CSE 3380 – Homework #8

Assigned: Thursday, April 5, 2018
Due: Thursday, April 12, 2018 at the end of class

Note the following about the homework:

1. You must show your work to receive credit.
2. If your submission has more than one page, staple the pages. If I have to staple it, the cost is 10 points.

Assignment:
Process

1. Given vectors

$$\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

what is the orthogonal projection of $\vec{y}$ onto $\vec{u}$?

2. Find the closest point to $\vec{y}$ in the subspace spanned by $\vec{v}_1$ and $\vec{v}_2$.

$$\vec{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

This is asking for the orthogonal projection of $\vec{y}$ onto the subspace spanned by $\vec{v}_i$.

3. Orthogonalize the following set of vectors using the Gram-Schmidt procedure.

$$\left\{ \begin{bmatrix} 4 \\ -10 \\ -4 \\ -8 \end{bmatrix}, \begin{bmatrix} -6 \\ -14 \\ 4 \\ -12 \end{bmatrix} \right\}$$

4. Orthogonalize the following set of vectors using the Gram-Schmidt procedure.

$$\left\{ \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 36 \\ 0 \\ 0 \end{bmatrix} \right\}$$

5. Orthogonalize the following set of vectors using the Gram-Schmidt procedure.

$$\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 3 \\ -3 \end{bmatrix} \right\}$$

6. (hand solution) If

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 0 \\ 8 \\ 20 \end{bmatrix}$$

find the least squares solution $\vec{x}^*$.

7. (hand solution) If

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

find the least squares solution $\vec{x}^*$.

8. (hand solution) Use linear regression to determine the coefficients of $y_i = \beta_0 + \beta_1 x_i$ for the following set of x and y values.

x	1	2	4	7
y	-1	0	1	1

Theory

9. If we have an $m \times n$ real matrix A, show that $A^T A$ is symmetric. Do this for the general case, which means for a generic $m \times n$ matrix. Hint: this is easier using dot products.

10. In order to find the least squares solution of a system of equations using the normal equations, we must be able to produce $(A^T A)^{-1}$. This requires that A be of full column rank. If A is an $m \times n$ matrix and $m < n$ (i.e., less rows than columns), then it cannot be of full column rank. Show why $A^T A$ is noninvertible in this case. You can consider the case where A is a general 2×3 matrix instead of $m \times n$; this doesn't mean to use specific numbers.

    Hint: if A is wider than it is tall, then we know that at least one of the columns must be a linear combination of the other columns.

Applications

11. **(CS application: data modeling)** (Python solution) One application of least squares is to find a function that best fits a set of data. For example, given a set of t values and the corresponding $f(t)$ values, we might want to find the coefficients of $f(t) = a_0 + a_1 t + \cdots + a_n t^n$. However, we might not know in advance the degree of the polynomial that best fits the data. We can try polynomials of different degrees and compare the residuals to see which polynomial best fits the data.

    (a) On the course website is a file, `leastSquaresMain.py`, that contains two sets of data. Each set contains a set of t values and the corresponding $f(t)$ values. This function will give each set of values to a function, `leastSquaresStudent.py`, with the signature

    ```
    x_lin, norm_l, x_q, norm_q = leastSquaresStudent(t, ft)
    ```

    (b) Write the Python function `leastSquaresStudent()` such that it

    i. sets up the appropriate A matrix and $\vec{b}$ vector for the linear model and the quadric model to be used below.

ii. uses least squares to fit the data to a linear model, $f(t) = a_0 + a_1 t$. These values will be returned as the vector

$$\text{x\_lin} = \left[\begin{array}{cc} a_0 & a_1 \end{array}\right]$$

iii. calculates the norm of the residual for the linear model, that is,

$$\text{norm\_lin} = ||\vec{b} - A\vec{x}||_2$$

iv. uses least squares to fit the data to a quadratic model, $f(t) = a_0 + a_1 t + a_2 t^2$. These values will be returned as the vector

$$\text{x\_q} = \left[\begin{array}{ccc} a_0 & a_1 & a_2 \end{array}\right]$$

v. calculates the norm of the residual for the quadratic model, that is,

$$\text{norm\_q} = ||\vec{b} - A\vec{x}||_2$$

(c) As an exercise for yourself, use the norms and the plots to determine which model you think best fits the data.

General requirements about the Python problems:

a) As a comment in your source code, include your name.

b) The Python program should do the work. Don't perform the calculations and then hard-code the values in the code or look at the data and hard-code to this data unless instructed to do so.

c) Your function should use the data passed to it and should work if I were to change the data.

d) The program should not prompt the user for values or read from files unless instructed to do so.

e) I don't want your function to cause anything to print; leave it to the `XXXmain.py` file to do the printing of what is returned.

To submit the Python portion, do the following:

a) Create a directory using your net ID in lowercase characters. This should be something of the form `abc1234`.

b) Place your .py files in this directory.

c) Zip the directory, not just the files within the directory. You must use the zip format and the name of the file, assuming your net ID is `abc1234`, will be `abc1234.zip`.

d) Upload the zip'd file to Blackboard.

Review Questions

These are not for credit, but instead are intended to test your understanding of the concepts. You should be able to answer these without simply regurgitating equations.

1. What does it mean for two vectors to be orthogonal?

2. What is the difference between orthogonal and orthonormal sets?
3. What is the orthogonal projection of $\vec{x}$ onto $\vec{v}$?
4. What makes a matrix an 'orthogonal matrix'?
5. What problem is least squares intended to solve? That is, why did we need to learn it?
6. What is the relationship between least squares and Col(A)?
7. What is the relationship between $\vec{b}$ and Col(A) when $A\vec{x} \neq \vec{b}$.