Week 04

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1 Ex1:

We have:

$$\langle x, y\rangle = x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2$$
$$\langle y, x\rangle = y_1x_1 - (y_1x_2 + y_2x_1) + 2y_2x_2$$
$$=> \langle x, y\rangle = \langle y, x\rangle \quad (*)$$
$$\langle x, x\rangle = x_1x_1 - (x_1x_2 + x_2x_1) + 2x_2x_2$$
$$<=> \langle x, x\rangle = x_1^2 - (2x_1x_2) + 2x_2^2$$
$$<=> \langle x, x\rangle = (x_1^2 - x_2^2) + x_2^2 > 0 \quad (**)$$

From (*) and (**) : $\langle x, y\rangle$ *is inner product*

2 Ex2:

We have distance between two vectors (x,y) is length of the difference vector x-y.

$$d_{\langle}x, y\rangle = \sqrt{\langle (x-y)(x-y)\rangle}$$
$$= \sqrt{\langle \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \rangle}$$

a,

$$d_{\langle}x, y\rangle = \sqrt{x^Ty} = \sqrt{\begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}} = \sqrt{22}$$

b,

$$d_{\langle}x, y\rangle = \sqrt{x^T A y} = \sqrt{\begin{bmatrix}2 & 3 & 3\end{bmatrix} \begin{bmatrix}2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2\end{bmatrix} \begin{bmatrix}2 \\ 3 \\ 3\end{bmatrix}}$$

$$= \sqrt{\begin{bmatrix}7 & 8 & 3\end{bmatrix} \begin{bmatrix}2 \\ 3 \\ 3\end{bmatrix}} = \sqrt{47}$$

3 Ex3:

a, $\langle x, y\rangle = x^T y$
we have:

$$cos(w) = \frac{x^T y}{\|x\|\|y\|}$$

$$= \frac{\begin{bmatrix}1 & 2\end{bmatrix} \begin{bmatrix}-1 \\ -1\end{bmatrix}}{\|x\|\|y\|} = \frac{-3}{\sqrt{5}\sqrt{2}} = \frac{-3}{\sqrt{10}}$$

$$=> w = cos^-1(\frac{-3}{\sqrt{10}})$$

$$\text{b, } \langle x, y\rangle = x^T B y, \quad B = \begin{bmatrix}2 & 1 \\ 1 & 3\end{bmatrix}$$

4 Ex4:

a,

$$U = \begin{bmatrix}0 & 1 & -3 & -1 \\ -1 & -3 & 4 & -3 \\ 2 & 1 & 1 & 5 \\ 0 & -1 & 2 & 0 \\ 2 & 2 & 1 & 7\end{bmatrix}$$

$$= \begin{bmatrix}6 & 0 & 0 & -1 & 9 \\ 0 & 6 & 0 & 2 & -6 \\ 0 & 0 & 6 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$=> rank(U) = 3$$

$$=> basic(U) = B = \begin{bmatrix}0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1\end{bmatrix}$$

We have:

$$B^TB = \begin{bmatrix} 9 & 9 & 0 \\ 9 & 16 & -14 \\ 0 & -14 & 31 \end{bmatrix}$$

$$B^Tx = \begin{bmatrix} 9 \\ 23 \\ -25 \end{bmatrix}$$

We also have:

$$\lambda = (B^TB)^{-1}Bx$$

$$\lambda = \begin{bmatrix} \frac{100}{63} & \frac{-31}{21} & \frac{-2}{3} \\ \frac{-31}{21} & \frac{31}{21} & \frac{2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 23 \\ -25 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$=> \pi_u(x) = B\lambda = \begin{bmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix}$$

b, We have:

$$d(x, U) = d(x, \pi_u(x) = \parallel x - \pi_u(x) \parallel$$

$$= \left\| \begin{vmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{vmatrix} - \begin{vmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{vmatrix} \right\| = \left\| \begin{matrix} -2 \\ -4 \\ 0 \\ 6 \\ -2 \end{matrix} \right\|$$

$$=> d(x, U) = \sqrt{60} = 2\sqrt{15}$$