CHALMERS, GÖTEBORGS UNIVERSITET

SOLUTIONS to RE-EXAM for
ARTIFICIAL NEURAL NETWORKS

COURSE CODES: **FFR 135, FIM 720 GU, PhD**

Time:	January 20, 2018, at $8^{30} - 12^{30}$
Place:	SB Multisal
Teachers:	Bernhard Mehlig, 073-420 0988 (mobile)
	Johan Fries, 070-370 1272 (mobile), visits once at 9^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	Any other written material, calculator

Maximum score on this exam: 12 points.
Maximum score for homework problems: 12 points.
To pass the course it is necessary to score at least 5 points on this written exam.
CTH $\geq$14 passed; $\geq$17.5 grade 4; $\geq$22 grade 5,
GU $\geq$14 grade G; $\geq$ 20 grade VG.

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1. Recognition of one pattern.

a) Define

$$Q^{(\mu,\nu)} = \sum_{j=1}^{j=42} \zeta_j^{(\mu)} \zeta_j^{(\nu)}. \tag{1}$$

The bit j constributes with $+1$ to $Q^{(\mu,\nu)}$ if $\zeta_j^{(\mu)} = \zeta_j^{(\nu)}$, and with -1 if $\zeta_j^{(\mu)} \neq \zeta_j^{(\nu)}$. Since the number of bits are 42, we have $Q^{(\mu,\nu)} = 42 - 2 \cdot H^{(\mu,\nu)}$, where $H^{(\mu,\nu)}$ is the number of bits that are different in pattern μ and pattern ν (the Hamming distance). We find:

- $H^{(1,1)} = 0 \Rightarrow Q^{(1,1)} = 42$
- $H^{(1,2)} = 10 \Rightarrow Q^{(1,2)} = 22$
- $H^{(1,3)} = 2 \Rightarrow Q^{(1,3)} = 38$
- $H^{(1,4)} = 42 \Rightarrow Q^{(1,4)} = -42$
- $H^{(1,5)} = 21 \Rightarrow Q^{(1,5)} = 0$

- $H^{(2,1)} = H^{(1,2)} = 10 \Rightarrow Q^{(2,1)} = 22$
- $H^{(2,2)} = 0 \Rightarrow Q^{(2,2)} = 42$
- $H^{(2,3)} = 10 \Rightarrow Q^{(2,3)} = 22$
- $H^{(2,4)} = 42 - H^{(2,1)} = 32 \Rightarrow Q^{(2,4)} = -22$
- $H^{(2,5)} = 11 \Rightarrow Q^{(2,5)} = 20$

b) We have that

$$\begin{aligned}
b_i^{(\nu)} &= \sum_j w_{ij}\zeta_j^{(\nu)} = \sum_j \frac{1}{42}\left(\zeta_i^{(1)}\zeta_j^{(1)} + \zeta_i^{(2)}\zeta_j^{(2)}\right)\zeta_j^{(\nu)} \\
&= \frac{1}{42}\zeta_i^{(1)}\sum_j \zeta_j^{(1)}\zeta_j^{(\nu)} + \frac{1}{42}\zeta_i^{(2)}\sum_j \zeta_j^{(2)}\zeta_j^{(\nu)} \\
&= \frac{1}{42}\zeta_i^{(1)}Q^{(1,\nu)} + \frac{1}{42}\zeta_i^{(2)}Q^{(2,\nu)}.
\end{aligned} \tag{2}$$

From a), we have that:

$$\begin{aligned}
b_i^{(1)} &= \frac{Q^{(1,1)}}{42}\zeta_i^{(1)} + \frac{Q^{(2,1)}}{42}\zeta_i^{(2)} = \zeta_i^{(1)} + \frac{22}{42}\zeta_i^{(2)}, \\
b_i^{(2)} &= \frac{Q^{(1,2)}}{42}\zeta_i^{(1)} + \frac{Q^{(2,2)}}{42}\zeta_i^{(2)} = \frac{22}{42}\zeta_i^{(1)} + \zeta_i^{(2)}, \\
b_i^{(3)} &= \frac{Q^{(1,3)}}{42}\zeta_i^{(1)} + \frac{Q^{(2,3)}}{42}\zeta_i^{(2)} = \frac{38}{42}\zeta_i^{(1)} + \frac{22}{42}\zeta_i^{(2)}, \\
b_i^{(4)} &= \frac{Q^{(1,4)}}{42}\zeta_i^{(1)} + \frac{Q^{(2,4)}}{42}\zeta_i^{(2)} = -\zeta_i^{(1)} - \frac{22}{42}\zeta_i^{(2)}, \\
b_i^{(5)} &= \frac{Q^{(1,5)}}{42}\zeta_i^{(1)} + \frac{Q^{(2,5)}}{42}\zeta_i^{(2)} = \frac{20}{42}\zeta_i^{(2)}.
\end{aligned} \tag{3}$$

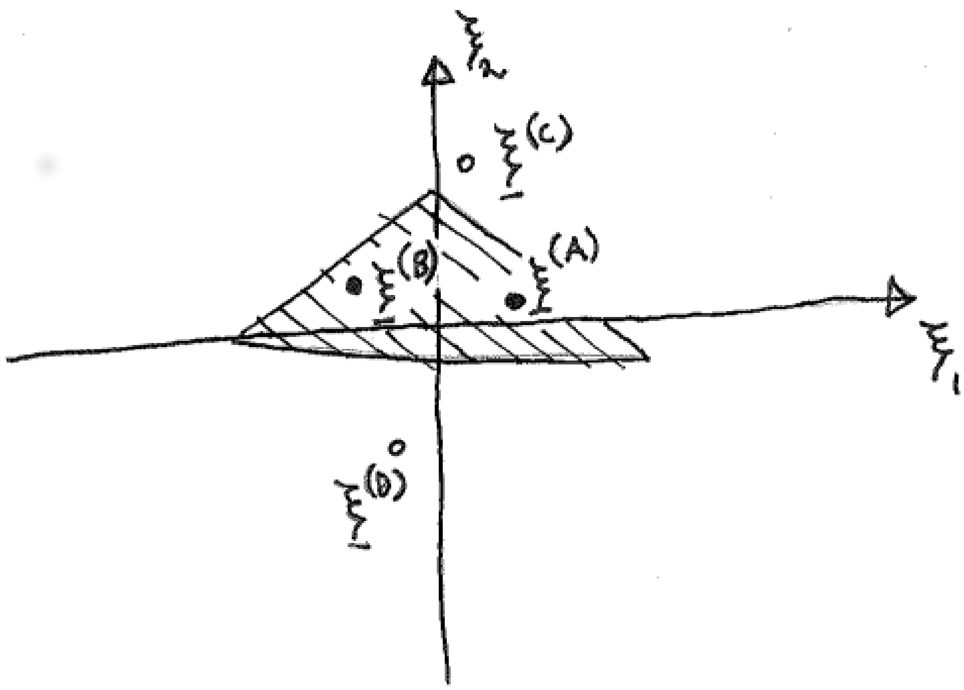
c) We have that From b), we find that:

$$\begin{aligned}
\zeta_i^{(1)} &\to \mathrm{sgn}(b_i^{(1)}) = \zeta_i^{(1)}, \\
\zeta_i^{(2)} &\to \mathrm{sgn}(b_i^{(2)}) = \zeta_i^{(2)}, \\
\zeta_i^{(3)} &\to \mathrm{sgn}(b_i^{(3)}) = \zeta_i^{(1)}, \\
\zeta_i^{(4)} &\to \mathrm{sgn}(b_i^{(4)}) = -\zeta_i^{(1)} = \zeta_i^{(4)}, \\
\zeta_i^{(5)} &\to \mathrm{sgn}(b_i^{(5)}) = \zeta_i^{(2)}.
\end{aligned} \tag{4}$$

Thus patterns $\zeta_i^{(1)}$, $\zeta_i^{(2)}$ and $\zeta_i^{(4)}$ are stable.

2. Linearly inseparable problem.

a) In the figure below $\boldsymbol{\xi}^{(A)}$ and $\boldsymbol{\xi}^{(B)}$ are to have output 1 and $\boldsymbol{\xi}^{(C)}$ and $\boldsymbol{\xi}^{(D)}$ are to have output 0. There is no straight line that can separate patterns $\boldsymbol{\xi}^{(A)}$ and $\boldsymbol{\xi}^{(B)}$ from patterns $\boldsymbol{\xi}^{(C)}$ and $\boldsymbol{\xi}^{(D)}$.



b) The triangle corners are:

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} -4 \\ 0 \end{bmatrix} \quad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \boldsymbol{\xi}^{(3)} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}. \tag{5}$$

- Let $v_1 = 0$ at $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$. This implies

$$\begin{cases} 0 = & w_{11}\xi_1^{(1)} + w_{12}\xi_2^{(1)} - \theta_1 = -4w_{11} - \theta_1 \\ 0 = & w_{11}\xi_1^{(2)} + w_{12}\xi_2^{(2)} - \theta_1 = 4w_{11} - w_{12} - \theta_1 \end{cases}$$
$$\Rightarrow \theta_1 = -4w_{11} \quad \text{and} \quad w_{12} = 4w_{11} - \theta_1 = 8w_{11}. \tag{6}$$

We choose $w_{11} = 1, w_{12} = 8 \quad \text{and} \quad \theta_1 = -4.$

- Let $v_2 = 0$ at $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$. This implies

$$\begin{cases} 0 = & w_{21}\xi_1^{(2)} + w_{22}\xi_2^{(2)} - \theta_2 = 4w_{21} - w_{22} - \theta_2 \\ 0 = & w_{21}\xi_1^{(3)} + w_{22}\xi_2^{(3)} - \theta_2 = 3w_{22} - \theta_2 \end{cases}$$
$$\Rightarrow w_{22} = 4w_{21} - \theta_2 = 4w_{21} - 3w_{22} \Rightarrow w_{22} = w_{21}$$
$$\text{and} \quad \theta_2 = 3w_{22}. \tag{7}$$

We choose $w_{21} = w_{22} = 1 \quad \text{and} \quad \theta_2 = 3$.

- Let $v_3 = 0$ at $\boldsymbol{\xi}^{(3)}$ and $\boldsymbol{\xi}^{(1)}$. This implies

$$\begin{cases} 0 = & w_{31}\xi_1^{(3)} + w_{32}\xi_2^{(3)} - \theta_3 = 3w_{32} - \theta_3 \\ 0 = & w_{31}\xi_1^{(1)} + w_{32}\xi_2^{(1)} - \theta_3 = -4w_{31} - \theta_3 \end{cases}$$
$$\Rightarrow 3w_{32} = -4w_{31} \quad \text{and} \quad \theta_3 = 3w_{32}. \tag{8}$$

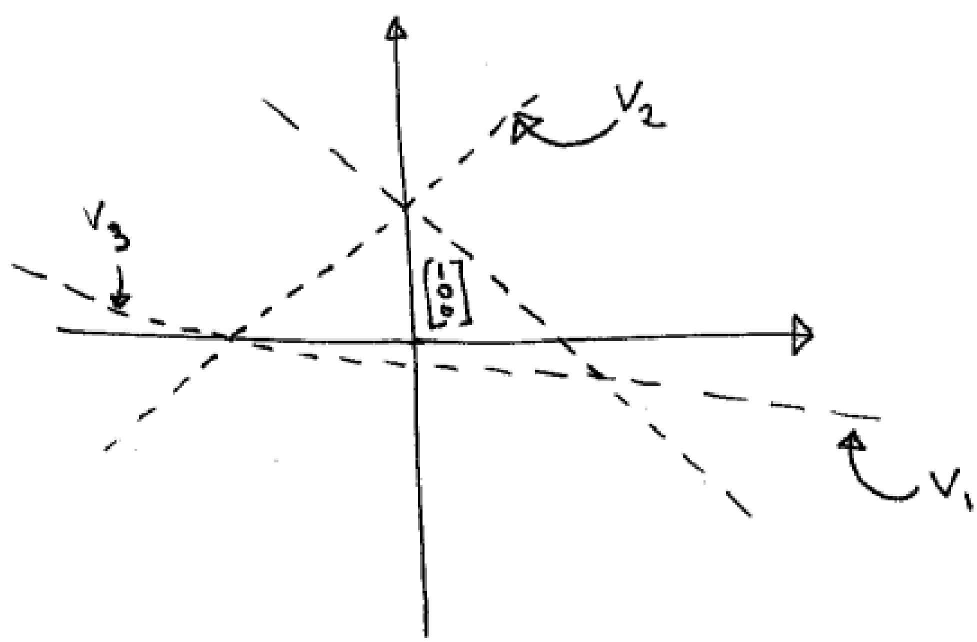
We choose $w_{32} = 4, w_{31} = -3 \quad \text{and} \quad \theta_3 = 12$. In summary:

$$\boldsymbol{w} = \begin{bmatrix} 1 & 8 \\ 1 & 1 \\ -3 & 4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\theta} = \begin{bmatrix} -4 \\ 3 \\ 12 \end{bmatrix}. \tag{9}$$

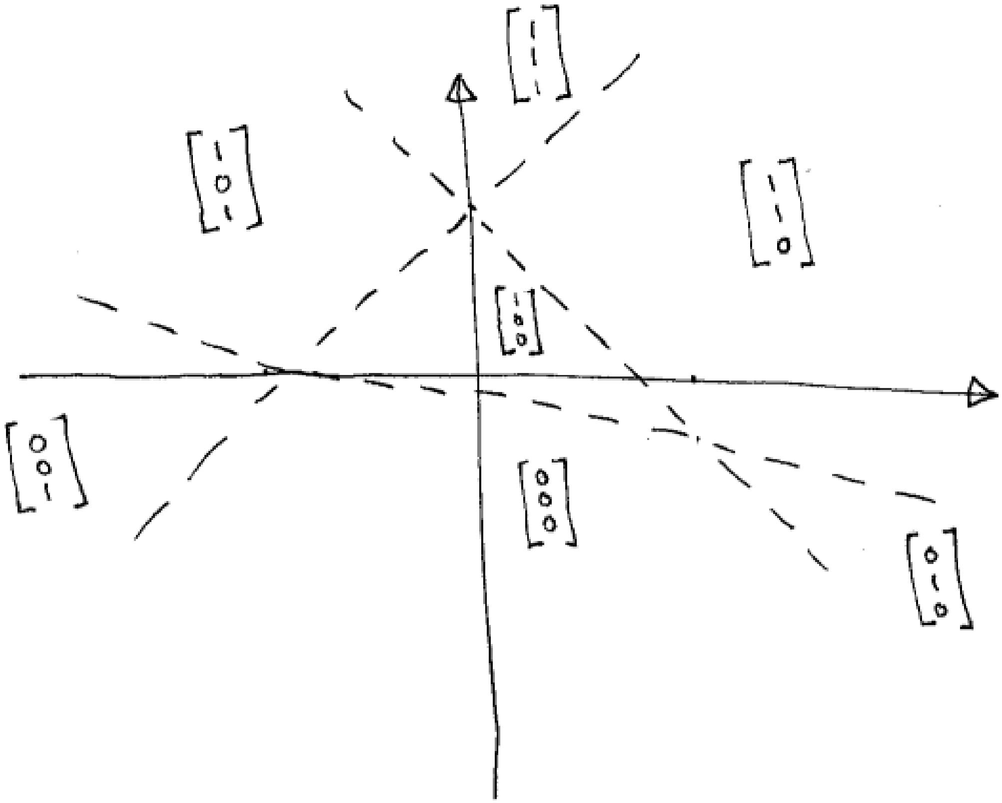
The origin maps to

$$\boldsymbol{v} = H\left(\boldsymbol{w}\mathbf{0} - \boldsymbol{\theta}\right) = H\left(\begin{bmatrix} 4 \\ -3 \\ -12 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{10}$$

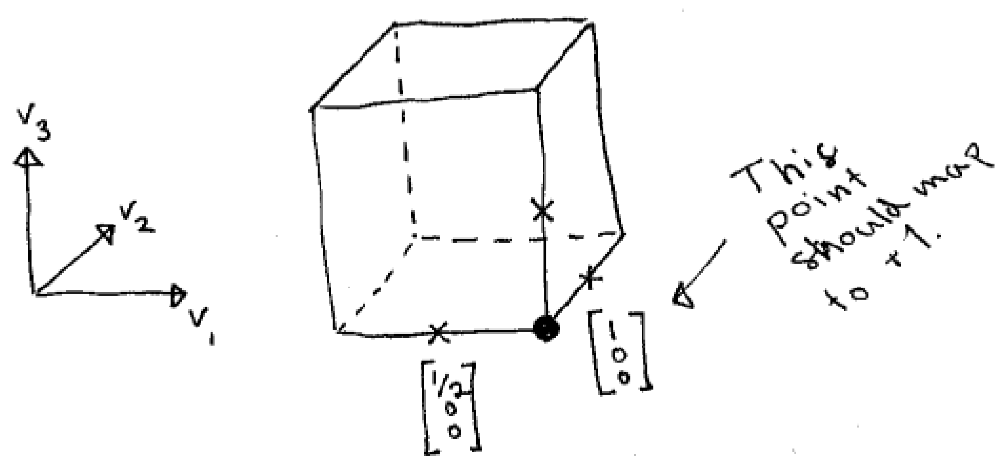
We know that the origin maps to $\boldsymbol{v} = [1, 0, 0]^T$ and that the hidden neurons change values at the dashed lines:



Thus we can conclude that the regions in input space maps to these regions in the hidden space:



We want $\boldsymbol{v} = [1, 0, 0]^T$ to map to 1 and all other possible values of $\boldsymbol{v}$ to map to 0. The hidden space can be illustrated as this:



$\boldsymbol{W}$ must be normal to the plane passing through the crosses in the picture above. Also, $\boldsymbol{W}$ points to $\boldsymbol{v} = [1,0,0]^T$ from $\boldsymbol{v} = [0,1,1]^T$. We may choose

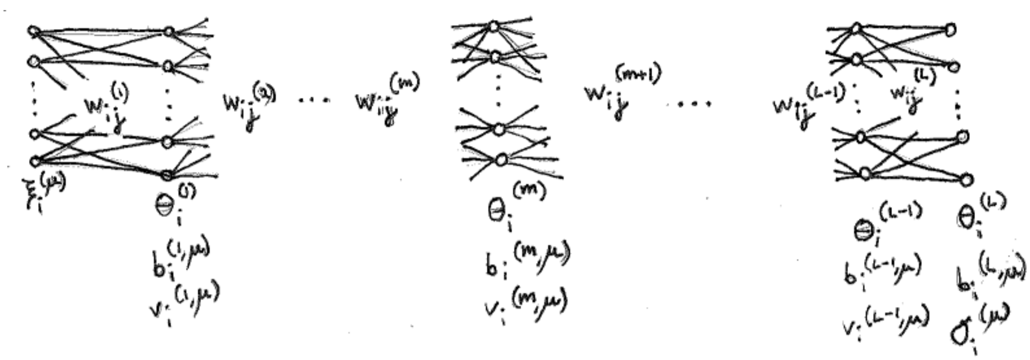
$$\boldsymbol{W} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\-1\end{bmatrix}. \tag{11}$$

We know that the point $\boldsymbol{v} = [1/2,0,0]^T$ lies on the decision boundary we are looking for. So

$$\boldsymbol{W}^T \begin{bmatrix}1/2\\0\\0\end{bmatrix} - T = 0 \Rightarrow T = \frac{1}{2}. \tag{12}$$

3. Backpropagation.

a)



Let N_m denote the number of weights $w_{ij}^{(m)}$. Let n_m denote the number of hidden units $v_i^{(m,\mu)}$ for $i = 1,\ldots,L-1$, let n_0 denote the number of input units and let n_L denote the number of output units. Find that the number of weights are

$$\sum_{m=1}^{L} N_m = \sum_{m=1}^{L} n_{m-1} n_m, \tag{13}$$

and that the number of thresholds are

$$\sum_{m=1}^{L} n_m. \tag{14}$$

b)

$$\begin{aligned}
\frac{\partial}{\partial w_{qr}^{(p)}} v_i^{(m,\mu)} =& \frac{\partial}{\partial w_{qr}^{(p)}} g\left(b_i^{(m,\mu)}\right) = g'\left(b_i^{(m,\mu)}\right) \frac{\partial}{\partial w_{qr}^{(p)}} b_i^{(m,\mu)} \\
=& g'\left(b_i^{(m,\mu)}\right) \frac{\partial}{\partial w_{qr}^{(p)}} \left(-\theta_i^{(m)} + \sum_j w_{ij}^{(m)} v_j^{(m-1,\mu)}\right) \\
=& g'\left(b_i^{(m,\mu)}\right) \left(\sum_j \frac{\partial}{\partial w_{qr}^{(p)}} w_{ij}^{(m)} v_j^{(m-1,\mu)}\right).
\end{aligned} \tag{15}$$

Using that $p < m$, we find:

$$\frac{\partial v_i^{(m,\mu)}}{\partial w_{qr}^{(p)}} = g'\left(b_i^{(m,\mu)}\right) \sum_j w_{ij}^{(m)} \frac{\partial v_j^{(m-1,\mu)}}{\partial w_{qr}^{(p)}}. \tag{16}$$

c) From b), we have:

$$\frac{\partial}{\partial w_{qr}^{(p)}} v_i^{(m,\mu)} = g'\left(b_i^{(m,\mu)}\right) \sum_j \frac{\partial}{\partial w_{qr}^{(p)}} w_{ij}^{(m)} v_j^{(m-1,\mu)}. \tag{17}$$

But since $p = m$, we find:

$$\frac{\partial}{\partial w_{qr}^{(p)}} v_i^{(m,\mu)} = g'\left(b_i^{(m,\mu)}\right) \sum_j \delta_{qi} \delta_{rj} v_j^{(m-1,\mu)} = g'\left(b_i^{(m,\mu)}\right) \delta_{qi} v_r^{(m-1,\mu)}. \tag{18}$$

d) We have $w_{qr}^{(L-2)} \leftarrow w_{qr}^{(L-2)} + \delta w_{qr}^{(L-2)}$, where

$$\delta w_{qr}^{(L-2)} = -\eta \frac{\partial H}{\partial w_{qr}^{(L-2)}}. \tag{19}$$

We derive the energy function:

$$\begin{aligned}
\frac{\partial H}{\partial w_{qr}^{(L-2)}} =& \frac{\partial}{\partial w_{qr}^{(L-2)}} \frac{1}{2} \sum_\mu \sum_i \left(O_i^{(\mu)} - \zeta_i^{(\mu)}\right)^2 \\
=& \sum_\mu \sum_i \left(O_i^{(\mu)} - \zeta_i^{(\mu)}\right) \frac{\partial O_i^{(\mu)}}{\partial w_{qr}^{(L-2)}}
\end{aligned} \tag{20}$$

From 3b) and 3c), we have:

$$\frac{\partial v_i^{(m,\mu)}}{\partial w_{qr}^{(p)}} = \begin{cases} g'\left(b_i^{(m,\mu)}\right) \sum_j w_{ij}^{(m)} \frac{\partial v_j^{(m-1,\mu)}}{\partial w_{qr}^{(p)}} & \text{if} \quad p < m \\ g'\left(b_i^{(m,\mu)}\right) \delta_{qi} v_r^{(m-1,\mu)} & \text{if} \quad p = m. \end{cases} \tag{21}$$

Define $v_i^{(L,\mu)} = O_i^{(\mu)}$. We have:

$$
\begin{aligned}
\frac{\partial O_i^{(\mu)}}{\partial w_{qr}^{(L-2)}} =& \frac{\partial v_i^{(L,\mu)}}{\partial w_{qr}^{(L-2)}} \\
&\{\text{Insert from eq. (21). Use that } L-2 < L.\ \} \\
=& g'\left(b_i^{(L,\mu)}\right) \sum_j w_{ij}^{(L)} \frac{\partial v_j^{(L-1,\mu)}}{\partial w_{qr}^{(L-2)}} \\
&\{\text{Insert from eq. (21). Use that } L-2 < L-1.\ \} \\
=& g'\left(b_i^{(L,\mu)}\right) \sum_j w_{ij}^{(L)} g'\left(b_j^{(L-1,\mu)}\right) \sum_k w_{jk}^{(L-1)} \frac{\partial v_k^{(L-2,\mu)}}{\partial w_{qr}^{(L-2)}} \\
&\{\text{Insert from eq. (21). Use that } L-2 = L-2.\ \} \\
=& g'\left(b_i^{(L,\mu)}\right) \sum_j w_{ij}^{(L)} g'\left(b_j^{(L-1,\mu)}\right) \sum_k w_{jk}^{(L-1)} g'\left(b_k^{(L-2,\mu)}\right) \delta_{qk} v_r^{(L-3,\mu)} \\
=& g'\left(b_i^{(L,\mu)}\right) \sum_j w_{ij}^{(L)} g'\left(b_j^{(L-1,\mu)}\right) w_{jq}^{(L-1)} g'\left(b_q^{(L-2,\mu)}\right) v_r^{(L-3,\mu)}. \qquad (22)
\end{aligned}
$$

The update rule is eq. (19) with the derivative of the energy function given by eqs. (20) and (22).

4. True/False questions. Indicate whether the following statements are true or false. 13-14 correct answers give 2 points, 11-12 correct answers give 1.5 points, 9-10 correct answers gives 1 point and, 8 correct answers give 0.5 points and 0-7 correct answers give zero points. (**2** p)

1. You need access to the state of all neurons in a multilayer perceptron when updating all weights through backpropagation. **TRUE (the update of a weight in layer depends on the value of the neuron in the layer before).**

2. Consider the Hopfield network. If a pattern is stable it must be an eigenvector of the weight matrix.**FALSE (due to the step-function).**

3. If you store two orthogonal patterns in a Hopfield network, they will always turn out unstable. **FALSE (the crosstalk term is zero).**

4. Kohonens algorithm learns convex distributions better than concave ones. **TRUE (concave corners can cause problems).**

5. The number of N-dimensional Boolean functions is 2^N. **FALSE (it is $2^{(2^N)}$).**

6. The weight matrices in a perceptron are symmetric. **FALSE (they may not even be square matrices).**

7. Using $g(b) = b$ as activation function and putting all thresholds to zero in a multilayer perceptron, allows you to solve some linearly inseparable problems. **FALSE (you have effectively one weight matrix that is the product of all your original ones).**

8. You need at least four radial basis functions for the XOR-problem to be linearly separable in the space of the radial basis functions. **FALSE (two are enough).**

9. Consider $p > 2$ patterns uniformly distributed on a circle. None of the eigenvalues of the covariance matrix of the patterns is zero. **TRUE (zero eigenvalue indicates patterns on a line).**

10. Even if the weight vector in Oja's rule equals its stable steady state at one iteration, it may change in the following iterations. **TRUE (it is only a statistically steady state).**

11. If your Kohonen network is supposed to learn the distribution $P(\boldsymbol{\xi})$, it is important to generate the patterns $\boldsymbol{\xi}^{(\mu)}$ before you start training the network. **FALSE (training your network does not affect which pattern you draw from your distribution).**

12. All one-dimensional Boolean problems are linearly separable. **TRUE (two different points can always be separated by a line).**

13. In Kohonen's algorithm, the neurons have fixed positions in the output space. **TRUE (it is the weights, in the input space , that are updated).**

14. Some elements of the covariance matrix are variances. **TRUE (the diagonal elements).**

5. Oja's rule.

a)

$$\begin{aligned}\mathbf{0} = \langle \boldsymbol{\delta w} \rangle &= \langle \eta\zeta\left(\boldsymbol{\xi} - \zeta\boldsymbol{w}\right)\rangle \\ &\Rightarrow \langle \boldsymbol{\xi}\zeta\rangle = \langle \zeta\zeta\boldsymbol{w}\rangle .\end{aligned} \tag{23}$$

Insert

$$\zeta = \boldsymbol{\xi}^T\boldsymbol{w} = \boldsymbol{w}^T\boldsymbol{\xi} : \tag{24}$$

$$\begin{aligned}\mathbf{0} = \langle \boldsymbol{\delta w}\rangle &\Rightarrow \left\langle \boldsymbol{\xi}\boldsymbol{\xi}^T\boldsymbol{w}\right\rangle = \left\langle \boldsymbol{w}^T\boldsymbol{\xi}\boldsymbol{\xi}^T\boldsymbol{w}\boldsymbol{w}\right\rangle \\ &\Rightarrow \left\langle \boldsymbol{\xi}\boldsymbol{\xi}^T\right\rangle \boldsymbol{w} = \boldsymbol{w}^T\left\langle \boldsymbol{\xi}\boldsymbol{\xi}^T\right\rangle \boldsymbol{w}\boldsymbol{w}.\end{aligned} \tag{25}$$

$\left\langle \boldsymbol{\xi}\boldsymbol{\xi}^T\right\rangle = \boldsymbol{C}$ is a matrix, so:

$$\mathbf{0} = \langle \boldsymbol{\delta w}\rangle \Rightarrow \boldsymbol{C}\boldsymbol{w} = \boldsymbol{w}^T\boldsymbol{C}\boldsymbol{w}\boldsymbol{w}. \tag{26}$$

We see that $\langle \boldsymbol{\delta w}\rangle = \mathbf{0}$ implies that $\boldsymbol{w}$ is an eigenvector of $\boldsymbol{C}$ with eigenvalue $\lambda = \boldsymbol{w}^T\boldsymbol{C}\boldsymbol{w}$:

$$\lambda = \boldsymbol{w}^T\boldsymbol{C}\boldsymbol{w} = \boldsymbol{w}^T\lambda\boldsymbol{w} = \lambda\boldsymbol{w}^T\boldsymbol{w} \Rightarrow \boldsymbol{w}^T\boldsymbol{w} = 1. \tag{27}$$

(note that $\boldsymbol{w}^T\boldsymbol{w} = \sum_i w_i w_i$).

b) Are the patterns centered?

$$\sum_{\mu=1}^{5} \xi_1^{(\mu)} = -6 - 2 + 1 + 1 + 5 = 0 \tag{28}$$

$$\sum_{\mu=1}^{5} \xi_2^{(\mu)} = -5 - 4 + 2 + 3 + 4 = 0. \tag{29}$$

So $\langle \boldsymbol{\xi}\rangle = \mathbf{0}$, and the patterns are centered. This means that the covariance matrix is:

$$\boldsymbol{C} = \frac{1}{5}\sum_{\mu=1}^{5} \boldsymbol{\xi}^{(\mu)}\boldsymbol{\xi}^{(\mu)\,T} = \left\langle \boldsymbol{\xi}\boldsymbol{\xi}^T\right\rangle . \tag{30}$$

We have

$$\begin{aligned}\boldsymbol{\xi}^{(1)}\boldsymbol{\xi}^{(1)\,T} &= \begin{bmatrix}36 & 30\\30 & 25\end{bmatrix}\\ \boldsymbol{\xi}^{(2)}\boldsymbol{\xi}^{(2)\,T} &= \begin{bmatrix}4 & 8\\8 & 16\end{bmatrix}\\ \boldsymbol{\xi}^{(3)}\boldsymbol{\xi}^{(3)\,T} &= \begin{bmatrix}4 & 4\\4 & 4\end{bmatrix}\\ \boldsymbol{\xi}^{(4)}\boldsymbol{\xi}^{(4)\,T} &= \begin{bmatrix}1 & 3\\3 & 9\end{bmatrix}\\ \boldsymbol{\xi}^{(5)}\boldsymbol{\xi}^{(5)\,T} &= \begin{bmatrix}25 & 20\\20 & 16\end{bmatrix}.\end{aligned} \tag{31}$$

We compute the elements of $\boldsymbol{C}$:

$$\begin{aligned}
5C_{11} =& 36+4+4+1+25=70 \\
5C_{12} =& 5C_{21}=30+8+4+3+20=65 \\
5C_{22} =& 25+16+4+9+16=70.
\end{aligned} \tag{32}$$

We find that

$$\boldsymbol{C}=\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}. \tag{33}$$

Maximal eigenvalue:

$$\begin{aligned}
0=\begin{vmatrix} 14-\lambda & 13 \\ 13 & 14-\lambda \end{vmatrix} =& (14-\lambda)^2-13^2=\lambda^2-28\lambda+14^2-13^2=\lambda^2-28\lambda+27 \\
\Rightarrow& \lambda=14\pm\sqrt{14^2-27}=14\pm\sqrt{169}=14\pm 13 \\
\Rightarrow& \lambda_{\text{max}}=27.
\end{aligned} \tag{34}$$

Eigenvector $\boldsymbol{u}$:

$$\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 27 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow u_1=u_2. \tag{35}$$

So an eigenvector corresponging to the largest eigenvalue of $\boldsymbol{C}$ is given by

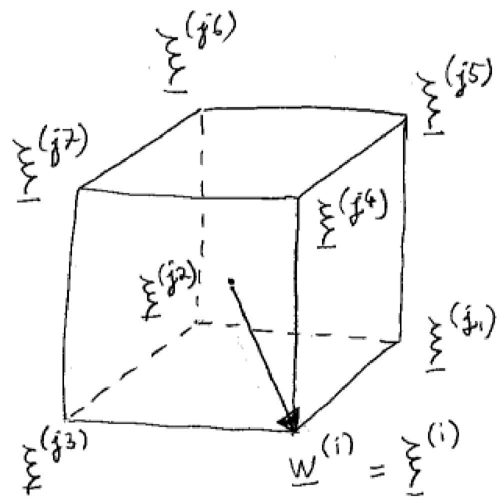
$$\boldsymbol{u}=t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{36}$$

for an arbitrary t. This is the principal component.

6. General Boolean problems. There was a typo in Eqn. (18) of the exam. The correct equation is:

$$v_i^{(\mu)} = \begin{cases} 1 & \text{if} \quad -\theta_i + \sum_j w_{ij}\xi_j^{(\mu)} > 0 \\ 0 & \text{if} \quad -\theta_i + \sum_j w_{ij}\xi_j^{(\mu)} \leq 0 \end{cases} .$$

a) The solution uses $w_{ij} = \xi_j^{(i)}$. This means that the i^{th} row of the weight matrix $\boldsymbol{w}$ is a vector $\boldsymbol{w}^{(i)} = \boldsymbol{\xi}^{(i)}$:



From the figure above, we see that:

- $\boldsymbol{w}^{(i)T}\boldsymbol{\xi}^{(i)} = 1 + 1 + 1 = 3$.
- $\boldsymbol{w}^{(i)T}\boldsymbol{\xi}^{(\mu)} = 1 + 1 - 1 = 1$ for $\mu = j_1, j_4$ and j_3.
- $\boldsymbol{w}^{(i)T}\boldsymbol{\xi}^{(\mu)} = 1 - 1 - 1 = -1$ for $\mu = j_2, j_5$ and j_7.
- $\boldsymbol{w}^{(i)T}\boldsymbol{\xi}^{(j_6)} = -1 - 1 - 1 = -3$.

Using that $\theta_i = 2$, we note that:

$$\boldsymbol{w}^{(i)T}\boldsymbol{\xi}^{(i)} - \theta_i \quad \text{is} \quad \begin{cases} > 0 & \text{if} \quad i = \mu \\ < 0 & \text{if} \quad i \neq \mu \end{cases} . \tag{37}$$

So we have:

$$v^{(i,\mu)} = \begin{cases} 1 & \text{if} \quad i = \mu \\ 0 & \text{if} \quad i \neq \mu \end{cases} . \tag{38}$$

We can understant that the corner μ of the cube of possible inputs is separated from the other corners by that it assigns 1 to the μ^{th} hidden neuron and 0 to the others.

From Figure 4 in the exam, we see that there are exactly 4 of the 8 possible inputs $\boldsymbol{\xi}^{(\mu)}$ that are to be mapped to $O^{\mu} = 1$. These are $\boldsymbol{\xi}^{(\mu)}$ for $\mu = 2, 4, 5$ and 7.
These inputs will assign, respectively:

$$\boldsymbol{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}^{(5)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{v}^{(7)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{39}$$

The weights $\boldsymbol{W}$ are now to 'detect' these and only these patterns, so that

$$O^{(\mu)} = \boldsymbol{W}^T \boldsymbol{w} = \begin{cases} 1 & \text{for} \quad \mu \in \{2, 4, 5, 7\} \\ 0 & \text{for} \quad \mu \in \{1, 3, 6, 8\} \end{cases}. \tag{40}$$

This is achieved by letting:

$$\boldsymbol{W} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{41}$$

b) The solution in 6a) implies separating each corner $\boldsymbol{\xi}^{(\mu)}$ of the cube of input patterns by letting

$$v^{(i,\mu)} = \begin{cases} 1 & \text{if} \quad i = \mu \\ 0 & \text{if} \quad i \neq \mu \end{cases} \quad . \tag{42}$$

Thus the solution requires $2^3 = 8$ hidden neurons. The analogous solution in 2D is to separate each corner of a square, and it requires $2^N = 2^2 = 4$ neurons. The decision boundaries of the hidden neurons are shown here: