# 142Z Day 28: Brownian Motion

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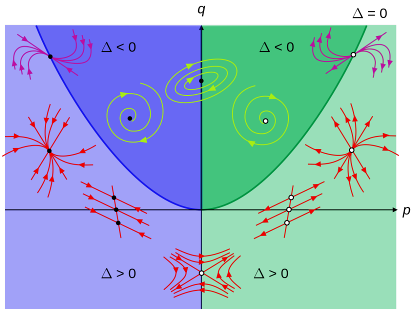
# Wrapping up dynamics

- We've systematically covered linear dynamics (local modeling) in one and two dimensional state spaces.
- We've talked a little bit about nonlinear dynamics (global modeling) using low-order polynomials.
- The next logical topic in terms of importance of use would be **linear dynamics with inputs**, e.g. the transient movement of a car suspension when hitting a series of bumps in the road or the principles for building a control system for dealing with such inputs. You will see this in "systems engineering" type courses, but we don't have time to cover it here in CalcZ. (Which is to say, "We have more important things to cover that traditionally are not we'll treated in downstream engineering courses even though the concepts of crucial to any kind of quantitative design work in fields from engineering to public health.)

# Are you where you need to be?

If you can make sense of the following diagram, you are in a good place.




$$\frac{dx}{dt} = Ax + By \qquad p = A + D$$
$$\frac{dy}{dt} = Cx + Dy \qquad q = AD - BC$$
$$\Delta = p^2 - 4q$$

Source

# Today's topic: Diffusion and random processes

- The classical topics of calculus were developed from about 1700 to 1850.
- We've added some others that are more recent (and so don't appear in most calculus texts), e.g.
  - eigenvalues and eigenvectors
  - linear combinations and least squares
  - computing

# Is diffusive motion calculus?

Today we're going to cover a new topic that isn't in calculus textbooks, **diffusive motion**.

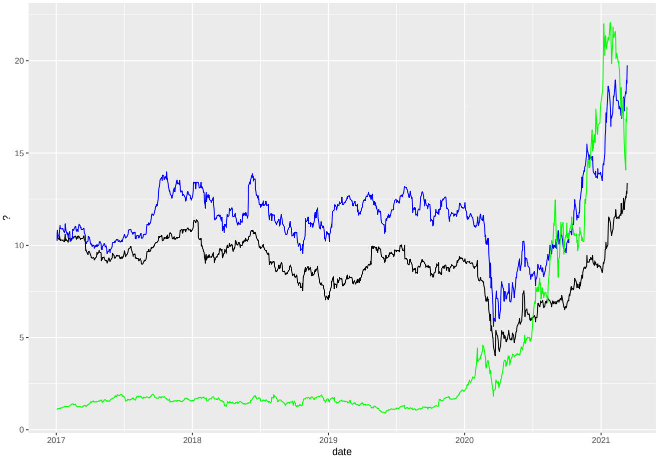
This was not anticipated by the people who developed calculus in 1700 to 1850 for a simple reason: Diffusion occurs on very small length and mass scales: from $100 \mu m$ (0.1 mm) down to molecular size.

- The molecular theory of matter started to be accepted broadly only around 1900.
- Experimental techniques for directly observing diffusive motion was developed only after 1900. (The 1926 Nobel Prize in Physics was awarded to Jean Baptiste Perrin (1870–1942) for this work, which was based on a revolutionary 1905 theoretical paper by Albert Einstein.)

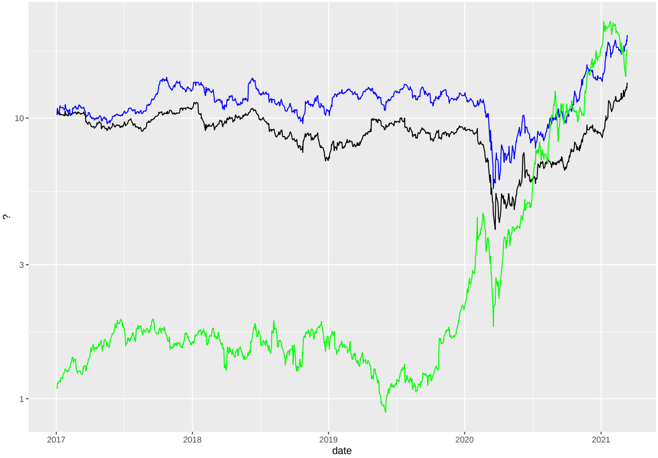
This kind of process turns out to be fundamental to **statistics** and to **finance**.

# What kind of dynamics?

What kind of dynamics might be behind this time series?



# Using semi-log axes



# Tools for understanding such randomness

1. The "normal" distribution, which we have adopted in CalcZ for our standard hump function.
   a. parameters: mean and variance (and standard deviation $= \sqrt{\text{variance}}$)
   b. shape: bell-shaped, characteristic length of tails. Z-scores
2. A new differential equation, the **diffusion equation**: $\partial_t C(x,t) = \partial_{xx} C(x,t)$
   a. The shape of the "normal" distribution is like a fixed point: It stays the same. But the variance changes with time in a simple, characteristic way.
3. A "molecular-level" perspective on diffusion: random walks.
   a. Applications to matter.
   b. Applications to finance. ("*volatility*")
4. Combining deterministic exponential dynamics with random walks: the basis of Black-Sholes.
5. Black Swans

# Our standard hump function

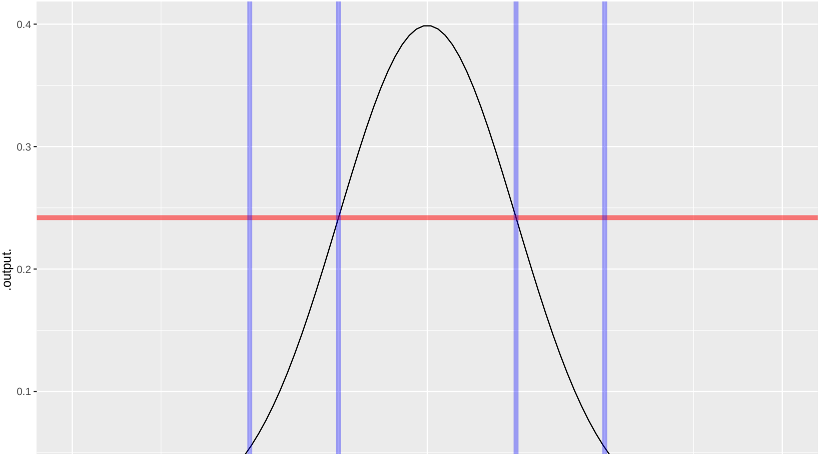
Hump functions are all of a kind, so as long as they are smooth and local there's not much reason to worry about the details.

- The hump function we have been using, which we typically call `hump()` is actually called a **Gaussian** which has this algebraic form:

$$g(x) = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}_{\text{Just a number}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$$

# What you need to know ... 1. Bell shaped

```
g <- makeFun((1/sqrt(2*pi*sigma^2)) *
                  exp(- (x - m)^2 / (2*sigma^2)) ~ x,
                  m = 2, sigma = 1)
```



## 2. Two parameters

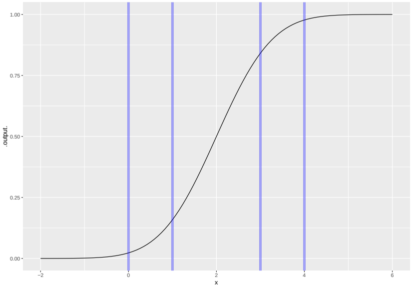
i. mean $m$: location of peak (that is, $\mathrm{argmax}_x\, g(x)$)
ii. variance $\sigma^2$ - Standard deviation $= \sqrt{\text{variance}}$ is half-width of mountain at height $exp(-1/2)$ of peak height.

## 3. Often called “normal distribution”

Particular in the social sciences

# 4. $\int_{-\infty}^{x} g(x)dx$ is a lovely sigmoid.

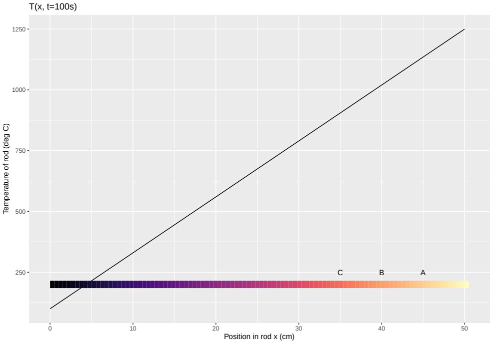
```
G <- antiD(g(x) ~ x, lower.bound = -Inf)
slice_plot(G(x) ~ x, domain(x=c(-2, 6))) %>%
  gf_vline(xintercept = c(0, 1, 3, 4), color="blue", size=2
```



# Differential dynamics of heat diffusion

*Source of image*

# Temperature as a function of x




The temperature changes with position $x$ and time $t$ and is a function $T(x, t)$.

# Time evolution of $T(x, t)$

We're interested in $\partial_t T(x, t)$

- $\partial_t T(x, t)$ is set by a balance between "fluxes"
  - heat moving from neighboring hotter places to $x$ \$$\partial_x T(x + h, t)$
  - heat moving from $x$ to neighboring cooler places.

$$\partial_x T(x, t)$$

- Net flux is $\partial_{xx} T(x, t)$

# Diffusion Equation

$$\partial_t T(x,t) = D\partial_{xx} T(x,t)$$

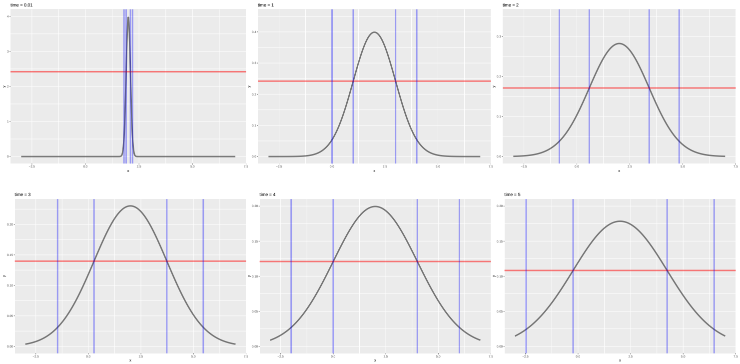
Constant $D$ is called the **diffusion coefficient**.

[Note, you are not expected to be able to solve this algebraically. Such equations are dealt with under the name *partial differential equations* which is an advanced college-level math course.]

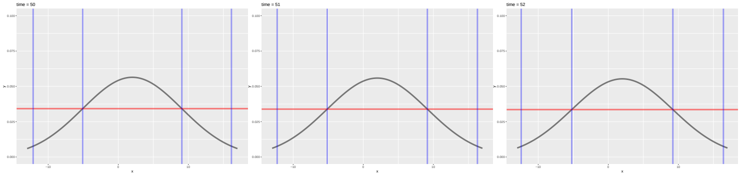
But you can understand the dynamics *qualitatively*:

# Where will it go up? down?

# At several times



# Longer times




[Link to Movie showing evolution]{www/gaussian.gif}

# Take-home point

- Typical distance travelled by a diffusing particle is

$$\sqrt{D\,t}$$

  - D is **diffusion coefficient**
- Velocity over this time is

$$\sqrt{\frac{D}{t}}$$

Diffusion becomes very *slow* as time goes by.

# Example: Ionic and Molecular diffusion

The diffusion coefficient for **sodium ions** in water is about

$$2 \times 10^{-5} cm^2/s$$

whereas for a big molecule like **glucose** the coefficient is about

$$6 \times 10^{-6} cm^2/s$$

If diffusion were the sole mechanism of spread, how long would it take a sugar molecule to travel 1cm, about half the distance from a sugar cube to the side of a teacup?

**Paradox** Nothing changes in the dynamics of an individual particle, but statistically it gets slower!

## Random Walks microscopically

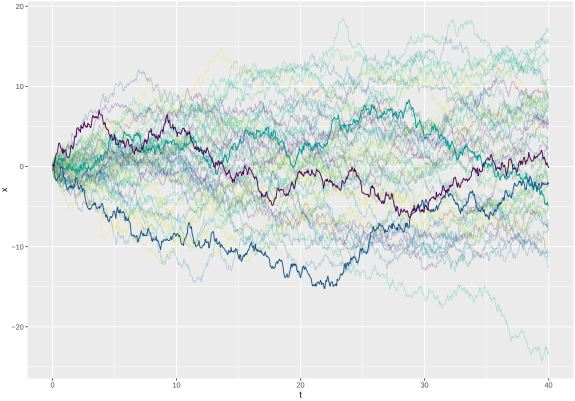
Taking the perspective of individual particles . . .

Videos of particles undergoing random walks

Early 1900s movie here

Diffusion of pollen grains

# Random walks mathematically



# Example: Stock Prices

- A random walk is the standard model of stock prices.
- Usually, the deterministic trend per day (if any) is much smaller than the random daily variation.
- Typical day-to-day variation in stock prices is about 1%.
- Typical year-to-year variation is therefore . . . ? (250 trading days in a year)

# Volatility: $\sigma$

The *volatility* of an asset (such as a stock price) is the standard deviation of the daily return over many days:

$$\text{daily return} \equiv \text{price}_{t+1}/\text{price}_t$$

Or, in terms of diffusion coefficient,

$$\text{volatility} \equiv \sqrt{D \cdot 1\text{day}}$$

Volatility depends on the time scale

# Deterministic growth

Our simplest, non-trivial differential equation is

$$\dot{x} = \alpha x \quad \Longrightarrow \quad x(t) = Ae^{\alpha t}$$

In terms of investment, $\alpha$ is the daily return.

# Random shock growth

The shocks accumulate to a random walk, with dynamics:

$$\dot{x} = \sigma W$$

where $W$ is the derivative of a random walk.

# Exponential growth with random shocks

$$\dot{x} = (\alpha + \sigma W)x$$

# The solution to $\dot{x} = (\alpha + \sigma W)x$

Solving this "stochastic differential equation" is beyond the scope of this course, but you should be able to understand the result.

You might think it would be

$$x(t) = Ae^{\alpha t + \text{random walk}}$$

But this is not the case. The actual solution is

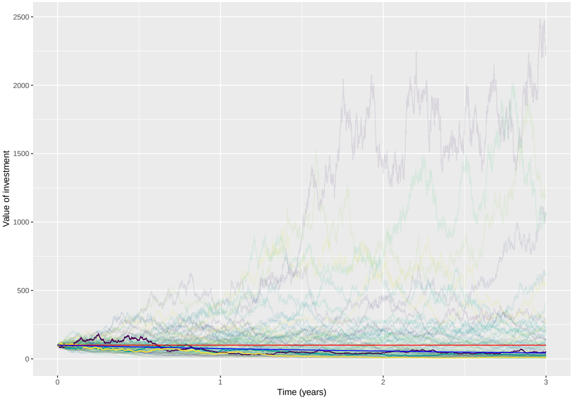
$$x(t) = A \exp\left[(\alpha - \frac{\sigma^2}{2})t\right] + \text{random walk}$$

Volatility takes a systematic bite out of deterministic returns.

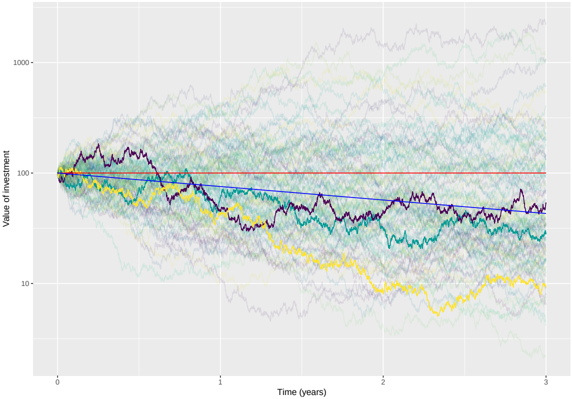
# Simulating many investments $\alpha = 0\%$ and $\sigma = 75\%$.

- red line is deterministic return (no gain or loss)
- blue line shows expected return with volatility included.

# On linear axes



# On semi-log axes



**Options traders**, or at least the systematically successful ones, use the blue line to evaluate the value of an option. The method is called the **Black-Sholes** formula.

In 1997, Myron Sholes and Robert Merton won the Nobel Prize in Economics for this work.

# Black Swans