3-dimensional Boolean functions

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I am using the following notation: (10000001) represents the boolean function t such that that $t^{(\mu)} = 1$ for $\mu \in \{1, 8\}$, $t^{(\mu)} = 0$ for $\mu \in \{2, 3, 4, 5, 6, 7\}$, where the μ values are visualised in figure 1. As in the task description on OpenTA, this is also represented by the ball $\mu = 1$ and the ball $\mu = 8$ being black.

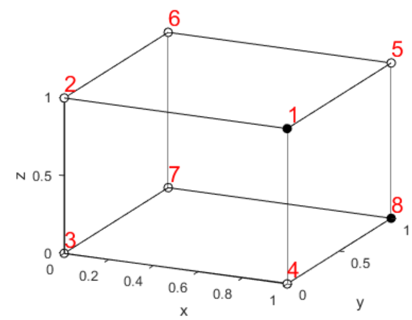

Figure 1: The location of μ in 3d-space and the function (10000001). For example, $\mu = 1$ represents the point $(1, 0, 1)$. Black balls indicate $t^{(\mu)} = 0$, white balls indicate $t^{(\mu)} = 1$.

Next, we split up the problem into looking at cases where the functions map exactly k input patterns are matched to 1. We know that the number of such functions is $N_k = \binom{8}{k}$. These functions can be grouped into symmetries whose "cubes" can be mapped onto each other by reflection or rotation. Let us call the number of functions belonging to such a symmetry as N_k^j with $j = 1, 2, ..., J$, where J is the number of symmetries for a given k. Next, let us call the number of linearly separable functions that map k patterns to 1 as n_k. By symmetry, we have $n_k = n_{(8-k)}$ (switch colors). Therefore, we only need to analyze $k \in \{0, 1, 2, 3, 4\}$.

Let's begin with $k = 0$. Trivially, there is only 1 such function, and it is linearly separable: $n_0 = n_8 = 1$.

Next, we analyze $k = 1$. This is the same as choosing one corner. Therefore, we have only one symmetry ($J = 1$), which is also linearly separable, so $n_1 = n_7 = N_1^1 = 8$.

Next, we have $k = 2$. Here, we have $J = 3$: $j = 1$ is choosing the black balls to be along one edge: $N_2^1 = N_{edges} = 12$, $j = 2$ is choosing the black balls to be along a face diagonal: $N_2^2 = 2N_{faces} = 12$, and $j = 3$ is choosing the black balls to be along a cube diagonal: $N_2^3 = 4$. We double check: $\sum_{j=1}^{J=3} N_2^j = 12 + 12 + 4 = 28 = \binom{8}{2}$. Only the symmetry $j = 1$ is linearly separable, so $n_2 = n_6 = N_2^1 = 12$.

Next, we have $k = 3$. Here, the symmetries are difficult to explain geometrically, so I will explain the linearly separable ones, but only give examples of the others. The only linearly separable symmetry is when we choose all three black balls to be on a single face. We have 6 faces, and each face has $\binom{4}{3} = 4$ ways to choose three balls, so we have $N_3^1 = 4 \cdot 6 = 24$. Next, we have the second symmetry (11000001) with $N_3^2 = 24$, and the third symmetry (01011000) with $N_3^3 = 8$. Double check: $\sum_{j=1}^{J=3} N_3^j = 24 + 24 + 8 = 56 = \binom{8}{3}$. We have $n_3 = n_5 = N_3^1 = 24$.

Lastly, we have $k = 4$. Again, the symmetries are difficult to explain geometrically. There are two linearly separable ones: firstly, when the black balls are along edges belonging to one face (e.g. (11110000)), with $N_4^1 = N_{faces} = 6$, and secondly, when the black balls are along the edges that are connected to one common corner (e.g. (11100100)), with $N_4^2 = N_{corners} = 8$. There are four other symmetries: (01110001) with $N_4^3 = 24$, (11000011) (here, the black balls are chosen along opposing edges) with $N_4^4 = 6$, (01011010) with $N_4^5 = 2$, and (01111000) with $N_4^6 = 24$. Double check: $\sum_{j=1}^{J=6} N_3^j = 6 + 8 + 24 + 6 + 2 + 24 = 70 = \binom{8}{4}$. We have $n_4 = N_4^1 + N_4^2 = 14$.

Finally, we have the number of linearly separable functions

$$n = \sum_{k=0}^{k=8} n_k = 1 + 8 + 12 + 24 + 14 + 24 + 12 + 8 + 1 = 104. \tag{1}$$