3. Hopfield network

a) Define

$$Q^{(\mu,\nu)} = \sum_{j=1}^{j=42} \zeta_j^{(\mu)} \zeta_j^{(\nu)}. \tag{1}$$

The bit j constributes with $+1$ to $Q^{(\mu,\nu)}$ if $\zeta_j^{(\mu)} = \zeta_j^{(\nu)}$, and with -1 if $\zeta_j^{(\mu)} \neq \zeta_j^{(\nu)}$. Since the number of bits are 42, we have $Q^{(\mu,\nu)} = 42 - 2 \cdot H^{(\mu,\nu)}$, where $H^{(\mu,\nu)}$ is the number of bits that are different in pattern μ and pattern ν (the Hamming distance). We find:

- $H^{(1,1)} = 0 \Rightarrow Q^{(1,1)} = 42$
- $H^{(1,2)} = 10 \Rightarrow Q^{(1,2)} = 22$
- $H^{(1,3)} = 2 \Rightarrow Q^{(1,3)} = 38$
- $H^{(1,4)} = 42 \Rightarrow Q^{(1,4)} = -42$
- $H^{(1,5)} = 21 \Rightarrow Q^{(1,5)} = 0$

- $H^{(2,1)} = H^{(1,2)} = 10 \Rightarrow Q^{(2,1)} = 22$
- $H^{(2,2)} = 0 \Rightarrow Q^{(2,2)} = 42$
- $H^{(2,3)} = 10 \Rightarrow Q^{(2,3)} = 22$
- $H^{(2,4)} = 42 - H^{(2,1)} = 32 \Rightarrow Q^{(2,4)} = -22$
- $H^{(2,5)} = 11 \Rightarrow Q^{(2,5)} = 20$

b) We have that

$$
\begin{aligned}
b_i^{(\nu)} &= \sum_j w_{ij}\zeta_j^{(\nu)} = \sum_j \frac{1}{42}\left(\zeta_i^{(1)}\zeta_j^{(1)} + \zeta_i^{(2)}\zeta_j^{(2)}\right)\zeta_j^{(\nu)} \\
&= \frac{1}{42}\zeta_i^{(1)}\sum_j \zeta_j^{(1)}\zeta_j^{(\nu)} + \frac{1}{42}\zeta_i^{(2)}\sum_j \zeta_j^{(2)}\zeta_j^{(\nu)} \\
&= \frac{1}{42}\zeta_i^{(1)}Q^{(1,\nu)} + \frac{1}{42}\zeta_i^{(2)}Q^{(2,\nu)}.
\end{aligned}
\tag{2}
$$

From a), we have that:

$$
\begin{aligned}
b_i^{(1)} &= \frac{Q^{(1,1)}}{42}\zeta_i^{(1)} + \frac{Q^{(2,1)}}{42}\zeta_i^{(2)} = \zeta_i^{(1)} + \frac{22}{42}\zeta_i^{(2)}, \\
b_i^{(2)} &= \frac{Q^{(1,2)}}{42}\zeta_i^{(1)} + \frac{Q^{(2,2)}}{42}\zeta_i^{(2)} = \frac{22}{42}\zeta_i^{(1)} + \zeta_i^{(2)}, \\
b_i^{(3)} &= \frac{Q^{(1,3)}}{42}\zeta_i^{(1)} + \frac{Q^{(2,3)}}{42}\zeta_i^{(2)} = \frac{38}{42}\zeta_i^{(1)} + \frac{22}{42}\zeta_i^{(2)}, \\
b_i^{(4)} &= \frac{Q^{(1,4)}}{42}\zeta_i^{(1)} + \frac{Q^{(2,4)}}{42}\zeta_i^{(2)} = -\zeta_i^{(1)} - \frac{22}{42}\zeta_i^{(2)}, \\
b_i^{(5)} &= \frac{Q^{(1,5)}}{42}\zeta_i^{(1)} + \frac{Q^{(2,5)}}{42}\zeta_i^{(2)} = \frac{20}{42}\zeta_i^{(2)}.
\end{aligned}
\tag{3}
$$

c) We have that From b), we find that:

$$
\begin{aligned}
\zeta_i^{(1)} &\to \mathrm{sgn}(b_i^{(1)}) = \zeta_i^{(1)}, \\
\zeta_i^{(2)} &\to \mathrm{sgn}(b_i^{(2)}) = \zeta_i^{(2)}, \\
\zeta_i^{(3)} &\to \mathrm{sgn}(b_i^{(3)}) = \zeta_i^{(1)}, \\
\zeta_i^{(4)} &\to \mathrm{sgn}(b_i^{(4)}) = -\zeta_i^{(1)} = \zeta_i^{(4)}, \\
\zeta_i^{(5)} &\to \mathrm{sgn}(b_i^{(5)}) = \zeta_i^{(2)}.
\end{aligned}
\tag{4}
$$

Thus patterns $\zeta_i^{(1)}$, $\zeta_i^{(2)}$ and $\zeta_i^{(4)}$ are stable.

4. Energy function for deterministic Hopfield network

Consider updating S_k into S_k'.
After the update we have

$$S_i' = \begin{cases} S_i & \text{if } i \neq k \\ S_k' & \text{if } i = k \end{cases}$$

The energy fcn updates to

$$H' = -\frac{1}{2}\sum_{ij} w_{ij} S_i' S_j'$$

$$\Rightarrow \Delta H = H' - H = -\frac{1}{2}\sum_{ij} S_i' w_{ij} S_j' + \frac{1}{2}\sum_{ij} S_i w_{ij} S_j$$

$$= \frac{1}{2}\sum_{ij} w_{ij}(S_i S_j - S_i' S_j')$$

$$= \frac{1}{2}\sum_{\substack{i\neq k\\ j\neq k}} w_{ij}\underbrace{(S_i S_j - \underbrace{S_i' S_j'}_{=S_i S_j})}_{0} + \frac{1}{2}\sum_{\substack{i=k\\ j\neq k}} w_{ij}(S_i S_j - \underbrace{S_i' S_j'}_{=S_k' S_j})$$

$$+ \frac{1}{2}\sum_{\substack{i\neq k\\ j=k}} w_{ij}(S_i S_j - \underbrace{S_i' S_j'}_{S_i S_k'}) + \frac{1}{2}\underbrace{\sum_{\substack{i=k\\ j=k}} w_{ij}(\underbrace{S_i S_j - S_i' S_j'}_{\substack{= S_k S_k - S_k' S_k' \\ = 1 - 1 = 0}})}_{0}$$

$$= \frac{1}{2}\sum_{j\neq k} w_{kj}(s_k s_j - s_k' s_j)$$

$$+ \frac{1}{2}\sum_{i\neq k} \underbrace{w_{ik}}_{=w_{ki}}(s_i s_k - s_i s_k') \quad \{j \to i\} \quad (*)$$

$$= \sum_{j\neq k} w_{kj}(s_k s_j - s_k' s_j)$$

$$= (s_k - s_k')\sum_{j\neq k} w_{kj} s_j \quad (**)$$

If $s_k' = s_k$, then $\Delta H = 0$. Otherwise $s_k' = -s_k$, and

$$\Delta H = -2 s_k' \sum_{j\neq k} w_{kj} s_j = -2 s_k'(h_k' - w_{kk} s_k)$$

$$= -2 s_k' h_k' + 2\underbrace{w_{kk}}_{=0} s_k s_k' = -2 s_k' h_k' < 0.$$

Which was to be shown. While showing this, we used that $\underline{\underline{w}}$ is symmetric ($\underline{\underline{w}} = \underline{\underline{w}}^T$) at (*).

b) Calculation proceeds as in a) until

If $s_k' = s_k$, then $\Delta H = 0$. Otherwise $s_k' = -s_k$ and:

$$\Delta H = -2\underbrace{s_k' s_k'}_{=1} + 2\underbrace{w_{kk}}_{=\frac{P}{N}}\underbrace{s_k s_k'}_{=-1}$$

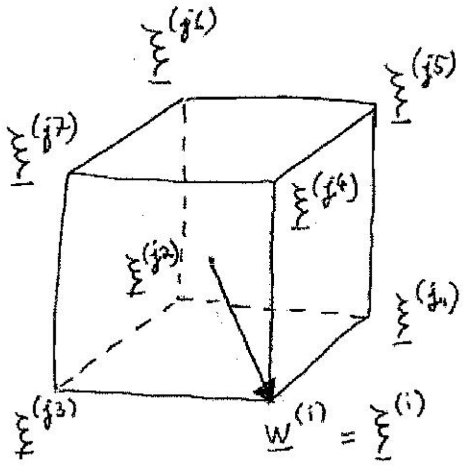
$$= -2 - 2\frac{P}{N} < 0.$$

Thus H either stays constant or decreases using weights given by Eqn. 4.

This property is the same as the energy fcn in 2a.

8. Boolean functions

a) The solution uses $w_{ij} = \xi_j^{(i)}$. This means that the i^{th} row of the weight matrix $\boldsymbol{w}$ is a vector $\boldsymbol{w}^{(i)} = \boldsymbol{\xi}^{(i)}$:



From the figure above, we see that:

- $\boldsymbol{w}^{(i)T}\boldsymbol{\xi}^{(i)} = 1 + 1 + 1 = 3$.
- $\boldsymbol{w}^{(i)T}\boldsymbol{\xi}^{(\mu)} = 1 + 1 - 1 = 1$ for $\mu = j_1, j_4$ and j_3.
- $\boldsymbol{w}^{(i)T}\boldsymbol{\xi}^{(\mu)} = 1 - 1 - 1 = -1$ for $\mu = j_2, j_5$ and j_7.
- $\boldsymbol{w}^{(i)T}\boldsymbol{\xi}^{(j_6)} = -1 - 1 - 1 = -3$.

Using that $\theta_i = 2$, we note that:

$$\boldsymbol{w}^{(i)T}\boldsymbol{\xi}^{(i)} - \theta_i \quad \text{is} \quad \begin{cases} > 0 & \text{if} \quad i = \mu \\ < 0 & \text{if} \quad i \neq \mu \end{cases} . \tag{37}$$

So we have:

$$v^{(i,\mu)} = \begin{cases} 1 & \text{if} \quad i = \mu \\ 0 & \text{if} \quad i \neq \mu \end{cases} . \tag{38}$$

We can understant that the corner μ of the cube of possible inputs is separated from the other corners by that it assigns 1 to the μ^{th} hidden neuron and 0 to the others.

From Figure 4 in the exam, we see that there are exactly 4 of the 8 possible inputs $\boldsymbol{\xi}^{(\mu)}$ that are to be mapped to $O^{\mu} = 1$. These are $\boldsymbol{\xi}^{(\mu)}$ for $\mu = 2, 4, 5$ and 7.
These inputs will assign, respectively:

$$\boldsymbol{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}^{(5)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{v}^{(7)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{39}$$

The weights $\boldsymbol{W}$ are now to 'detect' these and only these patterns, so that

$$O^{(\mu)} = \boldsymbol{W}^T \boldsymbol{w} = \begin{cases} 1 \quad \text{for} \quad \mu \in \{2, 4, 5, 7\} \\ -1 \quad \text{for} \quad \mu \in \{1, 3, 6, 8\} \end{cases}. \tag{40}$$

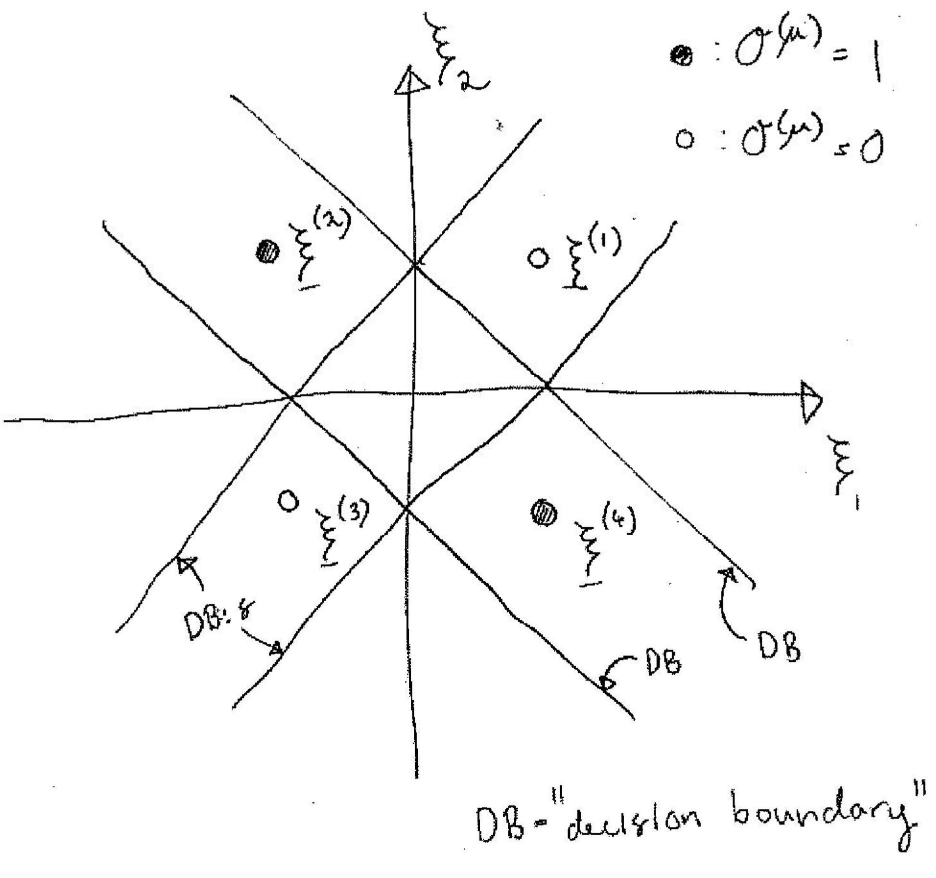
This is achieved by letting:

$$\boldsymbol{W} = \begin{bmatrix} \text{-1} \\ 1 \\ \text{-1} \\ 1 \\ 1 \\ \text{-1} \\ 1 \\ \text{-1} \end{bmatrix}. \tag{41}$$

b) The solution in 6a) implies separating each corner $\xi^{(\mu)}$ of the cube of input patterns by letting

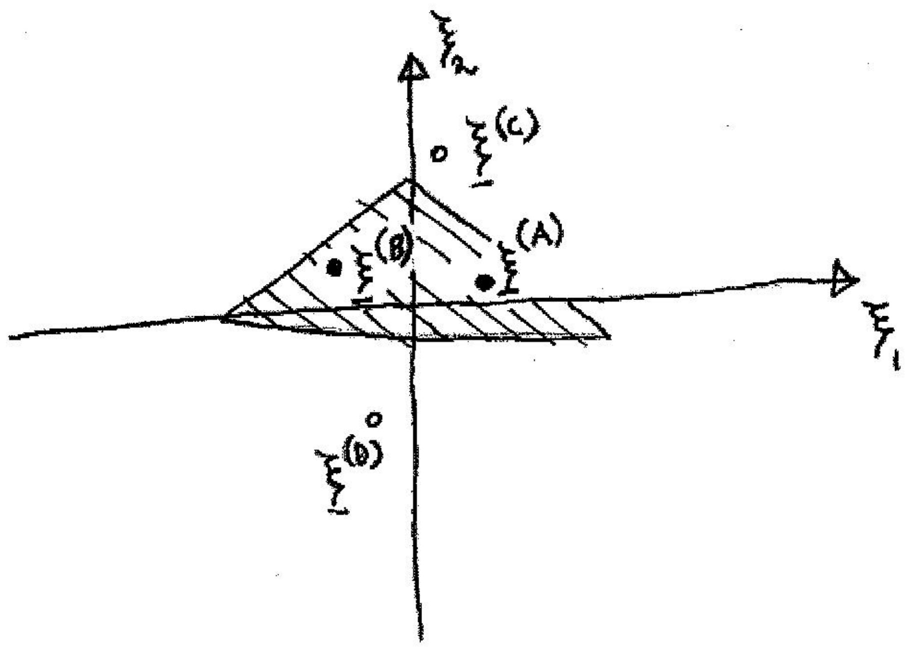
$$v^{(i,\mu)} = \begin{cases} 1 & \text{if} \quad i = \mu \\ 0 & \text{if} \quad i \neq \mu \end{cases} . \tag{42}$$

Thus the solution requires $2^3 = 8$ hidden neurons. The analogous solution in 2D is to separate each corner of a square, and it requires $2^N = 2^2 = 4$ neurons. The decision boundaries of the hidden neurons are shown here:



9. Linearly inseparable problem

a) In the figure below $\boldsymbol{\xi}^{(A)}$ and $\boldsymbol{\xi}^{(B)}$ are to have output 1 and $\boldsymbol{\xi}^{(C)}$ and $\boldsymbol{\xi}^{(D)}$ are to have output 0. There is no straight line that can separate patterns $\boldsymbol{\xi}^{(A)}$ and $\boldsymbol{\xi}^{(B)}$ from patterns $\boldsymbol{\xi}^{(C)}$ and $\boldsymbol{\xi}^{(D)}$.



b) The triangle corners are:

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} -4 \\ 0 \end{bmatrix} \quad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}. \tag{5}$$

• Let $v_1 = 0$ at $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$. This implies

$$\begin{cases} 0 = & w_{11}\xi_1^{(1)} + w_{12}\xi_2^{(1)} - \theta_1 = -4w_{11} - \theta_1 \\ 0 = & w_{11}\xi_1^{(2)} + w_{12}\xi_2^{(2)} - \theta_1 = 4w_{11} - w_{12} - \theta_1 \end{cases}$$
$$\Rightarrow \theta_1 = -4w_{11} \quad \text{and} \quad w_{12} = 4w_{11} - \theta_1 = 8w_{11}. \tag{6}$$

We choose $w_{11} = 1, w_{12} = 8$ and $\theta_1 = -4$.

- Let $v_2 = 0$ at $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$. This implies

$$\begin{aligned}
&\begin{cases} 0 = & w_{21}\xi_1^{(2)} + w_{22}\xi_2^{(2)} - \theta_2 = 4w_{21} - w_{22} - \theta_2 \\ 0 = & w_{21}\xi_1^{(3)} + w_{22}\xi_2^{(3)} - \theta_2 = 3w_{22} - \theta_2 \end{cases} \\
&\Rightarrow w_{22} = 4w_{21} - \theta_2 = 4w_{21} - 3w_{22} \Rightarrow w_{22} = w_{21} \\
&\qquad\qquad\qquad\qquad \text{and} \quad \theta_2 = 3w_{22}.
\end{aligned} \tag{7}$$

We choose $w_{21} = w_{22} = 1$ and $\theta_2 = 3$.

- Let $v_3 = 0$ at $\boldsymbol{\xi}^{(3)}$ and $\boldsymbol{\xi}^{(1)}$. This implies

$$\begin{aligned}
&\begin{cases} 0 = & w_{31}\xi_1^{(3)} + w_{32}\xi_2^{(3)} - \theta_3 = 3w_{32} - \theta_3 \\ 0 = & w_{31}\xi_1^{(1)} + w_{32}\xi_2^{(1)} - \theta_3 = -4w_{31} - \theta_3 \end{cases} \\
&\Rightarrow 3w_{32} = -4w_{31} \quad \text{and} \quad \theta_3 = 3w_{32}.
\end{aligned} \tag{8}$$

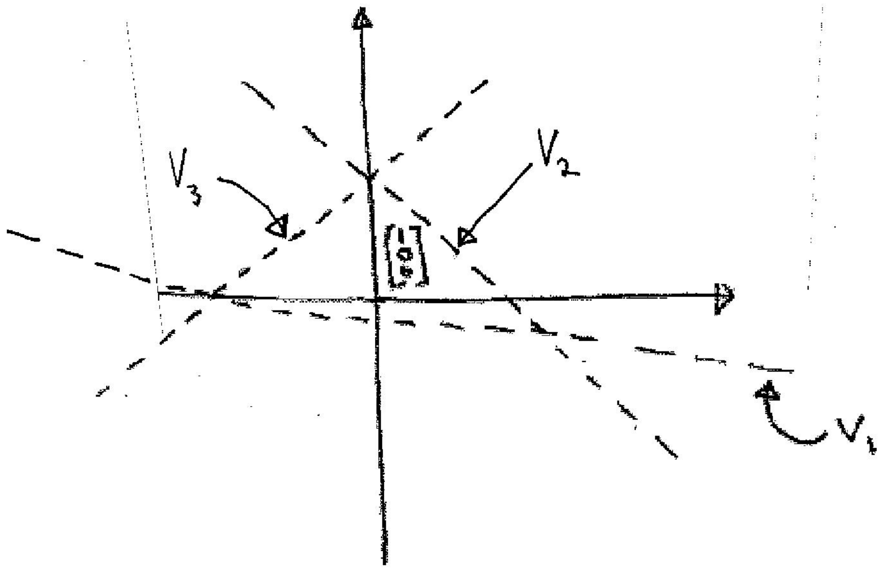
We choose $w_{32} = 4, w_{31} = -3$ and $\theta_3 = 12$. In summary:

$$\boldsymbol{w} = \begin{bmatrix} 1 & 8 \\ 1 & 1 \\ -3 & 4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\theta} = \begin{bmatrix} -4 \\ 3 \\ 12 \end{bmatrix}. \tag{9}$$

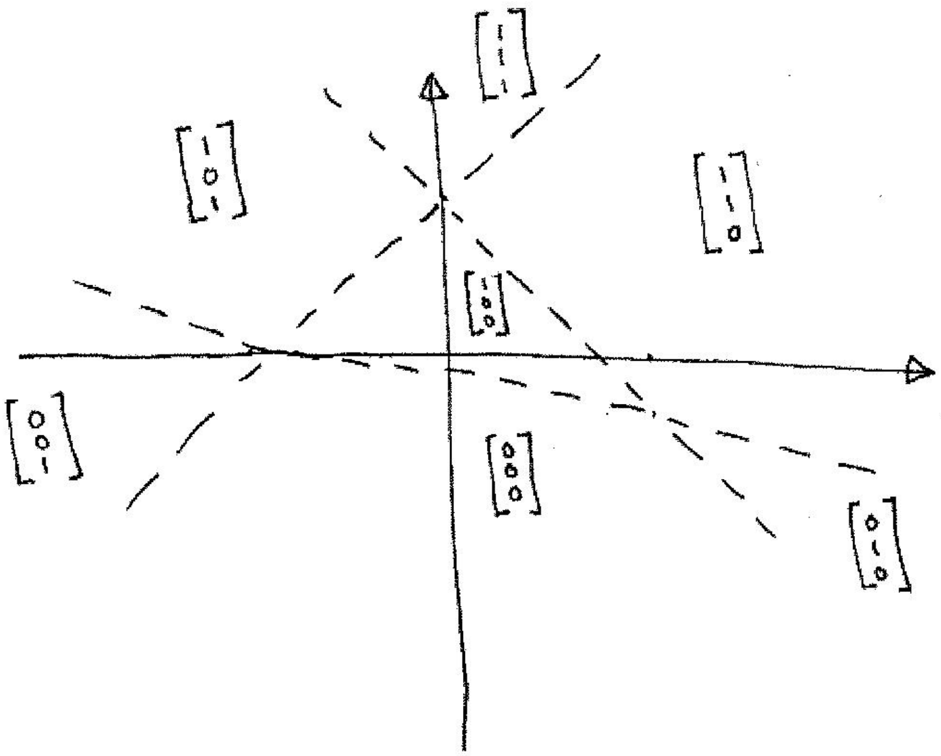
The origin maps to

$$\boldsymbol{v} = H\left(\boldsymbol{w}0 - \boldsymbol{\theta}\right) = H\left(\begin{bmatrix} 4 \\ -3 \\ -12 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{10}$$

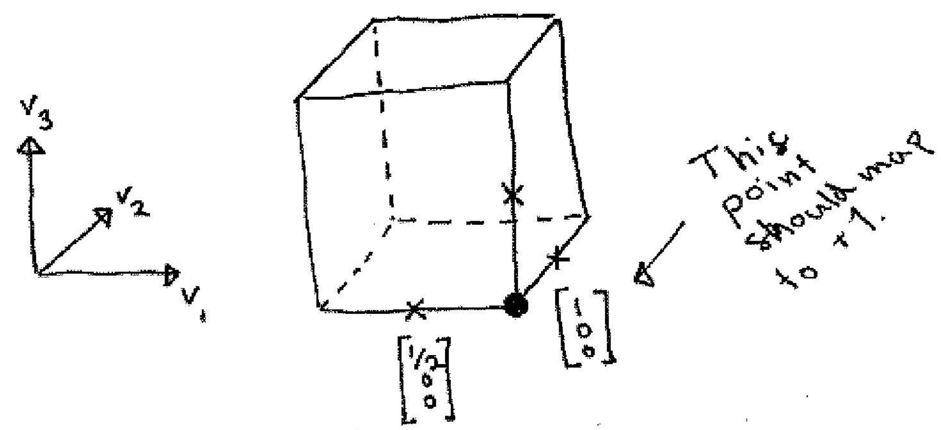
We know that the origin maps to $\boldsymbol{v} = [1, 0, 0]^T$ and that the hidden neurons change values at the dashed lines:



Thus we can conclude that the regions in input space maps to these regions in the hidden space:



We want $\boldsymbol{v} = [1, 0, 0]^T$ to map to 1 and all other possible values of $\boldsymbol{v}$ to map to 0. The hidden space can be illustrated as this:



$\boldsymbol{W}$ must be normal to the plane passing through the crosses in the picture above. Also, $\boldsymbol{W}$ points to $\boldsymbol{v} = [1,0,0]^T$ from $\boldsymbol{v} = [0,1,1]^T$. We may choose

$$\boldsymbol{W} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}. \tag{11}$$

We know that the point $\boldsymbol{v} = [1/2, 0, 0]^T$ lies on the decision boundary we are looking for. So

$$\boldsymbol{W}^T \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} - T = 0 \to T = \frac{1}{2}. \tag{12}$$

10. Perceptron with one hidden layer

$$\underline{W} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad T = \frac{1}{2}$$

$$\underline{W}^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - T = -\frac{1}{2} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ maps to } O = 0$$

$$\underline{W}^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - T = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ maps to } O = 1.$$

$$\underline{W}^T \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - T = 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \text{ maps to } O = 1$$

$$\underline{W}^T \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - T = -1 - \frac{1}{2} = -\frac{3}{2} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ maps to } O = 0$$

$$\underline{W}^T \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - T = 0 - \frac{1}{2} = -\frac{1}{2} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ maps to } O = 0$$

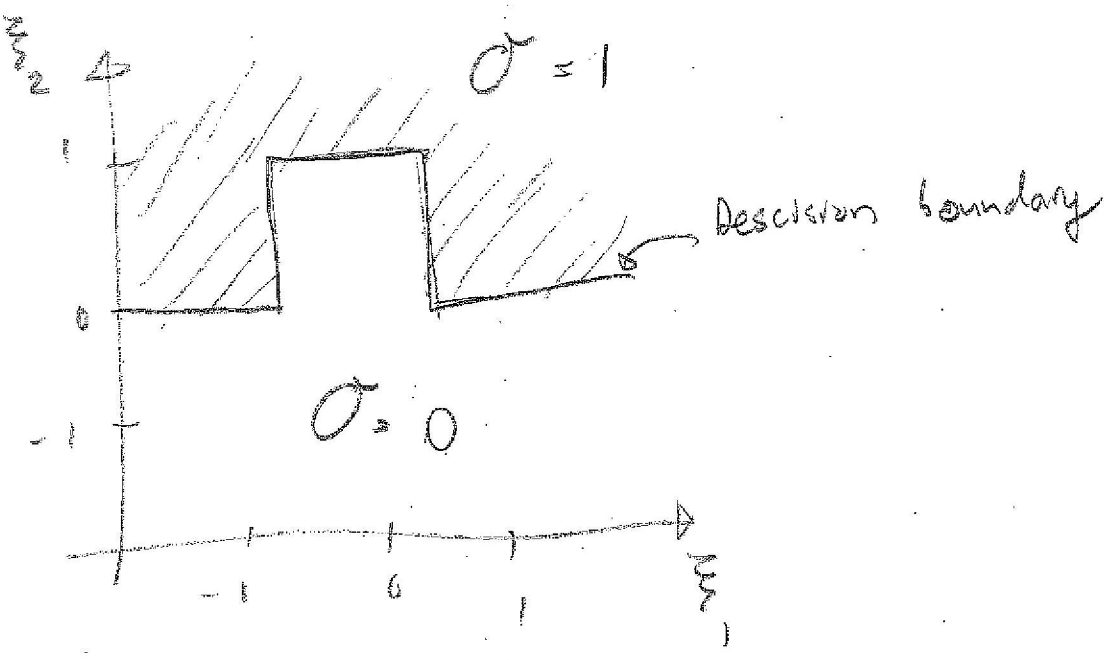
$$\underline{W}^T \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - T = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ maps to } O = 1$$

$$\underline{W}^T \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - T = 0 - \frac{1}{2} = -\frac{1}{2} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ maps to } \sigma = 0$$

$$\underline{W}^T \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - T = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ maps to } \sigma = 1$$

$$\underline{W}^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - T = 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ maps to } \sigma = 1$$

Illustration



b) Consider the sequence of patterns:

$$\underline{\xi}^{(1)} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}, \quad \underline{\xi}^{(2)} = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} \text{ and } \underline{\xi}^{(3)} = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}.$$

The corresponding sequence of $V_3^{(\mu)}$ is

$$V_3^{(1)} = 0, \quad V_3^{(2)} = 1 \text{ and } V_3^{(3)} = 0. \quad (*)$$

For a monotonously increasing sequence of patterns for which $\xi_1^{(\mu)}$ is constant, and $\xi_2^{(\mu)}$ is monotonously increasing the function $-\Theta_3 + \sum_j w_{3j}\xi_j^{(\mu)}$ is must increase (decrease) monotonously if w_{32} is positive (negative) or stay constant if $w_{32} = 0$. This is not the case for the sequence (*) above.

11. Multilayer perceptron

a)

No, since we can not draw a line that separate the points according to their target outputs.

b)

<u>Solid line</u> passes throug points $(0,1)$ and $(1,0)$.

$$(1,0) \Rightarrow w_{11} \cdot 1 + w_{12} \cdot 0 - \theta_1 = 0 \Rightarrow w_{11} = \theta_1$$

$$(0,1) \Rightarrow w_{11} \cdot 0 + w_{12} \cdot 1 - \theta_1 = 0 \Rightarrow w_{12} = \theta_1$$

So $w_{11} = w_{12} = \theta_1$. $(*)$

<u>Dash-dotted line</u>

$$(0.5, 0) \Rightarrow 0.5 w_{21} + 0 \cdot w_{22} - \theta_2 = 0 \Rightarrow \theta_2 = 0.5 w_{21}$$

$$(0.5, 1) \Rightarrow 0.5 w_{21} + w_{22} - \theta_2 = 0 \Rightarrow w_{22} = 0$$

So $\theta_2 = 0.5 w_{21}$ and $w_{22} = 0$ $(**)$

<u>Dashed line</u>

$$(0, 0.8) \Rightarrow 0 \cdot w_{31} + 0.8 \cdot w_{32} - \theta_3 = 0 \Rightarrow \theta_3 = 0.8 w_{32}$$

$$(1, 0.8) \Rightarrow 1 \cdot w_{31} + 0.8 w_{32} - \theta_3 = 0 \Rightarrow w_{31} = 0.$$

So: $\theta_3 = 0.8 w_{32}$ and $w_{31} = 0$. $(***)$

(*), (**) and (***) are realized by, for instance:

$$\underline{\underline{W}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{\theta} = \begin{bmatrix} 1 \\ 0.5 \\ 0.8 \end{bmatrix}$$

c/ Evaluate output at $(\xi_1, \xi_2) = (0, 0)$:

$$V_1 = \Theta\left[1\cdot 0 + 1\cdot 0 - 1\right] = 0$$

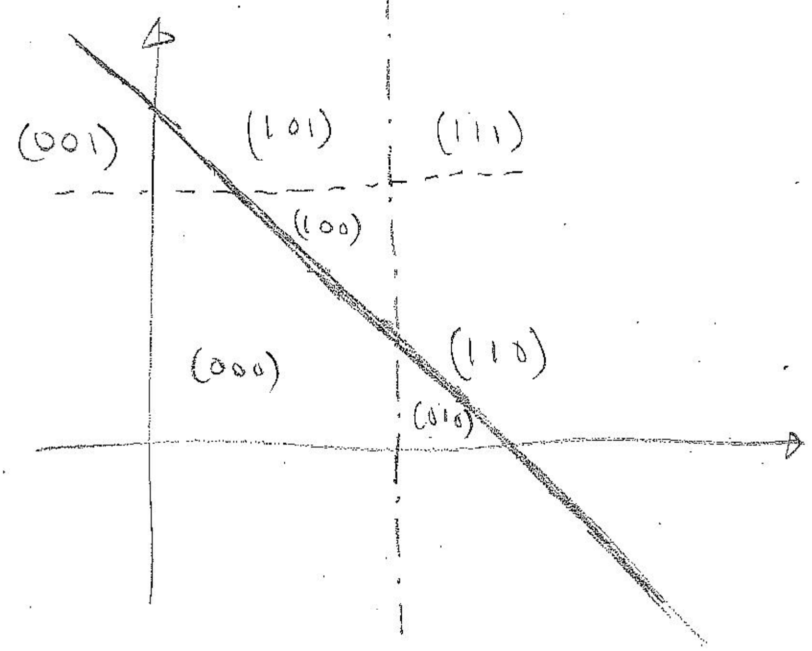
$$V_2 = \Theta\left[1\cdot 0 + 0\cdot 0 - 0.5\right] = 0$$

$$V_3 = \Theta\left[0\cdot 0 + 1\cdot 0 - 0.8\right] = 0$$

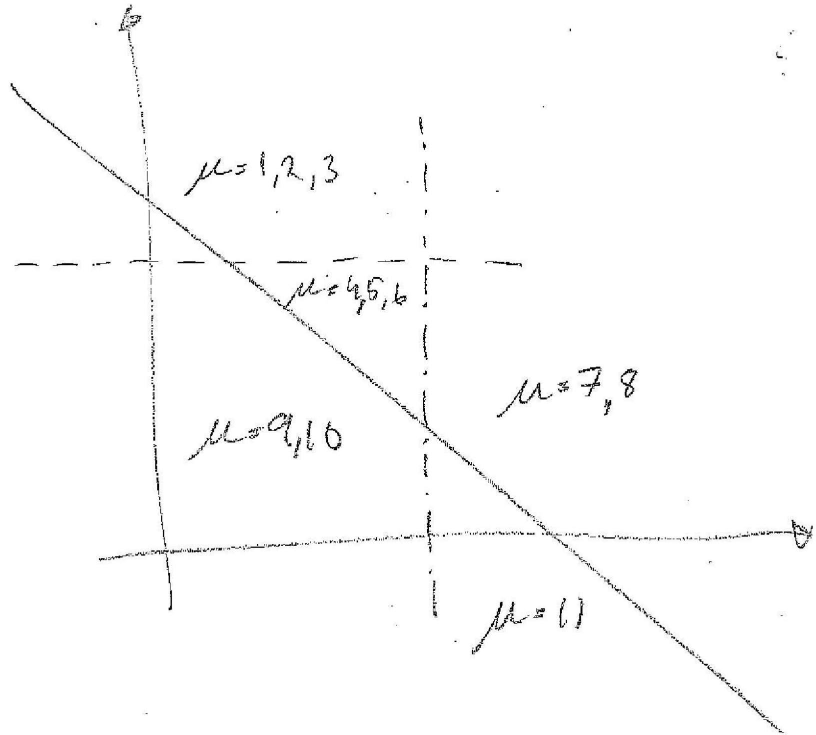
So the origin of the input space maps to $\underline{V} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Note that, as defined in 3b, V_1 flips at the solid line, V_2 flips at the dash-dotted line and V_3 flips at the dashed line.

Given that the origin in the input space is (0,0,0) we find:



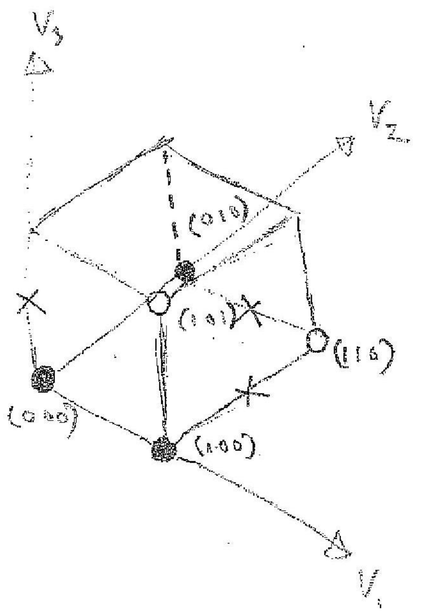
The input patterns are located according to:



So the coordinates in the transformed space are

μ	$\underline{V}$
1,2,3	(1,0,1)
4,5,6	(1,0,0)
7,8	(1 1 0)
9,10	(0,0,0)
11	(0,1,0)

d) Graphical illustration of hidden space:



Yes, the problem is linearly separable. We can separate the output with the plane passing through the crosses in the figure.

e)

The crosses drawn in 3d) are:

$P_1 : (1, \frac{1}{2}, 0)$

$P_2 : (\frac{1}{2}, 1, 0)$

$P_3 : (0, 0, \frac{1}{2})$

Decision boundary at plane containing these points imply $W_1 V_1 + W_2 V_2 + W_3 V_3 - \Theta = 0$.

$P_1 \Rightarrow W_1 + \frac{1}{2} W_2 = \Theta \quad \text{I}$

$P_2 \Rightarrow \frac{1}{2} W_1 + W_2 = \Theta \quad \text{II}$

$P_3 \Rightarrow \frac{1}{2} W_3 = \Theta \quad \text{III}$

$\text{I} + \text{II} \Rightarrow W_1 + \frac{1}{2} W_2 = \frac{1}{2} W_1 + W_2$

$\Rightarrow \frac{1}{2} W_1 = \frac{1}{2} W_2 \Rightarrow W_1 = W_2 \quad \text{IV}$

$\text{I} + \text{IV} \Rightarrow \frac{3}{2} W_1 = \frac{3}{2} W_2 = \Theta \quad \text{V}$

$\text{III} + \text{V} \Rightarrow \frac{1}{2} W_3 = \frac{3}{2} W_1 = \frac{3}{2} W_2 = \Theta$

Note that we want the origin of the transformed space to be mapped to +1 (from patterns 9 and 10). Thus

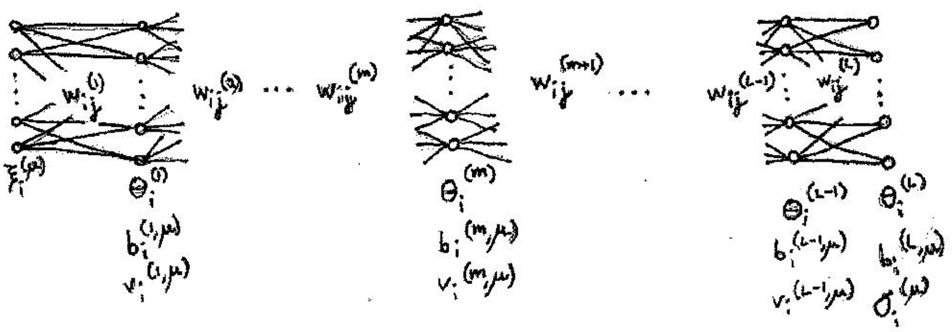
$$0 \cdot w_1 + 0 \cdot w_2 + 0 \cdot w_3 - \Theta > 1$$

$\Rightarrow \Theta < -1$. We may choose:

$$\underline{W} = \begin{bmatrix} -4/3 \\ -4/3 \\ -4 \end{bmatrix} \quad \text{and} \quad \Theta = -2$$

14. Backpropagation

a)



Let N_m denote the number of weights $w_{ij}^{(m)}$. Let n_m denote the number of hidden units $v_i^{(m,\mu)}$ for $i = 1, \ldots, L-1$, let n_0 denote the number of input units and let n_L denote the number of output units. Find that the number of weights are

$$\sum_{m=1}^{L} N_m = \sum_{m=1}^{L} n_{m-1} n_m, \tag{13}$$

and that the number of thresholds are

$$\sum_{m=1}^{L} n_m. \tag{14}$$

b)

$$
\begin{aligned}
\frac{\partial}{\partial w_{qr}^{(p)}} v_i^{(m,\mu)} =& \frac{\partial}{\partial w_{qr}^{(p)}} g\left(b_i^{(m,\mu)}\right) = g'\left(b_i^{(m,\mu)}\right) \frac{\partial}{\partial w_{qr}^{(p)}} b_i^{(m,\mu)} \\
=& g'\left(b_i^{(m,\mu)}\right) \frac{\partial}{\partial w_{qr}^{(p)}} \left(-\theta_i^{(m)} + \sum_j w_{ij}^{(m)} v_j^{(m-1,\mu)}\right) \\
=& g'\left(b_i^{(m,\mu)}\right) \left(\sum_j \frac{\partial}{\partial w_{qr}^{(p)}} w_{ij}^{(m)} v_j^{(m-1,\mu)}\right). \qquad (15)
\end{aligned}
$$

Using that $p < m$, we find:

$$
\frac{\partial v_i^{(m,\mu)}}{\partial w_{qr}^{(p)}} = g'\left(b_i^{(m,\mu)}\right) \sum_j w_{ij}^{(m)} \frac{\partial v_j^{(m-1,\mu)}}{\partial w_{qr}^{(p)}}. \qquad (16)
$$

c) From b), we have:

$$
\frac{\partial}{\partial w_{qr}^{(p)}} v_i^{(m,\mu)} = g'\left(b_i^{(m,\mu)}\right) \sum_j \frac{\partial}{\partial w_{qr}^{(p)}} w_{ij}^{(m)} v_j^{(m-1,\mu)}. \qquad (17)
$$

But since $p = m$, we find:

$$
\frac{\partial}{\partial w_{qr}^{(p)}} v_i^{(m,\mu)} = g'\left(b_i^{(m,\mu)}\right) \sum_j \delta_{qi}\delta_{rj} v_j^{(m-1,\mu)} = g'\left(b_i^{(m,\mu)}\right) \delta_{qi} v_r^{(m-1,\mu)}. \qquad (18)
$$

d) We have $w_{qr}^{(L-2)} \leftarrow w_{qr}^{(L-2)} + \delta w_{qr}^{(L-2)}$, where

$$
\delta w_{qr}^{(L-2)} = -\eta \frac{\partial H}{\partial w_{qr}^{(L-2)}}. \qquad (19)
$$

We derive the energy function:

$$
\begin{aligned}
\frac{\partial H}{\partial w_{qr}^{(L-2)}} =& \frac{\partial}{\partial w_{qr}^{(L-2)}} \frac{1}{2} \sum_\mu \sum_i \left(O_i^{(\mu)} - \zeta_i^{(\mu)}\right)^2 \\
=& \sum_\mu \sum_i \left(O_i^{(\mu)} - \zeta_i^{(\mu)}\right) \frac{\partial O_i^{(\mu)}}{\partial w_{qr}^{(L-2)}} \qquad (20)
\end{aligned}
$$

From 3b) and 3c), we have:

$$
\frac{\partial v_i^{(m,\mu)}}{\partial w_{qr}^{(p)}} = \begin{cases} g'\left(b_i^{(m,\mu)}\right) \sum_j w_{ij}^{(m)} \frac{\partial v_j^{(m-1,\mu)}}{\partial w_{qr}^{(p)}} & \text{if} \quad p < m \\ g'\left(b_i^{(m,\mu)}\right) \delta_{qi} v_r^{(m-1,\mu)} & \text{if} \quad p = m. \end{cases} \qquad (21)
$$

Define $v_i^{(L,\mu)} = O_i^{(\mu)}$. We have:

$$
\begin{aligned}
\frac{\partial O_i^{(\mu)}}{\partial w_{qr}^{(L-2)}} =& \frac{\partial v_i^{(L,\mu)}}{\partial w_{qr}^{(L-2)}} \\
& \{\text{Insert from eq. (21). Use that } L-2 < L. \} \\
=& g'\left(b_i^{(L,\mu)}\right) \sum_j w_{ij}^{(L)} \frac{\partial v_j^{(L-1,\mu)}}{\partial w_{qr}^{(L-2)}} \\
& \{\text{Insert from eq. (21). Use that } L-2 < L-1. \} \\
=& g'\left(b_i^{(L,\mu)}\right) \sum_j w_{ij}^{(L)} g'\left(b_j^{(L-1,\mu)}\right) \sum_k w_{jk}^{(L-1)} \frac{\partial v_k^{(L-2,\mu)}}{\partial w_{qr}^{(L-2)}} \\
& \{\text{Insert from eq. (21). Use that } L-2 = L-2. \} \\
=& g'\left(b_i^{(L,\mu)}\right) \sum_j w_{ij}^{(L)} g'\left(b_j^{(L-1,\mu)}\right) \sum_k w_{jk}^{(L-1)} g'\left(b_k^{(L-2,\mu)}\right) \delta_{qk} v_r^{(L-3,\mu)} \\
=& g'\left(b_i^{(L,\mu)}\right) \sum_j w_{ij}^{(L)} g'\left(b_j^{(L-1,\mu)}\right) w_{jq}^{(L-1)} g'\left(b_q^{(L-2,\mu)}\right) v_r^{(L-3,\mu)}. \qquad (22)
\end{aligned}
$$

The update rule is eq. (19) with the derivative of the energy function given by eqs. (20) and (22).

18. Oja's rule

a)

$$\begin{aligned}\mathbf{0} = \langle \boldsymbol{\delta w} \rangle &= \langle \eta\zeta\,(\boldsymbol{\xi} - \zeta \boldsymbol{w}) \rangle \\ &\Rightarrow \langle \boldsymbol{\xi}\zeta \rangle = \langle \zeta\zeta \boldsymbol{w} \rangle \,. \end{aligned} \tag{23}$$

Insert

$$\zeta = \boldsymbol{\xi}^T \boldsymbol{w} = \boldsymbol{w}^T \boldsymbol{\xi} : \tag{24}$$

$$\begin{aligned} 0 = \langle \boldsymbol{\delta w} \rangle &\Rightarrow \left\langle \boldsymbol{\xi}\boldsymbol{\xi}^T \boldsymbol{w} \right\rangle = \left\langle \boldsymbol{w}^T \boldsymbol{\xi}\boldsymbol{\xi}^T \boldsymbol{w}\boldsymbol{w} \right\rangle \\ &\Rightarrow \left\langle \boldsymbol{\xi}\boldsymbol{\xi}^T \right\rangle \boldsymbol{w} = \boldsymbol{w}^T \left\langle \boldsymbol{\xi}\boldsymbol{\xi}^T \right\rangle \boldsymbol{w}\boldsymbol{w}. \end{aligned} \tag{25}$$

$\left\langle \boldsymbol{\xi}\boldsymbol{\xi}^T \right\rangle = \boldsymbol{C}$ is a matrix, so:

$$0 = \langle \boldsymbol{\delta w} \rangle \Rightarrow \boldsymbol{C}\boldsymbol{w} = \boldsymbol{w}^T \boldsymbol{C}\boldsymbol{w}\boldsymbol{w}. \tag{26}$$

We see that $\langle \boldsymbol{\delta w} \rangle = \mathbf{0}$ implies that $\boldsymbol{w}$ is an eigenvector of $\boldsymbol{C}$ with eigenvalue $\lambda = \boldsymbol{w}^T \boldsymbol{C}\boldsymbol{w}$:

$$\lambda = \boldsymbol{w}^T \boldsymbol{C}\boldsymbol{w} = \boldsymbol{w}^T \lambda \boldsymbol{w} = \lambda \boldsymbol{w}^T \boldsymbol{w} \Rightarrow \boldsymbol{w}^T \boldsymbol{w} = 1. \tag{27}$$

(note that $\boldsymbol{w}^T \boldsymbol{w} = \sum_i w_i w_i$).

b) Are the patterns centered?

$$\sum_{\mu=1}^{5} \xi_1^{(\mu)} = -6 - 2 + 1 + 1 + 5 = 0 \tag{28}$$

$$\sum_{\mu=1}^{5} \xi_2^{(\mu)} = -5 - 4 + 2 + 3 + 4 = 0. \tag{29}$$

So $\langle \boldsymbol{\xi} \rangle = \mathbf{0}$, and the patterns are centered. This means that the covariance matrix is:

$$\boldsymbol{C} = \frac{1}{5} \sum_{\mu=1}^{5} \boldsymbol{\xi}^{(\mu)} \boldsymbol{\xi}^{(\mu)\,T} = \left\langle \boldsymbol{\xi}\boldsymbol{\xi}^T \right\rangle . \tag{30}$$

We have

$$\begin{aligned} \boldsymbol{\xi}^{(1)} \boldsymbol{\xi}^{(1)\,T} &= \begin{bmatrix} 36 & 30 \\ 30 & 25 \end{bmatrix} \\ \boldsymbol{\xi}^{(2)} \boldsymbol{\xi}^{(2)\,T} &= \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix} \\ \boldsymbol{\xi}^{(3)} \boldsymbol{\xi}^{(3)\,T} &= \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \\ \boldsymbol{\xi}^{(4)} \boldsymbol{\xi}^{(4)\,T} &= \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \\ \boldsymbol{\xi}^{(5)} \boldsymbol{\xi}^{(5)\,T} &= \begin{bmatrix} 25 & 20 \\ 20 & 16 \end{bmatrix} . \end{aligned} \tag{31}$$

We compute the elements of $\boldsymbol{C}$:

$$\begin{aligned} 5C_{11} =& 36+4+4+1+25=70 \\ 5C_{12} =& 5C_{21}=30+8+4+3+20=65 \\ 5C_{22} =& 25+16+4+9+16=70. \end{aligned} \tag{32}$$

We find that

$$\boldsymbol{C}=\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}. \tag{33}$$

Maximal eigenvalue:

$$\begin{aligned} 0=\begin{vmatrix} 14-\lambda & 13 \\ 13 & 14-\lambda \end{vmatrix} =& (14-\lambda)^2-13^2=\lambda^2-28\lambda+14^2-13^2=\lambda^2-28\lambda+27 \\ \Rightarrow& \lambda=14\pm\sqrt{14^2-27}=14\pm\sqrt{169}=14\pm 13 \\ \Rightarrow& \lambda_{\max}=27. \end{aligned} \tag{34}$$

Eigenvector $\boldsymbol{u}$:

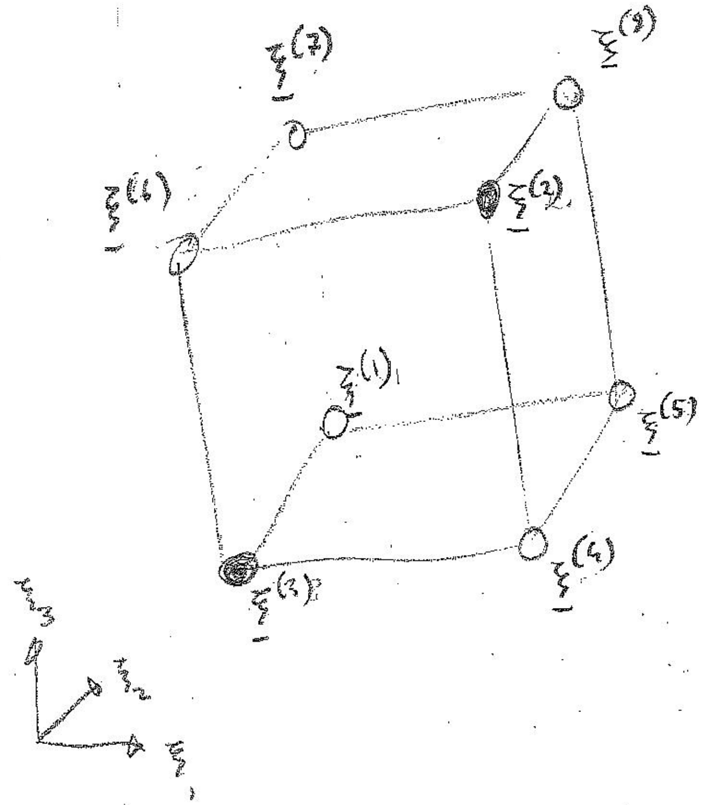
$$\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=27\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\Rightarrow u_1=u_2. \tag{35}$$

So an eigenvector corresponging to the largest eigenvalue of $\boldsymbol{C}$ is given by

$$\boldsymbol{u}=t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{36}$$

for an arbitrary t. This is the principal component.

22. Radial basis functions

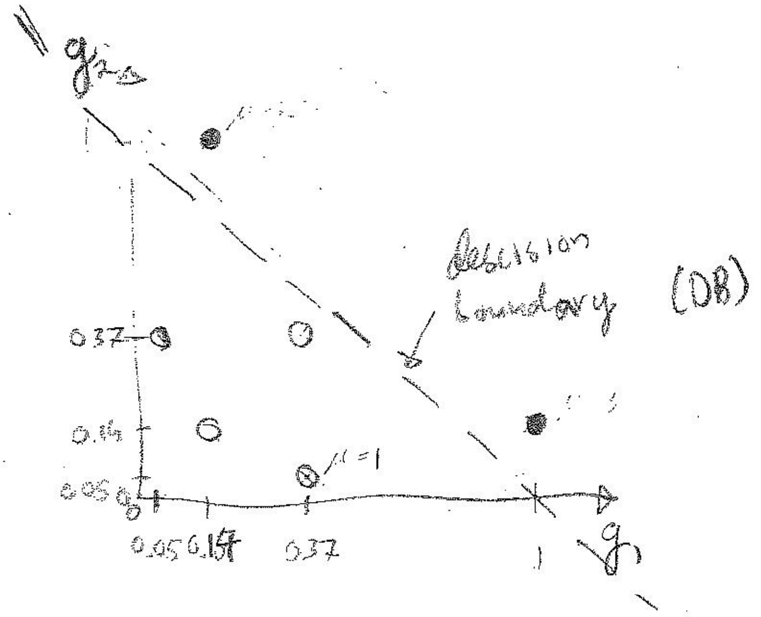


No, the problem can not be solved by a perceptron without hidden layers, since it is not linearly separable.

b)

μ	$\underline{\xi}^{(\mu)} - \underline{w}_1$	$\underline{\xi}^{(\mu)} - \underline{w}_2$	$\lvert \underline{\xi}^{(\mu)} - \underline{w}_1 \rvert^2$	$\lvert \underline{\xi}^{(\mu)} - \underline{w}_2 \rvert^2$	$g_1(\underline{\xi}^{(\mu)})$	$g_2(\underline{\xi}^{(\mu)})$
1	$(0\ 1\ 0)^T$	$(-1\ 1\ -1)^T$	1	3	0.37	0.05
2	$(1\ 0\ 1)^T$	$(0\ 0\ 0)^T$	2	0	0.14	1
3	$(0\ 0\ 0)^T$	$(-1\ 0\ -1)^T$	0	2	1	0.14
4	$(1\ 0\ 0)^T$	$(0\ 0\ -1)^T$	1	1	0.37	0.37
5	$(1\ 1\ 0)^T$	$(0\ 1\ -1)^T$	2	2	0.14	0.14
6	$(0\ 0\ 1)^T$	$(-1\ 0\ 0)^T$	1	1	0.37	0.37
7	$(0\ 1\ 1)^T$	$(-1\ 1\ 0)^T$	2	2	0.14	0.14
8	$(1\ 1\ 1)^T$	$(0\ 1\ 0)^T$	3	1	0.05	0.37

Transformed space:



Simple perceptron:

$$O^{(\mu)} = \text{sgn}\left[\sum_{i=1}^{2} W_i g_i - T\right]$$

Choose weight ~~vector~~ ~~orthogonal~~ to DB and with correct signs:

$$\underline{W} = \begin{bmatrix}1\\1\end{bmatrix}. \quad \text{DB at } \underline{g} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow O = \underline{W}^T\begin{bmatrix}1\\0\end{bmatrix} - T = 1 - T \Rightarrow T = 1.$$

$$\underline{W} = \begin{bmatrix}1\\1\end{bmatrix}, \quad T = 1.$$