Richardson's Model of Arms Races: Description, Critique, and an Alternative Model

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This paper will present a mathematical model of arms races between two nations. The model will be a generalization of the classical one developed by Richardson in a pioneering work and reviewed recently by Rapoport. Following a review of Richardson's model[1] we will propose a reinterpretation of its parameters, and this will lead to a suggested generalization.[2]

[1] We are referring to the first model presented in Lewis F. Richardson, *Arms and Insecurity* (Chicago: Quadrangle Books, Inc., 1960), Chapter II. Richardson's "rivalry" and "submissiveness" models (*Ibid.*, pp. 35-36 and chapter IV) are mentioned briefly in this paper on p. 69. Richardson's original work was *Generalized Foreign Politics* (*British Journal of Psychology*, Monograph Supplement No. 23, 1939). The review article referred to is Anatol Rapoport, "Lewis F. Richardson's Mathematical Theory of War," *Journal of Conflict Resolution*, I (1957), pp. 249-299.

[2] Several authors have proposed reinterpretations and generalizations of Richardson's two nation arms race model along lines complementary to the one presented here. These include Robert P. Abelson, "A 'Derivation' of Richardson's Equations," *Journal of Conflict Resolution*, VII (1963), pp. 13-20; Kenneth E. Boulding, *Conflict and Defense: A General Theory* (New York: Harper and Row, 1962); M. D. Intriligator, "Some Simple Models of Arms Races," *General Systems*, IX, pp. 143-147; and Paul Smoker, "The Arms Race as an Open and Closed System," Paper delivered at the Fourth North American Peace Research Conference, Peace Research Society (International), Chicago, Illinois, November 1966. Abelson, Boulding, and Intriligator have all stressed the limitations of any model restricted to two nations and a very simplified view of decision-making. Nonetheless, they have felt the work was suggestive enough to be worth some further development. The author of this paper shares these reservations and is, if anything, more pessimistic. The utility of the approach is also discussed by Rapoport, *op. cit.*, pp. 281-282, and in his more recent work

I. *Richardson's model*

Richardson's starting point is three hypotheses about conditions under which nations will increase or decrease their armaments. These hypotheses are:

1. Out of fear of military insecurity, country *A* will make increases in its "armaments"[3] proportional to the level of country *B*'s armaments. *B* will respond in a similar way to *A*'s armaments.

2. The burden of armaments upon the economy of the country imposes a restraint upon further expenditure. This restraint is proportional to the size of the existing force.

3. There are hostilities, ambitions, and grievances that drive nations to arm at a constant rate in the absence of a military threat from another nation.

Richardson has neatly sidestepped the traditional chicken-or-the-egg dispute over which comes first, armaments or tensions. He includes both effects in his model and waits to see what consequences they will have for the system. Along with these effects, both of which involve incentives to increase armaments, he has included the restraining effect of the economic burden of maintaining existing military forces.

Richardson would have been the first to admit that these three hypotheses constitute a very simplified view of national decision-making on national security requirements. This is exactly his intention. His experience as a physicist taught him that judicious simplification can often illuminate important properties of very complex systems.

The first step in moving from this verbal formulation to a mathematical one is the introduction of parameters expressing the magnitude of the three hypothesized effects. Let the "defense" coefficients, k and k' (for nations A and B, respectively), indicate how large a change one nation will make in its own armaments in response to a unit arms level in the other nation. Let the "expense" coefficients, a and a', indicate the magnitude of the restraining effect of a unit arms burden. Let the "grievance" constants, g and g', indicate the magnitude of the motive to arm independent of the armaments level of the other nation.[4]

Fights, Games, and Debates (Ann Arbor: University of Michigan Press, 1960), pp. 84-103.

[3] "Armaments" is the word used by Richardson when first presenting his hypothesis. This usage, however, does not remain consistent. See footnote 5.

[4] In Richardson's notation the parameters are k and l, α and β, and g and h (*Arms and Insecurity, op. cit.*)

If we call the armaments levels x and y,[5] we are now is a position to translate the English sentences in which the hypotheses were presented into mathematical expressions. The net effect of the incentives to arm and restraints against arming will be a rate of change of armaments. The mathematical expression for a rate of change over time of x is the time derivative dx/dt. Thus we have:

$$dx/dt = ky - ax + g$$

$$dy/dt = k'x - a'y + g'$$

It should be noted that in this formulation something has been added to the original hypotheses. This is the assumption that the three separate effects postulated have a net effect that is additive. This choice of an additive relation, as opposed to one in which the effects interact, is another application of the simplicity criterion on Richardson's part.

II. *Stability conditions*

Of great interest in any discussion of arms races are the conditions under which there will be an equilibrium rather than an unlimited increase in armaments levels. Equilibrium will occur when the rate of change of armaments is zero for both sides. Thus to obtain the equilibrium condition we set the right side of both equations equal to zero and solve for x and y.

$$ky - ax + g = 0$$

$$k'x - a'y + g' = 0$$

The stability of the equilibrium is shown by a graphical method. We examine the sign of the derivatives in the four regions into which the positive x, y plane is divided by the lines along which the derivatives are zero.

We can see in figure 1-A that for any point, (x,y), the rates of change of x and y are such that the motion of (x,y) will be toward the equilibrium point, (x_0, y_0). In figure 1-B any point (x,y) will be moving away from the equilibrium point. Thus the condition for

[5] When he introduces the variables x and y, Richardson switches to the term "defenses." In places, however, this is used interchangeably with his original term "armaments" (*Ibid.*, pp. 15, 19, 20). For example, he states: "unilateral disarmament corresponds to putting $y = 0$ . . ." (*Ibid.*, p. 17), etc. Later, he operationalizes x and y in terms of arms budgets (*Ibid.*, p. 32). We will retain the original usage expressed in the verbal hypotheses upon which the mathematical model is based.

stable equilibrium is the condition that differentiates figure 1-A from figure 1-B. That is, the slope of the line for nation A is greater than the slope of the line for nation B, or

$$kk' < aa'$$

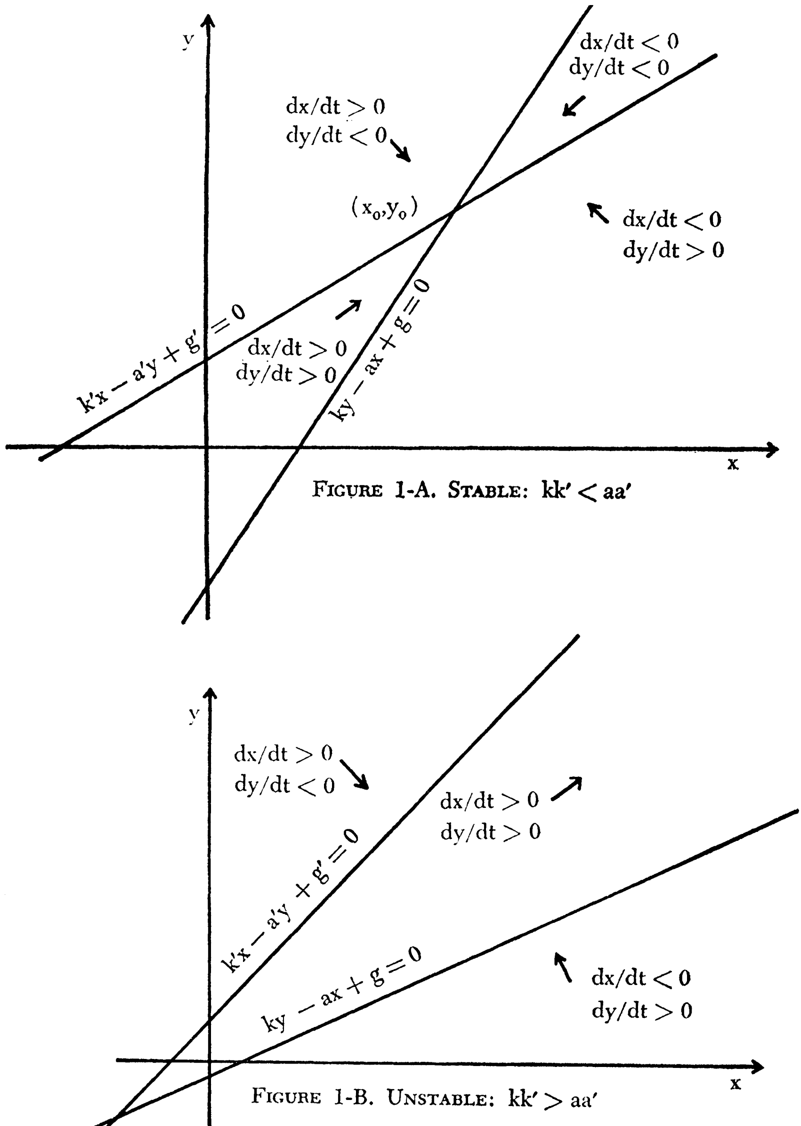

FIGURE 1-A. STABLE: $kk' < aa'$

FIGURE 1-B. UNSTABLE: $kk' > aa'$

FIGURE 1. Conditions for Stability

In other words, the product of the two defense coefficients is less than the product of the expense coefficients. Beyond the equilibrium point the burden of armaments in the system is greater than the incentive to arm. Below the equilibrium point the reverse is true.

This solution is completely independent of the grievance terms, g and g'. These terms do come into play, however, in determining the point at which equilibrium occurs, which is

$$x_0 = \frac{g'k + ga'}{aa' - kk'} \qquad y_0 = \frac{gk' + g'a}{aa' - kk'}$$

From this result we can see that the larger are g and g', the larger is the minimum level of armaments, (x_0, y_0), at which the equilibrium occurs.

In the unstable case, $kk' > aa'$, any initial value of x and y will lead to an arms race as long as g and g' are positive. If one admits negative values of g and g', however, a different result is possible. For values of x and y less than x_0 and y_0 there will be downward instability—a disarmament race. This makes intuitive sense. Below (x_0,y_0) the good will implied by negative values of the grievance terms is sufficient to restrain armaments competition. But above (x_0,y_0) security needs take over and each side arms in response to the other. (See figure 2.)

Thus the model has provided an answer to the debate over which comes first, armaments or tensions. It points to conditions—and specifies the conditions rather precisely—under which there can be an arms race despite good will. It also gives conditions when arms levels will be in equilibrium despite tensions. The key mechanism appears to be in the sensitivity of the nations to each other's arms levels. If the "defense" coefficients are large enough and the "expense" coefficients small enough, there can be a self generating arms race without either side having anything but defensive intent.

III. *Critique and reinterpretation*

If it produces nothing else, the mathematical effort would seem to be justified by these interesting qualitative conclusions. But are these conclusions sound? We do not mean to introduce here a host of reservations based on the complexity of the decision process or the incommensurability of the various elements of military power.

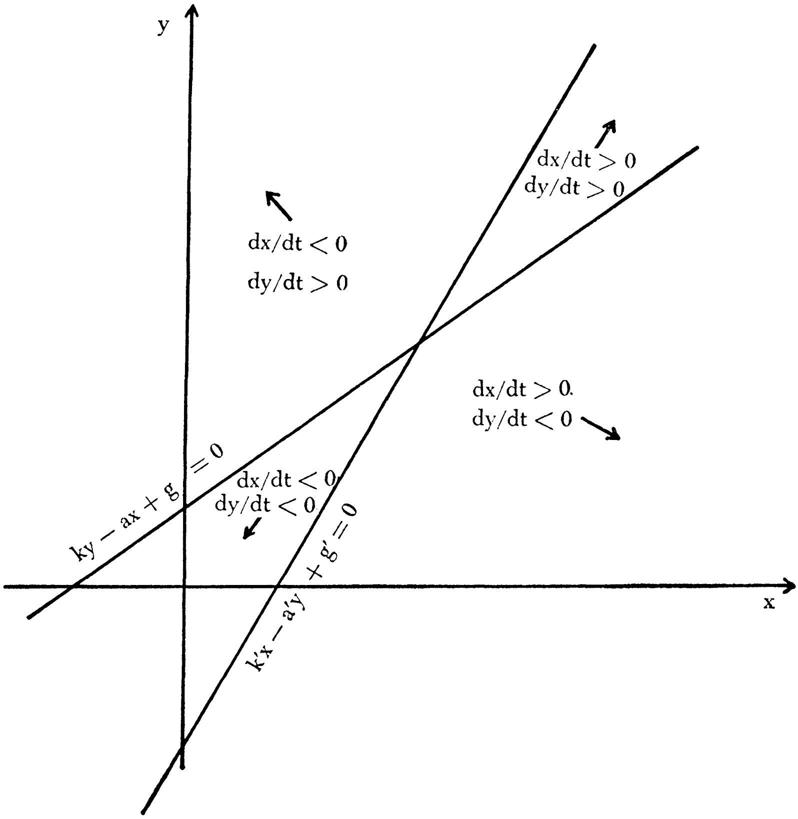

FIGURE 2. Conditions for Stability:
Unstable: kk' > aa', g and g' < 0

We merely want to ask whether Richardson's model adequately represents our simplest intuitions about the mechanisms of arms races, conventional or nuclear.

In particular, does it appear upon closer examination that Richardson's first hypothesis is really plausible? The intuitive basis for this hypothesis was the reasonable enough notion that a state will arm to preserve its security in the face of the armaments of another state. But Richardson expresses the incentive to arm as proportional to the *absolute* level of the other state's armaments. A nation's security, in fact, is dependent upon the *relative* size of its own and its opponent's forces. The opponent's forces are threatening only if the first nation's defenses are inadequate. Suppose, for example, that nation *B* is heavily armed and nation *A* is either disarmed or armed to the same level as *B.* Does it seem reasonable to suggest, as Richardson's first hypothesis does, that the rate of change of *A*'s armaments would be the same in both cases?

Richardson does appear to have had some misgivings about this, for he attempts to formulate a model of "rivalry" to take it into account.[6] To do this he substitutes the expressions $k(y - x)$ for the original ky, and $k'(x - y)$ for the original $k'x$. The resulting equations are

$$dx/dt = ky - (k + a)x + g$$

$$dy/dt = k'x - (k' + a')y + g$$

This results in the condition for stability being

$$kk' < (k + a)(k' + a')$$

$$kk' < kk' + ak' + a'k + aa'$$

$$0 < aa' + ak' + a'k$$

But since the constants are all positive, there would always be stability. Richardson satisfies himself with the conclusion "rivalry as formulated here is not an adequate description of the interaction between nations" and returns to his original model.[7]

Should we be so easily satisfied? And if not, then how can we find another way to formulate the hypothesis of "rivalry"? Let us take another look at Richardson's model.

$$dx/dt = ky - ax + g$$

$$dy/dt = k'x - a'y + g'$$

We can see that in fact the rate of change, dx/dt, is not dependent upon y alone. It is a function of both x and y because of the presence of the term $- ax$. This becomes even clearer if we rearrange the expression as follows.

$$dx/dt = a(ky/a - x + g/a)$$

If we think of the expression in parentheses as the desired increase in arms, we do have a situation in which the rate of change depends upon the relative arms levels.

[6] *Ibid.*, pp. 35-36. Richardson also develops a "submissiveness" model which includes a term in the difference between the arms levels. But in this model, the larger the difference, the greater the incentive to disarm, which is the opposite of what we are discussing here (*Ibid.*, Chapter IV).

[7] *Ibid.*, p. 36.

To be exact, this desired arms level for nation A is the difference between its own arms level, x, and a constant, k/a, times y plus another constant, g/a. Thus the nation is represented as not simply seeking arms equality, but striving to maintain at least a constant ratio between its forces and its opponent's. Since a certain ratio of offensive to defensive firepower is generally necessary to win—or to defend—an objective, this representation is not unreasonable. Richardson's model, therefore, reduces to what Intriligator terms a "ratio goal model."[8]

But what becomes of the economic restraints which were represented in the model by the term $(-ax)$? The constant a still plays the role of the reciprocal of the "relaxation time."[9] That is, when y is zero, the size of a determines how fast x will move toward zero, indicating how burdensome a given level of arms is. But when dx/dt is positive, a now determines how fast x catches up with $\frac{k}{a}y$. This is a very different role than Richardson intended the constant to play. A high value of a, indicating a heavy economic burden, now results in rapid increases in arms when the nation gets behind. This anomaly can be resolved by assigning different values to a when the derivative is positive and when it is negative. The adequacy of that solution will be discussed further in a later section.

In the light of these reinterpretations, what significance can be attached to the constants, g/a and g'/a'? If we set y equal to zero and look at the condition $dx/dt = 0$, we get

$$dx/dt = 0 = a(-x + g/a)$$

$$x = g/a$$

Thus g/a and g'/a' can be interpreted as the minimum acceptable arms levels. This is not inconsistent with Richardson's interpretation of them as "grievance" constants. The minimum level acceptable to an aggressor nation might be that necessary for conquest. On the other hand, g/a and g'/a' might also represent the minimum of forces required for internal security.

In any case, the presence of the constant terms does not seem to be sufficient to represent aggressive motivations. The level, g/a or g'/a', might be sufficient for conquest when the other nation is

[8] Intriligator, *op. cit.* He recognizes the identity in form between Richardson's model and the ratio goal model but not the possibility of their identity in substance.

[9] Richardson, *Arms and Insecurity*, *op. cit.*, p. 20.

disarmed, but not when it is well defended. In that case a certain ratio of forces is necessary to overcome those defenses. Thus the ratios, k/a and k'/a', may express aggressive as well as defensive motives. At a given state of military technology there is, in principle, some minimum ratio of forces that is sufficient for defense. The ratios, k/a and k'/a', can take on higher values, however, and a nation's choice of these higher values would be indicative of aggressive intentions.

By this time, although we have retained the mathematical formalism of Richardson's model, we have attached an entirely new meaning to it. Richardson's original three hypotheses have been called into question and have been replaced by the following hypotheses:

1. In order to maintain national security, nations will strive to maintain some minimum safe ratio between their forces and their opponents' forces.
2. If nations have aggressive motives, they will strive to achieve some ratio of forces large enough for victory in war. In case of disarmament by the other nation the aggressor will seek to retain a sufficient level for conquest.
3. A certain minimum of forces for internal security will be maintained even if the other side totally disarms.

Let us take another look at Richardson's stability conditions in the light of these changes. The condition for stability is

$$kk' < aa'$$

This can be re-expressed as

$$(k/a)(k'/a') < 1$$

The condition can now be seen as a restriction on the sizes of the ratios of forces sought by the two nations. This is a fairly severe condition. Since the product must be less than one, at least one of the ratios must be less than one. If one nation demands even a slight strategic advantage, the other must be content with an even greater strategic disadvantage. These conditions could be met, however, at times when defense has the advantage over offense.

The location of the equilibrium point also can be seen in a new light. If the two nations demand minimum levels of arms even when the other side is disarmed, this sets a lower limit. When both sides have these minimum force levels, each will add slightly more to secure itself against the other. Thus we obtain the above expressions for x_0 and y_0.

What of Richardson's conclusion that stability is independent of the "grievance" levels? Since we have now expressed aggressive motives through the constants, k/a and k'/a', as well as g/a and g'/a', this interesting finding must be relinquished. We must settle for a less novel but more plausible conclusion: (a) that security needs alone might be enough to generate an arms race under conditions which give an advantage to the offense in war; but (b) that strong enough aggressive motivation on the part of one or both of the nations will also generate an arms race in an otherwise stable situation.

We must also give up the conclusion that negative values of g' and g, representing good will, would lead to a disarmament race in the unstable case for low enough values of x and y. If g'/a' and g/a represent minimum levels needed for internal security, they can never be negative. Richardson's conclusion is plausible enough, however, for us to seek some way to restore it. This could be done by introducing threshold values of the opponent's arms below which a nation wouldn't respond. In other words, for y less than y_t, the constant, k/a, would drop to zero. Similarly, for x less than x_t, k'/a' would be zero. The height of the threshold would represent the amount of good will or trust.

An even more striking departure from Richardson's original conclusions is seen when we notice that the stability conditions involve only the desired force ratios and are unaffected by economic restraints. The ratios, k/a and k'/a', it should be emphasized, are not functions of a and a', they are new constants determined independently on military and political grounds. a and a' remain in the equations as multiplicative constants, but they have no effect on the conditions, $dx/dt = 0$, and $dy/dt = 0$.

IV. *Economic restraints*

Can anything be done to restore economic restraints in the model to a role in determining equilibrium and stability? We have taken over for other purposes Richardson's term, $(-ax)$, but his "expense coefficient," a, is still present. As well as determining the relaxation time when the other side's armaments go to zero, a determines the catching up time when the other side is ahead. Can this be seen as an adequate expression of economic restraints? The catching up time would be affected by bottlenecks in a country's raw materials supply and production system. But this effect would be negligible for low production demands and would impose an absolute maximum on yearly increases. Thus a simple constant times the desired arms level won't represent the bottleneck phenomenon. The catching up time will have to be viewed, then, as reflecting other—presumably military—considerations.

What is still missing, in any case, is some representation of Richardson's original notion that the maintenance of existing arms levels is itself a burden. Can this be represented by restoring the original term, $(-ax)$, which we have been using for other purposes? That this representation was insufficient is clear if we realize that Richardson's equations admit of unlimited arms races. At the very least the model must have some ceiling representing the gross national product of the country less a minimum amount for domestic consumption. This limitation may not be important in practice if most arms races occur at levels well below the ceiling.

But should the economic restraints enter only at the exhaustion point? In addition to this abrupt cutoff, a "law of diminishing returns" seems to be needed here. As the arms costs press more and more upon scarce resources, it should be harder and harder to extract budget increases from the society.

A look at the data for the European arms race in 1913 shows that defense costs[10] were running at roughly five percent of national income.[11] At present, United States military expenditures are running at about 10% of gross national product with the Soviet percentage probably being somewhat higher. These data do not indicate that either of these two major arms races came close to the point at which diminishing returns might have halted them. Thus the absence of economic restraints may be a rather good approximation after all. On the other hand, it is easy to imagine arms races among countries not so prosperous as the European powers and the United States and the Soviet Union. Indeed we are witnessing arms races today in underdeveloped areas (though somewhat distorted by outside aid). It seems worthwhile, then, to think further about how we might represent the effects of economic scarcity.

[10] *Ibid.*, p. 32.

[11] Paul Studenski, *The Income of Nations: Theory, Measurement, and Analysis* (New York: New York University Press, 1958).

V. *A generalized model*

At this point we will proceed, as promised in the introduction to this paper, to construct a new model. It will be a generalized model, expressing diminishing budget returns for high levels of arms and reducing to Richardson's equations for low levels.

Let C be the "ceiling" or "cutoff" value of resources available to nation A for military purposes and have the dimensions of dollars per year. Let M be the cost per unit of maintaining existing forces. Then $(C - Mx)$ represents the amount of resources available for new procurement. The desired arms increase must be paid for out of the amount $(C - Mx)$. For values of the desired arms increase whose cost approaches or exceeds $(C - Mx)$, the actual rate of change, dx/dt, will be considerably below the desired level because of diminishing returns. The same will be true for nation B, with $(C' - M'y)$ representing the available national resources.

To represent these features we will make use of a function that has the general form, $f(z) = C(1 - e^{-z/C})$. This function has several properties that are particularly desirable for our purposes. For values of z/C which are very much less than one, $f(z)$ is approximately equal to z.[12] This feature can be used to represent the hypothesis that for very low levels of arms, the burden of maintaining existing forces will be negligible. For moderate values of z/C, $f(z)$ gets to be less than z, representing the phenomenon of diminishing returns.[13] Finally, for very large values of z, the expo-

[12] The behavior of the function for small values of z/C can be seen by taking the power series expansion of the exponential and neglecting all the terms beyond the first two because they get very small.

$$e^{-z/C} = 1 - (z/C) + \frac{1}{2!}(z/C)^2 - \frac{1}{3!}(z/C)^3 +, -, \dots$$
$$e^{-z/C} \approx 1 - (z/C)$$
$$f(z) = C(1 - e^{-z/C}) \approx C(1 - 1 + z/C)$$
$$f(z) \approx z$$

[13] To show that $f(z)$ becomes less than z as z increases we show that

$$\frac{d}{dz} f(z) < \frac{d}{dz}(z) = 1$$
$$\frac{d}{dz} C(1 - e^{-z/C}) = -C \frac{d}{dz} e^{-z/C}$$
$$= -C\left(-\frac{1}{C}\right) e^{-z/C}$$
$$= e^{-z/C}$$

which is less than one. This is the desired result.

nential term, $e^{-z/C}$, gets very small and the whole expression approaches C. This can be used to represent the phenomenon of an absolute ceiling on arms expenditures.

Despite the fact that the equations will now be nonlinear, the choice of the exponential function here is made under Richardson's canon of simplicity. The function is simple in the sense that it embodies only one parameter, C. It is also simple in that the exponential function is a familiar one with a number of well known properties. The exponential function is also a good choice for a first guess since it is known to characterize many physical and social phenomena.[14]

Other possible choices with the same properties would have been the functions, $f(z) = C[1 - 1/(1 + z)]$, and $f(z) = \tanh(z)$ —the hyperbolic tangent of z. The first of these would impose more severe diminishing returns at lower levels of arms than the one we have chosen. The second function would impose less severe diminishing returns. If one violates the simplicity criterion by allowing curves characterized by several parameters, then a whole host of other functions becomes possible.

By substituting the desired increase in arms for z, and the available resources for C in our function, $f(z)$, we get a new expression for dx/dt and dy/dt. This expression represents economic restraints and reduces to Richardson's equations for small values of x and y. The constant, a, is still playing the same role, determining the relaxation time and catching up time.

$$p\, dx/dt = a(C - Mx)\ (1 - e^{-ND/C})$$

$$p' dy/dt = a'(C' - M'y)\ (1 - e^{-N'D'/C'})$$

where D and D' are desired arms increases given by

$$D = ky/a - x + g/a$$

$$D' = k'x/a' - y + g'/a'$$

N and N' are the costs of new arms procurement per armaments unit. Under some technological and economic conditions N may be quite different from the maintenance cost, M. The constants, p and p', are needed to change the dimensions of the left side of the equations from arms per unit time to dollars per unit time. When D is less than zero and the nation is disarming, p will be equal to the maintenance cost per unit, M. For D greater than zero, p equals

[14] Rapoport, *Fights, Games, and Debates, op. cit.*, p. 39.

the cost per unit of new procurement, N. We shall refer to this pair of equations as Model 2 to distinguish it from Richardson's model.

Model 2 is complicated by the introduction of several new parameters, but not enough, we feel, to obscure the model's implications or even make computation especially difficult. The new parameters represent attributes that are naturally quantified in gross national product figures and defense budget documents. Thus they present less measurement difficulties than such intangibles as "grievances."

An arms race characterized by the equations of Model 2 will proceed more slowly than one characterized by the Richardson equations. This is guaranteed by the expression $(C - Mx)$ which gets small for large x, and the exponential expression which gets smaller for large desired increases. Does this damping of the arms competition have any implications for the stability properties of the system? Does it introduce any new equilibrium points or make less restrictive the conditions required for stability? Surprisingly enough, the equilibrium conditions remain precisely the same (except for the additional equilibrium at the saturation point). The condition for equilibrium is that the rates of change be zero.

$$dx/dt = 0 = a(C - Mx)\ (1 - e^{-ND/C})$$

$$dy/dt = 0 = a'(C' - M'y)\ (1 - e^{-N'D'/C'})$$

Since the equations are identical in form we need only consider the first one. If the derivative is zero, either $(C - Mx)$ equals zero—which gives the equilibrium at the saturation point, or

$$(1 - e^{-ND/C}) = 0$$

$$e^{-ND/C}) = 1$$

$$D = 0$$

That is, the condition, $D = 0$, is the equilibrium condition for both Richardson's model and Model 2.

As for the stability condition, Richardson's derivation can be taken over directly. This derivation depends entirely upon the sign of the derivatives—which is to say, the sign of D and D′—for different values of x and y. It does not depend upon the magnitude of the derivatives. But in Model 2, the sign of the derivative is always the same as the sign of D or D'. For D positive the negative exponential term is less than one, so that one minus this term is positive.

For D negative, the negative exponential term is greater than one and the whole expression becomes negative.

Why is equilibrium in Model 2 not easier to attain given the fact that economic restraints have been introduced? The answer seems to be that the diminishing returns property operates only on projected increases in weaponry. That is, only D is included in the exponent. Once an arms level is reached it is maintained unless a reduction in the opposing forces makes part of it militarily superfluous (i.e., unless D is less than zero). Since any spending program has a certain inertia due to vested bureaucratic and clientele interests this may be an accurate picture of reality. If this is so, then the very restrictive Richardson conditions on stable equilibrium are still justified despite the stronger economic restraints in Model 2.

VI. *A model with stronger economic restraints*

A somewhat different result is obtained if we make the whole projected armaments level—including the existing level plus desired new procurement—subject to diminishing returns. That is, we put $(ND + Mx)$ and $(N'D + M'y)$ into the exponents. The model then takes the form

$$\frac{p}{a}(dx/dt) = C(1 - e^{-(ND+Mx)/C}) - Mx$$

$$\frac{p'}{a'}(dy/dt) = C'(1 - e^{-(N'D'+M'y)/C}) - M'y$$

This pair of equations will be referred to as Model 3. Like Model 2 it is a generalization of Richardson's model, reducing to that model for small values of x and y.

Let us examine the equilibrium properties of this model. We have already shown that

$$C(1 - e^{-z/C}) < z$$

Therefore

$$\begin{aligned}\frac{p}{a}(dx/dt) &= C(1 - e^{-(ND+Mx)/C}) - Mx \\ &< (ND + Mx) - Mx \\ &< ND\end{aligned}$$

At $dx/dt = 0$, the equilibrium condition for nation *A*, we have

$$0 < D$$

Similarly, at $dy/dt = 0$, we have

$$0 < D'$$

Thus the equilibrium condition is less strict than in Richardson's model which required that *D* and *D′* be equal to zero. This makes possible a new equilibrium point at intermediate levels between minimum armaments and saturation. The exact values of *x* and *y* at this equilibrium are a function of the size of the parameters. In particular the equilibrium level gets higher as the ratios, *k/a* and *k′/a′*, get larger.

Let us look now at the exact solution of the equilibrium equations. Again, since they are identical in form, we will treat the equation for nation *A* only.

$$\frac{p}{a}(dx/dt) = 0 = C(1 - e^{-(ND+Mx)/C}) - Mx$$

$$e^{-(ND+Mx)/C} = 1 - Mx/C$$

Let $Mx/c = X$. This new variable gives the current arms budget as a fraction of the maximum budget, *C*, so it takes on values from zero to one.

$$e^{-(X+ND/C)} = 1 - X$$

Taking the natural logarithm of both sides we get

$$-X - ND/C = \log(1 - X)$$

$$\frac{N}{C}(ky/a - x + g/a) = -\log(1 - X) - X$$

$$\frac{N}{C}\frac{k}{a}y - \frac{N}{M}X + \frac{Ng}{Ca} = \log\frac{1}{(1-X)} - X$$

$$\frac{N}{C}\frac{k}{a}y = \left(\frac{N}{M} - 1\right)X + \log\frac{1}{(1-X)} - \frac{N}{C}\frac{g}{a}$$

Let $M'y/C' = Y, \frac{M}{C}\frac{g}{a} = G, \frac{M}{C}\frac{g'}{a'} = G'$; Then,

$$Y = \frac{M'}{N}\frac{C}{C'}\frac{a}{k}\left[\left(\frac{N}{M} - 1\right)X + \log\frac{1}{(1-X)} - \frac{N}{M}G\right]$$

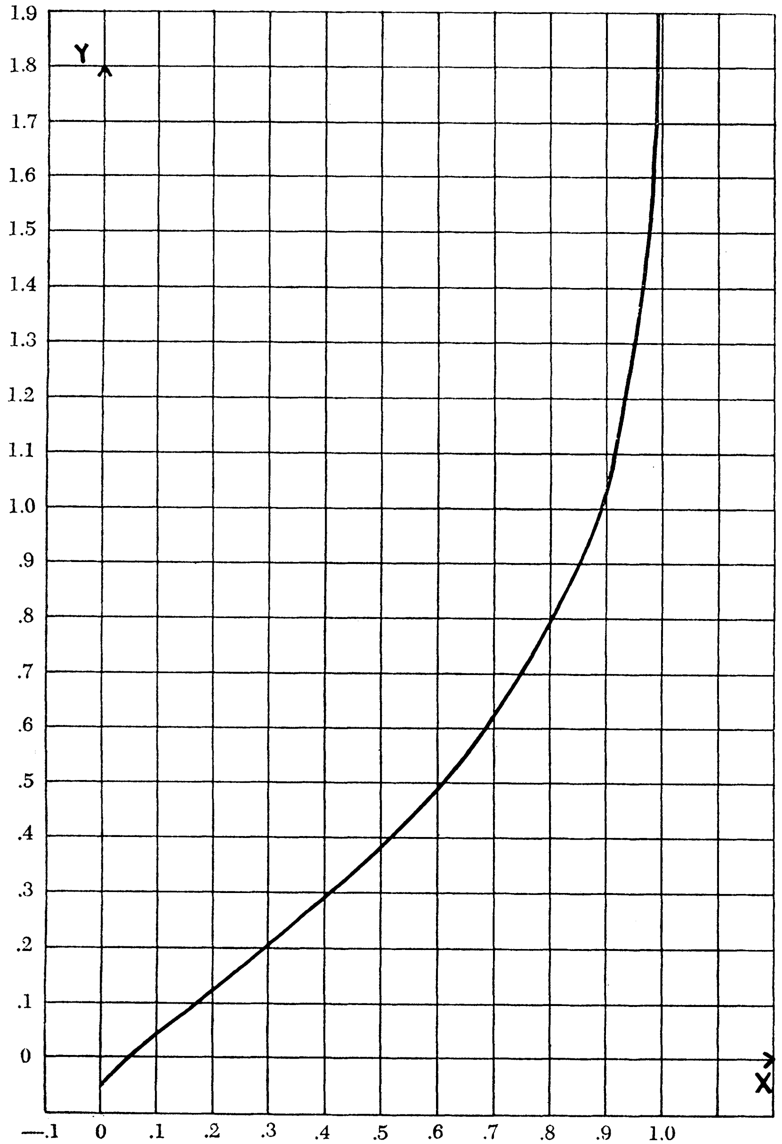

FIGURE 3. Equilibrium Condition for Nation *A*

This expression of Y as a function of X is the solution of the first equilibrium condition. A graph of it is shown in figure 3.[15] The graph has a positive X-intercept which represents nation *A*'s minimum acceptable arms level. The function goes to infinity as *X*

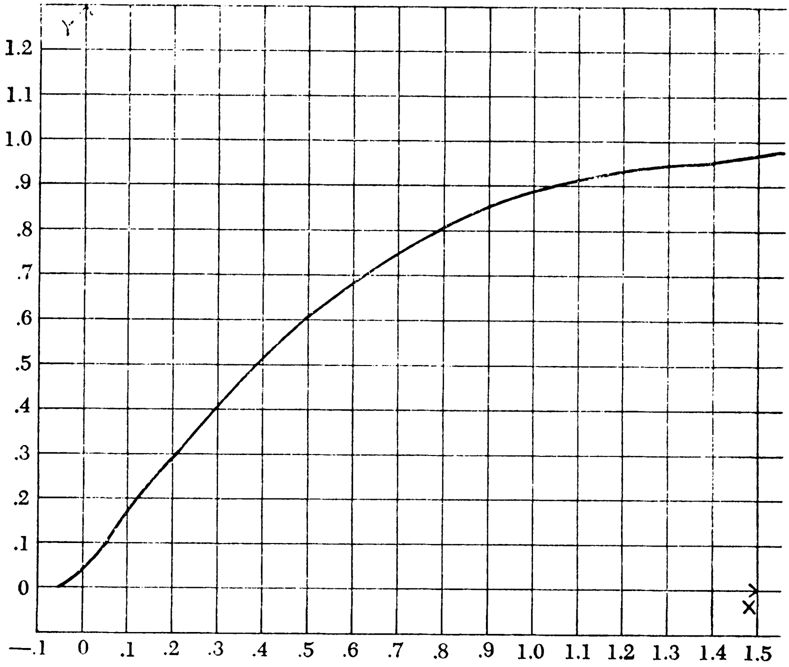

FIGURE 4. Equilibrium Condition for Nation *B*

approaches 1. This indicates that *Y* must get terribly large before nation *A* will attempt to maintain values of *X* near the maximum despite the economic burden.

Similarly we get for nation B

$$X = \frac{M}{N'}\frac{C'}{C}\frac{a'}{k'}\left[\left(\frac{N'}{M'} - 1\right)Y + \log\frac{1}{(1-Y)} - \frac{N'}{M'}G'\right]$$

The graph of this function is shown in figure 4. In form it is the inverse of the equation for nation *A*, although the parameters are different.

The intersection of the two curves, as shown in figure 5, gives

[15] The curves shown in figures 3, 4, and 5 are based on the parameter values used in the example given on pp. 19-20 with the additional constraint that $k/a = k'/a' = 1.5$ (see the intersection of the fourth row and fourth column in table 1, p. 20).

the equilibrium point.[16] An analytic solution of the pair of equilibrium equations is not possible, but we can calculate the equilibrium values of x and y numerically. There is an analytic proof, however, of the existence and uniqueness of the equilibrium point.[17]

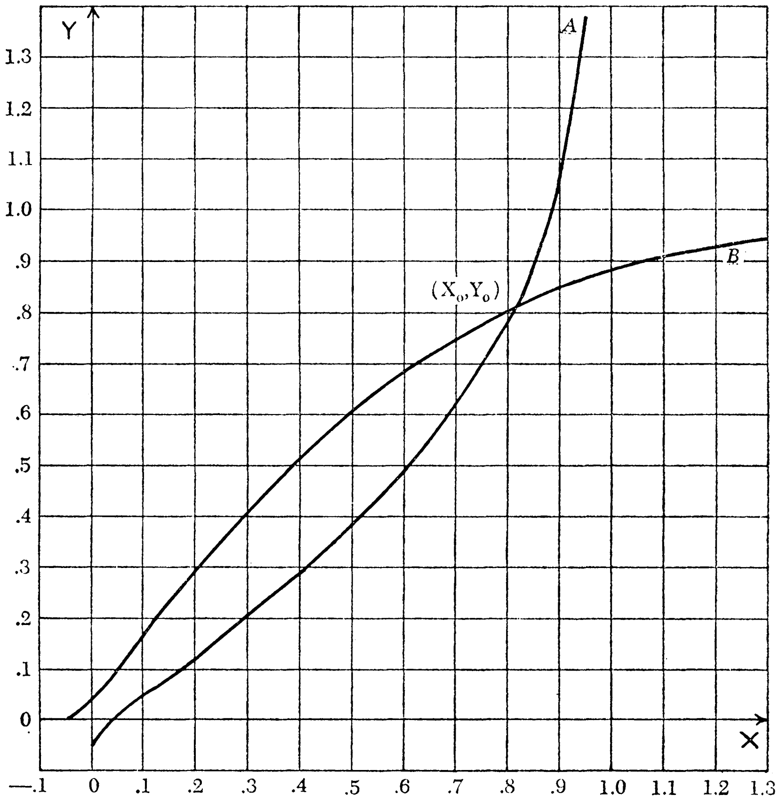

FIGURE 5. Equilibrium Point at Intersection of Curves *A* and *B*

The stability of the equilibrium is easily shown in the same graphical manner that Richardson used. We examine the sign of the de-

[16] On qualitative grounds Intriligator has argued for the inclusion in an arms race model of what he calls "constraint curves." These have roughly the same substantive basis and the same form as the curves of figure 5. We have, then, provided a mathematical derivation for Intriligator's constraint curves.

[17] The proof follows from the fact that for curve *A*, the slope is increasing throughout the interval zero to one. That is, d^2Y/dX^2 is everywhere positive. And for curve *B*, the slope is decreasing throughout the interval zero to one. That is, d^2Y/dX^2 is negative everywhere. Although we can't solve for *Y* as a function of *X* for nation *B*, we can ascertain dY/dX because it is the inverse of dX/dY.

rivatives in the four regions into which the plane is divided by the two curves. This shows that any point, (x,y), not identical with the equilibrium point, (x_0, y_0), will be moving toward the equilibrium point.

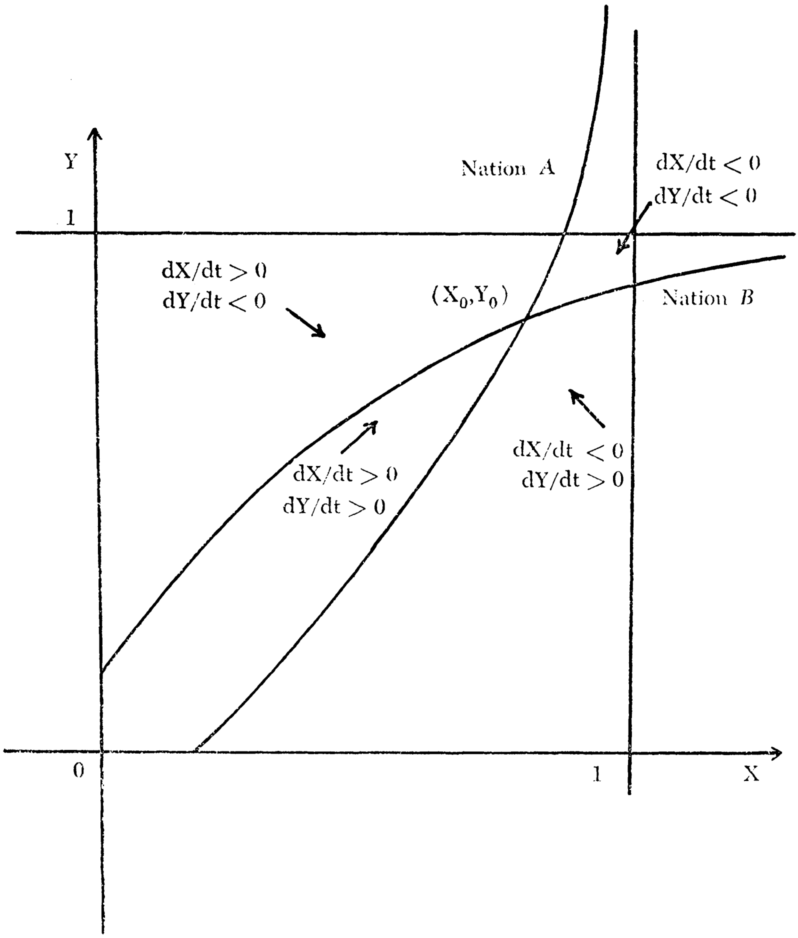

FIGURE 6. Stability in Model 3

VII. *Conclusions from the model*

Given that there exists a stable equilibrium point, where will this point fall for various values of the parameters? For simplicity's sake, let us first consider a symmetrical case in which the parameters are the same for both nations. We can examine departures from this idealized situation later. Let this be a case in which the force-ratios sought only slightly exceed the Richardson stability condition, say, $k/a = k'/a' = 1.05$. Suppose that these are wealthy nations so that the costs of the minimum acceptable arms levels are small compared to the maximum, say, $G = G' = .05$. If we are dealing with technologically sophisticated weapons systems for which the cost of research and development are very high, this would suggest a large value of N, new equipment cost, as compared to M, maintenance cost. Let $N/M = 2$.

The equilibrium point, obtained by a graphical method, has the value, $X = Y = .36$. To the author this was a surprising result. It suggests that if the ratios are only a slight amount above the Richardson equilibrium condition then the arms race will proceed to a point beyond one third of the countries' maximum available resources. For wealthy nations this allows considerable scope for an arms race. In short, the effects of economic restraints are a lot less limiting than might have been expected. It is such unanticipated consequences of a formal representation on which the case for the use of mathematical models in social science is based.

Let us now examine the range of positions of the equilibrium as the two ratios, a/k and a'/k', are varied. This result is presented in Table 1.

Several conclusions can be drawn from the figures in this table. First, the range of variation is very wide. For values of the ratios ranging only from .8 to 2.0, the location of the equilibrium point ranges from (.16C, .18C) to (.94C, .94C). Second, for values of the ratios on the order of 2 or more, the diminishing returns property has only marginal effects, as indicated in the (.94C, .94C) value. Third, the diminishing returns effect tends to narrow the gap between the two force levels. In the case in which the ratios are 2.0 and 0.8, the resulting force levels are .75C and .52C. Thus the leading nation has an advantage of less than 50% instead of 100%.

If we vary the constant, g/a, this affects the size of the desired force level and therefore the location of the equilibrium. The results are shown in Table 2. The values of the other parameters are the same as in the first example with the exception of $k/a = k'/a' = 1.2$. Increasing the size of these constants raises the equilibrium point. The magnitude of the effect, however, is fairly small. That is, if we vary the constants all the way from zero to one third of all possible resources the resulting range of equilibrium values is only .21 out of a possible 1.00.

The equilibrium point also varies with the ratio of the cost of

TABLE 1

EQUILIBRIUM POINT OF ARMS LEVELS, (X_0, Y_0), AS A FUNCTION OF THE DESIRED ARMS RATIOS, k/a AND k'/a' *

k'/a'	k/a .8	1.0	1.2	1.5	2.0
.8		.18, .16	.30, .25	.49, .37	.75, .52
1.0	.16, .18	.27	.44, .41	.65, .55	.84, .66
1.2	.25, .30	.41, .44	.58	.74, .68	.89, .76
1.5	.37, .49	.55, .65	.68, .74	.81	.92, .86
2.0	.52, .75	.66, .84	.76, .89	.86, .92	.94

* The curves in figure 5 are calculated for the case k/a = k'a' = 1.5 in this table.

TABLE 2
EQUILIBRIUM POINT OF ARMS LEVELS, (X_0, Y_0), AS A FUNCTION OF THE MINIMUM ACCEPTABLE ARMS LEVELS, G AND G' (FOR THE SYMMETRICAL CASE $G = G'$)

G = G′ =	0	.05	.10	.15	.20	.25	.30
X = Y =	.53	.58	.62	.66	.69	.72	.74

new arms to maintenance costs, N/M. This effect is also fairly small, though by no means negligible, as can be seen in Table 3.

TABLE 3
EQUILIBRIUM POINT OF ARMS LEVELS, (X_0, Y_0), AS A FUNCTION OF THE RATIO OF NEW ARMS COSTS TO MAINTENANCE COSTS (FOR THE SYMMETRICAL CASE, $G = G'$)

N′/M′ = N/M =	1.00	1.25	1.50	1.75	2.00	2.50	3.00
X = Y =	.45	.49	.52	.55	.58	.63	.68

Although a higher value of N is a brake on the speed of arms accumulation, it tends to make the equilibrium point higher rather than lower. This is because for a given desired arms level, a large value of N gives a high desired budget. Even though in the equations the desired budget is cut down, still the higher the demand, the higher the actual value.

So far we have been representing the equilibrium points in terms of X and Y, which are measures of cost. How much arms these numbers represent is a function of C/M, the ratio of maximum available funds to cost per unit. For poor countries characterized by small C/M, the location of the equilibrium point may be crucial. A sufficiently low equilibrium may prevent war altogether by keeping military forces below the minimum strength necessary for waging war. Thus small differences in parameter values take on added significance for this case.

Of course, theoretical questions about arms races include the question of when they break down into war, as well as the question we have been considering about their laws of motion. Though the breakdown requires a separate theory of its own, we can guess that the likelihood of breakdown would, all other things in the environment being equal, be a function of both the absolute levels and the size of the arms gap. If this is so, then the present equations have supplied results to serve as input to a theory of breakdown.

We have pointed out several times that the form of the model and the size of the parameters affect the speed of the arms race as well as the equilibrium and stability conditions. Let us compare Richardson's model with Model 3 for the same values of the parameters as used in the previous example. That is, let $k/a = 1.2$, $M/N = \frac{1}{2}$, $a = \frac{1}{2}$, $M(g/a)/C = .05$. In order to be able to calculate the changes over time without a complete solution of the equations, let us imagine a discrete case in which arms budgets change only at the end of the year (which is, of course, like the budget process, but leaves out the supplemental appropriations, carryover of unexpended funds, etc.). The results of the calculation are shown in Table 4 which shows the arms level as a function of time in years for Richardson's model and Model 3.

TABLE 4
THE PACE AND DURATION OF ARMS RACES IN RICHARDSON'S MODEL & MODEL 3: ARMS LEVELS, *X* AND *Y* AS A FUNCTION OF TIME

		Time in years						
Time in years		0	3	5	10	15	20	25
X and Y	Richardson's model	.05	.15	.23	.46	1.00	1.77	3.03
	Model 3	.05	.11	.19	.33	.44	.52	.56

It can be seen from the table that both arms races start slowly and pick up speed. But in Model 3 the diminishing returns property is already evident by the third year and results in a marked slowing down after the twentieth year. For Model 3, *X* and *Y* approach assymptotically the equilibrium value of .58. For Richardson's model, *X* and *Y* increase without limit. The slow pace of this arms race is due to our choice of the small value of $k/a = 1.2$.

VIII. *Future directions*

Once the simple models have been fully explored, both theoretically and empirically, we will be justified in moving on to more complex models which might better satisfy our intuition and account for more of the data. The process of exploration of the simple Richardson model is well under way now. In addition to the present paper and the theoretical works cited, a number of attempts at empirical application have been made.[18]

It is not hard to envision some of the directions which the elaboration of the model might take. First, the simple ratio goal might

[18] Richardson, *Arms and Insecurity, op. cit.*; Paul Smoker, "A Mathematical Study of the Present Arms Race," *General Systems,* VIII (1963), pp. 51-60; same author, "A Pilot Study of the Present Arms Race," *Ibid.*, pp. 61-76; same author, "The Arms Race: A Wave Model," *Peace Research Society (International), Papers,* IV (1965), pp. 151-192; same author, "The Arms Race as an Open and Closed System," *op. cit.*

be replaced by a function expressing the requirements in the nuclear age for various strategies such as finite deterrence, graduated deterrence, infinite deterrence, and preventive war. Second, an affective dimension could be added. At present emotions find a place in the model in high desired ratios for purposes of conquest. But the affective element is static—a fixed grievance or aggressive feeling. In fact, one suspects that emotions are a dynamic part of the system, stimulated by weapons levels and in turn stimulating weapons production.

Third, anticipation could be accounted for. That is, desired levels would be based not solely upon the present relative levels but also upon the anticipated levels of the opponent at some future time. In practice, this would make dx/dt a function of dy/dt as well as of y. The anticipation process appeared very strikingly in the United States fears in 1958 of a missile gap which would come in the early 1960's given the expected rate of Soviet missile production.

Fourth, provision would be made for the fact that the economies of the two nations might be growing. If the economies are simply growing at a steady rate this is accomplished by replacing C with $C(1 + rt)$ where t is the time, and r is a rate constant. A more interesting problem is the interaction between arms production and economic growth. Under certain conditions arms production might damage an economy, but we also have evidence that it can be a stimulant to economic growth.

IX. *Summary and conclusions*

An examination of Richardson's hypotheses has led us to question some of the assumptions on which his model is based. We suggested that it was satisfactory to retain Richardson's mathematical formulation if different interpretations were given to its parameters and terms. In particular we interpreted the equations as indicating that each nation was trying to maintain a certain ratio plus a constant between the forces of the two sides. This reinterpretation exposed Richardson's inadequate representation of economic restraints imposed by the burden of maintaining existing forces. The model building effort in this paper, therefore, was devoted to representing the process of economic restraint.

On the basis of the reinterpretation of his equations, we discounted several of Richardson's conclusions about stability. In particular, we considered untenable the claim that the occurrence of stable equilibrium will be independent of grievances or aggressive motives. The construction of a new model led to some further conclusions. We found that arms races could end in equilibrium at levels of arms between minimum levels and the maximum created by the limits of a nation's wealth. In particular we noted that these equilibrium values came out rather higher than might have been anticipated from the verbal theory alone.