PHẦN THỰC HÀNH
(Tính các tích phân xác định)

I. Phương pháp đổi biến số

1) $I = \int_1^e \frac{\ln^2 x}{x} dx$

Dấu hiệu $\left(\frac{1}{x}, \ln x\right) \rightarrow$ Đặt $t = \ln x$

Đặt $t = \ln x \rightarrow dt = \frac{1}{x} dx$

Đổi cận :

x	1	e
t	0	1

$\Rightarrow I = \int_0^1 t^2 dt = \left.\frac{t^3}{3}\right|_0^1 = \frac{1}{3}(1^3 - 0^3) = \frac{1}{3}$

2) $I = \int_0^{\pi/4} \frac{\tan x}{\cos^2 x} dx$

Đặt $t = \tan x \rightarrow dt = \frac{dx}{\cos^2 x}$

Đổi cận :

x	0	$\pi/4$
t	0	1

$\Rightarrow I = \int_0^1 t\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}$

3) $I = \int_0^r \sqrt{r^2 - x^2}\,dx$ với $r > 0$

Đặt $x = r\sin t$ $\quad t \in \left[\frac{-\pi}{2}; \frac{\pi}{2}\right]$

$\rightarrow dx = r\cos t\,dt$

Dấu hiệu $\sqrt{a^2 - x^2}$

$\rightarrow$ đặt $x = |a|\sin t$ $\quad t \in \left[\frac{-\pi}{2}; \frac{\pi}{2}\right]$

Đổi cận:

x	0	r
t	0	$\pi/2$

$$I = \int_0^{\pi/2} \sqrt{r^2 - r^2.\sin^2 t}\,.r\cos t\,dt = \int_0^{\pi/2} \sqrt{r^2.(1 - \sin^2 t)}\,.r\cos t\,dt$$

$$= \int_0^{\pi/2} r.\sqrt{1 - \sin^2 t}\,.r\cos t\,dt = \int_0^{\pi/2} r^2 \sqrt{\cos^2 t}\,.\cos t\,dt = r^2 \int_0^{\pi/2} \cos^2 t\,dt$$

$$= r^2 \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2t)dt = \frac{r^2}{2} \int_0^{\pi/2} (1 + \cos 2t)\,dt = \frac{r^2}{2}\left(t + \frac{\sin 2t}{2}\right)\Bigg|_0^{\pi/2}$$

$$= \frac{r^2}{2}\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right] = \frac{\pi r^2}{4}$$

4) $I = \int_0^{\pi/6} (1-\cos 3x)\sin 3x\,dx$

Đặt $t = 1 - \cos 3x \rightarrow dt = 3\sin 3x\,dx \rightarrow \sin 3x\,dx = \frac{dt}{3}$

Đổi cận:

x	0	$\pi/6$
t	0	1

$$I = \int_0^1 t\cdot\frac{dt}{3} = \frac{1}{3}\int_0^1 t\,dt = \frac{1}{3}\cdot\left(\frac{t^2}{2}\Big|_0^1\right) = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

5) $I = \int_0^2 \sqrt{4-x^2}\,dx$

Đặt $x = 2\sin t \qquad t \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

$\rightarrow dx = 2\cos t\,dt$

Đổi cận

x	0	2
t	0	$\pi/2$

$$I = \int \sqrt{4-4\sin^2 t}\,.\,2\cos t\,dt = \int \sqrt{4\underbrace{(1-\sin^2 t)}_{=\cos^2 t}}\,.\,2\cos t\,dt = \int 2\cos t\,.\,2\cos t\,dt$$

$$= \int 4\cos^2 t\,dt = \int 4\,.\,\frac{1}{2}(1+\cos 2t)\,dt$$

$$\left(\cos 2x = 2\cos^2 x - 1 \quad \frac{\cos 2x + 1}{2} = \cos^2 x\right)$$

$$= \int (2 + 2\cos 2t)\,dt = (2t + \sin 2t)\Big|_0^{\pi/2}$$

$$= (\pi + 0) - (0 + 0) = \pi$$

Dấu hiệu $\sqrt{\varphi(x)}$

$\rightarrow$ Đặt $t = \sqrt{\varphi(x)}$

6) $I = \int_0^3 x\sqrt{1+x}\,dx$

Đặt $t = \sqrt{1+x} \rightarrow x = t^2 - 1 \rightarrow dx = 2t\,dt$

Đổi cận

x	0	3
t	1	2

$$I = \int_1^2 (t^2-1).2.t^2\,dt = 2\int_1^2 (t^4 - t^2)\,dt = 2\left(\frac{t^5}{5} - \frac{t^3}{3}\right)\Big|_1^2$$

$$= 2\left[\left(\frac{2^5}{5} - \frac{2^3}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)\right] = \frac{116}{15}$$

7) $I = \int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx$

Đặt $t = \sin x \rightarrow dt = \cos x\,dx$

Đổi cận

x	0	$\pi/2$
t	0	1

$$I = \int_0^1 \frac{dt}{1+t^2} = \arctan t\Big|_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4}$$

8) $I = \int_1^2 \frac{e^{1/x}}{x^2}\,dx$

Đặt $t = \frac{1}{x} \rightarrow dt = \frac{-1}{x^2}dx \rightarrow -dt = \frac{dx}{x^2}$

Đổi cận

x	1	2
t	1	$1/2$

$\Rightarrow$ $I = \int_1^{1/2} (-e^t)\,dt = -e^t\Big|_1^{1/2} = -e^{1/2} + e = -\sqrt{e} + e = e - \sqrt{e}$

9) $I = \int_0^1 \frac{2x}{(x^2+4)^2}\,dx$

Dấu hiệu: hàm số có mẫu
$\rightarrow$ đặt t là mẫu số

Đặt $t = x^2 + 4 \rightarrow dt = 2x\,dx$

Đổi cận

x	0	1
t	4	5

$I = \int_4^5 \frac{dt}{t^2} = \int_4^5 t^{-2}\,dt = \int_4^5 \frac{t^{-1}}{-1} = \frac{-1}{t}\Big|_4^5 = \frac{-1}{5} + \frac{1}{4} = \frac{1}{20}$

10) $I = \int_0^{\pi/2} \sin^2 x . \cos^3 x \, dx = \int_0^{\pi/2} \sin^2 x . \cos^2 x . \cos x \, dx$

$= \int_0^{\pi/2} \sin^2 x . (1 - \sin^2 x) \cos x \, dx$

Đặt $t = \sin x \rightarrow dt = \cos x \, dx$

Đổi cận

x	0	$\pi/2$
t	0	1

$I = \int_0^1 t^2 (1 - t^2) \, dt = \int_0^1 (t^2 - t^4) dt = \left(\frac{t^3}{3} - \frac{t^5}{5} \right) \Big|_0^1 = \frac{2}{15}$

II. Phương pháp tích phân từng phần

11) $I = \int_1^2 (x+1)\ln x \, dx$

Đặt $\begin{cases} u = \ln x \\ dv = (x+1)dx \end{cases} \longrightarrow \begin{cases} du = \dfrac{1}{x}dx \\ v = \dfrac{x^2}{2} + 1 \end{cases}$

$$I = \left(\frac{x^2}{2}+1\right)\ln x \Big|_1^2 - \int_1^2 \left(\frac{x^2}{2}+1\right)\frac{1}{x}dx = \left(\frac{2^2}{2}+1\right)\ln 2 - \int_1^2 \left(\frac{x}{2}+\frac{1}{x}\right)dx$$

$$= 3\ln 2 - \left(\frac{x^2}{4} + \ln|x|\right)\Big|_1^2 = 3\ln 2 - \frac{2^2}{4} - \ln 2 + \frac{1}{4} = 2\ln 2 - \frac{3}{4}$$

12) $I = \int_0^1 x e^{-x} dx$

Đặt $\begin{cases} u = x \\ dv = e^{-x} dx \end{cases} \rightarrow \begin{cases} du = dx \\ v = -e^{-x} \end{cases}$

$$I = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x} dx = -e^{-1} - e^{-x}\Big|_0^1 = \frac{e-2}{e}$$

13) $I = \int_0^1 x e^{-2x} dx$

Đặt $\begin{cases} u = x \\ dv = e^{-2x} dx \end{cases} \rightarrow \begin{cases} du = dx \\ v = \dfrac{-e^{-2x}}{2} \end{cases}$

$$I = \frac{-x \cdot e^{-2x}}{2}\Big|_0^1 - \int_0^1 \frac{-e^{-2x}}{2} dx = \frac{-e^{-2}}{2} + \int_0^1 \frac{e^{-2x}}{2} dx$$

$$= \frac{-e^{-2}}{2} - \frac{e^{-2x}}{4}\Big|_0^1 = \frac{-e^{-2}}{2} - \frac{e^{-2}}{4} + \frac{1}{4} = \frac{-2e^{-2} - e^{-2} + 1}{4} = \frac{-3e^{-2}+1}{4}$$

14) $I = \int_0^{\pi/2} (x+1)\sin x\,dx$

Đặt $\begin{cases} u = x+1 \\ dv = \sin x\,dx \end{cases} \rightarrow \begin{cases} du = dx \\ v = -\cos x \end{cases}$

$$I = -(x+1).\cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} -\cos x\,dx = 1 + \sin x\Big|_0^{\pi/2} = 1+1-0 = 2$$

15) $I = \int_1^{e} \ln x\,dx$

Đặt $\begin{cases} u = \ln x \\ dv = dx \end{cases} \rightarrow \begin{cases} du = \dfrac{dx}{x} \\ v = x \end{cases}$

$$I = x\ln x\Big|_1^{e} - \int_1^{e} x\frac{dx}{x} = e - x\Big|_1^{e} = e - e + 1 = 1$$

16) $I = \int_0^{2\pi} x\cos x\,dx$

Đặt $\begin{cases} u = x \\ dv = \cos x\,dx \end{cases} \rightarrow \begin{cases} du = dx \\ v = \sin x \end{cases}$

$$I = x\sin x\Big|_0^{2\pi} - \int_0^{2\pi} \sin x\,dx = \cos x\Big|_0^{2\pi} = 0$$

17) $I = \int_0^{\pi/2} x(x+\cos x)\,dx = \int_0^{\pi/2} x^2 + \int_0^{\pi/2} x\cos x\,dx$

$$= \frac{x^3}{3}\Bigg|_0^{\pi/2} + \int_0^{\pi/2} x\cos x\,dx = \frac{\left(\frac{\pi}{2}\right)^3}{3} + \int_0^{\pi/2} x\cos x\,dx = \frac{\pi^3}{24} + \underbrace{\int_0^{\pi/2} x\cos x\,dx}_{A}$$

☆ Tính A:

Đặt $\begin{cases} u = x \\ dv = \cos x\,dx \end{cases} \longrightarrow \begin{cases} du = dx \\ v = \sin x \end{cases}$

$$A = x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x\,dx = \frac{\pi}{2} + \cos x\Big|_0^{\pi/2} = \frac{\pi}{2} - 1$$

Vậy: $I = \dfrac{\pi^3}{24} + \dfrac{\pi}{2} - 1$

18) $I = \int_0^{\pi/2} e^x \cos x\,dx$

Đặt $\begin{cases} u = e^x \\ dv = \cos x\,dx \end{cases} \longrightarrow \begin{cases} du = e^x dx \\ v = \sin x \end{cases}$

$$I = e^x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} e^x \sin x\,dx = e^{\pi/2} - \underbrace{\int_0^{\pi/2} e^x \sin x\,dx}_{I_1}$$

☆ Tính I_1:

Đặt $\begin{cases} u_1 = e^x \\ dv_1 = \sin x\,dx \end{cases} \longrightarrow \begin{cases} du_1 = e^x dx \\ v_1 = -\cos x \end{cases} \Longrightarrow I_1 = -e^x \cos x \Big|_0^{\pi/2} + \int_0^{\pi/2} e^x \cos x\,dx = 1 + I$

Vậy: $I = e^{\pi/2} - (1 + I) \Longleftrightarrow I = \dfrac{e^{\pi/2} - 1}{2}$

19) $I = \int_0^1 \arctan x \, dx$

Đặt $\begin{cases} u = \arctan x \\ dv = dx \end{cases} \rightarrow \begin{cases} du = \dfrac{dx}{1+x^2} \\ v = x \end{cases}$

$$I = x\arctan x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2}dx = \frac{\pi}{4} - \underbrace{\int_0^1 \frac{x}{1+x^2}dx}_{A}$$

☆ Tính A: Đặt $u = 1+x^2 \rightarrow du = 2x\,dx \rightarrow \dfrac{du}{2} = x\,dx$

$$A = \int_0^1 \frac{1}{u}\cdot\frac{du}{2} = \frac{1}{2}\int_0^1 \frac{du}{u} = \frac{1}{2}\ln|u|\Big|_0^1 = \frac{1}{2}\ln|1+x^2|\Big|_0^1$$

Vậy: $I = \dfrac{\pi}{4} - \dfrac{1}{2}\ln|1+x^2|\Big|_0^1 = \dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$

20) $I = \int_0^1 \frac{x\arctan x}{(1+x^2)^2}dx$

Đặt $\begin{cases} u = \arctan x \\ dv = \frac{xdx}{(1+x^2)^2} \end{cases} \rightarrow \begin{cases} du = \frac{dx}{1+x^2} \\ v = \frac{-1}{2(1+x^2)} \end{cases}$

$$I = \arctan x\left(\frac{-1}{2(1+x^2)}\right)\Big|_0^1 + \frac{1}{2}\int_0^1 \frac{dx}{(1+x^2)^2} = \frac{-\pi}{16} + \frac{1}{2}\underbrace{\int_0^1 \frac{dx}{(1+x^2)^2}}_{A}$$

☆ Tính A: Đặt $x = \tan t \rightarrow dx = (1+\tan^2 t)dt$

Đổi cận

x	0	1
t	0	$\pi/4$

$$A = \int_0^{\pi/4} \frac{(1+\tan^2 t)dt}{(1+\tan^2 t)^2} = \int_0^{\pi/4} \frac{dt}{1+\tan^2 t} = \int_0^{\pi/4} \cos^2 t\,dt$$

$$\boxed{1+\tan^2 x = \frac{1}{\cos^2 x}}$$

$$= \int_0^{\pi/4} \frac{1+\cos 2t}{2}dt = \left(\frac{t}{2} + \frac{1}{4}\sin 2t\right)\Big|_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4}$$

Vậy: $I = \frac{-\pi}{16} + \frac{1}{2}\left(\frac{\pi}{8} + \frac{1}{4}\right) = \frac{-\pi}{16} + \frac{\pi}{16} + \frac{1}{8} = \frac{1}{8}$