CSE 3380 – Homework #3

Assigned: Thursday, February 8, 2018
Due: Thursday, February 15, 2018 at the end of class

Note the following about the homework:

1. You must show your work to receive credit.
2. If your submission has more than one page, staple the pages. If I have to staple it, the cost is 10 points.

Assignment:
Process

1. For each set of partitioned matrices below, find AB using multiplication of partitioned matrices. The partitions to use are given.

$$a)\ A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \end{array}\right], \qquad B = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \hline 1 & 3 \\ 2 & 4 \end{array}\right], \qquad b)\ A = \left[\begin{array}{c|cc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \end{array}\right], \qquad B = \left[\begin{array}{cc} 1 & 2 \\ \hline 3 & 4 \\ 1 & 3 \\ \hline 2 & 4 \end{array}\right]$$

2. Find the determinants of the following:

$$a)\ A = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}, \qquad b)\ A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix},$$

$$c)\ A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 6 & 0 \end{bmatrix}, \qquad d)\ A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 6 & 6 & -3 \\ 0 & 0 & 1 & 9 & 10 \\ -2 & -4 & -6 & -7 & 1 \\ 0 & 3 & 18 & 18 & -8 \end{bmatrix}$$

Theory

For each of the following, use the rules of matrix algebra to show that the relationship is true. Don't replace the matrices by matrices with actual numbers nor generic matrices in which you use dot products to perform the multiplication. You should be able to show these using the representation of a matrix by a single letter, which is one of the strengths of the matrix representation of the relationships between values.

3. If $A, B \in \mathbb{R}^{n\times n}$, show that

$$A(B^{-1}(A+B)A^{-1})B = A + B$$

4. If $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times q}$, and $D \in \mathbb{R}^{q \times t}$, show that

$$(ABCD)^T = D^T C^T B^T A^T$$

Note that your answer should not be that since

$$(AB)^T = B^T A^T$$

then it must be true for any number of matrices in the product. However, that rule may be useful for proving it. The point of this exercise is to begin to see how the rule we learned for transposing the product of two matrices generalizes to more than two matrices instead of just assuming it is true.

Applications

5. **(CS application: computational geometry)** Given a set of points that represent a geometrical object (e.g., circle, line, plane, etc.) and an additional point, how can we determine the positional relationship between the object and the point? In some cases, the determinant can help us. See [Eri05] and [Wee12] for more details.

Here are some examples for 2D objects, but there are also 3D versions.

(a) Given three points A, B, and C, with coordinates (a_x, a_y), (b_x, b_y), and (c_x, c_y), respectively, twice the area of the triangle ABC is

$$\begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix} = \begin{vmatrix} a_x - c_x & a_y - c_y \\ b_x - c_x & b_y - c_y \end{vmatrix}$$

This value can be positive or negative, so we would need to take the absolute value when using to find the area. If the determinant is

- negative, then points A, B, and C are in clockwise order.
- positive, then points A, B, and C are in counterclockwise order.
- zero, the three points are collinear.

A practical use for this is that if we have the line AB then we can determine where point C is relative to the line.

(b) Given four points A, B, C, and D, with coordinates (a_x, a_y), (b_x, b_y), (c_x, c_y), and (d_x, d_y), respectively, where points A, B, and C lie on the edge of a circle in counterclockwise order, the determinant can be used to determine the relationship of point D to the circle.

$$\begin{vmatrix} a_x & a_y & a_x^2 + a_y^2 & 1 \\ b_x & b_y & b_x^2 + b_y^2 & 1 \\ c_x & c_y & c_x^2 + c_y^2 & 1 \\ d_x & d_y & d_x^2 + d_y^2 & 1 \end{vmatrix} = \begin{vmatrix} a_x - d_x & a_y - d_y & (a_x - d_x)^2 + (a_y - d_y)^2 \\ b_x - d_x & b_y - d_y & (b_x - d_x)^2 + (b_y - d_y)^2 \\ c_x - d_x & c_y - d_y & (c_x - d_x)^2 + (c_y - d_y)^2 \end{vmatrix}$$

- If the determinant is zero, then point D lies on the circle.
- If the determinant is positive, then point D is inside the circle.
- If the determinant is negative, then point D is outside the circle.

Problems for part 5

(a) We have the following points: $A = (5, 6)$, $B = (2, 3)$, and $C = (6, 4)$. Use the formula above to determine if the points A, B, and C are clockwise, counterclockwise, or collinear. Also, draw the positions of the points to see where this says that C is relative to line AB.

(b) We have the following points: $A = (0, 2)$, $B = (-2, 0)$, $C = (0, -2)$, and $D = (4, 0)$. If points A, B, and C lie on a circle in counterclockwise order, use the formula above to determine where D is relative to the circle. Also, draw the circle that points A, B, and C lie on and indicate where point D is relative to the circle.

6. **(CS application: computational geometry)** An engineer deploys a set of sensors. Each sensor reports its location as a point in two-dimensional space. The points are $(2, 7)$, $(8, 2)$, $(6, 11)$, $(6, 5)$, and $(11, 6)$. Use determinants to calculate the area of the polygon formed by the sensors.

   Also, plot the polygon formed by the points (you'll need to do this anyway to see the proper order). The polygon that you produce should enclose the most area; no lines should cross. This will show you the proper order to use for the pair of points to use in each determinant.

   The area of the polygon is found by taking one half of the sum of the determinants formed by each successive pair of points. That is, if the points in counterclockwise order (clockwise will give you the negative of the area) are (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), then the area is found by

$$\text{area} = \frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \cdots + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix} \right)$$

References

[Eri05] Christer Ericson. *Real-Time Collision Detection*. Morgan Kaufmann, San Francisco, CA, 2005.

[Wee12] Bob Weems. CSE 5311 Notes 17: Computational Geometry. http://ranger.uta.edu/~weems/NOTES5311/cse5311.html, accessed September 16, 2012.