An Axiomatic Theory of Fairness in Network Resource Allocation

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***Abstract*—We present five axioms for fairness measures in resource allocation. A family of fairness measures satisfying the axioms is constructed. Special cases of this family include α-fairness, Jain's index, and entropy. Properties of fairness measures satisfying the axioms are proven, including Schur-concavity. Among the engineering implications is a generalized Jain's index that tunes the resolution of fairness measure, a new understanding of α-fair utility functions, and an interpretation of "larger α is more fair". We also construct an alternative set of axioms to capture system efficiency and feasibility constraints.**

I. Quantifying Fairness

Given a vector $\mathbf{x} \in \mathbb{R}^n_+$, where x_i is the resource allocated to user i, *how fair* is it?

One approach to quantify the degree of fairness associated with $\mathbf{x}$ is through a fairness measure, which is a function f that maps $\mathbf{x}$ into a real number. Various fairness measures have been proposed throughout the years, e.g., in [1]–[6]. These range from simple ones, e.g., the ratio between the smallest and the largest entries of $\mathbf{x}$, to more sophisticated functions, e.g., Jain's index and the entropy function. Some of these fairness measures map $\mathbf{x}$ to normalized ranges between 0 and 1, where 0 denotes the minimum fairness, 1 denotes the maximum fairness (often corresponding to an $\mathbf{x}$ where all x_i are the same) and a larger value indicates more fairness. For example, min-max ratio [1] is given by the maximum ratio of any two user's resource allocation, while Jain's index [3] computes a normalized square mean. How are these fairness measures are related? Is one measure "better" than any other? What other measures of fairness may be useful?

An alternative approach that has gained attention in the networking research community since [7], [8] is the optimization-theoretic approach of α-fairness and the associated utility maximization. Given a set of feasible allocations, a maximizer of the α-fair utility function satisfies the definition of α-fairness. Two well-known examples are as follows: a maximizer of the log utility function ($\alpha = 1$) is proportionally fair, and a maximizer of the α-fair utility function as $\alpha \to \infty$ is max-min fair. More recently, α-fair utility functions have been connected to divergence measures [9], and in [10], [11], the parameter α was viewed as a fairness measure in the sense that a fairer allocation is one that is the maximizer of an α-fair utility function with larger α — although the exact role of α in trading-off fairness and throughput can sometimes be surprising [12]. While it is often held that $\alpha \to \infty$ is more fair than $\alpha = 1$, which is in turn more fair than $\alpha = 0$, it remains unclear what it means to say, for example, that $\alpha = 3$ is more fair than $\alpha = 2$.

Clearly, these two approaches for quantifying fairness are different. On the one hand, α-fair utility functions are continuous and strictly increasing in each entry of $\mathbf{x}$, thus its maximization results in Pareto optimal resource allocations. On the other hand, scale-invariant fairness measures (ones that map $\mathbf{x}$ to the same value as a normalized $\mathbf{x}$) are unaffected by the magnitude of $\mathbf{x}$, and an allocation that does not use all the resources can be as fair as one that does. Can the two approaches be unified?

To address the above questions, we develop an axiomatic approach to fairness measures. We discover that a set of five axioms, each of which simple and intuitive, can lead to a useful family of fairness measures. The axioms are: the axiom of continuity, of homogeneity, of asymptotic saturation, of irrelevance of partition, and of monotonicity. Starting with these five axioms, we can *generate* a family of fairness measures from generator functions g: any increasing and continuous functions that lead to a well-defined "mean" function (i.e., from any Kolmogorov-Nagumo function [16]). For example, using power functions with exponent β as the generator function, we derive a unique family of fairness measures f_β that include all of the following as special cases, depending on the choice of β: Jain's index, maximum or minimum ratio, entropy, and α-fair utility, and reveals new fairness measures corresponding to other ranges of β.

In particular, for $\beta \leq 0$, well-known fairness measures (e.g., Jain's index and entropy) are special cases of our construction, and we generalize Jain's index to provide a flexible tradeoff between "resolution" and "strictness" of the fairness measure. For $\beta \geq 0$, α-fair utility functions can be factorized as the product of two components: our fairness measure with $\beta = \alpha$ and a function of the total throughput that captures the scale, or efficiency, of $\mathbf{x}$. Such a factorization also quantifies a tradeoff between fairness and efficiency in achieving Pareto dominance with the maximum possible α, and facilitates a clearer understanding of what it means to say that a larger α is "more fair" for general $\alpha \in [0, \infty)$.

The axiomatic construction of fairness measures also illuminates their engineering implications. Any fairness measure satisfying the five axioms can be proven to have many prop-

This work has been supported in part by NSF CNS-0905086 and CNS-0519880.

erties quantifying common beliefs about fairness, including Schur-concavity [14]. Consequently, any operation balancing resources between two users always results in a higher fairness value, extending previous results using majorization to characterize fairness [4], [13].

The development of an axiomatic theory of fairness takes another turn towards the end of the paper. By removing the Axiom of Homogeneity, we propose an alternative set of four axioms, which allows *efficiency* of resource allocation be *jointly captured* in the fairness measure. We show how this alternative system connects with constrained optimization based resource allocation, where magnitude matters due to the feasibility constraint and an objective function that favors efficiency.

Main notation is shown in Table I. Due to space limitation, all proofs can be found in the full online version [19], together with additional discussions, such as one contrasting this paper with the well-known axiomatic theories of Nash bargaining solution and Shapley value in economics.

Variable	**Meaning**
$\mathbf{x}$	Resource allocation vector of length n
$\mathbf{x}^{\uparrow}$	Sorted vector with smallest element being first
$w(\mathbf{x})$	Sum of all elements of $\mathbf{x}$
$f(\cdot), f_\beta(\cdot)$	Fairness measure (of parameter β)
$g(\cdot)$	Generator function
s_i	Positive weights for weighted mean
$\mathbf{1}_n$	Vector of all ones of length n
$\mathbf{x} \succeq \mathbf{y}$	Vector $\mathbf{x}$ majorizes vector $\mathbf{y}$
β	Parameter for power function $g(y) = y^\beta$
$U_\alpha(\cdot)$	α-fair utility with parameter α
$H(\cdot)$	Shannon entropy function
$J(\cdot)$	Jain's index
$\Phi_\lambda(\cdot)$	Our multicriteria (fairness and efficiency) utility function

TABLE I
MAIN NOTATION

II. AXIOMS

Let $\mathbf{x}$ be a resource allocation vector with n non-negative elements. A fairness measure $f(\mathbf{x})$ is a mapping from $\mathbf{x}$ to a real number, i.e., $f : \mathbb{R}^n_+ \to \mathbb{R}$, for all integer $n \geq 1$. We first introduce the following set of axioms about f, whose explanations and implications are given next.

1) *Axiom of Continuity.* Fairness measure $f(\mathbf{x})$ is continuous on $\mathbb{R}^n_+$ for all integer $n \geq 1$.
2) *Axiom of Homogeneity.* Fairness measure $f(\mathbf{x})$ is a homogeneous function of degree 0:
$$f(\mathbf{x}) = f(t \cdot \mathbf{x}), \quad \forall\ t > 0. \tag{1}$$
Without loss of generality, for a single user, we take $|f(x_1)| = 1$ for all $x_1 > 0$, i.e., fairness is a constant for $n = 1$.
3) *Axiom of Asymptotic Saturation.* Fairness measure $f(\mathbf{x})$ of equal resource allocations eventually becomes independent of the number of users:
$$\lim_{n\to\infty} \frac{f(\mathbf{1}_{n+1})}{f(\mathbf{1}_n)} = 1. \tag{2}$$
4) *Axiom of Irrelevance of Partition.* If we partition the elements of $\mathbf{x}$ into two parts $\mathbf{x} = \left[\mathbf{x}^1, \mathbf{x}^2\right]$, the fairness index $f(\mathbf{x}^1, \mathbf{x}^2)$ can be computed recursively (with respect to a *generator function* $g(y)$) and is independent of the partition, i.e.,
$$f(\mathbf{x}^1, \mathbf{x}^2) = f\left(w(\mathbf{x}^1), w(\mathbf{x}^2)\right) \cdot g^{-1}\left(\sum_{i=1}^{2} s_i \cdot g\left(f(\mathbf{x}^i)\right)\right), \tag{3}$$
where $w(\mathbf{x}^1)$ and $w(\mathbf{x}^2)$ denote the sum of resource vectors $\mathbf{x}^1$ and $\mathbf{x}^2$ respectively, and $g(y)$ is a continuous and strictly monotonic function that can generate the following function h:
$$h = g^{-1}\left(\sum_{i=1}^{2} s_i \cdot g\left(f(\mathbf{x}^i)\right)\right), \tag{4}$$
with positive weights satisfying $\sum_i s_i = 1$ such that h qualifies as a *mean* function [15] of $\{f(\mathbf{x}^i), \forall i\}$.
5) *Axiom of Monotonicity.* For $n = 2$ users, fairness measure $f(\theta, 1-\theta)$ is monotonically increasing as the absolute difference between the two elements (i.e., $|1-2\theta|$) shrinks to zero.

Axioms 1 and 2 are very intuitive. The Axiom of Continuity says that a slight change in resource allocation shows up as a slight change in the fairness measure. The Axiom of Homogeneity says that the fairness measure is independent of the unit of measurement or absolute magnitude of the resource allocation.

Due to the Axiom of of Homogeneity, for an optimization formulation of resource allocation, the fairness measure $f(\mathbf{x})$ alone cannot be used as the objective function if efficiency (which depends on magnitude $\sum_i x_i$) is to be captured. In Section VI, we will connect this fairness measure with an efficiency measure in α-fair utility function. In Section VII, we will remove the Axiom of of Homogeneity and propose an alternative set of axioms, which make measure $f(\mathbf{x})$ dependent on both magnitude and distribution of $\mathbf{x}$, thus capturing fairness and efficiency at the same time.

Axiom 3 is a technical condition used to ensure *uniqueness* of the fairness measure and invariance under change of variable by fixing a scaling. For example, suppose $f(\mathbf{x})$ is a fairness measure satisfying all axioms (with respect to a mean function $g(y)$) except Axiom 3. It is easy to see that by making a logarithmic change of variables, fairness measure $\log f(\mathbf{x})$ also satisfies all axioms, respect to a mean function $e^{g(y)}$, other than Axiom 3.

Initial:	$\sum_{i=1}^{n} x_i$	
	$\swarrow$	$\searrow$
Level 1:	$\sum_{i=1}^{k} x_i$	$\sum_{i=k+1}^{n} x_i$
	$\swarrow \quad \searrow$	$\swarrow \quad \searrow$
Level 2:	$x_1, \ldots, x_k$	$x_{k+1}, \ldots, x_n$

TABLE II
ILLUSTRATION OF THE HIERARCHICAL COMPUTATION OF FAIRNESS.

None of Axioms 1–3 concerns the *construction* of fairness measure as the number of users varies. A hierarchical construction of fairness is defined in Axiom 4, which allows us to derive a fairness measure $f : \mathbb{R}_+^n \to \mathbb{R}$ of n users recursively from lower dimensions, $f : \mathbb{R}_+^k \to \mathbb{R}$ and $f : \mathbb{R}_+^{n-k} \to \mathbb{R}$ for integer $0 < k < n$. The recursive computation is illustrated by a two-level representation in Table II. Let $\mathbf{x}^1 = [x_1, \ldots, x_k]$ and $\mathbf{x}^2 = [x_{k+1}, \ldots, x_n]$. The computation is performed as follows. At level 1, since the total resource is divided into two chunks, $w(\mathbf{x}^1)$ and $w(\mathbf{x}^1)$, fairness across the chunks obtained in this level is measured by $f\left(w(\mathbf{x}^1), w(\mathbf{x}^2)\right)$. At level 2, the two chunks of resources are further allocated to k and $n-k$ users, achieving fairness $f(\mathbf{x}^1)$ and $f(\mathbf{x}^2)$, respectively. To compute overall fairness of the resource allocation $\mathbf{x} = [x_1, x_2, \ldots, x_n]$, we combine the fairness obtained in the two levels using a multiplication in equation (3).

As we consider a continuous and strictly increasing generator function $g(y)$, the function (4) is a mean value [15] for $\{f(\mathbf{x}^i), \forall i\}$, which represents the average fairness of individual parts of $\mathbf{x}$. The set of generator functions giving rise to the same fairness measures may not unique, e.g., logarithm and power functions. The simplest case is when g is identity and $s_i = 1/n$ for all i. A natural choice of the weight s_i in (3) is to choose the value proportional to the sum resource of vector $\mathbf{x}^i$. More generally, we will consider the following weights

$$s_i = \frac{w^\rho(\mathbf{x}^i)}{\sum_j w^\rho(\mathbf{x}^j)}, \ \forall i \tag{5}$$

where $\rho \geq 0$ is an arbitrary exponent. When $\rho = 0$, weights in (5) are equal and lead to an un-weighted mean in Axiom 4. As shown in Section IV, the parameter ρ can be chosen such that the hierarchical computation is independent of partition as stated in Axiom 4. As a special case of Axiom 4, if we denote the resource allocation at level 1 by a vector $\mathbf{z} = [w(\mathbf{x}^1), w(\mathbf{x}^2)]$ and if the resource allocation at level 2 are equal $\mathbf{x}^1 = \mathbf{x}^2 = \mathbf{y}$, it is straight forward to verify that Axiom 4 implies

$$f(\mathbf{y} \otimes \mathbf{z}) = f(\mathbf{y}) \cdot f(\mathbf{z}), \tag{6}$$

where $\otimes$ is the direct product of two vectors. We later show in Section VII, an extension of equation (6) gives an alternative way of stating Axiom 4 and leads to a set of more general axioms on fairness.

Axiom 5 is the only axiom that involves a *value* statement on fairness: when there are just two users, more equalized is more fair. This axiom specifies an increasing direction of fairness and ensures uniqueness of $f(\mathbf{x})$. Consider the allocation of a unit resource to two users as $\mathbf{x} = [\theta, 1-\theta]$. It is intuitive that fairness strictly improves as $\theta \to \frac{1}{2}$, since the difference between the two resource shares tends to be smaller. This intuition also holds for all existing fairness measures, e.g., various, spread, deviation, max-min ratio, Jain's index, α-fair utility, and entropy.

By definition, axioms are true, as long as they are consistent. They should also be non-redundant. However, not all sets of axioms are useful: unifying known notions, discovering new measures and properties, and providing useful insights. We first demonstrate the following existence (the axioms are consistent) and uniqueness results.

Theorem 1: (*Existence.*) There exists a fairness measure $f(\mathbf{x})$ satisfying Axioms 1–5. Furthermore, the fairness achieved by equal-resource allocations $\mathbf{1}_n$ is independent of the choice of $g(y)$, i.e.,

$$f(\mathbf{1}_n) = n^r \cdot f(1), \tag{7}$$

where r is a constant exponent.

Theorem 2: (*Uniqueness.*) Given a generator function g, the resulting $f(\mathbf{x})$ satisfying Axioms 1–5 is unique.

Our proof techniques for Theorems 1–2 extends those in proving Renyi's entropy [16], which is a family of functionals quantifying the uncertainty or randomness of generalized probability distributions. For instance, the Axiom of Irrelevance of Partition for fairness can be interpreted a replacement for the Mean-Value Axiom of entropy in [16]. To prove the existence in Theorems 1, we first apply the Axioms of Homogeneity and of Irrelevance of Partition to derive an explicit expression of fairness measure $f(\mathbf{x})$ for the case of two users and resource vectors of rational numbers. The result is then generalized to resource vectors of real numbers using the Axiom of Continuity, and to arbitrary number of users based on a recursive equation of partitions. The Axiom of Asymptotic Saturation and the Axiom of Monotonicity are then used to ensure uniqueness of fairness measures.

III. Properties of Fairness Measures

We first prove an intuitive corollary from the five axioms that will be useful for the rest of the presentation.

Corollary 1: (Symmetry.) A fairness measure satisfying Axioms 1–5 is symmetric over $\mathbf{x}$:

$$f(x_1, x_2, \ldots, x_n) = f(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), \tag{8}$$

where $i_1, \ldots, i_n$ is an arbitrary permutation of indices $1, \ldots, n$.

The symmetry property shows that the fairness measure $f(\mathbf{x})$ satisfying Axioms 1–5 is irrelevant of labeling of users.

We now connect of our axiomatic theory to a line of work on measuring statistical dispersion by vector majorization, including the popular Gini Coefficient [18]. Majorization [14] is a partial order over vectors to study whether the elements of vector $\mathbf{x}$ are less spread out than the elements of vector $\mathbf{y}$. We say that $\mathbf{x}$ is majorized by $\mathbf{y}$, and we write $\mathbf{x} \preceq \mathbf{y}$, if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ (always satisfied due to Axiom 2) and

$$\sum_{i=1}^{d} x_i^{\uparrow} \leq \sum_{i=1}^{d} y_i^{\uparrow}, \text{ for } d = 1, \dots, n, \tag{9}$$

where $x_i^{\uparrow}$ and $y_i^{\uparrow}$ are the ith elements of $\mathbf{x}^{\uparrow}$ and $\mathbf{y}^{\uparrow}$, sorted in ascending order. According to this definition, among the vectors with the same sum of elements, one with the equal elements is the most majorizing vector.

Intuitively, $\mathbf{x} \preceq \mathbf{y}$ can be interpreted as $\mathbf{y}$ being a fairer allocation than $\mathbf{x}$. It is a classical result [14] that $\mathbf{x}$ is majorized by $\mathbf{y}$, if and only if, from $\mathbf{x}$ we can produce $\mathbf{y}$ by a finite sequence of Robin Hood operations.[1]

Majorization alone cannot be used to define a fairness measure since it is a partial order and fails to compare vectors in certain cases. Still, if resource allocation $\mathbf{x}$ is majorized by $\mathbf{y}$, it is desirable to have a fairness measure f such that $f(\mathbf{x}) \leq f(\mathbf{y})$. A function satisfying this property is known as Schur-concave. In statistics and economics, many measures of statistical dispersion are known to be Schur-concave, e.g., Gini Coefficient and Robin Hood Ratio [18], and we show our fairness measure also is Schur-concave:

Theorem 3: (*Schur-concavity.*) A fairness measure satisfying Axioms 1–5 is Schur-concave:

$$f(\mathbf{x}) \leq f(\mathbf{y}), \text{ if } \mathbf{x} \preceq \mathbf{y}. \tag{10}$$

Next we present several properties of fairness measures satisfying the axioms, whose proofs rely on Schur-concavity.

Corollary 2: (Equal-resource allocation is fairest.) A fairness measure $f(\mathbf{x})$ satisfying Axioms 1–5 is maximized by equal-resource allocations, i.e.,

$$f(\mathbf{1}_n) = \max_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}). \tag{11}$$

Corollary 3: (Collecting a fixed-tax is unfair.) If a fixed amount $c > 0$ of the resource is subtracted from each user (i.e., $x_i - c$ for all i), the resulting fairness measure decreases

$$f(\mathbf{x} - c \cdot \mathbf{1}_n) \leq f(\mathbf{x}), \quad \forall c > 0, \tag{12}$$

where $c > 0$ must be small enough such that all elements of $\mathbf{x} - c \cdot \mathbf{1}_n$ are positive.

Corollary 4: (Inactive user achieves no fairness.) When a fairness measure $f(\mathbf{x})$ satisfying Axioms 1–5 is generated by by $\rho > 0$ in 5, Removing users with zero resources does not change fairness:

$$f(\mathbf{x}, \mathbf{0}_n) = f(\mathbf{x}), \quad \forall n \geq 1. \tag{13}$$

[1] In a Robin Hood operation, we replace two elements x_i and $x_j < x_i$ with $x_i - \epsilon$ and $x_j + \epsilon$, respectively, for some $\epsilon \in (0, x_i - x_j)$. In other words, we take from the rich (x_i), and give to the poor (x_j).

IV. A Family of Fairness Measures

A. Constructing Fairness Measures

From any function $g(y)$ satisfying the condition in Axiom 4, we can generate a unique $f(\mathbf{x})$. Such an $f(\mathbf{x})$ is a well-defined fairness measure if it also satisfies Axioms 1–5. We then refer to the corresponding $g(y)$ as a generator of the fairness measure.

Definition 1: Function $g(y)$ is a generator if there exists a $f(\mathbf{x})$ satisfying Axioms 1–5 with respect to $g(y)$.

We note, however, that different generator functions may generate the same fairness measure. Although it is difficult to find the entire set of generators $g(y)$, we have found that many forms of $g(y)$ functions (e.g., logarithm, polynomial, exponential, and their combinations) result in fairness measures equivalent to those generated by the family of power functions. It remains to be determined if all fairness measures satisfying Axioms 1–5 can be generated by power functions.

In this section, we consider power functions, $g(y) = |y|^{\beta}$, parameterized by β and derive the resulting family of fairness measures, which indeed satisfy all the axioms. The absolute value ensures that $g(y)$ is non-increasing over $\mathbb{R}_+$ for $\beta \geq 0$, and over $\mathbb{R}_-$ for $\beta < 0$. From here on, we replace Equation (3) in Axiom 4 by

$$f(\mathbf{x}^1, \mathbf{x}^2) = f\left(w(\mathbf{x}^1), w(\mathbf{x}^2)\right) \cdot \left(\sum_{i=1}^{2} s_i \cdot f^{\beta}(\mathbf{x}^i)\right)^{\frac{1}{\beta}},$$

where the weights s_i are given by (5).

Theorem 4: (*Fairness measures generated by power functions*) For power mean ($g(y) = |y|^{\beta}$ with parameter β), Axioms 1–5 define a unique family of fairness measures as follows

$$f(\mathbf{x}) = \left[\sum_{i=1}^{n} \left(\frac{x_i}{\sum_j x_j}\right)^{1-\beta r}\right]^{\frac{1}{\beta}}, \text{ for } \beta r \leq 1 \tag{14}$$

$$f(\mathbf{x}) = -\left[\sum_{i=1}^{n} \left(\frac{x_i}{\sum_j x_j}\right)^{1-\beta r}\right]^{\frac{1}{\beta}}, \text{ for } \beta r \geq 1, \tag{15}$$

where $r = \frac{1-\rho}{\beta}$ is a constant exponent, which determines the growth rate of maximum fairness as population size n increases:

$$f(\mathbf{1}_n) = n^r \cdot f(1). \tag{16}$$

For different values of parameter β, the fairness measures derived above are equivalent up to a constant exponent r:

$$f_{\beta,r}(\mathbf{x}) = \left[f_{\beta r,1}\right]^r(\mathbf{x}), \tag{17}$$

if we denote $f_{\beta,r}$ as the fairness measure with parameters β and r. According to Theorem 1, r determines the growth rate of maximum fairness as population size n increases. Without loss of generality, we choose $r = 1$ such that the maximum average fairness per user is a constant, i.e., $\frac{f(\mathbf{1}_n)}{n} = f(1)$.

From a user's perspective, her perception of maximum fairness is independent of the population size of the system. From now on, we will use a unified representation of the constructed fairness measures:

$$f_\beta(\mathbf{x}) = \text{sign}(1-\beta) \cdot \left[\sum_{i=1}^{n} \left(\frac{x_i}{\sum_j x_j} \right)^{1-\beta} \right]^{\frac{1}{\beta}}. \quad (18)$$

We summarize the special cases in Table III, where β sweeps from $-\infty$ to ∞ and $H(\cdot)$ denotes the entropy function. For some values of β, the corresponding mean function h has a standard name, and for some, known approaches to measure fairness are recovered, while for $\beta \in (0,-1)$ and $\beta \in (-1,-\infty)$, new fairness measures are discovered. For a fixed resource allocation vector $\mathbf{x} = [1,2,5]$, we plot fairness $f(\mathbf{x})$ for different values of β in Figure 1.

Value of β	Our Fairness Measure	Known Names
$\beta \to \infty$	$-\max_i \left\{ \frac{\sum_i x_i}{x_i} \right\}$	Max ratio
$\beta \in (1,\infty)$	$-\left[(1-\beta) U_{\alpha=\beta} \left(\frac{\mathbf{x}}{w(\mathbf{x})} \right) \right]^{\frac{1}{\beta}}$	α-fair utility
$\beta \in (0,1)$	$\left[(1-\beta) U_{\alpha=\beta} \left(\frac{\mathbf{x}}{w(\mathbf{x})} \right) \right]^{\frac{1}{\beta}}$	α-fair utility
$\beta \to 0$	$e^{H\left(\frac{\mathbf{x}}{w(\mathbf{x})}\right)}$	Entropy
$\beta \in (0,-1)$	$\left[\sum_{i=1}^{n} \left(\frac{x_i}{w(\mathbf{x})} \right)^{1-\beta r} \right]^{\frac{1}{\beta}}$	No name
$\beta = -1$	$\frac{(\sum_i x_i)^2}{\sum_i x_i{}^2} = n \cdot J(\mathbf{x})$	Jain's index
$\beta \in (-1,-\infty)$	$\left[\sum_{i=1}^{n} \left(\frac{x_i}{w(\mathbf{x})} \right)^{1-\beta r} \right]^{\frac{1}{\beta}}$	No name
$\beta \to -\infty$	$\min_i \left\{ \frac{\sum_i x_i}{x_i} \right\}$	Min ratio

TABLE III
PREVIOUS RESULTS ARE RECOVERED AS SPECIAL CASES OF OUR AXIOMATIC CONSTRUCTION. FOR $\beta \in (0,-1)$ AND $\beta \in (-1,-\infty)$, NEW FAIRNESS MEASURES OF GENERALIZED JAIN'S INDEX ARE REVEALED.

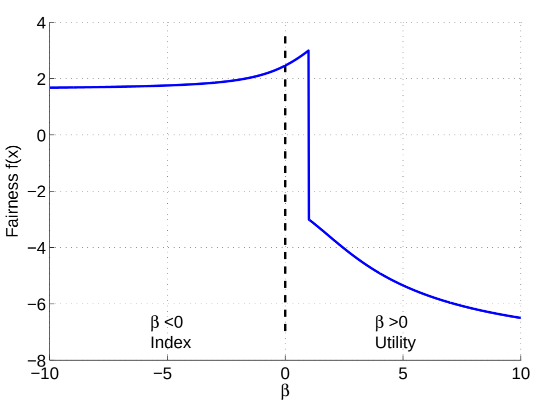

Fig. 1. Plot of fairness $f_\beta(\mathbf{x})$ for different values of β: $\beta > 0$ recovers the utility-based approach, and $\beta < 0$ recovers the index-based approach.

B. *Engineering Implications*

The fairness measures f_β in (18) corresponding to the generator function $g(y) = |y|^\beta$ satisfy a number of properties, which give interesting engineering implications.

Corollary 5: (Number of inactive users.) The fairness measures in (18) also count the number of inactive users in the system. When $f_\beta < 0$, $f(\mathbf{x}) \to -\infty$ if any user is assigned zero resource. When $f > 0$,

$$\text{Number of users with zero resource} \leq n - f(\mathbf{x}), \quad (19)$$

$$\text{Maximum resource to a user} \geq \frac{\sum_i x_i}{f(\mathbf{x})}. \quad (20)$$

Corollary 6: (Threshold level of resource.) If we increase resource allocation to user i by a small amount ϵ, while not changing other users' allocation, the fairness measures in (18) increases if and only if $x_i < \bar{x} = \left(\frac{\sum_j x_j}{\sum_j x_j^{1-\beta}} \right)^{\frac{1}{\beta}}$ and $0 < \epsilon < \bar{x} - x_i$.

Corollary 7: (Lower bound under box-constraints.) If a resource allocation $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ satisfies box-constraints, i.e., $x_{min} \leq x_i \leq x_{max}$ for all i, the fairness measures in (18) is lower bounded by a constant that only depends on β, x_{min}, x_{max}:

$$f(\mathbf{x}) \geq \text{sign}(1-\beta) \cdot \frac{\left(\mu \Gamma^{1-\beta} + 1 - \mu\right)^{\frac{1}{\beta}}}{(\mu\Gamma + 1 - \mu)^{\frac{1}{\beta}-1}}, \quad (21)$$

where $\Gamma = \frac{x_{max}}{x_{min}}$ and $\mu = \frac{\Gamma - \Gamma^{1-\beta} - \beta(\Gamma-1)}{\beta(\Gamma-1)(\Gamma^{1-\beta}-1)}$. The bound is tight when a μ fraction of users have $x_i = x_{max}$ and the remaining $1-\mu$ fraction of users have $x_i = x_{min}$.

These results provide possible interpretations and applications of the family of fairness measures. Through Corollary 5, by specifying a level of fairness, we can limit the number of starved users in a system. Corollary 6 implies that $\bar{x}$ serves as a threshold for identifying "poor" and "rich" users, since assigning additional resources to user i improves fairness if $x_i < \bar{x}$, and reduces fairness if $x_i > \bar{x}$.

V. IMPLICATION 1: GENERALIZING JAIN'S INDEX

When $\beta = -1$ (i.e., harmonic mean is used in Axiom 4), we get a scalar multiple of the widely used Jain's index $J(\mathbf{x}) = \frac{1}{n} f(\mathbf{x})$. Upon inspection of (18) and the specific cases noted in Table III, we note that any $(0,-\infty) \cup \beta \in (0,1)$ the range of fairness measure $f_\beta(\mathbf{x})$ lies between 1 and n. Equivalently, we can say that the fairness *per user* resides in the interval $\left[\frac{1}{n}, 1\right]$. When the limit as $\beta \to 0$ is considered, the resulting fairness measure can also be shown to have this property. Because $f_\beta(\mathbf{x})$ for $\beta < 1$ has this characteristic, we refer to this subclass of our family of fairness measures as the generalization of Jain's index.

Definition 2: $J_\beta(\mathbf{x}) = \frac{1}{n} f_\beta(\mathbf{x})$ is a generalized Jain's index parameterized by $\beta \leq 1$.

The common properties of our fairness index proven in Section III and IV carry over to this generalized Jain's index. For $\beta = -1$, $J_{-1}(\mathbf{x})$ reduces to the original Jain's index.

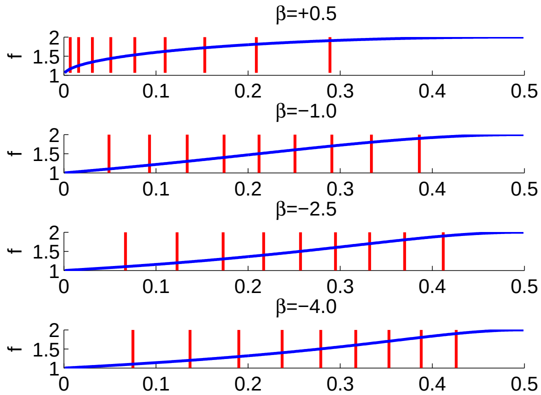

Fig. 2. Plot of the fairness measure $f_\beta(\theta, 1-\theta)$ against θ, for resource allocation $\mathbf{x} = [\theta, 1-\theta]$ and different choices of $\beta = \{-4.0, -2.5, -1.0, 0.5\}$. It can be observed that $f_\beta(\theta, 1-\theta)$ is monotonic as $\beta \to 1$. Further, smaller values of $|1-\beta|$ results in a steeper incline over small θ, i.e., the low-fairness region.

Theorem 5: (*Monotonicity with respect to β.*) The fairness measures in (18) is negative and decreasing for $\beta \in (1, \infty)$, and positive and increasing for $\beta \in (-\infty, 1)$:

$$\frac{\partial f_\beta(\mathbf{x})}{\partial \beta} \le 0 \text{ for } \beta \in (1, \infty), \tag{22}$$

$$\frac{\partial f_\beta(\mathbf{x})}{\partial \beta} \ge 0 \text{ for } \beta \in (-\infty, 1). \tag{23}$$

The monotonicity of fairness measures $f_\beta(\mathbf{x})$ on $\beta \in (-\infty, 1)$ gives an engineering interpretation of β. Figure 2 plots fairness $f_\beta(\theta, 1-\theta)$ for resource allocation $\mathbf{x} = [\theta, 1-\theta]$ and different choices of $\beta = \{-4.0, -2.5, -1.0, 0.5\}$. The vertical bars in the figure represent the level sets of function f, for values $f_\beta(\theta_i, 1-\theta_i) = \frac{i}{10}\left(f_{max} - f_{min}\right), i = 1, 2, \ldots, 9$. For fixed resource allocations, since f increases as β approaches 1, the level sets of f are pushed toward the region with small θ (i.e., the low-fairness region), resulting in a steeper incline in the region. In the extreme case of $\beta = 1$, all level set boundaries align with the y-axis in the plot. The fairness measure f point-wise converges to step functions $f_\beta(\theta, 1-\theta) = 2$. Therefore, parameter β characterizes the shape of the fairness measures: a smaller value of $|1-\beta|$ (i.e., β closer to 1) causes the level sets to be condensed in the low-fairness region.

Since the fairness measure must still evaluate to a number between 1 and n here, the monotonicty and resulting shift in granularity of the fairness measure associated with varying β suggests differences in evaluating unfairness. At one extreme, $\beta \to 1$ any solution where no user receives an allocation of zero is fairest. On the other hand, as $\beta \to -\infty$ the relationship between $f_\beta(\mathbf{x})$ and θ becomes linear, suggesting a stricter concept of fairness – for the same allocation, as $\beta \to -\infty$ more fairness is lost. Therefore, the parameter β can tune the generalization of Jain's index f for different tradeoffs between the resolution and the strictness of fairness measure.

VI. Implication 2: Understanding α-Fairness

Due to Axiom 2, the Axiom of Homogeneity, our fairness measures only express desirability over the $(n-1)$-dimension subspace orthogonal to the $\mathbf{1}_n$ vector. Hence, they do not capture any notion of efficiency of an allocation.

We focus in this section on the widely applied α-fair utility function:

$$\sum_i U_\alpha(x_i), \text{ where } U_\alpha(x) = \begin{cases} \frac{x^{1-\alpha}}{1-\alpha} & \alpha \ge 0,\ \alpha \ne 1 \\ \log(x) & \alpha = 1 \end{cases}. \tag{24}$$

We first show that the α-fairness network utility function can be *factored* into two components: one corresponding to the family of fairness measures we constructed and one corresponding to efficiency. We then demonstrate that, for a fixed α, the factorization can be viewed as a single point on the optimal tradeoff curve between fairness and efficiency. Furthermore, this particular point is one where maximum emphasis is placed on fairness while maintaining Pareto optimality of the allocation. This allows us to quantitatively interpret the belief of "larger α is more fair" *across all* $\alpha \ge 0$.

A. Factorization of α-fair Utility Function

Re-arranging the terms of the equation in Table III, we have

$$\begin{aligned} U_{\alpha=\beta}(\mathbf{x}) &= \frac{1}{1-\beta}\left|f_\beta(\mathbf{x})\right|^\beta \left(\sum_i x_i\right)^{1-\beta} \\ &= \left|f_\beta(\mathbf{x})\right|^\beta \cdot U_\beta\left(\sum_i x_i\right), \end{aligned} \tag{25}$$

where $U_\beta\left(\sum_i x_i\right)$ is the one-dimensional version of the α-fair utility function with $\alpha = \beta$. For $\beta \to 1$, it is easy to show that our fairness measure $f_\beta(\mathbf{x})$, multiplied by a function of throughput $\sum_i x_i$, equals α-fair utility function with $\alpha = 1$. Similarly, for $\beta \to \infty$, it equals α-fair utility function as $\alpha \to \infty$. Therefore, Equation (25) also holds for proportional fairness at $\alpha = 1$ and max-min fairness at $\alpha \to \infty$.

Equation (25) demonstrates that the α-fair utility functions can be factorized as the product of two components: a fairness measure, $\left|f_\beta(\mathbf{x})\right|^\beta$, and an efficiency measure, $U_\beta\left(\sum_i x_i\right)$. The fairness measure $\left|f_\beta(\mathbf{x})\right|^\beta$ only depends on the normalized distribution, $\mathbf{x}/(\sum_i x_i)$, of resources (due to Axiom 2), while the efficiency measure is a function of the sum resource $\sum_i x_i$.

The factorization of α-fair utility functions is illustrated in Table IV and decouples the two components to tackle issues such as fairness-efficiency tradeoff and feasibility of $\mathbf{x}$ under a given constraint set. For example, it helps to explain the counter-intuitive throughput behavior in [12]: an allocation vector that maximizes the α-fair utility with a larger α may not be less efficient, because the α-fair utility incorporates both fairness and efficiency at the same time.

Although Corollary 2 states equal allocation is fairest, an α-fair allocation may not have an equal distribution. This is because the additional efficiency component in (25) can skew the optimizer (i.e., the resource allocation resulting from α-fair utility maximization) away from an equal distribution. For

Allocation:	$\mathbf{x}$	
	$\swarrow$	$\searrow$
Factorize:	$\mathbf{x}/\sum_i x_i$	$\sum_i x_i$
	$\downarrow$	$\downarrow$
Measure:	$f_\beta\left(\mathbf{x}/\sum_i x_i\right)$	$U_\beta\left(\sum_i x_i\right)$
	$\searrow$	$\swarrow$
Combine:	$U_{\alpha=\beta}(\mathbf{x})$	

TABLE IV
ILLUSTRATION OF THE FACTORIZATION OF THE α-FAIR UTILITY FUNCTIONS INTO A FAIRNESS COMPONENT OF THE NORMALIZED RESOURCE DISTRIBUTION AND A EFFICIENCY COMPONENT OF THE SUM RESOURCE.

this to happen there must exist an allocation that is feasible (within the constraint set of realizable allocations) with a large enough gain in efficiency over all equal distribution allocations. Hence, the magnitude of this skewing depends on the fairness parameter ($\alpha = \beta$), the constraint set of $\mathbf{x}$, and the relative importance of fairness and efficiency.

Guided by the product form of (25), we consider a scalarization of the maximization of the two objectives: fairness and efficiency:

$$\Phi_\lambda(\mathbf{x}) = \lambda\ell\left(f_\beta\left(\mathbf{x}\right)\right) + \ell\left(\sum_i x_i\right), \tag{26}$$

where $\beta \in (0,1) \cup (1,\infty)$ is fixed, $\lambda \in [0,\infty)$ absorbs the exponent β in the fairness component of (25) and is a weight specifying the relative emphasis placed on the fairness, and

$$\ell(y) = \mathrm{sign}(y)\log(|y|). \tag{27}$$

The use of the log function later recovers the product in the factorization of (25) from the sum in the scalarized (26).

B. What Does "Larger α is More Fair" Mean?

It is commonly believed that larger α is more fair, but it is not exactly clear what this statement means for general $\alpha \in [0,\infty]$. Guided by the factorization above and the axiomatic construction of fairness measures, we provide two interpretation of this statement that justify it from the viewpoints of Pareto optimality and geometry of the constraint set.

An allocation vector $\mathbf{x}$ is said to be Pareto dominated by $\mathbf{y}$ if $x_i \le y_i$ for all i and $x_i < y_i$ for at least some i. An allocation is called Pareto optimal if it is not Pareto dominated by any other feasible allocation. If the relative emphasis on efficiency is sufficiently high, Pareto optimality of the solution can be maintained. To preserve Pareto optimality, we require that if $\mathbf{y}$ Pareto dominates $\mathbf{x}$, then $\Phi_\lambda(\mathbf{y}) > \Phi_\lambda(\mathbf{x})$.

Theorem 6: (*Preserving Pareto optimality.*) The necessary and sufficient condition on λ such that $\Phi_\lambda(\mathbf{y}) > \Phi_\lambda(\mathbf{x})$ if $\mathbf{y}$ Pareto dominates $\mathbf{x}$ is

$$\lambda \le \left|\frac{\beta}{1-\beta}\right|. \tag{28}$$

Consider the set of maximizers of (26) for λ in the range in Theorem 6:

$$\mathbb{P} = \left\{\mathbf{x} : \mathbf{x} = \arg\max_{\mathbf{x}\in\mathbb{R}} \Phi_\lambda(\mathbf{x}),\ \forall\lambda \le \left|\frac{\beta}{1-\beta}\right|\right\}. \tag{29}$$

When weight $\lambda = 0$, the corresponding points in $\mathbb{P}$ is most efficient. When weight $\lambda = \left|\frac{\beta}{1-\beta}\right|$, it can be shown that the factorization in (25) is equivalent to (26). Therefore, α-fairness corresponds to the solution of an optimization that places the *maximum emphasis* on the fairness measure parameterized by $\beta = \alpha$ while *preserving Pareto optimality*. Allocations in $\mathbb{P}$ corresponding to other values of λ achieve a tradeoff between fairness and efficiency, while Pareto optimality is preserved.

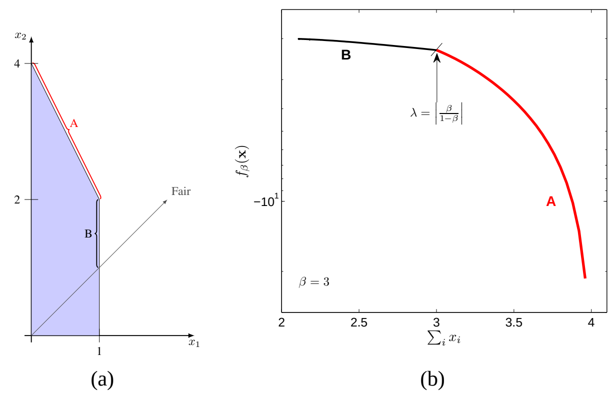

Fig. 3. (a) Feasible region (i.e., the constraint set of the utility maximization problem) where overemphasis of fairness violates Pareto dominance, and (b) its fairness-efficiency tradeoff for $\beta = 3$. Region A corresponds to Pareto optimal solutions. Region B is when the condition of Theorem 6 is violated, and solutions are more fair, but no longer Pareto optimal.

Figure 3(b) illustrates an optimal fairness-efficiency tradeoff curve $\left\{\left[f_\beta(\mathbf{x}), \Sigma_i x_i\right],\ \forall\mathbf{x} = \arg\max_{\mathbf{x}\in\mathbb{R}} \Phi_\lambda(\mathbf{x}),\ \forall\lambda\right\}$ corresponding to the constraint set shown in Figure 3(a). The set of optimizers $\mathbb{P}$ in (29), which is obtained by maximizing Pareto optimal utilities (26), is shown by curve A in Figure 3(b).

We just demonstrated the factorization (25) is an extreme point on the tradeoff curve between fairness and efficiency for fixed $\beta = \alpha$. What happens when α becomes bigger?

We denote by $\bigtriangledown_{\mathbf{x}}$ the gradient operator with respect to the vector $\mathbf{x}$. For a differentiable function, we use the standard inner product ($\langle\mathbf{x},\mathbf{y}\rangle = \sum_i x_i y_i$) between the gradient of the function and a normalized vector to denote the directional derivative of the function.

Theorem 7: (*Monotonicity of fairness-efficiency reward ratio.*) Let allocation $\mathbf{x}$ be given. Define $\boldsymbol{\eta} = \frac{1}{n}\mathbf{1}_n - \frac{\mathbf{x}}{\sum x_i}$ as the vector pointing from the allocation to the nearest fairness maximizing solution. Then the fairness-efficiency reward ratio:

$$\frac{\left\langle \bigtriangledown_{\mathbf{x}} U_{\alpha=\beta}(\mathbf{x}), \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}\right\rangle}{\left\langle \bigtriangledown_{\mathbf{x}} U_{\alpha=\beta}(\mathbf{x}), \frac{\mathbf{1}_n}{\|\mathbf{1}_n\|}\right\rangle}, \tag{30}$$

is non-decreasing with α, i.e., higher α gives a greater relative reward for fairer solutions.

The the choice of direction $\boldsymbol{\eta}$ is a direct result of Axiom 2 and Corollary 2, which together imply that $\boldsymbol{\eta}$ is the direction that most increases fairness and is orthogonal to increases in efficiency.

An increase in either fairness or efficiency is a "desirable" outcome. The choice of α dictates exactly *how desirable* one objective is relative to the other (for a fixed allocation). Theorem 7 states that, with a larger α, there is a larger component of the utility function gradient in the direction of fairer solutions, relative to the component in the direction of more efficiency. Notice, however, that comparison is in terms of the ratio between these two gradient components rather than the magnitude of the gradient, and both fairness and efficiency may increase simultaneously.

This result provides a justification for the belief that larger α is "more fair", not just for $\alpha \in \{0, 1, \infty\}$, but for any $\alpha \in [0, \infty)$. Figure 4 depicts how this ratio increases with $\alpha = \beta$ for some examples allocations.

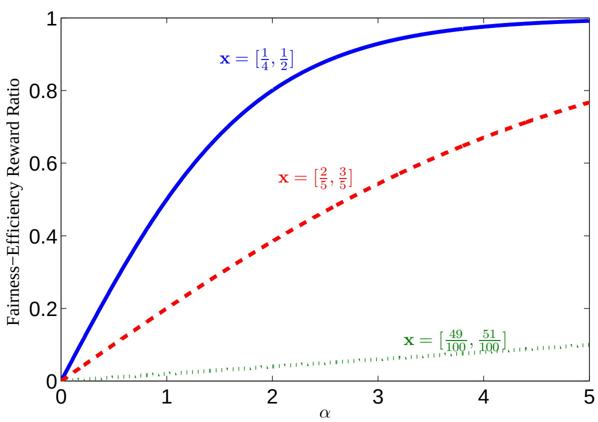

Fig. 4. Monotonic behavior of the ratio (30) as a function of α. Three fixed allocations are considered, and solutions that are already more fair have a lower ratio.

VII. Alternative Axioms

Given a set of useful axioms, it is important to ask if other useful axiomatic systems are possible. By removing or modifying some of the five axioms here — for example, Axiom 2 that decouples the concern on efficiency from fairness — what kind of fairness measures would result? Could an alternative set of axioms lead to the construction of fairness measures that do not automatically decouple from the notions of efficiency and feasibility of resource allocation?

In this section, we propose a set of alternative axioms, which includes Axioms 1-5 as a special case. Let $F : \mathbb{R}_+^n \to \mathbb{R}$ be a general fairness measure satisfying four axioms as follows.

1′) *Axiom of Continuity.* Fairness measure $F(\mathbf{x})$ is continuous on $\mathbb{R}_+^n$ for all integer $n \geq 1$.

2′) *Axiom of Asymptotic Saturation.* Fairness measure $f(\mathbf{x})$ of equal resource allocations eventually becomes independent of the number of users:

$$\lim_{n\to\infty} \frac{F(\mathbf{1}_{n+1})}{F(\mathbf{1}_n)} = 1. \tag{31}$$

3′) *Axiom of Irrelevance of Splitting.* For an allocation vector $\mathbf{x} = [x_1, x_2]$, we split each element x_i into multiple elements by a direct product $x_i\mathbf{y}^{\mathbf{i}}$, where $\mathbf{y}^{\mathbf{i}}$ is a non-negative vector. If the splitting vectors have equal weights $w(\mathbf{y}^{\mathbf{1}}) = w(\mathbf{y}^{\mathbf{2}})$, the fairness of the new allocation vector $[x_1\mathbf{y}^{\mathbf{1}}, x_2\mathbf{y}^{\mathbf{2}}]$ is given by

$$F(x_1\mathbf{y}^{\mathbf{1}}, x_2\mathbf{y}^{\mathbf{2}}) = F(\mathbf{x}) \cdot g^{-1}\left(\sum_{i=1}^{2} s_i \cdot g\left(F(\mathbf{y}^i)\right)\right), \tag{32}$$

where $\sum_i s_i = 1$ are positive weights and $g(y)$ is a continuous and strictly increasing function.

4′) *Axiom of Monotonicity.* For $n = 2$ users, fairness measure $F(x_1, x_2)$ is increasing as ratio x_1/x_2 goes to 1 when sum resource $x_1 + x_2$ is fixed.

Axioms $1'$ and $2'$ remain the same as Axioms 1 and 3 before. Axiom $4'$ is equivalent to Axiom 5 with the additional qualification that the sum-resource does not change. This qualification was previously unnecessary due to Axiom 2 — $f(\mathbf{x})$ did not vary with the amount of total resources — however, is now required in the new set of axioms. Axiom $3'$ is used to recursively construct fairness measure $F(\mathbf{x})$ from lower dimensions and is similar to the Axiom 4. The vector $[x_1\mathbf{y}^{\mathbf{1}}, x_2\mathbf{y}^{\mathbf{2}}]$ can be viewed as a generalized direct product of vector $\mathbf{x}$ with two different vectors $\mathbf{y}^{\mathbf{1}}$ and $\mathbf{y}^{\mathbf{2}}$, which split the resource of each element of $\mathbf{x}$ to multiple users. If $\mathbf{y}^{\mathbf{1}} = \mathbf{y}^{\mathbf{2}}$, this splitting reduces to a direct product.

Since the Axiom 2 of Homogeneity is removed, fairness measure $F(\mathbf{x})$ may depend on the absolute magnitude of resource vector $\mathbf{x}$. Using Axiom $3'$, we can prove that $F(\mathbf{x})$ is a homogeneous function of real degree. Furthermore, the two sets of axioms are equivalent, if the order of homogeneity is zero. This means that the new axiomatic system is more general than the original one.

Theorem 8: (*Existence and Uniqueness.*) For each generator $g(y)$, there exists a unique fairness measure $F(\mathbf{x})$ satisfying Axioms $1' - 4'$. We have,

$$F(\mathbf{x}) = f(\mathbf{x}) \cdot \left(\sum_i x_i\right)^{\frac{1}{\lambda}}, \tag{33}$$

where $\frac{1}{\lambda} \in \mathbb{R}$ is the degree of homogeneity and $f(\mathbf{x})$ is a fairness measure satisfying Axioms 1–5 with respect to the same generator $g(y)$.

While it is easy to verify that some properties, like that of symmetry, in Section III also hold for fairness measure $F(\mathbf{x})$, some properties of fairness measures satisfying Axioms 1–5 are lost in the generalization. For instance, we can *no longer* guarantee that the fairness measure will be Schur-concave, nor can we say that equal allocations are the best.

When power generators $g(y) = |y|^\beta$ are considered, from Axioms $1'$–$4'$ we can derive fairness measure $F_{\beta,\lambda}(\mathbf{x})$, which is parameterized by both λ and β,

$$F_{\beta,\lambda}(\mathbf{x}) = f_\beta(\mathbf{x}) \cdot \left(\sum_i x_i \right)^{\frac{1}{\lambda}}. \tag{34}$$

This unifies our results in Sections IV-VI: Generalized Jain's index is a special case of $F_{\beta,\lambda}(\mathbf{x})$ for $1/\lambda = 0$ and $\beta < 1$; fairness measure $f_\beta(\mathbf{x})$ is a subclass of $F_{\beta,\lambda}(\mathbf{x})$ for $\lambda = 0$; and α-utility is obtained for $1/\lambda = \beta/(1-\beta)$ and $\beta > 0$ by comparing (34) and (26). The degree of homogeneity $1/\lambda$ determines how $F_{\beta,\lambda}(\mathbf{x})$ scales as throughput increases. The decomposition of fairness and efficiency in Section VI is now an immediate consequence from Axioms $1' - 4'$.

There is a useful connection with the characterization of α-fair utility function in the last section. The absolute value $|\lambda|$ is exactly equivalent to the parameter used for defining the utility function (26) in Section VI.B. From Theorem 6, we can conclude that fairness measure $F_{\beta,\lambda}(\mathbf{x})$ is Pareto optimal if and only if

$$\frac{1}{|\lambda|} \geq \left| \frac{1-\beta}{\beta} \right|. \tag{35}$$

For every β, there is a minimum degree of homogeneity such that Pareto optimality can be achieved. When inequality (35) is not satisfied, $F_{\beta,\lambda}(\mathbf{x})$ loses Pareto optimality and produces less throughput-efficient solutions if it is used as an objective function in utility optimization. Fairness measures with small degree of homogeneity $1/\lambda$ are more suitable for computing index values of fairness.

The degree of homogeneity of a fairness measure satisfying Axioms $1'$–$4'$ parameterizes a tradeoff between the concept of fairness and efficiency. Moreover, when power functions are used as generating functions, the degree of homogeneity is exactly equivalent to $\frac{1}{\lambda}$ in (26). Therefore, the intuition behind our result on a maximum $|\lambda|$ (minimum degree of homogeneity) to ensure Pareto optimality can be extended to the general optimization-theoretic approach to fairness, i.e., for a fairness measure F generated from any g, there is a minimum degree of homogeneity $\frac{1}{\lambda}$ to produce a Pareto optimal solution. In the way our first set of axioms generalized Jain's index and revealed new fairness measures with desirable properties, the second set of axioms offers a rich family of objective functions.

To summarize, in removing Axiom 2 and adapting the set of axioms accordingly, we have shown that Axioms $1'$–$4'$ include Axioms 1–5 as special cases. The resulting measures are now affected by notions of both efficiency and fairness, with the balance between the two governed by the degree of homogeneity, $\frac{1}{\lambda}$.

VIII. Concluding Remarks

An axiomatic approach to the fundamental concepts of fairness illuminates many issues in network resource allocation research. This paper is far from the end of axiomatic *theories* of fairness. One way to re-examine axioms is to refute their corollaries in the context of network resource allocation. Perhaps all x_i being the same should not a maximizer of fairness measure, and a fairness measure need not be Schur-concave. Instead, making some x_i bigger should be called more fair if the resulting $\mathbf{x}$ is bigger in all coordinates, i.e., those contributing to the overall efficiency should "fairly" receive more resources. Perhaps the fairness measure should be a function dependent on the feasible region of allocations. These possibilities that alternative sets of axioms of fairness, ones with a value statement different from that in Axiom 5 or Axiom $4'$ in this paper, deserves further exploration.

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