TrachtPS4

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1 Problem Set 4

1.0.1 by Daniel Tracht, February 2019

1.1 Problem 1

For this problem, we use the 200 data points provided in the incomes.txt file, containing incomes reported in U.S. dollars. We will be using the log normal distribtion

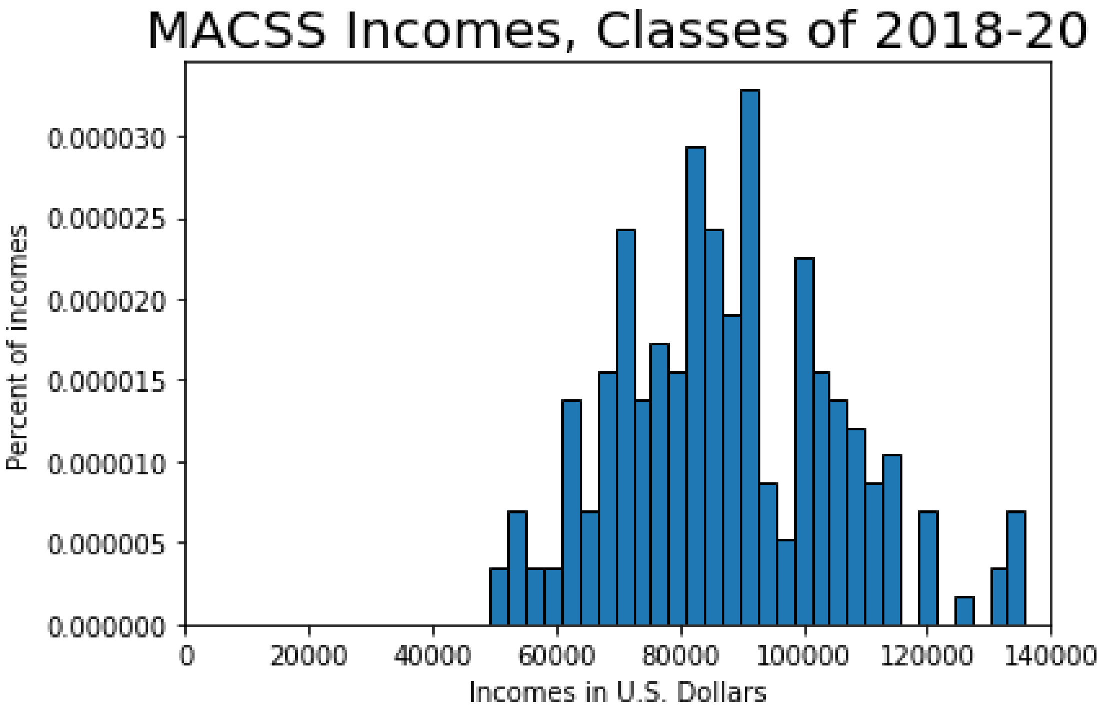
1.1.1 Part a

We wish to plot a histogram of the percentages from our data with 30 bins:

```
In [1]: # Import the necessary libraries
        import numpy as np
        import scipy.stats as sts
        import requests

        # Download and save the data file incomes.txt
        url = ('https://raw.githubusercontent.com/dtracht/persp-model-econ_W19/' +
               'master/ProblemSets/PS4/data/incomes.txt')
        data_file = requests.get(url, allow_redirects=True)
        open('data/incomes.txt', 'wb').write(data_file.content)

        # Load the data as a NumPy array
        pts = np.loadtxt('data/incomes.txt')
```

```
In [3]: import matplotlib.pyplot as plt

        num_bins = 30
        # normed option has been deprecated
        # using density=True instead
        count, bins, ignored = plt.hist(pts, num_bins, density=True,
                                        edgecolor='k')
        plt.title("MACSS Incomes, Classes of 2018-20", fontsize=20)
        plt.xlabel("Incomes in U.S. Dollars")
        plt.ylabel("Percent of incomes")
        plt.xlim([0,140000])
```

```
Out[3]: (0, 140000)
```



1.1.2 Part b

We wish to plot the probability density function of the lognormal distribution,

$$f(x|\mu = 11.0, \sigma = 0.5)$$

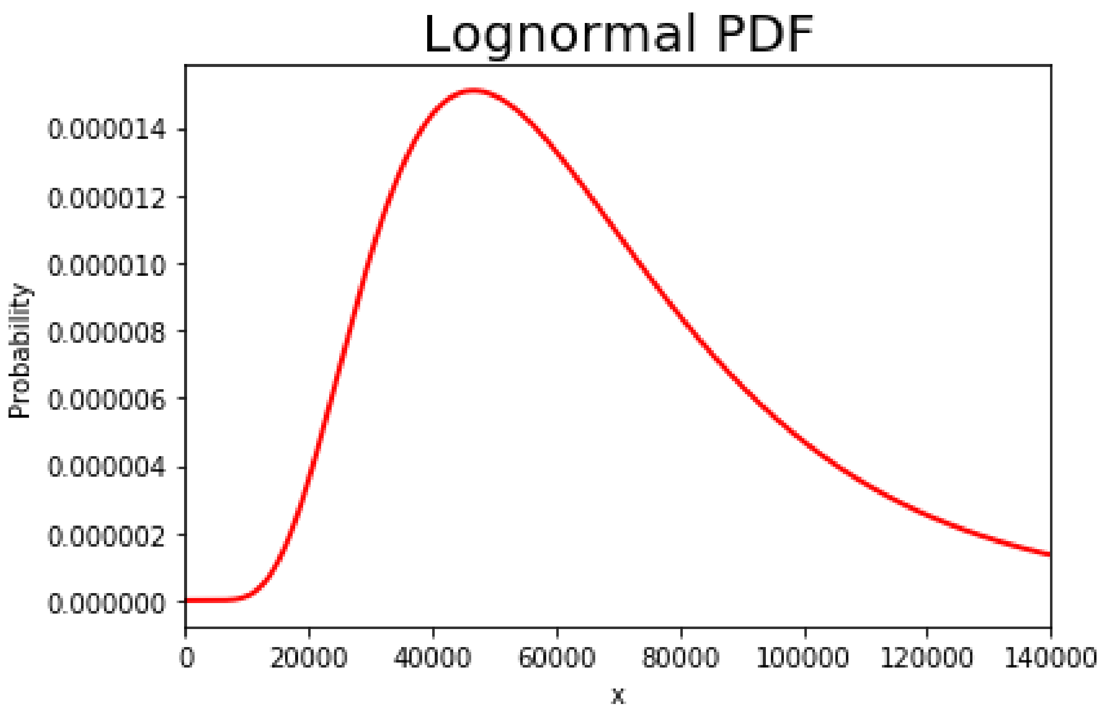
for

$$0 \leq x \leq 150000$$

:

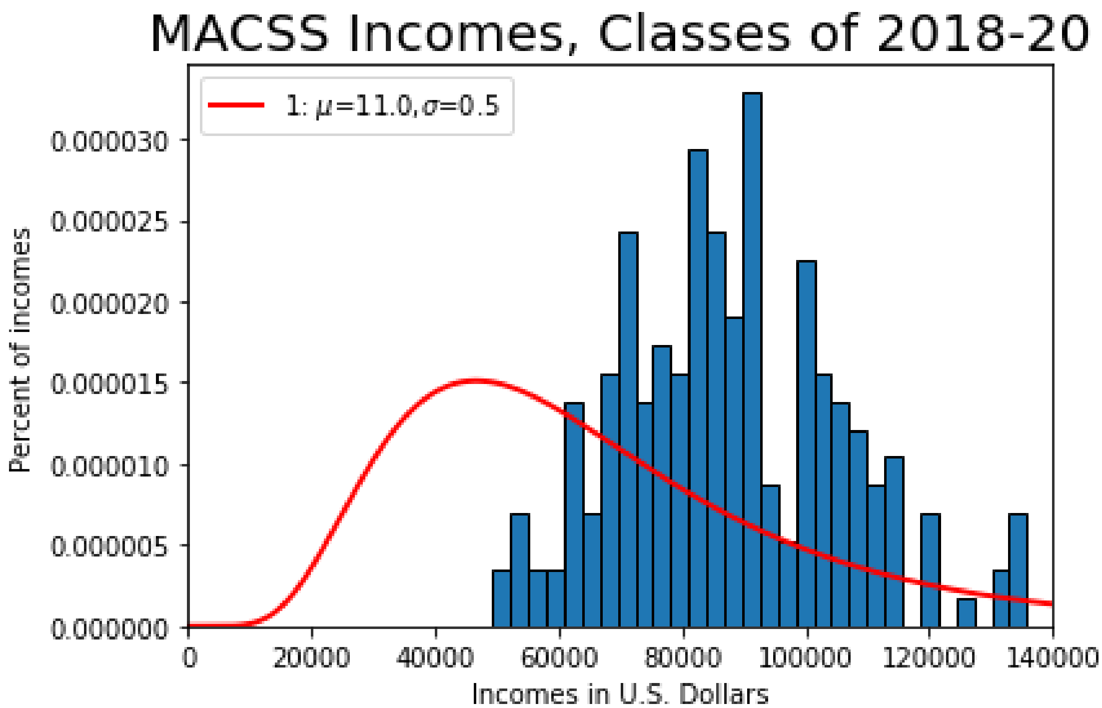
```
In [4]: # takes vector of x values, parameters for mu and sigma
        # returns vector of pdf values for the lognormal
        from scipy.stats import lognorm
        def lognormal_pdf (xvals, mu, sigma):
            pdf_vals = (1 / (xvals * sigma * np.sqrt(2 * np.pi))) * np.exp (-(np.log(xvals)-mu
            #pdf_vals = lognorm.pdf(xvals, sigma, loc=mu, scale=np.exp(mu))
            #take pdf values that are rounded to zero and reset them to very low
            pdf_zero = pdf_vals==0
            pdf_vals[pdf_zero] = 1e-16
            return pdf_vals
```

In [5]:
```
# cannot actually start at zero
dist_pts = np.linspace(1e-16, 150000, 1000)
mu_1 = 11
sig_1 = 0.5
plt.plot(dist_pts, lognormal_pdf(dist_pts, mu_1, sig_1),
         linewidth=2, color='r', label='1: $\mu$=11.0,$\sigma$=0.5')
plt.title("Lognormal PDF", fontsize=20)
plt.xlabel("x")
plt.ylabel("Probability")
plt.xlim([0,140000])
plt.show()
```



In [6]:
```
# Plot histogram with the PDF overlaid
num_bins = 30
# normed option has been deprecated
# using density=True instead
count, bins, ignored = plt.hist(pts, num_bins, density=True,
                                edgecolor='k')
plt.title("MACSS Incomes, Classes of 2018-20", fontsize=20)
plt.xlabel("Incomes in U.S. Dollars")
plt.ylabel("Percent of incomes")
plt.xlim([0,140000])

plt.plot(dist_pts, lognormal_pdf(dist_pts, mu_1, sig_1),
```

```
             linewidth=2, color='r', label='1: $\mu$=11.0,$\sigma$=0.5')
    plt.legend(loc='upper left')
    plt.show()
```



We wish to find the log likelihood value for this particular parameteriziation of this function form and the given data:

```
In [7]: # function that takes a vector of x values, and parameters mu and sigma
        # returns the log likelihood for that vector for the lognormal distribution
        def log_likelihood(xvals, mu, sigma):
            pdf_vals = lognormal_pdf(xvals, mu, sigma)
            log_pdf_vals = np.log(pdf_vals)
            log_likelihood = log_pdf_vals.sum()
            return log_likelihood
```

```
In [8]: print('Log-likelihood:', log_likelihood(pts, mu_1, sig_1))
```

```
Log-likelihood: -2385.856997808558
```

1.1.3 Part c

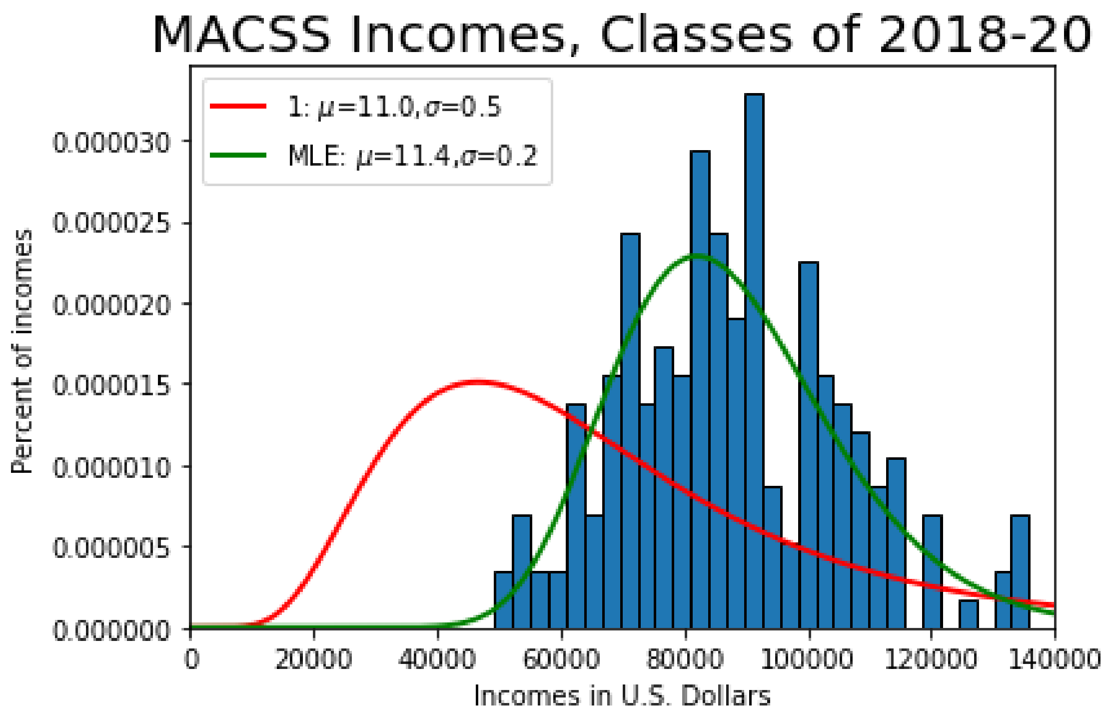
We wish to estimate the parameters for a log normal distribution to fit this data by the method of maxmimum likelihood estimation:

```
In [9]: # a critereon function to be passed to the minimizer
        def crit(params, *args):
            mu, sigma = params
            xvals, junk = args
            log_likelihood_value = log_likelihood(xvals, mu, sigma)
            neg_log_likelihood_value = -log_likelihood_value

            return neg_log_likelihood_value
```

```
In [10]: import scipy.optimize as opt

         # starting from our values in part b
         mu_init = 11
         sig_init = 0.5
         params_init = np.array([mu_init, sig_init])
         mle_args = (pts, 0)
         # constraining sigma to be positive
         bnds = ((None, None), (1e-16, None))
         results = opt.minimize(crit, params_init, args=(mle_args), method='SLSQP', bounds=bnds
         mu_MLE, sig_MLE = results.x
```

We wish to plot our new distribution as well as our original guess and the data:

```
In [11]: # Plot histogram with the old PDF and new PDF overlaid
         num_bins = 30
         # normed option has been deprecated
         # using density=True instead
         count, bins, ignored = plt.hist(pts, num_bins, density=True,
                                         edgecolor='k')
         plt.title("MACSS Incomes, Classes of 2018-20", fontsize=20)
         plt.xlabel("Incomes in U.S. Dollars")
         plt.ylabel("Percent of incomes")
         plt.xlim([0,140000])

         plt.plot(dist_pts, lognormal_pdf(dist_pts, mu_1, sig_1),
                  linewidth=2, color='r', label='1: $\mu$=11.0,$\sigma$=0.5')
         plt.legend(loc='upper left')

         plt.plot(dist_pts, lognormal_pdf(dist_pts, mu_MLE, sig_MLE),
                  linewidth=2, color='g', label='MLE: $\mu$=11.4,$\sigma$=0.2')
         plt.legend(loc='upper left')

         plt.show()
```



We wish to get the variance-covariance matrix for out estimates. However, using the contstrained method above, the optimizer does not return an inverse Hessian object. To get it, we can simply use a method that returns one beginning at the solution to the contstrained problem.

In [12]:
```
# starting from our values in the constrained case
mu_init = mu_MLE
sig_init = sig_MLE
params_init = np.array([mu_init, sig_init])
mle_args = (pts, 0)

results = opt.minimize(crit, params_init, args=(mle_args))
mu_MLE_2, sig_MLE_2 = results.x

print('mu_MLE=', mu_MLE_2, ' sig_MLE=', sig_MLE_2)
print('MLE Log-likelihood:', log_likelihood(pts, mu_MLE_2, sig_MLE_2))

vcv_mle = results.hess_inv

stderr_mu_mle = np.sqrt(vcv_mle[0,0])
stderr_sig_mle = np.sqrt(vcv_mle[1,1])
print('VCV(MLE) = ', vcv_mle)
print('Standard error for mu estimate = ', stderr_mu_mle)
print('Standard error for sigma estimate = ', stderr_sig_mle)
```

```
mu_MLE= 11.359022994120751  sig_MLE= 0.20817731910735596
MLE Log-likelihood: -2241.7193013573587
```

```
VCV(MLE) =  [[1.94308215e-04 1.27192004e-05]
 [1.27192004e-05 1.24567598e-04]]
Standard error for mu estimate =  0.013939448157069998
Standard error for sigma estimate =  0.01116098553849848
```

1.1.4 Part d

We wish to preform a likelihood ratio test to determine the probability that the data came from the distribution using the parameters from part b:

```
In [13]: # null hypothesis coming from the parameters in part b
         mu_new, sig_new = np.array([11, 0.5])
         log_lik_h0 = log_likelihood(pts, mu_new, sig_new)
         print('hypothesis value log likelihood', log_lik_h0)
         log_lik_mle = log_likelihood(pts, mu_MLE, sig_MLE)
         print('MLE log likelihood', log_lik_mle)
         LR_val = 2 * (log_lik_mle - log_lik_h0)
         print('likelihood ratio value', LR_val)
         pval_h0 = 1.0 - sts.chi2.cdf(LR_val, 2)
         print('chi squared of H0 with 2 degrees of freedom p-value = ', pval_h0)
```

```
hypothesis value log likelihood -2385.856997808558
MLE log likelihood -2241.7193013574033
likelihood ratio value 288.27539290230925
chi squared of H0 with 2 degrees of freedom p-value =  0.0
```

The probability of the data being observed under the assumption of the functional form and parameters in part b is so small it rounds to zero in floating point operation.

1.1.5 Part e

We wish to estimate the probability that a graduate of MACSS will earn more than 100,000 dollars and the probability that a graduate of MACSS will ear less than 75,000 dollars using the model and parameters we have estimated. For this we can go to the CDF of the log normal distribution

```
In [14]: prob_under75 = lognorm.cdf(75000, sig_MLE, loc=mu_MLE, scale=np.exp(mu_MLE))
         prob_over100 = 1-lognorm.cdf(100000, sig_MLE, loc=mu_MLE, scale=np.exp(mu_MLE))
         print("Probability of earning less than $75,000:", prob_under75)
         print("Probability of earning more than $100,000:", prob_over100)
```

```
Probability of earning less than $75,000: 0.2599983218978105
Probability of earning more than $100,000: 0.23003229599994834
```

1.2 Problem 2

In this problem we will be solving a linear regression using maximum likelihood estimation. Our model is

$$sick_i = \beta_0 + \beta_1 age_i + \beta_2 children_i + \beta_3 temp_w inter_i + \varepsilon_i$$

where

$$\varepsilon_i \sim N\left(0, \sigma^2\right)$$

1.2.1 Part a

We begin by estimating the parameters through maximum likelihood estimation:

```
In [15]: # Download and save the data file incomes.txt
         url = ('https://raw.githubusercontent.com/dtracht/persp-model-econ_W19/' +
                'master/ProblemSets/PS4/data/sick.txt')
         data_file = requests.get(url, allow_redirects=True)
         open('data/sick.txt', 'wb').write(data_file.content)

         # Load the data as a NumPy array
         pts = np.genfromtxt("data/sick.txt", delimiter=",", skip_header=1)

         y_pts = pts[:,0]
         x1_pts = pts[:,1]
         x2_pts = pts[:,2]
         x3_pts = pts[:,3]
```

```
In [16]: # takes vector of x values, sigma
         # returns vector of pdf values for the normal with mean 0
         from scipy.stats import norm
         def normal_pdf (xvals, sigma):
             # reset sigma to absolute value
             # get around negative issue here as opposed to solving and then getting the VCV m
             # don't see a way around this now, but it feels dirty
             sigma = abs(sigma)
             pdf_vals = (1 / np.sqrt(2 * np.pi * sigma**2)) * np.exp(-(xvals) ** 2 / (2 * sigma
             #pdf_vals = norm.pdf(xvals, loc=0, scale=sigma)
             # take values rounded to zero and reset to very low
             pdf_zero = pdf_vals == 0
             pdf_vals[pdf_zero] = 1e-16
             return pdf_vals
```

```
In [17]: # function that takes an array of data, vector of betas, and sigma parameter
         # returns the log likelihood for that vector for the normal distribution with mean 0
         def log_likelihood_normal(beta_0, beta_1, beta_2, beta_3, sigma, y_vals, x1_vals, x2_
             # calculates epsilon values for beta vector and data
             epsilon_vals = y_vals - beta_0 - beta_1*x1_vals - beta_2*x2_vals - beta_3*x3_vals
             pdf_vals = normal_pdf(epsilon_vals, sigma)
             log_pdf_vals = np.log(pdf_vals)
```

```
        log_likelihood = log_pdf_vals.sum()
        return log_likelihood
```

```
In [18]: # a critereon function to be passed to the minimizer
         def crit_normal(params, *args):
             beta_0, beta_1, beta_2, beta_3, sigma = params
             y_vals, x1_vals, x2_vals, x3_vals = args
             log_likelihood_value = log_likelihood_normal(beta_0, beta_1, beta_2, beta_3, sigma
             neg_log_likelihood_value = -log_likelihood_value
             return neg_log_likelihood_value
```

```
In [19]: import scipy.optimize as opt

         # starting from visual inspection
         beta_0_init = 0
         beta_1_init = 0
         beta_2_init = 0
         beta_3_init = 0
         sigma_init = 1
         params_init = np.array([beta_0_init, beta_1_init, beta_2_init, beta_3_init, sigma_ini
         mle_args = (y_pts, x1_pts, x2_pts, x3_pts)

         # constraining sigma to be positive
         #bnds = ((None, None), (None, None), (None, None), (None, None), (1e-16, None))
         #results = opt.minimize(crit_normal, params_init, args=(mle_args), method='L-BFGS-B',
         results = opt.minimize(crit_normal, params_init, args=(mle_args))
         beta_0_MLE, beta_1_MLE, beta_2_MLE, beta_3_MLE, sigma_MLE = results.x

         print('beta_0_MLE=', beta_0_MLE)
         print('beta_1_MLE=', beta_1_MLE)
         print("beta_2_MLE=", beta_2_MLE)
         print("beta_3_MLE=", beta_3_MLE)
         print("sigma_MLE=", sigma_MLE)
         print('MLE Log-likelihood:', log_likelihood_normal(beta_0_MLE, beta_1_MLE, beta_2_MLE

         vcv_mle = results.hess_inv
         print('VCV(MLE) = ', vcv_mle)
```

```
beta_0_MLE= 0.25164638362713215
beta_1_MLE= 0.012933350045131211
beta_2_MLE= 0.400502048301034
beta_3_MLE= -0.009991673036254978
sigma_MLE= 0.003017682175869847
MLE Log-likelihood: 876.8650462887678
VCV(MLE) =  [[ 9.21562214e-08 -6.95529550e-10 -5.02282376e-09 -9.62673307e-10
  -2.62646033e-08]
 [-6.95529550e-10  6.61646149e-10 -2.29838661e-09 -4.87517749e-10
   1.64799761e-09]
```

```
 [-5.02282376e-09 -2.29838661e-09  2.31900554e-08  1.25339295e-09
  -1.30889427e-08]
 [-9.62673307e-10 -4.87517749e-10  1.25339295e-09  4.29373073e-10
  -4.75934351e-10]
 [-2.62646033e-08  1.64799761e-09 -1.30889427e-08 -4.75934351e-10
   1.66995820e-08]]
```

```
In [ ]: '''
        # starting from our values in the constrained case
        beta_0_init_2 = beta_0_MLE
        beta_1_init_2 = beta_1_MLE
        beta_2_init_2 = beta_2_MLE
        beta_3_init_2 = beta_3_MLE
        sigma_init_2 = sigma_MLE

        params_init_2 = np.array([beta_0_init_2, beta_1_init_2, beta_2_init_2, beta_3_init_2, 
        print(params_init_2)
        mle_args = (y_pts, x1_pts, x2_pts, x3_pts)

        results = opt.minimize(crit_normal, params_init_2, args=(mle_args))
        beta_0_MLE_2, beta_1_MLE_2, beta_2_MLE_2, beta_3_MLE_2, sigma_MLE_2 = results.x

        print('beta_0_MLE=', beta_0_MLE_2)
        print('beta_1_MLE=', beta_1_MLE_2)
        print("beta_2_MLE=", beta_2_MLE_2)
        print("beta_3_MLE=", beta_3_MLE_2)
        print("sigma_MLE=", sigma_MLE_2)
        print('MLE Log-likelihood:', log_likelihood_normal(beta_0_MLE_2, beta_1_MLE_2, beta_2_

        vcv_mle = results.hess_inv
        print('VCV(MLE) = ', vcv_mle)
        '''
```

1.2.2 Part b

We wish to use a likelihood ratio test to determine the probability that $\beta_0 = 1.0$, $\sigma^2 = 0.01$, and $\beta_1 = \beta_2 = \beta_3 = 0$:

```
In [20]: beta_0_null = 1
         beta_1_null = 0
         beta_2_null = 0
         beta_3_null = 0
         sigma_null = 0.01
         log_lik_h0 = log_likelihood_normal(beta_0_null, beta_1_null, beta_2_null, beta_3_null
         print('hypothesis value log likelihood', log_lik_h0)
         log_lik_mle = log_likelihood_normal(beta_0_MLE, beta_1_MLE, beta_2_MLE, beta_3_MLE, si
         print('MLE log likelihood', log_lik_mle)
```

```
        LR_val = 2 * (log_lik_mle - log_lik_h0)
        print('likelihood ratio value', LR_val)
        pval_h0 = 1.0 - sts.chi2.cdf(LR_val, 2)
        print('chi squared of H0 with 2 degrees of freedom p-value = ', pval_h0)
```

```
hypothesis value log likelihood -30663.099086401762
MLE log likelihood 876.8650462887678
likelihood ratio value 63079.92826538106
chi squared of H0 with 2 degrees of freedom p-value =  0.0
```

We strongly reject the hypothesis that the data was generated with a true process of the given function and parameters.