Week - 11

Gaussian Distribution

Bernoulli distribution.

$$P(X=x) = p^x (1-p)^{1-x}$$

Proof normalization:

$$\sum p(X=x) = p^1(1-p)^{1-1} + p^0(1-p)^{1-0}$$
$$= p + (1-p) = 1$$

Expected:

$$E[x] = \sum_x x\, p(X=x) = 1.\, p^1(1-p)^{1-1} + 0.\, p^0(1-p)^{1-0}$$
$$= p$$

$$Var[X] = E\left[(X - E[X])^2\right] = E[X^2] - E^2[X]$$
$$= \sum_x x^2 p(x) - p^2$$
$$= p - p^2 = p.(1-p) = p.q$$

Gaussian distribution:

$$p(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Proof normalization:

set $a = x - \mu \rightarrow da = dx$

$$\Rightarrow \int \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{a^2}{2\sigma^2}} da = 1$$

$$\Leftrightarrow \int \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \left(\frac{-a^2}{2\sigma^2}\right) \cdot e^{-\frac{a^2}{2\sigma^2}} = 1$$

$$\Leftrightarrow \int e^{\frac{-a^2}{2\sigma^2}} da = \sqrt{2\pi\sigma^2}$$

Set : $a = x \cdot \cos\theta$
$b = x \cdot \sin\theta$

$$da\,db = \begin{vmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial \theta} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial \theta} \end{vmatrix} dx\, d\theta$$

$$\Leftrightarrow \iint e^{\frac{-a^2}{2\sigma^2}} \cdot e^{\frac{-b^2}{2\sigma^2}} da\,db = 2\pi\sigma^2$$

$$\Leftrightarrow \int_0^{2\pi} \int_0^{\infty} e^{-x^2/2\sigma^2} \cdot x \cdot dx \cdot d\theta = 2\pi\sigma^2$$

$$\Leftrightarrow \int_0^{2\pi} -\sigma^2 \cdot e^{\frac{-x^2}{2\sigma^2}} \Big|_0^{+\infty} d\theta = 2\pi\sigma^2$$

$$\Leftrightarrow \int_0^{2\pi} (0 - -\sigma^2)\, d\theta = 2\pi\sigma^2$$

$$\Leftrightarrow \int_0^{2\pi} \sigma^2\, d\theta = 2\pi\sigma^2$$

$$\Leftrightarrow 2\pi\sigma^2 = 2\pi\sigma^2$$

$$X \sim N(\mu, 6) \Rightarrow Z = \frac{X-\mu}{6} \sim N(0,1)$$

⊕ Multivariate Gaussian Distribution.

D _ dimensional vector x

Σ _ a $D \times D$ convariance matrix

$|\Sigma|$ _ denotes the determinant of Σ

$$p(x|\mu, 6^2) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} e^{-1/2 (x-\mu)^T \Sigma^{-1}(x-\mu)}$$

$(\Sigma^{-1} = \Delta^{-1})$ → covarian matrix → đối xứng

Set: $\Delta^2 = \frac{-1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)$

$$= -\frac{1}{2}(x^T - \mu^T).\Delta^{-1}(x-\mu)$$

$$= \frac{-1}{2}x^T\Delta^{-1}x + \frac{1}{2}(\underline{\underline{x^T\Delta^{-1}\mu + \mu^T\Delta^{-1}x}}) - \frac{1}{2}\mu\Delta^{-1}\mu^T$$

→ số ⇒ $x^T = x$

$$= -\frac{1}{2}x^T\Delta^{-1}x + \frac{1}{2}(x^T\Delta^{-1}\mu + x^T\Delta^{-1}\mu) - \frac{1}{2}\mu\Delta^{-1}\mu^T$$

$$= x^T\Delta^{-1}\mu - \frac{1}{2}x^T\Delta^{-1}x + const$$

We have:

$$\Sigma = \sum_{i=1}^{D} \lambda_i u_i u_i^T \Rightarrow \Sigma^{-1} = \sum_{i=1}^{D} \frac{1}{\lambda_i} u_i u_i^T$$

So that:

$$\Delta^2 = (x-\mu)^T \Sigma^{-1}(x-\mu) = \sum_{i=1}^{D} \frac{1}{\lambda_i}(x-\mu)^T u_i u_i^T (x-\mu)$$

$$= \sum_{i=1}^{D} \frac{y_i^2}{\lambda_i} \text{, with } y_i = u_i^T(x-\mu$$

$$|\Sigma|^{1/2} = \prod_{j=1}^{D} \lambda_j^{1/2}$$

$$P(y) = \prod_{j=1}^{D} \frac{1}{2\pi\lambda_j}^{1/2} e^{-\frac{y_j^2}{2\lambda_j}}$$

$$\Rightarrow \int_{-\infty}^{+\infty} p(y)\, dy = \prod_{j=1}^{D} \int_{-\infty}^{\infty} \frac{1}{(2\pi\lambda_j)}^{1/2} e^{-\frac{y_j^2}{2\lambda_j}} dy_j$$

$$= 1.$$

trace (Σ) = trace (λ)

y_j = số thực

tích tích phân vì biến univariate

$\Rightarrow = 1$

⊛ Condition Gaussian Distribution

$$x = \begin{pmatrix} x_a \\ x_b \end{pmatrix} \qquad \mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$$

$x_a, \mu_a \rightarrow M$ chiều

$x_b, \mu_b \rightarrow D-M$ chiều

$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix} \Rightarrow A = \Sigma^{-1} = \begin{pmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{pmatrix}$$

Σ is symmetric so Σ_{aa} and Σ_{bb} are symmetric while $\Sigma_{ab} = \Sigma_{ba}^T$

Condition <u>Tìm 2 tham số</u> $\mu, \Sigma \rightarrow p(x_a|x_b)$ xđ

$p(x_a|x_b)$

We have:

$$-\frac{1}{2}(x-\mu)^T A(x-\mu) = -\frac{1}{2}\begin{bmatrix} x_a - \mu_a \\ x_b - \mu_b \end{bmatrix}^T A \begin{bmatrix} x_a - \mu_a \\ x_b - \mu_b \end{bmatrix}$$

$$= -\frac{1}{2}\Big[(x_a-\mu_a)^T A_{aa}(x_a-\mu_a) + (x_a-\mu_a)^T A_{ab}(x_b-\mu_b) + (x_b-\mu_b)^T A_{ba}(x_a-\mu_a) + (x_b-\mu_b)^T A_{bb}(x_b-\mu_b)\Big]$$

$$= -\frac{1}{2} x_a^T A_{aa}^{-1} x_a + x_a^T \left(A_{aa}\mu_a - A_{ab}(x_b-\mu_b)\right) + \text{const}$$

Đối xứng với CT chuẩn

$$\rightarrow \Sigma_{a|b} = A_{aa}^{-1}$$

$$\mu_{a|b} = \Sigma_{ab}\left(A_{aa}\mu_a - A_{ab}(x_b-\mu_b)\right)$$

Schur complement

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} M & -MBD^{-1} \\ -D^{-1}CMD^{-1} & D^{-1}CMBD^{-1} \end{pmatrix} \quad M = (A - BD^{-1}C)^{-1}$$

$$\to \Lambda_{aa} = (\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba})^{-1}$$

$$\Lambda_{ab} = -(\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba})^{-1}\Sigma_{ab}\Sigma_{bb}^{-1}$$

result:

$$\mu_{a|b} = \mu_a + \Sigma_{ab}\Sigma_{bb}^{-1}(x_b - \mu_b)$$

$$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}$$

$$\Rightarrow p(x_a|x_b) = N(x_{a|b} | \mu_{a|b}, \Sigma_{a|b})$$

Marginal Gaussian distribution

$$p(x_a) = \int p(x_a, x_b)\,dx_b$$

We have:

$$-\frac{1}{2}x_b^T \Lambda_{bb} x_b + x_b^T m = -\frac{1}{2}(x_b - \Lambda_{bb}^{-1}m)^T \Lambda_{bb}(x_b - \Lambda_{bb}^{-1}m) + \frac{1}{2}m^T \Lambda_{bb}^{-1} m$$

$$m = \Lambda_{bb}\mu_b - \Lambda_{ba}(x_a - \mu_a)$$

Because Gaussian Distribution normalization

$$\to \int \exp\left(-\frac{1}{2}(x_b - \Lambda_{bb}^{-1}m)^T \Lambda_{bb}(x_b - \Lambda_{bb}^{-1}m)\right\}dx_b$$

$$= \frac{1}{}$$