Align-and-Forward Relaying for Two-hop Erasure Broadcast Channels

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***Abstract*—We consider the problem of broadcast over wireless erasure networks. To understand the challenges and opportunities of these setups, we study a two-hop erasure broadcast channel consisting of a single source, two relays, and two destinations desiring independent messages. In our network, no transmitter has channel state knowledge of erasures on outgoing links (i.e., no CSIT): The source has no knowledge of any channel state, each relay only has knowledge of the channel states of its incoming link, and destinations are provided with full channel knowledge.**

We propose a scheme, referred to as *Align-and-Forward*, that exploits the (unknown) common subspace of received signals at the relays, which results from the source-to-relay broadcast, in order to minimize the dimension of the interference subspace at each destination. We show that Align-and-Forward outperforms available alternative schemes in terms of sum-rate. We also present new outer-bounds and demonstrate the optimality of Align-and-Forward in certain regimes.

I. Introduction

Capacity analysis of erasure networks can provide valuable insight into protocol design for packetized wireless networks and thus has been the subject of much research. For instance, in [1] it was shown that the capacity of unicast or multicast in wireless erasure networks is achievable through random linear network coding at all nodes, implying that it is optimal for all intermediate relays to act as "dumb" mixers to enable all destinations to decode the packets. However, as one goes beyond multicast to multiple unicast or even broadcast scenarios, the capacity of packetized wireless networks and their optimal communication protocols remain, in general, an open problem.

In order to gain broader insight into the challenges posed by general wireless erasure networks, we consider broadcast in a two-hop erasure network, containing a single source, two relays, and two destinations (shown in Figure 1). We assume all transmitting nodes (source and relays) are provided with no knowledge of erasures affecting their own transmissions; thus, our model is one of no channel state information at transmitters (i.e., no CSIT) and no feedback. Relays have knowledge only of erasures on their incoming link, whereas destinations have full knowledge of all channel states. This problem remains one of the simplest unsolved cases in wireless erasure networks (for one-hop broadcast, the full capacity region is achievable through time-division and random linear codes [2]) and captures communication challenges posed by: 1) multi-hop communication paths, 2) multiple simultaneous communication sessions, and 3) distributed relaying.

For the two-hop erasure broadcast channel described above, we propose a transmission scheme which relies on a novel relaying strategy referred to as Align-and-Forward. In all network regimes, Align-and-Forward either matches or outperforms the best known scheme proposed in [2]. Align-and-Forward relies on a form of intersession coding that exploits the (unknown) overlap in bits transmitted by the source and received by both relays in the first hop of the network. Relays use these overlapping bits to form and broadcast bits of common interest to both destinations. It is important to note that relays are able to exploit the overlap despite not knowing which bits are overlapping at both relays. Simply stated, Align-and-Forward implements a probabilistic form of interference alignment at the relays without explicit knowledge of the targeted alignment subspace. Although it is known that intersession coding can provide capacity gain in erasure broadcast networks, all prior works have relied on *feedback* to achieve the gains (e.g., [3]–[5] and references therein). Quite interestingly, Align-and-Forward demonstrates a gain from intersession coding without requiring feedback.

We also develop new upper bounds on the sum-capacity of the two-hop erasure broadcast channel. Our bounds leverage a recent result of [6] which captures the entropy "leakage" of information from a transmitter to an unintended receiver in erasure networks with no CSIT. We use the lemma, along with a genie-aided construction, to develop complementary inequalities, that when summed, yield two novel upper bounds.

II. Network Model

We study the wireless erasure network depicted in Figure 1, which consists of a source, two relays, and two destinations. Each node's transmission is broadcast to all connected receiving nodes, while each received sequence is corrupted by random symbol (packet) erasures. At a receiving node, sequences from different transmitters are received orthogonally (i.e, without interference). Channel input symbols are binary and erasures occur independently on each transmit-receive link. As shown in Figure 1, in this paper, we consider a symmetric network, parametrized by the erasure probability tuple, $(\epsilon_1, \epsilon_2, \epsilon_3)$. Without loss of generality, we assume $\epsilon_2 \leq \epsilon_3$.

Formally, at time t, let $X_A[t] \in \{1, 2\}$ denote the channel input at node A and $Y_{BA}[t] \in \{0, 1, 2\}$ denote the channel output on the link from node A to node B. Let $G_{BA}[t] \in \{0, 1\}$ denote a channel state, indicating whether an erasure occurs (i.e., $G_{BA}[t] = 0$) on the link from node A to node B

at time t. Channel input-output relationships are given by

$$Y_{BA}[t] = G_{BA}[t]X_A[t]. \quad (1)$$

We assume codes of length n, and we use the notation X_A^n to refer to the vector of channel inputs $X_A^n = [X_A[1] \ \ldots \ X_A[n]]$. Accordingly, we define Y_{BA}^n and G_{BA}^n. For each pair of nodes A and B, G_{BA}^n is an i.i.d. Bernoulli process with parameter $1-\epsilon_k$, where ϵ_k is specified in Figure 1. Finally, we denote the tuple of all channel state variables as $\overline{G}^n = (G_{\mathsf{R_1S}}^n, G_{\mathsf{R_2S}}^n, G_{\mathsf{D_1R_1}}^n, G_{\mathsf{D_1R_2}}^n, G_{\mathsf{D_2R_1}}^n, G_{\mathsf{D_2R_2}}^n)$.

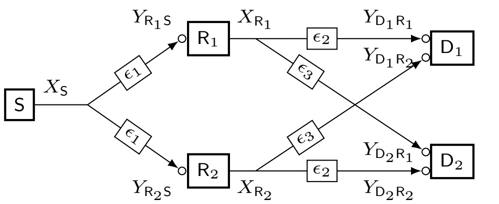

Fig. 1. The symmetric two-hop erasure broadcast channel.

We assume destinations have full channel state information (CSI) for the entire the network ($\overline{G}^n$), and relays have CSI only of their own incoming link, i.e., Relay 1 knows only $G_{\mathsf{R_1S}}^n$ and Relay 2 knows only $G_{\mathsf{R_2S}}^n$. This means that each relay is unaware of the erasures within the other's received signal. The source has no CSI whatsoever.[1]

We are interested in the sum-capacity of the broadcast messaging scenario, where the messages desired by Destination 1 and Destination 2, M_1 and M_2 respectively, are independent.

III. Achievable Scheme

In this section, we present a new scheme for the two-hop erasure broadcast channel. The innovative aspect of our approach is a relaying strategy referred to as Align-and-Forward. As a base line, we use the best known scheme, proposed in [2], which is based on dedicating a fraction of time at each node to each session (i.e., message), and applying random coding within each session. We refer to this scheme as time-division (TD) approach. For our model, the optimized TD approach achieves the sum-rate

$$r_\Sigma^{\mathsf{TD}} = \min\left\{1-\epsilon_1^2, (1-\epsilon_1)+(1-\epsilon_3)\left[2-\frac{1-\epsilon_1}{1-\epsilon_2}\right]_+, 2-2\epsilon_2\right\}.$$

As an example, when $\epsilon_1 = \epsilon_2 = \frac{1}{2}$ and $\epsilon_3 = \frac{4}{5}$, the TD approach dedicates half the time at each node to each message, and achieves a sum rate of $r_\Sigma^{\mathsf{TD}} = \frac{7}{10}$ bits.

Our scheme also utilizes random linear encoding and time-division at the source, but achieves a sum-rate gain by improving the approach taken at the relays: Intersession encoding is used to create transmissions simultaneously useful to both destinations. For these "bits of common interest" from Relay 1, the component that is undesired by Destination 2 (i.e., bits about M_1) is *aligned* with what Destination 2 receives from Relay 2. Similarly, Relay 2 aligns parts of M_2 with what Destination 1 receives from Relay 1. Because each relay is unaware of what the other has received, relays do not know the precise subspace that should be targeted for alignment. Surprisingly, our scheme demonstrates that alignment is possible without relays even knowing which of their transmissions are aligned.

Because the general Align-and-Forward scheme involves many steps, we first present its construction and sum-rate gain with an illustrative example. A comprehensive presentation may be found in [7].

A. Illustrative Example

Let $\epsilon_1 = \epsilon_2 = \frac{1}{2}$ and $\epsilon_3 = \frac{4}{5}$. Recall that for this network, the TD scheme achieves $r_\Sigma^{\mathsf{TD}} = \frac{7}{10}$ bits. We show that our scheme improves upon this, achieving a sum rate of $r_\Sigma^{\mathsf{A\&F}} = \frac{3}{4}$ bits, which is indeed the sum-capacity for this example.

Let messages M_1 and M_2 both consist of $\frac{3n}{8}$ bits, where n is a large integer. For ease of exposition, we ignore terms of order $o(n)$, which vanish in the calculation of rate per channel use. The source applies random linear encoding on each message to create $\frac{n}{2}$ random linear combination (RLC) bits. Denote with $\mathcal{X}_{\mathsf{S}}^i$ the vector of RLC bits created for message M_i. Elements of $\mathcal{X}_{\mathsf{S}}^1$ and $\mathcal{X}_{\mathsf{S}}^2$ are broadcast sequentially by the source, as shown in Figure 2. Each relay receives approximately $\frac{n}{4}$ unerased bits per message, M_i, denoted with the vectors $\mathcal{Y}_{A\mathsf{S}}^i$ for $A \in \{\mathsf{R}_1, \mathsf{R}_2\}$. Because erasures occur independently, approximately $\frac{n}{8}$ RLC bits are received at both relays (overlapping) for each message. However, due to lack of knowledge, each relay is unaware of which RLC bits have been received by the other relay.

Relay 1 re-encodes received RLC bits using the following three-phase transmission scheme (depicted in Figure 3):

Phase 1 ($\frac{n}{4}$ channel uses): Relay 1 applies a random linear code upon $\mathcal{Y}_{\mathsf{R_1S}}^1$, and creates a vector, $\mathcal{Q}_{\mathsf{R_1}}$ depicted as the red striped block, of $\frac{n}{4}$ RLC bits describing M_1, and broadcasts these sequentially.

Phase 2 ($\frac{n}{2}$ channel uses): Relay 1 applies a random linear code upon $\mathcal{Y}_{\mathsf{R_1S}}^2$, and creates a vector, $\mathcal{W}_{\mathsf{R_1}}^1$ depicted as the blue striped block, of $\frac{n}{2}$ RLC bits describing M_2, and broadcasts these sequentially.

Phase 3 ($\frac{n}{4}$ channel uses): Relay 1 applies another random linear code upon $\mathcal{Y}_{\mathsf{R_1S}}^2$, and creates a vector, $\mathcal{W}_{\mathsf{R_1}}^2$, of $\frac{n}{4}$ RLC bits describing M_2. Each of these is XORed with one (*uncoded*) element of $\mathcal{Y}_{\mathsf{R_1S}}^1$ and broadcasted.

As we will see later, Phase 3 increases the efficiency of the scheme, versus TD approach, by creating messages which are interesting to both receivers. However, there are two subtle points in Phase 3. The first one is that not all bits created in this phase are of common interest. Indeed all of these bits created at Relay 1 will be useful to Destination 1 (as will be explained later), however *only part of them* are also useful for Destination 2. This part is shown in Fig. 3 in bold-dotted box. Remember that one part of $\mathcal{Y}_{\mathsf{R_1S}}^1$, colored by shaded red, is also available at Relay 2. Only the bits that are created by XORing this shaded-red part of $\mathcal{Y}_{\mathsf{R_1S}}^1$ and RLC of $\mathcal{W}_{\mathsf{R_1}}^1$ are

[1]The Align-and-Forward scheme only requires nodes have "downstream" CSI: knowledge of erasures occurring along paths connecting the node to the source. One might accomplish this by forwarding CSI from the source-relay links to both destinations, with overhead cost vanishing as n grows large.

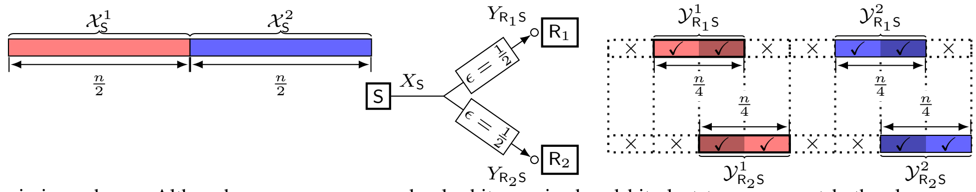

Fig. 2. First-hop transmission scheme: Although erasures occur randomly, bits received and bits lost to erasures at both relays are grouped into solid and dotted blocks respectively. Colors used to denote message content: red for M_1 and blue for M_2. Bits that overlap (received by both relays) are depicted as shaded blocks, but overlap is not known to either relay.

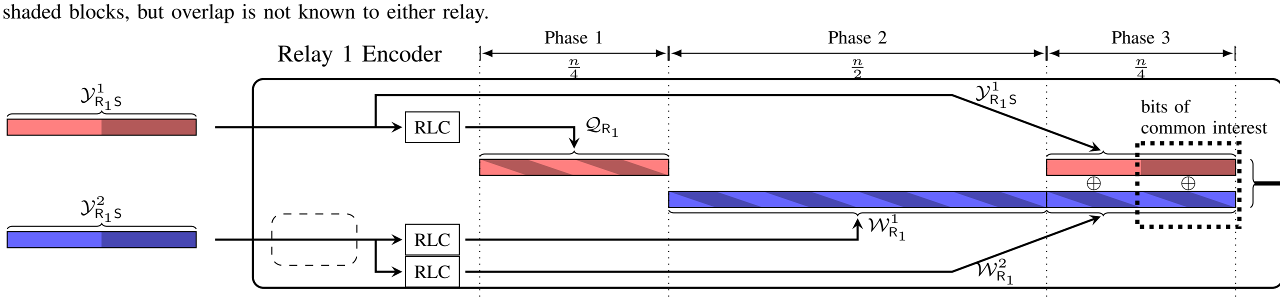

Fig. 3. Relaying scheme employed at Relay 1: Message M_1 content in red, M_2 content in blue. Shaded areas represent RLC bits overlapping at the relays. The lower left dashed box represents a processing block (shown in Figure 5) not needed for the example but necessary for general Align-and-Forward relaying.

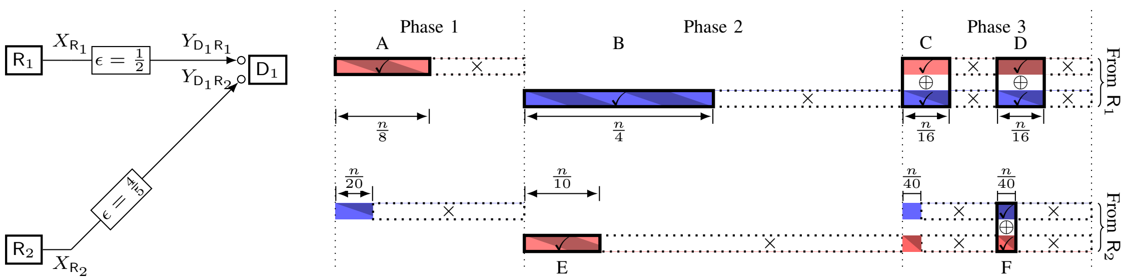

Fig. 4. Second-hop transmission and decoding at Destination 1: Message M_1 content in red and M_2 content in blue. Shaded areas represent RLC bits overlapping at the relays. Bold-outlined blocks with alphabetical labels are used to decode M_1.

of common interest. The reason is that some of these shaded-red bits have been previously communicated by Relay 2 in Phase 2, and overheard by Destination 2. Thus, Destination 2 can use them to clean some of the bits received from Relay 1 in Phase 3, and resolve its own (blue) message.

The second subtle point is that Relay 1 cannot identify the shaded-red part of $\mathcal{Y}_{\mathsf{R}_1\mathsf{S}}^1$, due to its local knowledge of the channel state. Otherwise, in Phase 3, Relay 1 could use just the shaded-red part of $\mathcal{Y}_{\mathsf{R}_1\mathsf{S}}^1$ and create messages that are all of common interest. This point justifies why we use *uncoded* versions of $\mathcal{Y}_{\mathsf{R}_1\mathsf{S}}^1$ in Phase 3: Applying RLC over $\mathcal{Y}_{\mathsf{R}_1\mathsf{S}}^1$ mixes the overlapping (shaded) part with the non-overlapping (unshaded) part. Since the unshaded part of $\mathcal{Y}_{\mathsf{R}_1\mathsf{S}}^1$ has not been overheard by Destination 2, it cannot be canceled.

Relay 2 (not depicted) uses the analogous three-phase scheme, but switches the roles of messages: a random linear code is applied to $\mathcal{Y}_{\mathsf{R}_2\mathsf{S}}^2$ during Phase 1 to create $\mathcal{Q}_{\mathsf{R}_2}$, two other random linear codes are applied to $\mathcal{Y}_{\mathsf{R}_2\mathsf{S}}^1$ during Phases 2 and 3 to create $\mathcal{W}_{\mathsf{R}_2}^1$ and $\mathcal{W}_{\mathsf{R}_2}^2$ respectively, and elements of $\mathcal{W}_{\mathsf{R}_2}^2$ are XORed with uncoded elements of $\mathcal{Y}_{\mathsf{R}_2\mathsf{S}}^2$ during Phase 3.

Focusing on Destination 1, we now describe the procedure to decode M_1 by clarifying the nature of bits received during each phase from each relay, while referring to Figure 4.

Relay 1, Phase 1 (Block A of Fig. 4): Destination 1 receives approximately $\frac{n}{8}$ bits from $\mathcal{Q}_{\mathsf{R}_1}$, which describe M_1.

Relay 1, Phase 2 (Block B of Fig. 4): Destination 1 receives approximately $\frac{n}{4}$ bits from $\mathcal{W}_{\mathsf{R}_1}^1$, which carry information about M_2. With high probability, Destination 1 can recreate the original $\frac{n}{4}$ bits in $\mathcal{Y}_{\mathsf{R}_1\mathsf{S}}^2$, *entirely*. Moreover, Destination 1 can use knowledge of all erasures in the network ($\overline{G}^n$) to identify approximately half of these ($\frac{n}{8}$) as overlapping bits (colored shaded-blue).

Relay 1, Phase 3 (Blocks C & D of Fig. 4): Recall that, from the previous phase, Destination 1 already knows $\mathcal{Y}_{\mathsf{R}_1\mathsf{S}}^2$. It uses this knowledge to remove the $\mathcal{W}_{\mathsf{R}_1}^2$ component from Phase 3 transmissions and receives approximately $\frac{n}{8}$ bits from $\mathcal{Y}_{\mathsf{R}_1\mathsf{S}}^1$, which all carry information about M_1. Therefore, as we mentioned before, all the bits sent by Relay 1 in this phase, and received by Destination 1, are useful to Destination 1.

Relay 2, Phase 1: Destination 1 ignores these packets.

Relay 2, Phase 2 (Block E of Fig. 4): Destination 1 receives approximately $\frac{n}{10}$ bits from $\mathcal{W}_{\mathsf{R}_2}^1$, which describe M_1.

Relay 2, Phase 3 (Block F of Fig. 4): The bits received from Relay 2 during this phase can be split into two parts. The part which is useful at Destination 1 is denoted by F. This part has been formed at Relay 2 by XORing RLC of $\mathcal{Y}_{\mathsf{R}_2\mathsf{S}}^1$ with the uncoded shaded-blue part of $\mathcal{Y}_{\mathsf{R}_2\mathsf{S}}^2$. The reason is as follows. Recall that Block B was enough for Destination 1 to resolve all of $\mathcal{Y}_{\mathsf{R}_1\mathsf{S}}^2$. Due its full access to channel state information, Destination 1 can recognize the shaded-blue part of $\mathcal{Y}^2_{\mathsf{R}_1\mathsf{S}}$ (M_2 bits overlapping at both relays). Therefore, such knowledge is used by Destination 1 to clean interference from Block F. The number that can be recovered is $(1-\epsilon_3)\frac{1}{2} \times \frac{n}{4} = \frac{n}{40}$. The coefficient $\frac{1}{2}$ comes from the fact only half of the messages received in this phase from Relay 2 falls in Block F.

In order to determine the rate achieved for M_1 we now determine the maximum number of linearly independent RLC bits received by Destination 1. First we observe that Destination 1 can decode all $\frac{n}{4}$ elements of $\mathcal{Y}^1_{\mathsf{R}_1\mathsf{S}}$ from what it receives in Phase 1 and Phase 3 from Relay 1. Also it receives $\frac{n}{10}+\frac{n}{40}=\frac{n}{8}$ RLC bits describing M_1 from Relay 2 in Blocks E and F. Therefore, in total, it has $\frac{3n}{8}$ RLC bits which can be shown to be independent and thus are enough to recover M_1. The same argument is valid for Destination 2. Therefore, we achieve the sum-rate of $r_\Sigma = 2 \times \frac{1}{n}\frac{3n}{8} = \frac{3}{4}$.

B. General Scheme

We now describe the general scheme, which requires an additional step at the relays. This step was unnecessary for the example, but it allows us to guarantee a rate either matching or outperforming the TD scheme for all network instances. A more comprehensive presentation may be found in [7].

Source Encoding & Transmission: Each message, M_1 and M_2, consists of $n\frac{r_\Sigma^{\mathsf{A\&F}}}{2}$ bits, where $r_\Sigma^{\mathsf{A\&F}}$ is specified later. For each M_i, the source applies a random linear code and creates $\frac{n}{2}$ RLC bits, and broadcasts these sequentially.

Align-and-Forward Relaying: As in the example, relays use a three-phase transmission with the proportion of channel uses allocated to Phases 1, 2, and 3 given by the parameters

$$\tau_1 \triangleq \min\left\{\frac{1+\epsilon_1}{2}, \frac{(1-\epsilon_1)\epsilon_2}{2(1-\epsilon_2)}\right\},\ \tau_2 \triangleq \left[1-\frac{1-\epsilon_1}{2(1-\epsilon_2)}\right]_+,\ \tau_3 \triangleq \frac{1-\epsilon_1}{2},$$

respectively. By construction, $\lfloor n\tau_1 \rfloor$, $\lfloor n\tau_2 \rfloor$, and $\lfloor n\tau_3 \rfloor$ also define cardinalities of $\mathcal{Q}_{\mathsf{R}_i}$, $\mathcal{W}^1_{\mathsf{R}_i}$, and $\mathcal{W}^2_{\mathsf{R}_i}$, respectively.

Phase 1 is as in the example: Relay i applies a random linear code to $\mathcal{Y}^i_{\mathsf{R}_i\mathsf{S}}$ to create $\mathcal{Q}_{\mathsf{R}_i}$, and broadcasts $\mathcal{Q}_{\mathsf{R}_i}$.

For Phases 2 and 3, we include an additional step to the scheme described in the example. When $\epsilon_2 > \epsilon_1$, the second-hop link is weaker than than the first, and therefore Relay i cannot communicate to Destination i everything it has received. In our scheme, each Relay i therefore reduces the number of bits communicated to Destination i about $\mathcal{Y}^{i'}_{\mathsf{R}_i\mathsf{S}}$ (where $i' = 3-i$). The block diagram for the approach is shown in Figure 5, which is inserted (for the general scheme) into the lower left dashed region of Figure 3. We first define two reduction parameters, ν_u and ν_c, and require

$$n\nu_u + n\nu_c \le \min\left\{\lfloor n\tau_2 \rfloor(1-\epsilon_2), \left|\mathcal{Y}^{i'}_{\mathsf{R}_i\mathsf{S}}\right|\right\}. \tag{2}$$

Two operations are applied at Relay i to $\mathcal{Y}^{i'}_{\mathsf{R}_i\mathsf{S}}$. In the first, $\lfloor n\nu_u \rfloor$ RLC bits are randomly *selected* from $\mathcal{Y}^{i'}_{\mathsf{R}_i\mathsf{S}}$ to create a subvector $\mathcal{V}^u_{\mathsf{R}_i}$. In the second, a random linear code is applied to $\mathcal{Y}^{i'}_{\mathsf{R}_i\mathsf{S}}$ to create a vector of $\lfloor n\nu_c \rfloor$ bits, denoted as $\mathcal{V}^c_{\mathsf{R}_i}$. The concatenated vector of bits $\mathcal{V}_{\mathsf{R}_i} = (\mathcal{V}^u_{\mathsf{R}_i}, \mathcal{V}^c_{\mathsf{R}_i})$ is passed serially into the random encoders as in Figure 3 to create the vectors $\mathcal{W}^1_{\mathsf{R}_i}$ and $\mathcal{W}^2_{\mathsf{R}_i}$, and Phases 2 and 3 proceed as in the example.

For our scheme, ν_u and ν_c are chosen such that they are non-negative, satisfy (2), and maximize the sum-rate. The constraint (2) ensures that Destination i can decode all of $\mathcal{V}_{\mathsf{R}_i}$. Note that the suboptimal choice where $\nu_u = 0$ and $n\nu_c = \min\left\{\lfloor n\tau_2 \rfloor(1-\epsilon_2), \left|\mathcal{Y}^{i'}_{\mathsf{R}_i\mathsf{S}}\right|\right\}$, results in the TD rate, since Phase 3 transmissions may no longer be aligned.

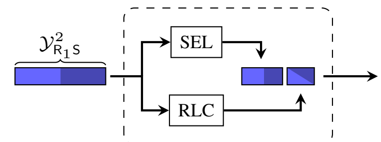

Fig. 5. Reduction processing block used by Relay 1. Notice that processing is applied to bits for M_2.

Decoding & Sum-Rate: The decoding process is as in the example. Due to restricted space, we omit the complete analysis of decoding error. However, note that to decode M_i, Destination i only uses RLC bits that are either a function of only M_i or where Destination i can cancel the $M_{i'}$ component using overheard bits. We count the (approximate) number of such bits received by Destination i that are linearly independent to compute the achieved sum rate. The computed rates are summarized as three separate network regimes in Table I, where reduction paramaters for Regime II are

$$\nu_u^{(II)} = (1-\epsilon_2-\frac{1-\epsilon_1}{2})\min\{1, \frac{\epsilon_3-\epsilon_2}{(1-\epsilon_1)(1-\epsilon_2)(2-\epsilon_3)}\},$$
$$\nu_c^{(II)} = (1-\epsilon_2-\frac{1-\epsilon_1}{2}) - \nu_u^{(II)}.$$

In [7], we present full analyses of error and achievable rates.

IV. Upper Bounds

We now present a novel capacity upper bound for our setup:

Theorem 1 (Upper Bounds)**.** *The sum-capacity, C_Σ, of the 2-hop erasure broadcast channel with $\epsilon_2 \le \epsilon_3$ satisfies*

$$C_\Sigma \le 1-\epsilon_1 + 2\frac{(1-\epsilon_2)(1-\epsilon_3)}{2-\epsilon_2-\epsilon_3}, \tag{3}$$
$$C_\Sigma \le \frac{(\epsilon_3-\epsilon_2)(1-\epsilon_1)+4(1-\epsilon_2)(1-\epsilon_3)}{2-\epsilon_2-\epsilon_3}. \tag{4}$$

Our bound is, in general, not tight with the sum-rate achieved in our scheme. However, in network regimes where Align-and Forward meets neither the cut-set nor the recent broadcast-cut upper bound of [5], our bound is often tighter than the existing bounds (see Figure 6).

To prove Theorem 1, we will require the following lemma, which is related to Lemma 3 in [6], and proven similarly in [7]:

Lemma 2 (Entropy Leakage with no CSIT)**.** *Let Y_A^n and Y_B^n be the channel outputs of a two-user (1-hop) erasure broadcast channel, with independent erasures occurring with probabilities ϵ_a and ϵ_b respectively, with $\epsilon_A \le \epsilon_B$. If U is a random variable such that the Markov relationships*

$$U - X^n \begin{array}{l} \nearrow Y_A^n \\ \searrow Y_B^n \end{array}$$

Regime	Network Conditions	Reduction Parameters	Sum-Rate ($r_\Sigma^{\mathsf{A\&F}}$)
I	$\epsilon_2 \leq \epsilon_1$	$\nu_u = \frac{1-\epsilon_1}{2}$, $\nu_c = 0$	$(1-\epsilon_1) + \min\left\{\epsilon_1(1-\epsilon_1), 2\frac{(1-\epsilon_2)(1-\epsilon_3)}{2-\epsilon_2-\epsilon_3}\right\}$
II	$2\epsilon_2 - 1 \leq \epsilon_1 < \epsilon_2$,	$\nu_u = \nu_u^{(II)}$, $\nu_c = \nu_c^{(II)}$	$(1-\epsilon_1) + \min\left\{\epsilon_1(1-\epsilon_1),\ 2(1-\epsilon_3)\left[1 - \frac{1-\epsilon_1}{2(1-\epsilon_2)} + (1-\epsilon_1)\nu_u^{(II)}\right]\right\}$
III	$\epsilon_1 < 2\epsilon_2 - 1$	$\nu_u = 0$, $\nu_c = 0$	$2 - 2\epsilon_2$

TABLE I

ACHIEVABLE SUM RATES USING ALIGN-AND-FORWARD RELAYING.

hold and $\overline{G}^n$ is knowledge of all erasures in the network, then

$$H(Y_B^n|U,\overline{G}^n) \geq \frac{1-\epsilon_B}{1-\epsilon_A} H(Y_A^n|U,\overline{G}^n). \tag{5}$$

Proof of Theorem 1: Assume communication rates of r_1 and r_2 from the source to Destinations 1 and 2 respectively are achievable. Then, we establish the following two inequalities:

$$\begin{aligned}
nr_1 &\overset{\text{(Fano)}}{\leq} I(M_1; Y^n_{\mathsf{D_1R_1}}, Y^n_{\mathsf{D_1R_2}}|\overline{G}^n) + n\varepsilon_n \\
&= I(M_1; Y^n_{\mathsf{D_1R_1}}|\overline{G}^n) + H(Y^n_{\mathsf{D_1R_2}}|Y^n_{\mathsf{D_1R_1}}, \overline{G}^n) \\
&\quad - H(Y^n_{\mathsf{D_1R_2}}|Y^n_{\mathsf{D_1R_1}}, \overline{G}^n, M_1) + n\varepsilon_n, \qquad (6)
\end{aligned}$$

$$\begin{aligned}
nr_2 &\overset{\text{(Fano)}}{\leq} I(M_2; Y^n_{\mathsf{D_2R_1}}, Y^n_{\mathsf{D_2R_2}}|\overline{G}^n) + n\varepsilon_n \\
&\overset{(a)}{\leq} I(M_2; Y^n_{\mathsf{D_1R_1}}, Y^n_{\mathsf{D_2R_2}}|\overline{G}^n, M_1) + n\varepsilon_n \\
&= I(M_2; Y^n_{\mathsf{D_1R_1}}|\overline{G}^n, M_1) \\
&\quad + H(Y^n_{\mathsf{D_2R_2}}|Y^n_{\mathsf{D_1R_1}}, \overline{G}^n, M_1) + n\varepsilon_n \\
&\overset{(b)}{\leq} I(M_2; Y^n_{\mathsf{D_1R_1}}|\overline{G}^n, M_1) \\
&\quad + \frac{1-\epsilon_2}{1-\epsilon_3} H(Y^n_{\mathsf{D_1R_2}}|Y^n_{\mathsf{D_1R_1}}, \overline{G}^n, M_1) + n\varepsilon_n. \qquad (7)
\end{aligned}$$

Step (a) is obtained by giving a genie signal, $Z^n_{\mathsf{D_2R_1}}, M_1$, to Destination 2 such that the combined tuple $(Y^n_{\mathsf{D_2R_1}}, Z^n_{\mathsf{D_2R_1}})$ and the random vector $Y^n_{\mathsf{D_1R_1}}$, conditioned on $\overline{G}^n$, are identically distributed random variables. In step (b) we applied Lemma 2 to the term $H(Y^n_{\mathsf{D_2R_2}}|Y^n_{\mathsf{D_1R_1}}, \overline{G}^n, M_1)$. We now drop the term ε_n to avoid confusion with erasure probabilities ϵ_k. Scaling (7) and summing with (6) we find

$$\begin{aligned}
n\left(r_1 + \frac{1-\epsilon_3}{1-\epsilon_2} r_2\right) &\leq \frac{\epsilon_3-\epsilon_2}{1-\epsilon_2} I(M_1; Y^n_{\mathsf{D_1R_1}}|\overline{G}^n) \\
&\quad + \frac{1-\epsilon_3}{1-\epsilon_2} \overbrace{\left(I(M_1; Y^n_{\mathsf{D_1R_1}}|\overline{G}^n) + I(M_2; Y^n_{\mathsf{D_1R_1}}|\overline{G}^n, M_1)\right)}^{I(M_1, M_2; Y^n_{\mathsf{D_1R_1}}|\overline{G}^n)} \\
&\quad + H(Y^n_{\mathsf{D_1R_2}}|Y^n_{\mathsf{D_1R_1}}, \overline{G}^n) \\
&\leq \frac{\epsilon_3-\epsilon_2}{1-\epsilon_2} I(M_1; Y^n_{\mathsf{R_1S}}|\overline{G}^n) \\
&\quad + n\frac{1-\epsilon_3}{1-\epsilon_2} \min\{(1-\epsilon_2), (1-\epsilon_1)\} + n(1-\epsilon_3). \qquad (8)
\end{aligned}$$

We may also derive a complementary bound for $\frac{1-\epsilon_3}{1-\epsilon_2} r_1 + r_2$:

$$\begin{aligned}
n\left(\frac{1-\epsilon_3}{1-\epsilon_2} r_1 + r_2\right) &\leq \frac{\epsilon_3-\epsilon_2}{1-\epsilon_2} I(M_2; Y^n_{\mathsf{R_2S}}|\overline{G}^n) \\
&\quad + n\frac{1-\epsilon_3}{1-\epsilon_2} \min\{(1-\epsilon_2), (1-\epsilon_1)\} + n(1-\epsilon_3). \qquad (9)
\end{aligned}$$

Since $Y^n_{\mathsf{R_2S}}$ and $Y^n_{\mathsf{R_1S}}$ are identically distributed, we also have

$$\begin{aligned}
&I(M_1; Y^n_{\mathsf{R_1S}}|\overline{G}^n) + I(M_2; Y^n_{\mathsf{R_2S}}|\overline{G}^n) \\
&\quad \leq I(M_1; Y^n_{\mathsf{R_1S}}|\overline{G}^n) + I(M_2; Y^n_{\mathsf{R_1S}}|\overline{G}^n, M_1) \\
&\quad = I(M_1, M_2; Y^n_{\mathsf{R_1S}}|\overline{G}^n) \leq n(1-\epsilon_1). \qquad (10)
\end{aligned}$$

Scaling and summing (8) and (9) , and noting (10), we find

$$\begin{aligned}
r_1 + r_2 &\leq \frac{2(1-\epsilon_3)}{2-\epsilon_2-\epsilon_3} \min\{(1-\epsilon_2), (1-\epsilon_1)\} \\
&\quad + \frac{\epsilon_3-\epsilon_2}{2-\epsilon_2-\epsilon_3}(1-\epsilon_1) + 2\frac{(1-\epsilon_3)(1-\epsilon_2)}{2-\epsilon_2-\epsilon_3}. \qquad (11)
\end{aligned}$$

Evaluating the min operation for the two possibilities and simplifying yields the bounds (3) and (4) respectively.

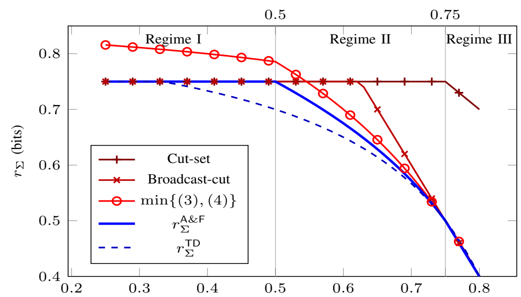

Fig. 6. Upper and lower bounds on sum capacity for $\epsilon_1 = 0.5$, $\epsilon_2 \in [0, 0.8]$, $\epsilon_3 = 0.8$. For this range of values, in Regime I, the cut set bound is tight with the sum rate achieved by Align-and-Forward ($r_\Sigma^{\mathsf{A\&F}}$). In Regime II, our new outer bound is significantly closer to the achieved rate. In Regime III, A&F and TD are both capacity achieving.

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