Dynamical systems, Home Exam
for January 17, 12.00 (lunch), 2021

1. Model for spreading of fabricated news [4 points] Consider the following simple model for how fabricated news may spread in a population

$$\dot{n} = (d+n)b - rnb^2 \tag{1}$$

with $b = 1-d-n$. Here d, n, and b are non-negative fractions of three populations: devoted (d), non-believers (n), and believers (b). Members of the devoted population are certain that the news is fake and will never change their opinion. d is therefore a constant parameter, $0 \leq d \leq 1$. Members of the populations of non-believers and believers of the news may change their opinion. Believers change their opinion (changing to non-believer) by becoming convinced by devotees or non-believers, their conversion rate is given by the first term in Eq. (1). Non-believers change their opinion by aggressive sharing of the news from bots (last term in Eq. (1)), with a conversion rate proportional to the fraction of non-believers, but also quadratically proportional to the fraction of believers (the fake news bots share more if the number of believers becomes large). The proportionality constant r is a finite positive parameter, $0 < r < \infty$.

a) For the case $d = 0$, analytically find all fixed points of the system (1) and determine their stability as a function of the parameter r. Make sure to only include valid fixed points, excluding fixed points outside of the intervals given in the problem formulation above.

b) For the case $d = 0$, plot the bifurcation diagram against r and label all bifurcations with their type and the value of the bifurcation point.

c) Write down all fixed points of the system (1) in terms of the parameters r and d.

d) Use Mathematica or an equivalent software to analytically determine all valid bifurcations in the dynamics (1) in terms of the parameters r and d. Make a plot in the (r, d) plane that shows the curves at which bifurcations occur, labeled with the type of bifurcation. Label different regions in the (r, d) space by the number of valid fixed points and how many of these are stable.

   Hint If you want to decrease the probability of mistakes, you can check that the limit $d \to 0$ in subtasks c) and d) gives the results in subtasks a) and b). In particular, check that the number of allowed fixed points comes out as expected.

e) Explain what the different stable fixed points mean for the number of believers in the long-time limit. Under which conditions do the entire population become believers? Does the system show catastrophes? Does it show hysteresis?

2. Classification of exotic fixed points [3 points] Consider the dynamical system

$$\begin{aligned} \dot{x} &= -\mu y + y(x - xy - y) \\ \dot{y} &= \mu x - x^2 + xy - y^3 \end{aligned} \tag{2}$$

where μ is a real parameter.

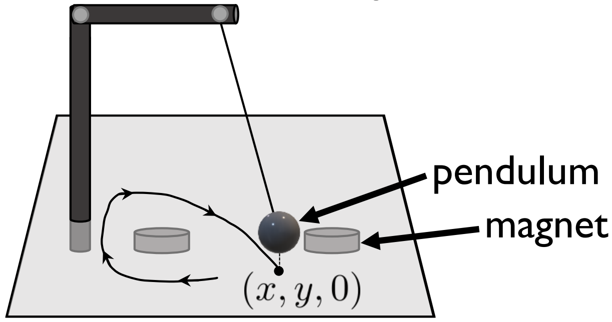
a) Analytically find all fixed points of the system (2) and classify them according to linear stability theory.

b) For the cases $\mu < 0$ and $\mu > 0$, describe and explain the dynamics close to the fixed points if non-linear terms in Eq. (2) are taken into account.

   You get full score (1.5p) on this subtask by deriving the behavior using an analytical approach, for example by analysing the system using suitable coordinate changes close to the fixed points. You get half score (0.75p) by deriving the behavior using convincing geometrical solutions, for example by plotting detailed phase portraits for different μ-values.

c) Describe the bifurcation that occurs when μ passes zero. Discuss, for example using sketches, how the basins of attractions of the two fixed points changes before and after the bifurcation.

3. Fractal dimension of magnetic pendulum [5 points] The figure below shows a damped magnetic pendulum interacting with two magnets.



Here $\boldsymbol{x} = (x, y, 0)$ are the projected coordinates of the pendulum on the plane with $z = 0$. The two magnets are located at $\boldsymbol{m}^{(1)} = L(-1, 0, -0.5)$ and $\boldsymbol{m}^{(2)} = L(1, 0, -0.5)$, where L is a constant length scale (note that the plane of the magnets is located $L/2$ below the plane of the pendulum). If the pendulum is long enough, its projected coordinates follow the dynamics

$$\begin{aligned} \ddot{x} &= -\gamma\dot{x} - \frac{g}{l}x + M\sum_{i=1}^{2}\frac{m_1^{(i)} - x}{|\boldsymbol{m}^{(i)} - \boldsymbol{x}|^5} \\ \ddot{y} &= -\gamma\dot{y} - \frac{g}{l}y + M\sum_{i=1}^{2}\frac{m_2^{(i)} - y}{|\boldsymbol{m}^{(i)} - \boldsymbol{x}|^5} \end{aligned}. \tag{3}$$

Here $\gamma \geq 0$ describes damping. When $\gamma = 0$ the system is Hamiltonian. g, l and M are positive parameters describing in order: gravitational acceleration, pendulum length and strength of magnetic interactions.

a) Write Eq. (3) as a dynamical system of dimensionality 4. Use suitable dimensionless coordinates and choose the length scale L in $\boldsymbol{m}^{(i)}$ to write the system in terms of a single dimensionless parameter α that multiplies the dimensionless versions of the damping terms in Eq. (3).

Clearly state your choice of L and the expression you obtain for α.

Hint You should find that your dimensionless coordinates satisfy

$$\begin{aligned} \ddot{x} &= -\alpha\dot{x} - x + \sum_{i=1}^{2}\frac{m_1^{(i)} - x}{|\boldsymbol{m}^{(i)} - \boldsymbol{x}|^5} \\ \ddot{y} &= -\alpha\dot{y} - y + \sum_{i=1}^{2}\frac{m_2^{(i)} - y}{|\boldsymbol{m}^{(i)} - \boldsymbol{x}|^5} \end{aligned}, \quad \text{with } \boldsymbol{m}^{(i)} = (\pm 1, 0, -0.5). \tag{4}$$

b) Consider the system you obtained in subtask a), i.e. Eq. (4). Use arguments about the physics of the problem (or analysis of the dynamical system) to rule out that the system has stable limit cycles or strange attractors for any parameter value $\alpha \geq 0$.

c) For the two cases $\alpha = 0$ and $\alpha = 0.15$, numerically find all fixed points of the system (4) and determine their stability. Explain your method, list your fixed points and explain their stability properties. For the case of $\alpha = 0.15$, you should find that two fixed points are stable.

d) For the two cases $\alpha = 0$ and $\alpha = 0.15$, numerically integrate Eq. (4), starting from the initial position $x_0 = y_0 = 0.5$ and at rest, i.e. $\dot{x}_0 = \dot{y}_0 = 0$. For each case show a plot of the trajectory in the x, y-plane for times from $t = 0$ to $t = 200$.

   Run additional trajectories from additional initial conditions to increase your understanding of the system and then discuss the two points below for both the cases of $\alpha = 0$ and $\alpha = 0.15$.

   - Describe the dynamics. Does the behavior depend on the initial position? What happens for initial conditions with $x_0 = 0$?
   - Does the system show regular behaviour, transient chaos, intermittent chaos, or full chaos? You don't need to calculate Lyapunov exponents. Argue instead from the observed behaviour between $t = 0$ and $t = 200$ and what you know from subtasks b) and c).

   Hint You may need to use high accuracy in the integrations with $\alpha = 0$.

e) In what follows, consider $\alpha = 0.15$. Numerically integrate the dynamics for initial positions $(x_0, y_0, 0)$ on a 512×512 grid in the region $-5 \leq x_0 \leq 5$ and $-5 \leq y_0 \leq 5$ with the pendulum starting from rest. Make two plots in the x_0, y_0-space of initial positions. The first plot should contain the basin of attraction for the different fixed points. Achieve this by coloring the initial position with different colors depending on which fixed point it ends up in. The second plot should contain the boundary of the basin of attraction for one of the attracting fixed points. You can obtain the boundary by, for each initial position on the 512×512 grid, color it if any of the four neighbouring initial conditions is attracted to the other fixed point.

   Make sure to store the data used to generate the plots as it is needed in the following subtasks. If it takes too long to generate the 512×512 grid, you can use a smaller grid (you can still obtain full score on the problem, but it will be harder to draw conclusions in following subtasks).

f) Use data from subtask e) to numerically estimate the box-counting dimension for the basin of attraction for one attracting fixed point. Plot the number of boxes against the box size ϵ in a loglog plot.

   Answer the following questions

   - Do you observe a power-law scaling? If so, what is the box-counting dimension? If not, explain possible reasons why you do not see it.
   - Does the data describe a fractal?

   Hint It is convenient to choose values of ϵ such that each direction is covered with 16, 32, 64, 128, 256, or 512 boxes.

g) Repeat the procedure in subtask f) for the boundary of the basin of attraction and answer the questions in subtask f).