}{\lambda}\log(z)$ must be a logarithmic function with an exponent $\frac{1}{\lambda}$. We have

$$F(z\mathbf{y}) = z^{\frac{1}{\lambda}} F(\mathbf{y}), \tag{99}$$

which is a homogenous function of order $\frac{1}{\lambda}$. Therefore, its normalization $f(\mathbf{x})$ in (96) is a homogenous function of order zero and satisfies Axiom 2.

Using the homogeneity property and Axioms 2′, we obtain

$$
\begin{aligned}
\lim_{n\to\infty}\frac{f(\mathbf{1}_{n+1})}{f(\mathbf{1}_n)} &= \lim_{n\to\infty}\frac{F(\mathbf{1}_{n+1})}{F(\mathbf{1}_n)} \cdot \left(1+\frac{1}{n}\right)^{-\frac{1}{\lambda}} \\
&= \lim_{n\to\infty}\frac{F(\mathbf{1}_{n+1})}{F(\mathbf{1}_n)} \\
&= 1.
\end{aligned} \tag{100}
$$

This is exactly Axiom 3. From Axiom 4′m, Axiom 5 is straight since

$$f(\theta, 1-\theta) = F(\theta, 1-\theta) \cdot (\theta + 1 - \theta)^{-\frac{1}{\lambda}} = F(\theta, 1-\theta).$$

Therefore, monotonicity holds for $f(\theta, 1-\theta)$ for $\theta \in [0, \frac{1}{2}]$ and $\theta \in [0, \frac{1}{2}]$, respectively. To prove Axiom 4, we choose $x_1 = w(\mathbf{y}^1)$ and $x_2 = w(\mathbf{y}^2)$ in Axiom 3$'$, which results in

$$\begin{aligned}
& f(\mathbf{y^1}, \mathbf{y^2}) \\
& = F(\mathbf{y^1}, \mathbf{y^2}) \cdot \left(w(\mathbf{y}^1) + w(\mathbf{y}^2)\right)^{-\frac{1}{\lambda}} \\
& = F\left(x_1, x_2\right) \cdot g^{-1}\left(\sum_{i=1}^{2} s_i \cdot g\left(F(\mathbf{y}^i / x_i)\right)\right) \cdot \left(x_1 + x_2\right)^{-\frac{1}{\lambda}} \\
& = F\left(x_1, x_2\right) \cdot \left(x_1 + x_2\right)^{-\frac{1}{\lambda}} \cdot g^{-1}\left(\sum_{i=1}^{2} s_i \cdot g\left(f(\mathbf{y}^i / x_i)\right)\right) \\
& = f\left(x_1, x_2\right) \cdot g^{-1}\left(\sum_{i=1}^{2} s_i \cdot g\left(f(\mathbf{y}^i)\right)\right) \\
& = f\left(w(\mathbf{y}^1, w(\mathbf{y}^2)\right) \cdot g^{-1}\left(\sum_{i=1}^{2} s_i \cdot g\left(f(\mathbf{y}^i)\right)\right)
\end{aligned} \tag{101}$$

where the second last step uses the fact that f is a homogenous function of order zero. This proves Axiom 4.

If $F(\mathbf{x})$ satisfies Axioms 1$'$-4$'$, we have shown that its normalization $f(x)$ is a fairness measure satisfying Axioms 1-5. Therefore, $F(x)$ is homogenous function of order $\frac{1}{\lambda}$ and is given by

$$F(\mathbf{x}) = f(\mathbf{x}) \cdot \left(\sum_{i} x_i\right)^{\frac{1}{\lambda}}. \tag{102}$$

Existence and unique of $F(\mathbf{x})$ is straightforward from that of $f(\mathbf{x})$ in Theorems 1 and 2.