Phần thực hành
(Tính các tích phân xác định)

I. Phương pháp đổi biến số

1) $I = \int_1^e \frac{\ln^2 x}{x}\,dx$

Đặt $t = \ln x \rightarrow dt = \frac{1}{x}dx$

Nếu $x = 1$ thì $t = 0$, $x = e$ thì $t = 1$

Khi đó ta có: $I = \int_1^e \frac{\ln^2 x}{x}\,dx = \int_0^1 t^2\,dt = \frac{1}{3}t^3 \Big|_0^1 = \frac{1}{3}(1^3 - 0^3) = \frac{1}{3}$

2) $I = \int_0^{0,75} \frac{dx}{(x+1)\sqrt{x^2+1}}$

Đặt $\frac{1}{1+x} = t \rightarrow x = \frac{1}{t} - 1 \rightarrow dx = -\frac{t}{t^2}dt$

Và $x^2 + 1 = \frac{1}{t^2} - \frac{1}{t} + 2$

Vậy : $I = \frac{1}{\sqrt{2}} \int_{4/7}^{1} \frac{d\left(t - \frac{1}{2}\right)}{\sqrt{\left(t - \frac{1}{2}\right)^2 + \frac{1}{4}}}$

$$= \frac{1}{\sqrt{2}} \ln\left|\left(t - \frac{1}{2}\right) + \sqrt{\left(t - \frac{1}{2}\right)^2 + \frac{1}{4}}\right|\Bigg|_{\frac{4}{7}}^{1} = \frac{1}{\sqrt{2}} \ln \frac{9 + 4\sqrt{2}}{7}$$

3) $I = \int_0^r \sqrt{r^2 + x^2}\, dx$ với $r > 0$

Đặt $x = r.\sin t \longrightarrow dx = r\cos t\, dt$

Nếu $x = 0$ thì $t = 0$, $x = r$ thì $t = \frac{\pi}{2}$

Vậy : $I = r^2 \int_0^{\pi/2} \cos^2 t\, dt = \frac{1}{2} r^2 \int_0^{\pi/2} (1 + \cos 2t)\, dt$

$$= \frac{1}{2} r^2 \left[t + \frac{\sin 2t}{2}\right]_0^{\pi/2} = \frac{r^2}{2}\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right] = \frac{\pi r^2}{4}$$

4) $I = \int_{-\pi}^{\pi} \sin^7 2x \, dx$

Ta có $f(x) = \sin^7 2x$ là hàm số lẻ vì vậy theo tính chất của tích phân xác định thì: $\int_{-\pi}^{\pi} \sin^7 2x \, dx = 0$

5) $I = \int_{0}^{2} \sqrt{4-x^2} \, dx$

Đặt $x = 2\sin t$ $\left(\frac{-\pi}{2} \leq t \leq \frac{\pi}{2} \right)$

Ta có: $dx = 2\cos t \, dt$; $\sqrt{4-x^2} = 2\cos t$

Nếu $x = 0$ thì $t = 0$, $x = 2$ thì $t = \frac{\pi}{2}$

Vậy $I = 4\int_{0}^{\pi/2} \cos^2 t \, dt = 4\int_{0}^{\pi/2} \frac{1+\cos 2t}{2} dt = 2\left[t + \frac{\sin 2t}{2} \right]_{0}^{\pi/2} = \pi$

6) $I = \int_0^3 x\sqrt{1+x}\,dx$

Đặt $t = \sqrt{1+x} \rightarrow x = t^2 - 1 \rightarrow dx = 2t\,dt$

Nếu $x = 0$ thì $t = 1$, $x = 3$ thì $t = 2$

Vậy : $I = \int_1^2 2(t^2-1)t^2\,dt = 2\left(\frac{t^5}{5} - \frac{t^3}{3}\right)\Big|_1^2 = \frac{116}{15}$

7) $I = \int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx$

Đặt $t = \sin x \rightarrow dt = \cos x\,dx$

Nếu $x = 0$ thì $t = 0$, $x = \frac{\pi}{2}$ thì $t = 1$

Vậy : $I = \int_0^1 \frac{dt}{1+t^2} = \operatorname{arctg}\Big|_0^1 = \frac{\pi}{4}$

8) $I = \int_1^2 \frac{e^{1/x}}{x^2} dx$

Đặt $t = \frac{1}{x} \rightarrow dt = \frac{-1}{x^2} dx$

Nếu $x = 1$ thì $t = 1$, $x = 2$ thì $t = \frac{1}{2}$

Vậy: $I = \int_1^{1/2} (-e^t) dt = -e^t \Big|_1^{1/2} = e - \sqrt{e}$

II. Phương pháp tích phân từng phần

9) $I = \int_0^1 x e^{-x} dx$

Đặt $\begin{cases} u = x \\ dv = e^{-x} dx \end{cases} \rightarrow \begin{cases} du = dx \\ v = -e^{-x} \end{cases}$

Vậy: $I = -xe^{-x} \Big|_0^1 + \int_0^1 e^{-x} dx = -e^{-1} - e^{-x} \Big|_0^1 = \frac{e-2}{e}$

10) $I = \int_{-\pi/3}^{\pi/3} x \arctan x \, dx$

Đặt $\begin{cases} u = \arctan x \\ dv = x dx \end{cases} \rightarrow \begin{cases} du = \dfrac{1}{x^2+1} dx \\ v = \dfrac{x^2}{2} \end{cases}$

Vậy : $I = \dfrac{x^2}{2} \arctan x \Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} x^2 \cdot \dfrac{1}{1+x^2} dx$

$= \dfrac{\pi}{2} - \dfrac{1}{2} \int_0^{\sqrt{3}} dx + \dfrac{1}{2} \int_0^{\sqrt{3}} \dfrac{dx}{1+x^2}$

$= \dfrac{\pi}{2} - \dfrac{\sqrt{3}}{2} + \dfrac{1}{2} \operatorname{arctg} x \Big|_0^{\sqrt{3}} = \dfrac{2\pi}{3} - \dfrac{\sqrt{3}}{2}$

11) $I = \displaystyle\int_{-\frac{\pi}{3}}^{\pi/3} \frac{x\sin x}{\cos^2 x}\,dx$

Vì hàm dưới dấu tích phân là hàm chẵn nên ta có: $I = \displaystyle\int_{0}^{\pi/3} \frac{x\sin x}{\cos^2 x}\,dx$

Đặt $\begin{cases} u = x \\ dv = \dfrac{\sin x}{\cos^2 x}\,dx \end{cases} \rightarrow \begin{cases} du = dx \\ v = \dfrac{-1}{\cos x} \end{cases}$

Vậy: $I = 2\left[-\dfrac{x}{\cos x}\Big|_0^{\pi/3} + \displaystyle\int_0^{\pi/3} \frac{dx}{\cos x}\right]$

$= 2\left(\dfrac{\pi}{3\cos\frac{\pi}{3}} - \ln\tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\Big|_0^{\pi/3}\right)$

$= 2\left(\dfrac{2\pi}{3} - \ln\tan\left(\dfrac{\pi}{6} + \dfrac{\pi}{4}\right) + \ln\tan\dfrac{\pi}{4}\right)$

$= 2\left(\dfrac{2\pi}{3} - \ln\tan\dfrac{5\pi}{2}\right)$

12) $I = \int_1^e \ln x \, dx$

Đặt $\begin{cases} u = \ln x \\ dv = dx \end{cases} \rightarrow \begin{cases} du = \dfrac{dx}{x} \\ v = x \end{cases}$

Vậy: $I = x\ln x \Big|_1^e - \int_1^e x \dfrac{dx}{x} = e - x\Big|_1^e = e - e + 1 = 1$

13) $I = \int_0^{2\pi} x\cos x \, dx$

Đặt $\begin{cases} u = x \\ dv = \cos x \, dx \end{cases} \rightarrow \begin{cases} du = dx \\ v = \sin x \end{cases}$

Vậy: $I = x\sin x \Big|_0^{2\pi} - \int_0^{2\pi} \sin x \, dx = \cos x \Big|_0^{2\pi} = 0$

14) $I = \int_0^{\pi/2} e^x \cos x\,dx$

Đặt $\begin{cases} u = e^x \\ dv = \cos x\,dx \end{cases} \rightarrow \begin{cases} du = e^x dx \\ v = \sin x \end{cases}$

$\rightarrow I = e^x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} e^x \sin x\,dx = e^{\pi/2} - \underbrace{\int_0^{\pi/2} e^x \sin x\,dx}_{I_1}$

Tính I_1: đặt $\begin{cases} u = e^x \\ dv = \sin x\,dx \end{cases} \rightarrow \begin{cases} du = e^x dx \\ v = -\cos x \end{cases}$

$\rightarrow I_1 = -e^x \cos x \Big|_0^{\pi/2} + \int_0^{\pi/2} e^x \cos x\,dx = 1 + I$

Vậy $I = e^{\pi/2} - (1 + I) \Leftrightarrow I = \dfrac{e^{\pi/2} - 1}{2}$

15) $I = \int_0^1 \operatorname{arctg} x \, dx$

Đặt $\begin{cases} u = \operatorname{arctg} x \\ dv = dx \end{cases} \longrightarrow \begin{cases} du = \dfrac{dx}{1+x^2} \\ v = x \end{cases}$

Vậy $I = x \operatorname{arctg} x \Big|_0^1 - \int_0^1 \dfrac{x}{1+x^2} dx$

$$= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{d(1+x^2)}{1+x^2} = \frac{\pi}{4} - \frac{1}{2} \ln|1+x^2|\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

16) $I = \int_0^1 \frac{x \operatorname{arctg} x}{(1+x^2)^2} dx$

Đặt $\begin{cases} u = \operatorname{arctg} x \\ dv = \frac{x dx}{(1+x^2)^2} \end{cases} \rightarrow \begin{cases} du = \frac{dx}{1+x^2} \\ v = \frac{1}{2(1+x^2)} \end{cases}$

$\rightarrow I = \operatorname{arctg} x \left(\frac{-1}{2(1+x^2)} \right) \Big|_0^1 + \frac{1}{2} \int_0^1 \frac{dx}{(1+x^2)^2} = \frac{-\pi}{16} + \frac{1}{2} J$

Xét $J = \int_0^1 \frac{dx}{(1+x^2)^2}$

Đặt $x = \operatorname{tg} t$ ta có $dx = (1 + \operatorname{tg}^2 t) dt$

Nếu $t = 0$ thì $x = 0$, $t = \frac{\pi}{4}$ thì $x = 1$

$J = \int_0^1 \frac{dx}{(1+x^2)^2} = \int_0^{\pi/4} \frac{(1+\operatorname{tg}^2 x) dx}{(1+\operatorname{tg}^2 x)^2} = \int_0^{\pi/4} \frac{dx}{1+\operatorname{tg}^2 x} = \int_0^{\pi/4} \cos^2 x dx$

$= \int_0^{\pi/4} \frac{1+\cos 2x}{2} dx = \left(\frac{t}{2} + \frac{1}{4} \sin 2t \right) \Big|_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4}$