GAUGES THEORIES

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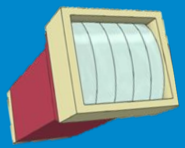

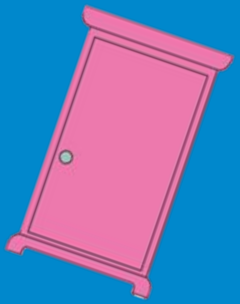
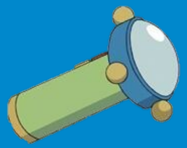

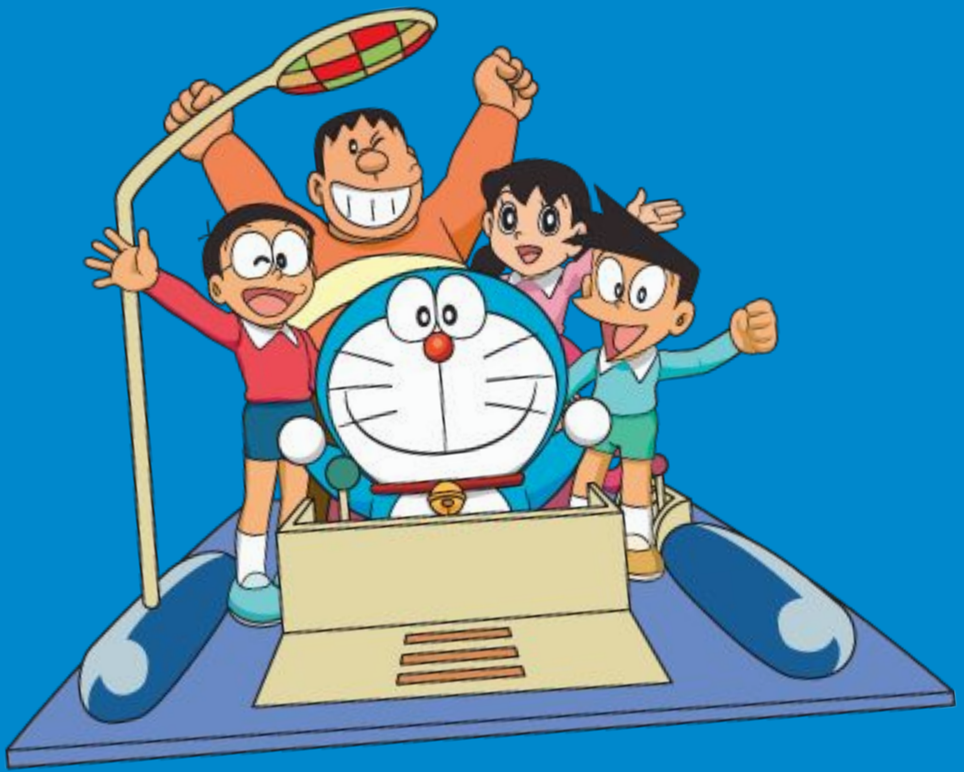

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Lagrangians in relativistic field theory

	classical mechanics	relativistic field
particle location	position	some region
variable	x(t),y(t),z(t)	Φ_i (x,y,z,t)
fuction	$L(q_i, \dot{q}_i)$	$£(\phi_i, \partial_\mu \phi_i)$

Euler-Lagrange

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right)=\frac{\partial\mathcal{L}}{\partial\phi_i}\qquad (i=1,2,3,\ldots)$$

Klein-Gordon Lagrangian: Scalar (Spin 0) Field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi \qquad \frac{\partial\mathcal{L}}{\partial\phi} = -\left(\frac{mc}{\hbar}\right)^2\phi$$

$$\Rightarrow \quad \partial_\mu\partial^\mu\phi + \left(\frac{mc}{\hbar}\right)^2\phi = 0$$

Dirac Lagrangian for a Spinor (Spin 1/2) Field

$$\text{h:} = i(\hbar c)\bar{\psi}\gamma^{\mu}\,\partial_{\mu}\psi - (mc^2)\bar{\psi}\psi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\bar{\psi})} = 0, \qquad \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = i\hbar c\gamma^{\mu}\,\partial_{\mu}\psi - mc^2\psi$$

$$\Rightarrow \quad i\gamma^{\mu}\,\partial_{\mu}\psi - \left(\frac{mc}{\hbar}\right)\psi = 0$$

Proca Lagrangian for a Vector (Spin 1) Field

$$\mathcal{L} = \frac{-1}{16\pi}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu A_\nu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{-1}{4\pi}(\partial^\mu A^\nu - \partial^\nu A^\mu) \qquad \frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{1}{4\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu$$

$$\Rightarrow \quad \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + \left(\frac{mc}{\hbar}\right)^2 \text{A”} = 0$$

Proca Lagrangian for a Vector (Spin 1) Field

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu A_\nu$$

$$\partial_\mu F^{\mu\nu} + \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0$$

Maxwell Lagrangian for a Massless Vector Field

$$\mathcal{L} = \frac{-1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J^{\mu} A_{\mu}$$

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

$$\partial_\nu(\partial_\mu F^{\mu\nu}) = \partial_\nu(4\pi J^\nu)$$

$$\partial_\nu \partial_\mu F^{\mu\nu} = 4\pi \partial_\nu J^\nu$$

$$\Rightarrow \quad \partial_\nu J^\nu = 0$$

Local Gauge Invariance

$$\mathcal{L} = i\hbar c\bar{\psi}\gamma^{\mu}\,\partial_{\mu}\psi - mc^2\bar{\psi}\psi$$

is invariant under global gauge tr.

$$\psi \rightarrow e^{i\theta}\psi$$

First Term:

$$\overline{\psi}' i\gamma^{\mu}\partial_{\mu}\psi' = \overline{\psi}e^{-i\theta}i\gamma^{\mu}\partial_{\mu}(e^{i\theta}\psi)$$

Using the product rule, we find:

$$\partial_{\mu}(e^{i\theta}\psi) = e^{i\theta}\partial_{\mu}\psi$$

Thus, substituting back we get:

$$= \overline{\psi}e^{-i\theta}i\gamma^{\mu}\left(e^{i\theta}\partial_{\mu}\psi\right) = \overline{\psi}i\gamma^{\mu}\partial_{\mu}\psi$$

Second Term:

$$-m\overline{\psi}'\psi' = -m(\overline{\psi}e^{-i\theta})(e^{i\theta}\psi) = -m\overline{\psi}\psi$$

Local Gauge Invariance

But what if the phase factor is different at different space–time points; that is, what *if* θ is a function of x^{μ}:

$$\psi \longrightarrow e^{i\theta(x)}\psi \qquad \text{(local gauge transformation)} \qquad (11.27)$$

$$\partial_\mu \psi \rightarrow \partial_\mu (e^{i\theta(x)}\psi) = e^{i\theta(x)}\partial_\mu\psi + \psi(i\partial_\mu\theta(x))e^{i\theta(x)}$$

$\Rightarrow$ not invariant

Local Gauge Invariance

$$\lambda(x) \equiv -\frac{\hbar c}{q}\theta(x)$$

local gauge tr.

$$\mathcal{L} \rightarrow \mathcal{L} + (q\bar{\psi}\gamma^{\mu}\psi)\partial_{\mu}\lambda$$

$$\mathcal{L} = [i\hbar c\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - mc^{2}\bar{\psi}\psi] - (q\bar{\psi}\gamma^{\mu}\psi)A_{\mu}$$

Local Gauge Invariance

$$\mathcal{L} = [i\hbar c\bar{\psi}\gamma^\mu \partial_\mu\psi - mc^2\bar{\psi}\psi] - (q\bar{\psi}\gamma^\mu\psi)A_\mu$$

gauge field

$$A_\mu \rightarrow A_\mu + \partial_\mu\lambda$$

invariant

Local Gauge Invariance

Proca Lagrangian

$$\mathcal{L} = \frac{-1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{m_A c}{\hbar}\right)^2 A^\nu A_\nu$$

$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ is invariant

$A^\nu A_\nu$ is not

→ Evidently, the gauge field must be massless (mA = 0), otherwise local gauge invariance will be lost

Local Gauge Invariance

if we start with the Dirac Lagrangian, and impose local gauge invariance -> complete Lagrangian

$$\mathcal{L} = [i\hbar c\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - mc^2\bar{\psi}\psi] + \left[\frac{-1}{16\pi}F^{\mu\nu}F_{\mu\nu}\right] - [(q\bar{\psi}\gamma^{\mu}\psi)A_{\mu}]$$

$$J^{\mu} = cq(\bar{\psi}\gamma^{\mu}\psi)$$

Local Gauge Invariance

difference $\quad \partial_\mu \psi \longrightarrow e^{-iq\lambda/\hbar c}\left[\partial_\mu - i\,\frac{q}{\hbar c}\,(\partial_\mu \lambda)\right]\psi$

covariant derivative $\quad \mathcal{D}_\mu \equiv \partial_\mu + i\,\frac{4}{\hbar c}\,\mathbf{A}$

$$\mathcal{D}_\mu \psi \longrightarrow e^{-iq\lambda/\hbar c}\mathcal{D}\ \psi$$

Yang - Mills theory

$$\mathcal{L} = [i\hbar c\bar{\psi}_1\gamma^\mu \partial_\mu\psi_1 - m_1c^2\bar{\psi}_1\psi_1] + [i\hbar c\bar{\psi}_2\gamma^\mu\partial_\mu\psi_2 - m_2c^2\bar{\psi}_2\psi_2]$$

$$\psi \equiv \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \qquad \bar{\psi} = (\bar{\psi}_1 \quad \bar{\psi}_2)$$

$$\Rightarrow \quad \mathcal{L} = i\hbar c\bar{\psi}\gamma^\mu \partial_\mu\psi - c^2\bar{\psi}M\psi$$

where

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2\end{pmatrix}$$

Yang - Mills theory

if 2 masses equal

$$\mathcal{L} = i\hbar c\bar{\psi}\gamma^{\mu}\,\partial_{\mu}\psi - mc^{2}\bar{\psi}\psi$$

- General global inv. (where U is 2×2 unitary matrix

$$\psi \to U\psi, \quad U^{\dagger}U = 1, \quad \bar{\psi} \to \bar{\psi}U^{\dagger}$$

We can write (where H is Hermitian)

$$U = e^{iH}, \quad H = \theta 1 + \tau \cdot a, \; U = e^{i\theta}\mathrm{e}^{\mathrm{i}\tau\cdot a}$$

$$\psi \to \mathrm{e}^{\mathrm{i}\tau\cdot a}\psi \quad \text{[global SU(2) trans.]}$$

invariant

Yang - Mills theory

$$\lambda(x) \equiv -(\hbar c/q)\mathbf{a}(x)$$

$\psi \longrightarrow S\psi, \qquad \text{where } S \equiv e^{-iq\boldsymbol{\tau}\cdot\boldsymbol{\lambda}(x)/\hbar c}$ [local $SU(2)$ transformation]

$\partial_\mu \psi$ [redacted] $S\,\partial_\mu \psi + (\partial_\mu S)\psi$

→ not invariant

Yang - Mills theory

$$\mathcal{D}_\mu \equiv \mathrm{d}_{,,} + i\,\frac{q}{\hbar c}\,\boldsymbol{\tau}\cdot\mathbf{A}_\mu$$

$$\mathcal{D}_\mu\psi - S(\mathcal{D}_\mu\psi)$$

$$\mathcal{L} = i\hbar c\bar{\psi}\gamma^\mu\,\partial_\mu\psi - mc^2\bar{\psi}\psi$$

is invariant

Yang - Mills theory

$$\boldsymbol{\tau}\cdot\mathbf{A}'_\mu = S(\boldsymbol{\tau}\cdot\mathbf{A}_\mu)S^{-1} + i\left(\frac{\hbar c}{q}\right)(\partial_\mu S)S^{-1}$$

S does not commute with $\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\lambda}$

$$\boldsymbol{\tau}\cdot\mathbf{A}'_\mu \cong \boldsymbol{\tau}\cdot\mathbf{A}_\mu + \frac{iq}{\hbar c}[\boldsymbol{\tau}\cdot\mathbf{A}_\mu, \boldsymbol{\tau}\cdot\boldsymbol{\lambda}] + \boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\lambda}$$

$$\mathbf{A}'_\mu \cong \mathbf{A}_\mu + \partial_\mu\boldsymbol{\lambda} + \frac{2q}{\hbar c}(\mathsf{A}\times\mathbf{A}_\mu)$$

$$\mathcal{L} = i\hbar c\bar{\psi}\gamma^\mu\mathcal{D}_\mu\psi - mc^2\bar{\psi}\psi = [i\hbar c\bar{\psi}\gamma^\mu\partial_\mu\psi - mc^2\bar{\psi}\psi] - (q\bar{\psi}\gamma^\mu\boldsymbol{\tau}\psi)\cdot\mathbf{A}_\mu$$

is invariant

Yang - Mills theory

$$\mathbf{A}^\mu = (A_1^\mu, A_2^\mu, A_3^\mu)$$

$$\mathcal{L}_A = -\frac{1}{16\pi}F_1^{\mu\nu}F_{\mu\nu 1} - \frac{1}{16\pi}F_2^{\mu\nu}F_{\mu\nu 2} - \frac{1}{16\pi}F_3^{\mu\nu}F_{\mu\nu 3} = -\frac{1}{16\pi}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu}$$

$$\Rightarrow \quad \mathcal{L} = [i\hbar c\bar{\psi}\gamma^\mu\partial_\mu\psi - mc^2\bar{\psi}\psi] - \frac{1}{16\pi}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} - (q\bar{\psi}\gamma^\mu\tau\psi)\cdot\mathbf{A}_\mu$$

Spontaneous Symmetry - Breaking

- The principle of local gauge invar. works beautifully for the strong and E.M. interactions.
- The application to weak interactions was stymied because gauge fields have to be massless.
- Can we make gauge theory to accommodate massive gauge fields?

Yes, by using spontaneous symmetry-breaking and the Higgs mechanism.

- Suppose

$$£ = \frac{1}{2}\left(\partial_{\mu}\phi\right)\left(\partial^{\mu}\phi\right) + e^{-(\alpha\phi)^2}$$

Spontaneous Symmetry - Breaking

- If we expand the exponential

$$\pounds = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + 1 - \alpha^2\phi^2 + \frac{1}{2}\alpha^4\phi^4 - \frac{1}{6}\alpha^6\phi^6 + \cdots$$

the second term looks like the mass term in the K.G. Lagrangian with $\alpha^2 = \frac{1}{2}m^2, \quad m = \sqrt{2}\alpha$

The higher-order terms represent couplings, of the form

This is not supposed to be a realistic theory

Spontaneous Symmetry - Breaking

- To identify how mass term in a Lagrangian may be disguised, we pick out the term propotional to Φ^2

$$\pounds = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda^2\phi^4$$

the second term looks like mass, and the third term like an interaction. If that is mass term, m is imaginary(nonsense)

- Feynman calculus about a perturbation start from the ground state(vacuum) and treat the fields as fluctuations about that state: Φ=0

But for above Lagrangian, Φ=0 is not the ground state.

To determine the true ground state, consider

$$\pounds = T - U$$

Spontaneous Symmetry - Breaking

- so, $U(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda^2\phi^4$

And the minimum occurs at $\phi = \pm\mu/\lambda$

- Introduce a new field variable $\eta \equiv \phi \pm \mu/\lambda$

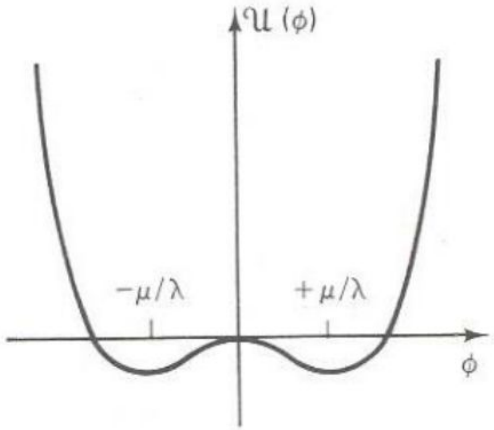
In terms of η

$$£ = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2\eta^2 \pm \mu\lambda\eta^3 - \frac{1}{4}\lambda^2\eta^4 + \frac{1}{4}(\mu^2/\lambda)^2$$

Now second term is a mass term, with the correct sign.

$$m = \sqrt{2}\mu$$

Spontaneous Symmetry - Breaking



[graph of U(Φ)]

- The third and fourth terms represend couplings of the form

Spontaneous Symmetry - Breaking

- From the mass term, the original Lagrangian is even in Φ

$$\pounds = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda^2\phi^4$$

- The reformulated Lagrangian is not even in η
- (the symmetry has been broken)

$$\pounds = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2\eta^2 \pm \mu\lambda\eta^3 - \frac{1}{4}\lambda^2\eta^4 + \frac{1}{4}(\lambda^2/\lambda)^2$$

- It happened because the vacuum does not share the symmetry of the Lagrangian

Spontaneous Symmetry - Breaking

- For example, the Lagrangian with spontaneously broken continuous symmetry

$$\pounds = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) + \frac{1}{2}\mu^2(\phi_1{}^2 + \phi_2{}^2) - \frac{1}{4}\lambda^2(\phi_1{}^2 + \phi_2{}^2)^2$$

(it is invar. under rotations in Φ_1 Φ_2 space)

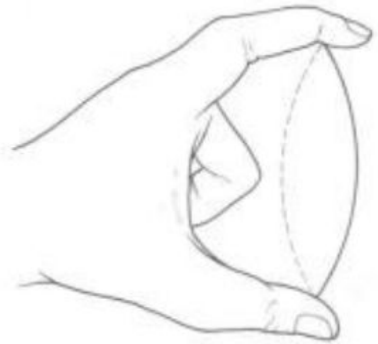
where, $U = -\frac{1}{2}\mu^2(\phi_1{}^2 + \phi_2{}^2) + \frac{1}{4}\lambda^2(\phi_1{}^2 + \phi_2{}^2)^2$

The minimum condition

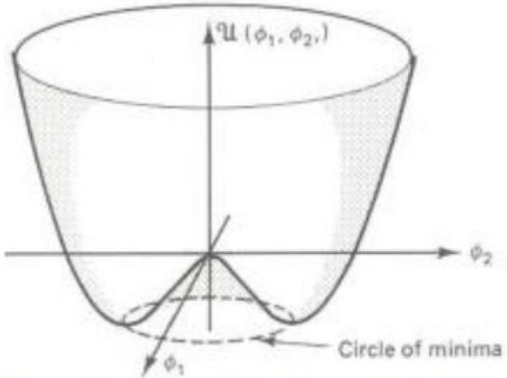
$$\phi_{1\min}{}^2 + \phi_{2\min}{}^2 = \mu^2 / \lambda^2$$

We may as well pick, $\phi_{1\min} = \mu / \lambda, \quad \phi_{2\min} = 0$

Spontaneous Symmetry - Breaking



- [spontaneous symmetry breaking in a plastic strip]



- [the potential function]

Spontaneous Symmetry - Breaking

- Introduce new fields

$$\eta \equiv \phi_1 + \mu / \lambda, \qquad \xi \equiv \phi_2$$

- Rewriting the Lagrangian in terms of new variables,

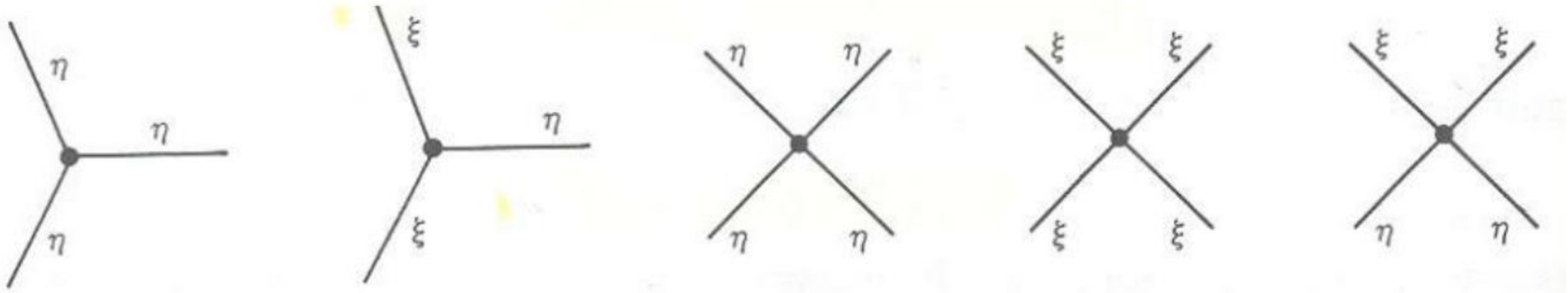
$$£ = \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2 \eta^2\right] + \left[\frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi)\right] + \left[\mu\lambda(\eta^3 + \eta\xi^2) - \frac{\lambda^2}{4}(\eta^4 + \xi^4 + 2\eta^2\xi^2)\right] + \frac{\mu^4}{4\lambda^2}$$

- The first term is a free K.G. Lagrangian for the field η
  the second term is a free Lagrangian for the field ξ

$$m_\eta = \sqrt{2}\mu, \qquad m_\xi = 0$$

Spontaneous Symmetry - Breaking

- The third term defines five couplings



- In this form, the Lagrangian doesn’t look symmetrical at all (the symmetry has been broken by the selection of a particular vacuum state)

- One of the fields(ξ) is automatically massless

The Higgs Mechanism

- If we combine the two real fields into a single complex field

$$\phi \equiv \phi_1 + i\phi_2 \qquad \phi^*\phi \equiv \phi_1^{\,2} + \phi_2^{\,2}$$

$$\pounds = \frac{1}{2}\left(\partial_\mu \phi\right)^*\left(\partial^\mu \phi\right) + \frac{1}{2}\mu^2\left(\phi^*\phi\right) - \frac{1}{4}\lambda^2\left(\phi^*\phi\right)^2$$

- The rotational(SO(2)) symmetry that was spontaneously broken becomes invar. under U(1) phase trans.

$$\phi \to e^{i\theta}\phi$$

- We can make the system invar. under local gauge trans.

$$\phi \to e^{i\theta(x)}\phi$$

The Higgs Mechanism

- Replace equations with covariant derivatives

$$Đ_\mu \equiv \partial_\mu + iqA_\mu$$

- Thus

$$£ = \frac{1}{2}\left[\left(\partial_\mu - iqA_\mu\right)\phi^*\right]\left[\left(\partial_\mu + iqA_\mu\right)\phi\right] + \frac{1}{2}\mu^2\left(\phi^*\phi\right) - \frac{1}{4}\lambda^2\left(\phi^*\phi\right)^2 - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}$$

Define the new fields

$$\eta \equiv \phi_1 - \mu/\lambda, \qquad \xi \equiv \phi_2$$

Lagrangian becomes

$$£ = \left[\frac{1}{2}\left(\partial_\mu\eta\right)\left(\partial^\mu\eta\right) - \mu^2\eta^2\right] + \left[\frac{1}{2}\left(\partial_\mu\xi\right)\left(\partial^\mu\xi\right)\right]$$
$$+\left[-\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\frac{q\mu}{\lambda}\right)^2 A_\mu A^\mu\right] - 2i\left(\frac{q\mu}{\lambda}\right)\left(\partial_\mu\xi\right)A^\mu$$
$$+\left\{q\left[\eta\left(\partial_\mu\xi\right) - \xi\left(\partial_\mu\eta\right)\right]A^\mu + \frac{\mu}{\lambda}(q)^2\eta\left(A_\mu A^\mu\right) + \frac{1}{2}(q)^2\left(\xi^2 + \eta^2\right)\left(A_\mu A^\mu\right) - \lambda\mu\left(\eta^3 + \eta\xi^2\right) - \frac{1}{4}\lambda^2\left(\eta^4 + 2\eta^2\xi^2 + \xi^4\right)\right\} + \left(\frac{\mu^2}{2\lambda}\right)^2$$

The Higgs Mechanism

- The first line describes a scalar particle $m_\eta = \sqrt{2}\mu$ and a massless Goldstone boson (ξ)
- The second line describes the free gauge field A^μ, **it has acquired a mass** $m_A = 2\sqrt{\pi}\left(\frac{q\mu}{\lambda}\right)$
- The term in curly brackets specifies various coupling of ξ,η, A^μ
- We still have unwanted Goldstone boson (ξ)

$$-2i\left(\frac{\mu q}{\lambda}\right)(\partial_\mu \xi)A^\mu$$

as interaction, it leads to a vertex of the form

The Higgs Mechanism

- Writing equation in terms of its real and imaginary parts

$$\phi \to \phi' = (\cos\theta + i\sin\theta)(\phi_1 + i\phi_2)$$
$$= (\phi_1\cos\theta - \phi_2\sin\theta) + i(\phi_1\sin\theta - \phi_2\cos\theta)$$

- Pick

$$\theta = -\tan^{-1}(\phi_2/\phi_1)$$

will render Φ' real, $\Phi_2{}' = 0$

In this particular gauge, (ξis zero)

$$\pounds = \left[\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \mu^2\eta^2\right] +$$
$$+\left[-\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\frac{q\mu}{\lambda}\right)^2 A_\mu A^\mu\right]$$
$$+\{\frac{\mu}{\lambda}(q)^2\eta(A_\mu A^\mu) + \frac{1}{2}(q)^2(\eta^2)(A_\mu A^\mu) - \lambda\mu(\eta^3) - \frac{1}{4}\lambda^2\eta^4\} + \left(\frac{\mu^2}{2\lambda}\right)^2$$

The Higgs Mechanism

- We have eliminated the Goldstone boson and the offending term in £; we are left with a single massive scalar η(the Higgs particle) and massive gauge field A^{μ}

- A massless vector field carries two degree of freedom (tranverse polarizations). When A^{μ} acquires mass, it picks up a third degree of freedom(longitudinal polarization)

Q: where did this extra degree of freedom come from?

A: it came from the Goldstone boson, which meanwhile disappeared from the theroy.

The gauge field ate the Goldstone boson, thereby acquiring both a mass and a third polarization state (**Higgs mechanism**)

References:

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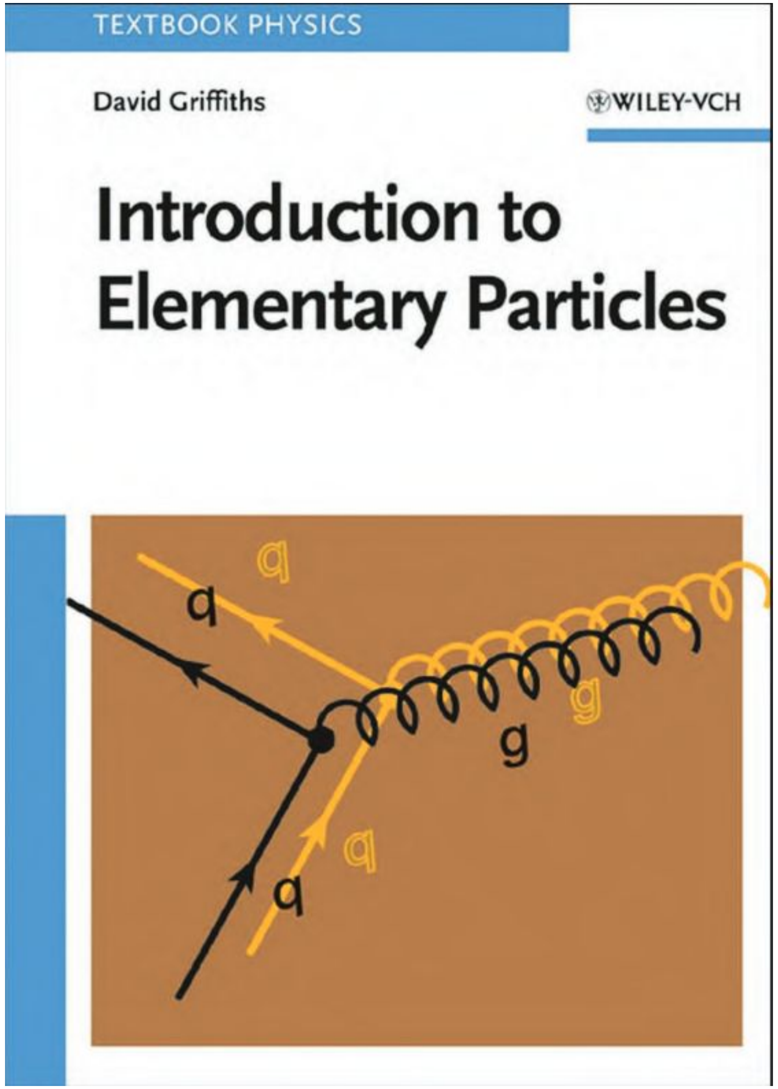
- Griffiths, David J. Introduction to Elementary Particles, Wiley



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