Dynamics overview

true

2022-03-10

Differential equations is rightly called “a language of science,” since so many fundamental scientific topics are expressed this way, from Newtonian motion to quantum physics to relativity. They are also a very important tool in engineering design and modeling generally. Block 6 provides an introduction to “ordinary differential equations.”

We use the term “dynamics” rather than “differential equations” to emphasize our concern with the **qualitative phenomena** seen in dynamical systems, especially the stability of fixed points. Analytic solutions are derived only in simple linear, constant coefficient systems and some straightforward integration of motion in a constant gravitational field. Often, we use numerical solutions to demonstrate or confirm some aspect of theory.

Following the usual pattern in *MOSAIC Calculus*, much of the focus is on two-dimensional systems. One reason for this emphasis is that graphics can be effective, providing a non-algebraic path to understanding many of the concepts. Another reason is physical: Fundamental concepts in Newtonian dynamics are position, velocity, and acceleration and these can be captured in two-dimensional systems.

A **state space** (a.k.a. “phase space” or “phase plane”) approach is used almost exclusively, up until Chapter 54. The components of a state space representation of dynamics are introduced at the beginning: Chapter 47. In a nutshell, each possible instantaneous configuration of the system is a point in the state space. The dynamics are represented as a flow in this state space, that is, a description of how a system in any given state changes state over time. For Newtonian dynamics, for example, simple ballistic motion (in one spatial dimension) has two state components: position x and velocity $v = \partial_t x$.

The dynamical flow is represented as a vector field: the assignment of a vector to every point in state space. Graphically, this is shown by drawing a representative collection of the vectors in the field.

Alternatively (and equivalently), the dynamical flow is presented as a **system** of differential equations, where there is one equation for each component of the vector flow. If x and v are the state variables, $\partial_t x$ and $\partial_t v$ are the components of the flow vector.

$$\partial_t x = f(x, u) \partial_t v = g(x, u)$$

The left side of this system is the instantaneous change in state. The functions on the right side, $f()$ and $g()$ here, are functions of the instantaneous state. We’re calling these functions the **dynamical functions** because they describe the dynamical rule.

Another way of representing a dynamical system with two state variables is with a **second-order** differential equation, which looks like this:

$$\partial_{tt} u = h(x, u) \partial_t u + w(x, u)\ .$$

In contrast to the system of first-order differential equations, where it’s very easy to draw the flow, there doesn’t seem to be any easy, general method of drawing a picture of the second-order differential equation. (Instead, what is presented is a graph of the **solution**, that is $u(t)$. This leads to a focus on solutions rather than the dynamical phenomenon.)

When we do use second-order DEs in *MOSAIC Calculus*, it is mainly to show students a format that they will encounter in other courses and, in the case of linear, second-order DEs, to demonstrate the characteristic equation and its link to eigenvalues (Chapter 54) or to examine the response to forcing (Chapter 55).

Role in modeling

Differential equations (or, equivalently, flow fields in state space) are an important framework for modeling. The flow-field/dynamical-function format is a way of organizing knowledge. Rather than having to keep in mind the whole time evolution of the system, the modeler can focus on the instantaneous change of state (as a function of state).

There's a saying:

> *Think globally, act locally.*

Differential equations work the other way round:

> **Model locally, let the global stuff sort itself out by the mechanics of flow.**

Review of material

Block 6 brings together many of the tools developed earlier in *MOSAIC Calculus*, and so provides an opportunity for review.

- Modeling, especially with low-order polynomials
- Integration
- Vectors and matrices
- Numerical approximation and iteration