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SPECIAL RELATIVITY: HOMEWORK

I. Lecture questions

1. 4-vectors

$$u^\mu = \frac{dx^\mu}{d\tau}$$ τ: proper time such that $ds^2 = d\tau^2$

a) We have the ~~ove~~ general form of invarient interval:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2 d\tau^2$$

Given: $ds^2 = d\tau^2$ (which means $c=1$)

$$\rightarrow dt^2 = \frac{d\tau^2 - dx^2 - dy^2 - dz^2}{c^2} = \frac{d\tau}{c}\sqrt{1 - \frac{dx^2 + dy^2 + dz^2}{c^2 d\tau^2}}$$

$$= \frac{d\tau}{c}\sqrt{1-\frac{v^2}{c^2}} = d\tau\sqrt{1-\frac{v^2}{c^2}}$$

b) Given $u^\mu = \frac{dx^\mu}{d\tau}$, Lorentz factor: $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

For $\mu = 0$ (time component): $u^0 = \frac{dx^0}{d\tau} = \frac{c\,dt}{d\tau} = \frac{c}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma c$ (= γ when $c=1$)

For $\mu = i$ (spatial component): $u^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\cdot\frac{dt}{d\tau} = v^i\gamma$

c) Given $A^\mu = \frac{du^\mu}{d\tau}$

For $\mu = 0$: $A^0 = \frac{d}{d\tau}(\gamma c) = c\frac{d\gamma}{d\tau} = c\frac{d\gamma}{dt}\frac{dt}{d\tau} = \gamma^3\frac{\vec{v}\cdot\vec{a}}{c}$ $\xrightarrow{}$ acceleration

For $\mu = i$: $A^i = \frac{d}{d\tau}(\gamma v^i) = \gamma^3 a^i + v^i\frac{d}{d\tau}\gamma = \gamma^3\left(a^i + \frac{v^i}{c^2}(\vec{v}\cdot\vec{a})\right)$

2. Invariant & type of intervals

2) General: $\Delta s^2 = -c^2\Delta t^2 + \Delta x^2$

when $c=1$

$$\Delta s^2 = -\Delta t^2 + \Delta x^2$$

3. $\Delta s'^2 = -c^2\Delta t'^2 + \Delta x'^2$

$$= -c^2(\gamma^2(\Delta t - \frac{v}{c^2}\Delta x)^2) + \gamma^2(\Delta x - v\Delta t)^2$$

$$= \gamma^2(-c^2\Delta t^2 + 2v\Delta t\Delta x - \frac{v^2}{c^2}\Delta x^2 + \Delta x^2 - 2v\Delta t\Delta x + v^2\Delta t^2)$$

$$= \gamma^2(-c^2\Delta t^2(1 - \frac{v^2}{c^2}) + \Delta x^2(1 - \frac{v^2}{c^2}))$$

$$= -c^2\Delta t^2 + \Delta x^2 = \Delta s^2$$

$\rightarrow$ Δs^2 is invariant under Lorentz trans.

4, 2 events are simultaneous in an inertial frame:

$\Delta t' = 0 \rightarrow \gamma(\Delta t - \frac{v}{c^2}\Delta x) = 0$

$\rightarrow \Delta t = \frac{v}{c^2}\Delta x$

For this condition; $\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 > 0$ (space like interval)

5, condition: $\Delta t = \frac{v}{c^2}\Delta x \rightarrow v = \frac{c^2\Delta t}{\Delta x}$

3. Muons

a) we have: $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ and given $\gamma = \sqrt{2}$

$\rightarrow \sqrt{2} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\sqrt{2}} \rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{2}$

$\rightarrow \frac{v^2}{c^2} = \frac{1}{2} \rightarrow v = \frac{c}{\sqrt{2}} \approx 0.7c$

b) time dilation $\rightarrow$ Δt in lab frame is: $\Delta t = \gamma\tau_\mu = \sqrt{2} \times 2e{-6} \approx 2.83e{-6}$ (s)

distance $d = v.\Delta t = \frac{c}{\sqrt{2}} \times 2.83e{-6} \approx 600m$

c) Given: $\vec{u} = -\frac{1}{3}\vec{v}$

$\rightarrow$ v of muon in astronaut's frame: $v' = \frac{u + v}{1 + \frac{uv}{c^2}}$

$\rightarrow \gamma' = \frac{1}{\sqrt{1 - \frac{v'^2}{c^2}}}$ $\rightarrow$ energy: $E = \gamma' \underset{\substack{\downarrow \\ \text{rest mass}}}{m_0} c^2$

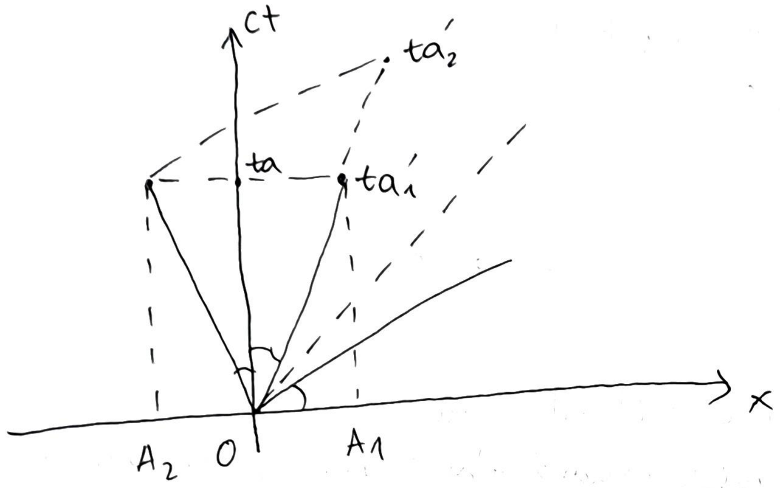
II. Runners

i) For C_1: $z=0$; $z_1 = L$

For C_2 : $z=0$; $z_2 = -L$

in R: $t_a = \frac{L}{v_a}$

ii)



iii) in C_1's rest frame, C_1 is stationary & C_2 is moving at v.

time dilation: $t'_{A_2} = \gamma t_a \rightarrow t'_{A_2} > t_a$

$\rightarrow$ C_1 thinks he won

iv) Similarly, in C_2's rest frame : $t'_{A_1} = \gamma t_a \rightarrow t'_{A_1} > t_a$

$\rightarrow$ C_2 thinks he reach A_2 before C_1 reaches A_1.

v) For C_1 moving towards A_1: $v_1 = v$

For C_2 moving towards A_2: $v_2 = -v$

General: $v_{B|A} = \frac{|v_B - v_A|}{1 - \frac{v_A v_B}{c^2}}$

$\rightarrow v_{C_2|C_1} = \frac{|v_2 - v_1|}{1 - \frac{v_2 v_1}{c^2}} = \frac{|-v-v|}{1 - \frac{-v^2}{c^2}} = \frac{2v}{1 + \frac{v^2}{c^2}}$

vi) if $v = 0.5$

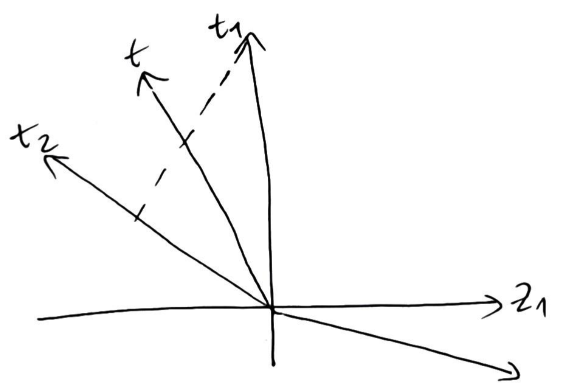
$\rightarrow$ For C_1: $v_1 = 0.5c$

For C_2 : $v_2 = -0.5c$

$\rightarrow v_{C_2|C_1} = \frac{2 \times 0.5c}{1 - \frac{(0.5c)(-0.5c)}{c^2}} = \frac{c}{1.25} = 0.8c$

it is smaller than c $\rightarrow$ we can conclude that no object with mass can reach or exceed the speed of light in any inertial frame

vii)

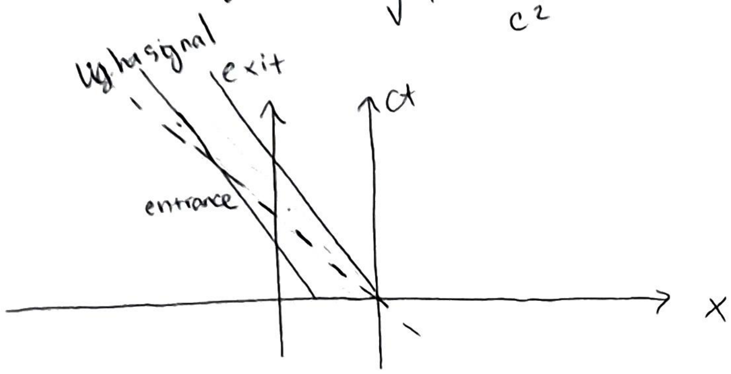


III. Train in a tunnel

1 In the train driver's frame, the tunnel length is contracted
length contraction

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}} \quad (l_0 : \text{proper length})$$

2.



3. From the pov of a person outside the train, the train is moving fast.
when the front cross the entrace of tunnel → light signal sent to close the barrier.

Relativity of simultaneity → the barrier closes after the train passed.

4. In the train's frame : $t_{close} = \frac{l_0}{c}$

5. The back of the train passes the entrance after: $t = \frac{l_0}{v}$.

6. Difference: $\Delta t = \frac{l_0}{c} - \frac{l_0}{v} = l_0 \left(\frac{1}{c} - \frac{1}{v} \right)$

4. Decay

1) $p^\mu = (E/c, \vec{p})$

$\swarrow$ energy $\quad\quad$ $\searrow$ 3 momentum

2) In the rest frame of the pion : $\vec{P}_\nu^{*} + \vec{P}_\mu^{*} = 0 \rightarrow \vec{P}_\nu^{*} = -\vec{P}_\mu^{*}$

3) Conservation of energy : $m_\pi^* c^2 = E_\mu^* + E_\nu^*$

4) General: $E^2 = (pc)^2 + (m_\mu c^2)^2$

$$E_\mu^{*2} = (\vec{P}_\mu^* c)^2 + (m_\mu^* c^2)^2$$

5) In the rest frame: $E_\nu^* = |\vec{p}_\nu^*| c$

$$E_\nu^* = E_\pi^* - E_\mu^* = \frac{m_\pi^{*2} - m_\mu^{*2}}{2 m_\pi^*} c^2$$

6) lab frame R, rest frame R^*

$$\gamma = \frac{E_\pi^*}{m_\pi^* c^2} \qquad \beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$

7) $E_\nu = \gamma (E_\nu^* + \beta |\vec{p}_\nu^*| c \cos\theta_\nu^*)$

neutrino has no mass

$\rightarrow E_\nu = \gamma E_\nu^* (1 + \beta \cos\theta_\nu^*)$

8) $E_\nu \cos\theta_\nu = \gamma E_\nu^* (\cos\theta_\nu^* + \beta)$

9) $$E_\nu = \frac{\gamma E_\nu^* (1 + \beta \cos\theta_\nu^*)}{1 - \beta^2 \cos^2\theta_\nu}$$