Week 20 In-Class Assessment

2025-06-12

Question 0.1: Consider this function, a low-order polynomial:

$$g(x, y) \equiv a + bx + cy + dxy + hx^2 + ky^2$$

Write down, in symbols, $\int g(x, y)dx$.

Question 0.2: For each of these basic modeling functions, write down the anti-derivative:

i. Straight-line $f(x) \equiv ax + b$.

$F(x)$ is ...

ii. Sinusoid $g(x) \equiv a\sin\left(\frac{2\pi}{P}x\right) + b$

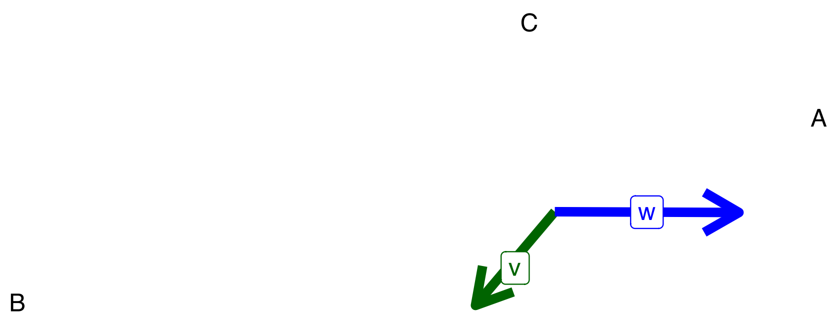
$G(x)$ is ...

iii. Exponential $h(x) \equiv ae^{kz} + c$

$H(x)$ is ...

Question 0.3: Any matrix $\mathbf{M}$ can be factored into a product of two special matrices $\mathbf{QR}$. Explain in a sentence or two what is the special structure of each of these matrices.

Question 0.4: Find the coefficients on $\vec{v}$ and $\vec{w}$ to reach each of the target points, A, B, C. (For ease, all the coefficients are either integers or half integers, e.g. 2 and −1.5.) Write down the coefficients next to the target point.



Question 0.5: Here are some relationships expressed in terms of dot products:

i. The square length $|\vec{v}|^2$ of a vector $\vec{v}$ is $\vec{v} \cdot \vec{v}$.
ii. The cosine of the angle between two vectors $\vec{v}$ and $\vec{w}$ is

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|\ |\vec{w}|}$$

iii. The projection of $\vec{v}$ onto $\vec{w}$ is

$$\left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right) \vec{w}$$

Here are two vectors:

$$\vec{v} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{w} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$$

For each of the following, do the arithmetic calculations. Make sure to show your calculations so that it's obvious which formula you are using.

a. What is the $|\vec{w}|^2$?

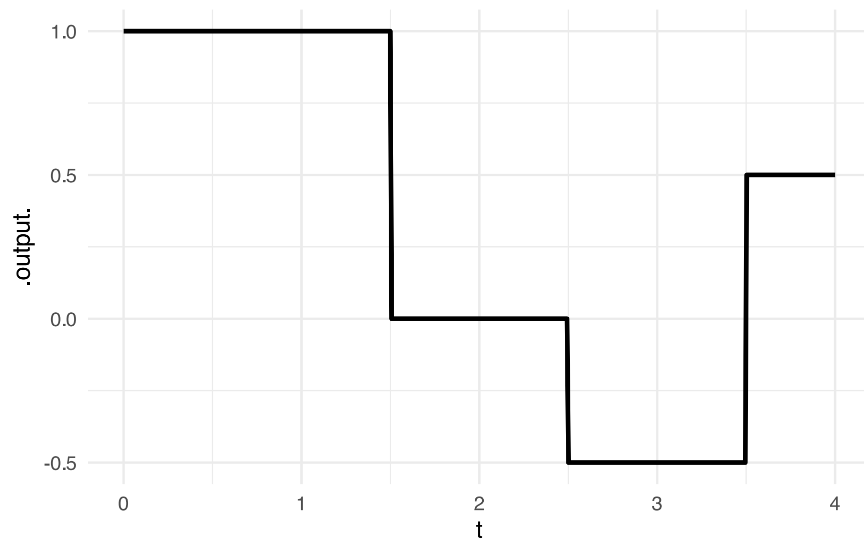
b. What is the angle between $\vec{v}$ and $\vec{w}$?

c. What is the scalar γ by which you would multiply $\vec{w}$ in order to produce a vector $\gamma\vec{w}$ that approximates $\vec{v}$ as closely as possible?

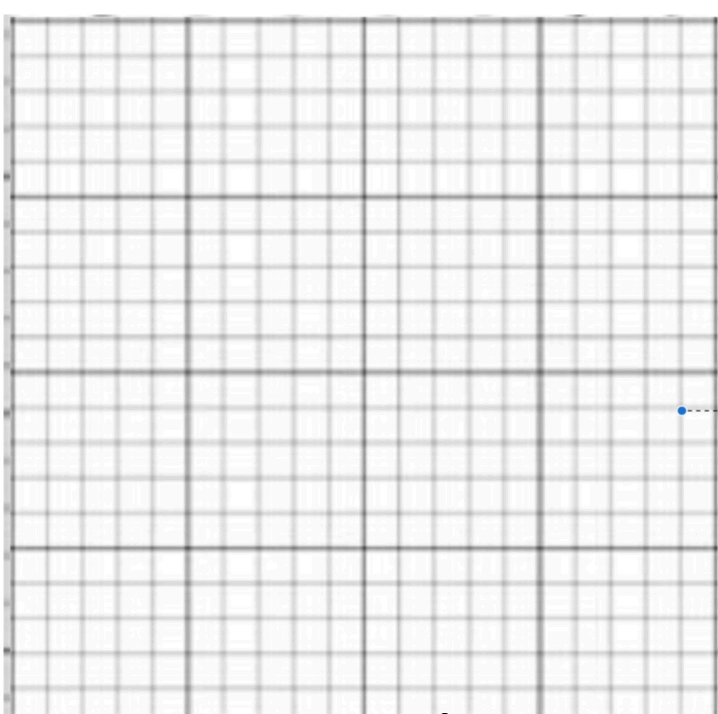
Question 0.6: Here is an R definition of a piecewise function $f(x)$.

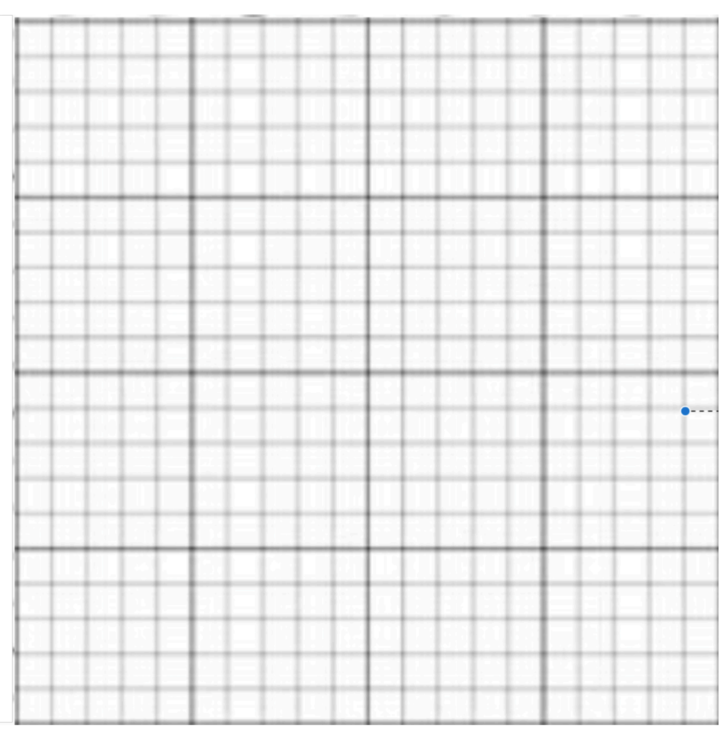
```
left <- function(x) {
  ifelse(x < 1.5, 1, 0)
}
right <- function(x) {
  ifelse(x < 3.5, -0.5, 0.5)
}
f <- function(x) {
  ifelse (x < 2.5, left(x), right(x))
}
```

Here's a graph of $f(t)$ versus t:



a. On the grid below, sketch $F(t)$, an anti-derivative of $f(t)$. Your sketch should be quantitatively correct (allowing for the natural imprecision of sketching by hand). You have two copies of the graph paper in case you make a mistake. Show clearly what is your final answer.

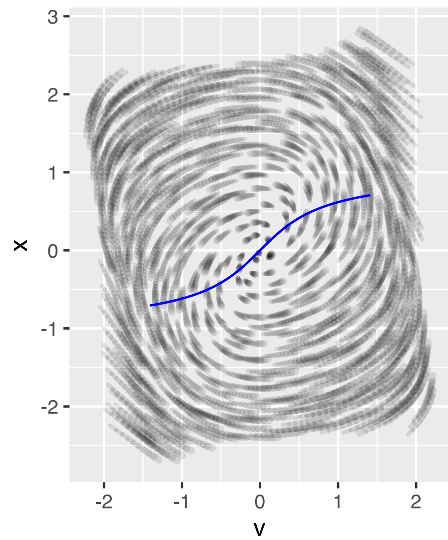




Question 0.7: Consider this dynamical system, whose flow is graphed below for $\mu = 1$:

$$\partial_t x = \mu\left(x - \frac{1}{3}x^3 - y\right)$$

$$\partial_t y = \frac{1}{\mu}x$$



a. There is only one fixed point in the system. Where is it?

b. Part of one of the nullclines is shown in the figure. Which is it?

c. The flow has places where nearby trajectories (i) get pushed together and places where nearby trajectories are (ii) pulled apart. Circle a small region of the state space where (i) applies and another where (ii) applies.

d. Starting from near (not at!) the fixed point draw a trajectory for the flow.

e. The Jacobian matrix corresponds to a linearization of the flow near the fixed point. It looks like this:

$$\begin{pmatrix} \partial_{xt}x & \partial_{yt}x \\ \partial_{xt}y & \partial_{yt}y \end{pmatrix} \text{ evaluated at the fixed point}$$

Use the dynamical functions to calculate the Jacobian and write it here:

f. The eigenvalues of the Jacobian are

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

i. Based on the eigenvalues, when $\mu = 1$ what is the stability of the fixed point? Use terms like these: Stable cycle, stable fixed point, unstable cycle, unstable fixed point,
ii. Think about increasing μ. At what value of μ does the stability undergo a *qualitative change* in stability from that in (i)?
iii. As μ increases above 2, what is the stability near the fixed point?