CSE 3380 – Homework #5

Assigned:	Thursday, March 1, 2018
Due:	Thursday, March 8, 2018 at the end of class

Note the following about the homework:

1. You must show your work to receive credit.
2. If your submission has more than one page, staple the pages. If I have to staple it, the cost is 10 points.

Assignment:
Process

1. For each of the following sets of vectors, determine which forms a basis for $\mathbb{R}^3$. For those that do not, which span $\mathbb{R}^3$?

$$a) \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad b) \quad \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$$

2. Find a basis for the column space of each set of vectors in question 1.
3. Find the nullspace of each set of vectors in question 1.
4. For matrices B and C, determine if they are in the subspace spanned by W and, if so, what is the linear combination.

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad W = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\}$$

5. Find a basis for all 2×2 matrices A such that

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} A = A \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

6. Find a basis for all 2×2 matrices A such that

$$A \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

7. For each of the following, use the conditions of linearity

   i) $T(\vec{v}) + T(\vec{w}) = T(\vec{v} + \vec{w})$

   ii) $kT(\vec{v}) = T(k\vec{v})$

   to determine if the transformation is linear or nonlinear.

$$a)\ T(\vec{x}) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}, \quad b)\ T(\vec{x}) = \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix}, \quad c)\ T(\vec{x}) = \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix}, \quad d)\ T(\vec{x}) = x_1 + x_2$$

Applications

8. **(CS application: data transmission, error-correcting codes)** Transmitted data always have the possibility of being corrupted. We would like to be able to detect errors and, better yet, correct them. One approach is to use redundant data, for example, submit three versions of the data so that if one version is corrupted the data can be retrieved from one of the two remaining versions. The downside of this approach is that in many situations the probability of an error is so low that it is a waste of resources to have this much redundancy.

   One alternative is to use use the Hamming (7, 4) [Wik12] code to encode 4 bits of information in 7 bits. The additional 3 bits are the parity bits, which allow us to check the validity of the data. Using 3 parity bits for 4 bits of data allows

   - single-bit errors to be detected and corrected
   - double-bit errors to be detected, but we don't know where they are

   To encode our four bits of data as a vector $\vec{p}$, we multiply it by a generator matrix G to form a 7-bit vector $\vec{r}$. The 7-bit vector $\vec{r}$ that is received is multiplied by the parity-check matrix H to produce a 3-bit vector $\vec{z}$. If there are no errors in the received vector then $\vec{z} = \vec{0}$.

   Example: Let the data we wish to submit be $\vec{p} = [1, 1, 0, 1]^T$. The code generator matrix G and parity check matrix H are

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

   When we multiply G by $\vec{p}$ we will get a vector with seven elements, some of which may have values other than 0 or 1. To produce $\vec{r}$, we apply modulo 2 arithmetic to each element of the vector.

$$G\vec{p} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{\text{mod } 2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \vec{r}$$

   The received vector $\vec{r}$ must be multiplied by H. Modulo 2 arithmetic must be applied to the result in order to produce $\vec{z}$.

$$H\vec{r} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \xrightarrow{\text{mod } 2} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{z}$$

If we determine that our received vector $\vec{r}$ is error-free, to transform it back to $\vec{p}$ we define

$$R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and produce

$$R\vec{r} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

What if there is a single bit error? Using our same example, let the first bit of $\vec{r}$ be corrupted to produce $\hat{r} = [0, 0, 1, 0, 1, 0, 1]^T$.

$$H\vec{r} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \xrightarrow{\text{mod } 2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{z}$$

which tells us that there is an error in the first bit since 001 ($\vec{z}$ read bottom to top) is 1 in base-2. Note that whether the position of the error can be read as the base-2 value of the bit position depends on the particular G and H matrices used, so it's not always the case.

For each of the following transmitted bit patterns, determine if the bit pattern is corrupted (you can assume that at most a single bit of the transmitted information is corrupted). If so, indicate which bit was corrupted and then determine what the original 4-bit data was before encoding (this will not be corrupted). If not, determine what the original 4-bit data was before encoding.

(a) 0011001

(b) 1111110

(c) 0100000

(d) 1011010

Review Questions

These are not for credit, but instead are intended to test your understanding of the concepts. You should be able to answer these without simply regurgitating equations.

1. What does it mean for a set of vectors to be linearly independent?

2. Given a set of vectors, can we say that some of them are the independent vectors and others are the dependent vectors?

3. What is a basis?

4. What is the difference between span and a basis?

5. What is the rank of a matrix? Which subspace is it related to–Col(A), Nul(A), etc.?

6. If V is a vector space, what is $\dim(V)$? What does this have to do with a basis? Can subspaces with differently shaped vectors (e.g., $\mathbb{R}^4$ and $\mathbb{R}^{2\times 2}$) have the same dimension?

References

[Wik12] Wikipedia. Hamming(7,4). URL: http://en.wikipedia.org/wiki/Hamming(7,4), 2012. retrieved 10/01/2012.