5. Diluted Hopfield network

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$$S_i = \operatorname{sgn}(b_i), \quad b_i = \sum_{j=1}^{N} w_{ij} S_j$$

$$w_{ij} = \frac{K_{ij}}{K} \sum_{\mu=1}^{p} x_i^{(\mu)} x_j^{(\mu)}$$

$$K_{ij} = \begin{cases} 1, \text{ with prob. } \frac{K}{N} \\ 0, \text{ with prob. } 1-\frac{K}{N} \end{cases}$$

a) $\underline{S} = \underline{x}^{(\nu)}$

$$b_i = \sum_{j=1}^{N} w_{ij} S_j = \sum_{j=1}^{N} w_{ij} x_j^{(\nu)}$$

$$b_i = \sum_{j=1}^{N} \frac{K_{ij}}{K} \sum_{\mu=1}^{p} x_i^{(\mu)} x_j^{(\mu)} x_j^{(\nu)}$$

$$= \sum_{j=1}^{N} \frac{K_{ij}}{K} x_i^{(\nu)} \underbrace{x_j^{(\nu)} x_j^{(\nu)}}_{1} + \sum_{j=1}^{N} \frac{K_{ij}}{K} \sum_{\substack{\mu=1 \\ \mu\neq\nu}}^{p} x_i^{(\mu)} x_j^{(\mu)} x_j^{(\nu)}$$

$$= \sum_{j=1}^{N} \frac{K_{ij}}{K} x_i^{(\nu)} + \sum_{j=1}^{N} \frac{K_{ij}}{K} \sum_{\substack{\mu=1 \\ \mu\neq\nu}}^{p} x_i^{(\mu)} x_j^{(\mu)} x_j^{(\nu)}$$

$$\langle b_i \rangle_{\text{connections}} = \sum_{j=1}^{N} \left[1 \cdot \frac{K}{N} + 0 \cdot \left(1-\frac{K}{N}\right)\right] \frac{x_i^{(\nu)}}{K} +$$

$$\underbrace{+ \frac{1}{K} \sum_{j=1}^{N} \left[1 \cdot \frac{K}{N} + 0 \cdot \left(1-\frac{K}{N}\right)\right] \sum_{\substack{\mu=1 \\ \mu\neq\nu}}^{p} x_i^{(\mu)} x_j^{(\mu)} x_j^{(\nu)}} =$$

$$= x_i^{(\nu)} + \langle \text{cross-talk term} \rangle$$

b) $\langle C_i^{(\nu)} \rangle_{\text{conn.}} = \langle \text{cross-talk term} \rangle =$ sum of $N \cdot (p-1) \cdot \frac{K}{N}$ random numbers that are $+1$ or -1 with equal probability, divided by $\underline{\underline{K}}$

(the remaining $N \cdot (p-1)\left(1-\frac{K}{N}\right)$ terms are equal to zero)

$\Rightarrow$ $\langle C_i^{(\nu)} \rangle_{\text{connections}}$ = sum of $K \cdot (p-1)$ random numbers ± 1 with equal prob., divided by K.

In the limit of $Kp \gg 1$, we make use of the CLT, and conclude:

$\langle C_i^{(\nu)} \rangle_{\text{connections}}$ is Gaussian distributed with mean $\underline{0}$, and variance that is approximately equal to

$$\approx \frac{K \cdot p}{K^2} = \frac{p}{K}$$

c) $$m_\nu = \frac{1}{N} \sum_{i=1}^{N} x_i^{(\nu)} \langle \langle S_i \rangle_{\text{connections}} \rangle_{\text{time}}$$

In the limit of $N \gg 1$, use the following approximation

$$\left(\langle \langle S_i \rangle_{\text{connections}} \rangle_{\text{time}} \approx \langle S_i \rangle_{\text{connections}} \approx \operatorname{sgn}(\langle b_i \rangle_{\text{connections}}) \right)$$

$$m_\nu = \frac{1}{N} \sum_{i=1}^{N} x_i^{(\nu)} \operatorname{sgn}(\langle b_i \rangle_{\text{connections}})$$

averaging over connections

$$m_\nu = \frac{1}{N} \sum_{i=1}^{N} \operatorname{sgn}\left(x_i^{(\nu)} \left\langle \sum_{j=1}^{N} w_{ij} S_j \right\rangle \right) =$$

$$= \frac{1}{N} \sum_{i=1}^{N} \operatorname{sgn}\left(x_i^{(\nu)} \left\langle \sum_{j=1}^{N} \frac{K_{ij}}{K} \sum_{\mu=1}^{p} x_i^{(\mu)} x_j^{(\mu)} S_j \right\rangle \right) \Rightarrow$$

$$m_\nu = \frac{1}{N}\sum_{i=1}^{N} \mathrm{sgn}\Big(\langle \sum_{j=1}^{N} \frac{K_{ij}}{K} x_i^{(\nu)} x_j^{(\nu)} x_i^{(\nu)} S_j \rangle +$$

$$+ x_i^{(\nu)} \underbrace{\langle \sum_{j=1}^{N} \frac{K_{ij}}{K} \sum_{\substack{\mu=1\\ \mu\neq\nu}}^{P} x_i^{(\mu)} x_j^{(\mu)} S_j \rangle}_{\langle C_i^{(\nu)}\rangle_{\text{connections}}}\Big)$$

$$m_\nu = \frac{1}{N}\sum_{i=1}^{N} \mathrm{sgn}\Big(\langle \sum_{j=1}^{N} \frac{K_{ij}}{K} x_j^{(\nu)} S_j \rangle_{\text{connections}} +$$

$$+ x_i^{(\nu)} \langle C_i^{(\nu)} \rangle_{\text{connections}} \Big)$$

The first term on the r.h.s is $\approx m_\nu$, that is

$$\langle \sum_{j=1}^{N} \frac{K_{ij}}{K} x_j^{(\nu)} S_j \rangle_{\text{connections}} \approx m_\nu \quad \text{in the limit of } K \gg 1$$

It follows

$$m_\nu \approx \frac{1}{N}\sum_{i=1}^{N} \mathrm{sgn}\big(m_\nu + x_i^{(\nu)} \langle C_i^{(\nu)} \rangle_{\text{connections}} \big)$$

$\downarrow$ Gaussian distributed with mean $\underline{0}$, and variance $\approx \frac{P}{K}$

$$\Rightarrow m_\nu \approx \int d\langle C_i^{(\nu)}\rangle_{\text{conn.}} \underbrace{P(\langle C_i^{(\nu)}\rangle_{\text{conn.}})}_{\frac{1}{\sqrt{2\pi \frac{P}{K}}} e^{-\frac{\langle C_i^{(\nu)}\rangle_{\text{conn.}}}{2\frac{P}{K}}}} \mathrm{sgn}\big(m_\nu + x_i^{(\nu)} \cdot \langle C_i^{(\nu)}\rangle_{\text{conn.}}\big)$$

Now need to split the integral above to account for the fact that $x_i^{(\nu)} = +1$ or -1 with prob. $\frac{1}{2}$

(4)

$$m_\nu \approx \frac{1}{2} \int d\langle C_i^{(\nu)}\rangle P(\langle C_i^{(\nu)}\rangle) \,\mathrm{sgn}(m_\nu + \langle C_i^{(\nu)}\rangle) +$$

$$+ \frac{1}{2} \int d\langle C_i^{(\nu)}\rangle P(C_i^{(\nu)}) \,\mathrm{sgn}(m_\nu - \langle C_i^{(\nu)}\rangle) =$$

// Assuming $m_\nu \geq 0$ // (*)

$$= \frac{1}{2}\left[-\int_{-\infty}^{-m_\nu} d\langle C_i^{(\nu)}\rangle P(\langle C_i^{(\nu)}\rangle) + \int_{-m_\nu}^{\infty} d\langle C_i^{(\nu)}\rangle P(\langle C_i^{(\nu)}\rangle) \right]$$

$$+ \frac{1}{2}\left[-\int_{m_\nu}^{\infty} d\langle C_i^{(\nu)}\rangle P(\langle C_i^{(\nu)}\rangle) + \int_{-\infty}^{m_\nu} d\langle C_i^{(\nu)}\rangle P(\langle C_i^{(\nu)}\rangle) \right]$$

$$= \frac{1}{2}\left[-\left(1 - \int_{-m_\nu}^{\infty} d\langle C_i^{(\nu)}\rangle P(\langle C_i^{(\nu)}\rangle)\right) + \int_{-m_\nu}^{\infty} d\langle C_i^{(\nu)}\rangle P(\langle C_i^{(\nu)}\rangle) \right]$$

$$+ \frac{1}{2}\left[-\int_{m_\nu}^{\infty} d\langle C_i^{(\nu)}\rangle P(\langle C_i^{(\nu)}\rangle) + 1 - \int_{m_\nu}^{\infty} d\langle C_i^{(\nu)}\rangle P(\langle C_i^{(\nu)}\rangle) \right] =$$

$$= \frac{1}{\cancel{2}}\left[2\int_{-m_\nu}^{\infty} d\langle C_i^{(\nu)}\rangle P(\langle C_i^{(\nu)}\rangle) \cancel{-1} \cancel{+1} - 2\int_{m_\nu}^{\infty} d\langle C_i^{(\nu)}\rangle P(\langle C_i^{(\nu)}\rangle) \right]$$

$$= \frac{1}{2}\left(1 - \mathrm{erf}\left(\frac{-m_\nu}{\sqrt{2\frac{P}{K}}}\right)\right) - \frac{1}{2}\left(1 - \mathrm{erf}\left(\frac{m_\nu}{\sqrt{2\frac{P}{K}}}\right)\right)$$

$$= \frac{1}{2}\left[\mathrm{erf}\left(\frac{m_\nu}{\sqrt{2\frac{P}{K}}}\right) - \underbrace{\mathrm{erf}\left(\frac{-m_\nu}{\sqrt{2\frac{P}{K}}}\right)}_{= -\mathrm{erf}\left(\frac{m_\nu}{\sqrt{2P/K}}\right)} \right]$$

It follows:

$$\boxed{m_v \approx \operatorname{erf}\left(\frac{m_v}{\sqrt{2\frac{p}{K}}}\right)}$$

The same conclusion holds if we instead assumed in (*) that $\underline{\underline{m_v < 0}}$.